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On simple Filippov superalgebras of type $B(0,n)$, II

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Abstract. It is proved that there exist no simple finite-dimensional Filippov superalgebras of type $B(0,n)$ over an algebraically closed field of characteristic 0.

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1. Introduction

The notion of *n*-Lie superalgebra was presented by Daletskii and Kushnirevich in [1] as a natural generalization of a notion of *n*-Lie algebra introduced by Filippov in 1985 (cf. [2]). Following [3] and [7], we use the terms Filippov superalgebra and Filippov algebra instead of *n*-Lie superalgebra and *n*-Lie algebra, respectively. Filippov algebras were also known before under the names of Nambu Lie algebras and Nambu algebras. We may also remark that Filippov algebras are a particular case of *n*-ary Malcev algebras (see, for example, [10]).

This work is one of the first steps on the way of the classification of finite-dimensional simple Filippov superalgebras over an algebraically closed field of characteristic 0. In [8], finite-dimensional commutative *n*-ary Leibniz algebras over a field of characteristic 0 were studied by the first author. There it was shown that there exist no simple ones. The finite-dimensional simple Filippov algebras over an algebraically closed field of characteristic 0 were classified earlier by Wuxue in [11]. Notice that an *n*-ary Leibniz algebra is exactly a Filippov superalgebra with trivial even part, and a Filippov algebra is exactly a Filippov

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superalgebra with trivial odd part. Bearing in mind these facts, in this article we consider the n-ary Filippov superalgebras with $n \geq 3$ and with nonzero even and odd parts. In [9], it was proved that there are no simple finite-dimensional Filippov superalgebras with multiplication Lie superalgebra isomorphic to $B(0,n)$ under the assumption that a generator of a module over $B(0,n)$ is even. The case of odd generator requires techniques different from the one that was used in the even case. In the present work we eliminate the assumption for the generator to be even, and prove a theorem (analogous to the main theorem of [9]) for the general case.

We start recalling some definitions. An Ω-*algebra* over a field k is a linear space over k equipped with a system of multilinear algebraic operations $\Omega = \{\omega_i \mid |\omega_i| = n_i \in \mathbb{N}, i \in I\}$, where $|\omega_i|$ denotes the arity of ω_i.

An n-*ary Leibniz algebra* over a field k is an Ω-algebra L over k with one n-ary operation $(x_1, \ldots, x_n)$ satisfying the identity

$$\big((x_1, \ldots, x_n), y_2, \ldots, y_n\big) = \sum_{i=1}^{n} \big(x_1, \ldots, (x_i, y_2, \ldots, y_n), \ldots, x_n\big).$$

If this operation is anticommutative, we obtain a definition of Filippov (n-Lie) algebra over a field.

An n-*ary superalgebra* over a field k is a $\mathbb{Z}_2$-graded n-ary algebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ over k, that is, if $x_i \in L_{\alpha_i}$, $\alpha_i \in \mathbb{Z}_2$, then $(x_1, \ldots, x_n) \in L_{\alpha_1 + \cdots + \alpha_n}$. An n-*ary Filippov superalgebra* over k is an n-ary superalgebra $\mathscr{F} = \mathscr{F}_{\bar{0}} \oplus \mathscr{F}_{\bar{1}}$ over k with one n-ary operation $[x_1, \ldots, x_n]$ satisfying the identities

$$[x_1, \ldots, x_{i-1}, x_i, \ldots, x_n] = -(-1)^{p(x_{i-1})p(x_i)}[x_1, \ldots, x_i, x_{i-1}, \ldots, x_n], \tag{1}$$

$$[[x_1, \ldots, x_n], y_2, \ldots, y_n] = \sum_{i=1}^{n} (-1)^{p\bar{q}_i}[x_1, \ldots, [x_i, y_2, \ldots, y_n], \ldots, x_n], \tag{2}$$

where $p(x) = l$ means that $x \in \mathscr{F}_{\bar{l}}$, $p = \sum_{i=2}^{n} p(y_i)$, $\bar{q}_i = \sum_{j=i+1}^{n} p(x_j)$, $\bar{q}_n = 0$. The identities (1) and (2) are called the anticommutativity and the generalized Jacobi identity, respectively. By (1), we can rewrite (2) as

$$[y_2, \ldots, y_n, [x_1, \ldots, x_n]] = \sum_{i=1}^{n} (-1)^{pq_i}[x_1, \ldots, [y_2, \ldots, y_n, x_i], \ldots, x_n], \tag{3}$$

where $q_i = \sum_{j=1}^{i-1} p(x_j)$, $q_1 = 0$. (Sometimes instead of using the long term "n-ary superalgebra" we simply say for short "superalgebra".) If we denote by $L_x = L(x_1, \ldots, x_{n-1})$ the operator of left multiplication $L_x y = [x_1, \ldots, x_{n-1}, y]$, then, by (3), we get

$$[L_y, L_x] = \sum_{i=1}^{n-1} (-1)^{pq_i} L(x_1, \dots, L_y x_i, \dots, x_{n-1}), \tag{4}$$

where L_y is an operator of left multiplication and p is its parity. (Here and afterwards, we denote by $[\,,]$ the supercommutator.)

Let $L = L_{\bar{0}} \oplus L_{\bar{1}}$ be an n-ary anticommutative superalgebra. A subalgebra $B = B_{\bar{0}} \oplus B_{\bar{1}}$ of the superalgebra L, $B_{\bar{i}} \subseteq L_{\bar{i}}$, is a $\mathbb{Z}_2$-graded vector subspace of L which is a superalgebra. A subalgebra I of L is called an *ideal* if $[I, L, \dots, L] \subseteq I$. The subalgebra (in fact, an ideal) $L^{(1)} = [L, \dots, L]$ of L is called the *derived algebra* of L. Put $L^{(i)} = [L^{(i-1)}, \dots, L^{(i-1)}]$, $i \in \mathbb{N}$, $i > 1$. The superalgebra L is called *solvable* if $L^{(k)} = 0$ for some k. Denote by $R(L)$ the maximal solvable ideal of L (if exists). If $R(L) = 0$, then the superalgebra L is called *semisimple*. The superalgebra L is called *simple* if $L^{(1)} \neq 0$ and L lacks ideals other than 0 or L.

The article is organized as follows. In the second section, we recall how to reduce the classification problem of the simple Filippov superalgebras to some question about Lie superalgebras, using the same ideas as in [11]. We reduce this question to an existence problem for some skew-symmetric homomorphisms of semisimple Lie superalgebras and their faithful irreducible modules.

In the last section, we restrict our attention to the case of the Lie superalgebra $B(0,n)$ (and an odd generator of a module over $B(0,n)$) and solve the existence problem of these skew-symmetric homomorphisms in this case. It turns out that the required homomorphisms do not exist. Therefore, there are no simple Filippov superalgebras of type $B(0,n)$ over an algebraically closed field of characteristic 0, as stated in the main result of this article (Theorem 3.1).

In what follows, by Φ we denote an algebraically closed field of characteristic 0, by F a field of characteristic 0, by k a field and by $\langle w_\upsilon; \upsilon \in \Upsilon \rangle$ a linear space over a field (the field is clear from the context) generated by the family of vectors $\{w_\upsilon; \upsilon \in \Upsilon\}$.

2. Reduction to Lie superalgebras

Let $\mathscr{F}$ be a Filippov superalgebra over k. Denote by $\mathscr{F}^*\big(L(\mathscr{F})\big)$ the associative (Lie) superalgebra generated by the operators $L(x_1, \dots, x_{n-1})$, $x_i \in \mathscr{F}$. The algebra $L(\mathscr{F})$ is called *the algebra of multiplications* of $\mathscr{F}$.

Lemma 2.1 ([9]). *Given $\mathscr{F} = \mathscr{F}_{\bar{0}} \oplus \mathscr{F}_{\bar{1}}$ a simple finite-dimensional Filippov superalgebra over a field of characteristic 0 with $\mathscr{F}_{\bar{1}} \neq 0$, the algebra $L = L(\mathscr{F}) = L_{\bar{0}} \oplus L_{\bar{1}}$ has nontrivial even and odd parts.*

Theorem 2.1 ([9]). *If $\mathscr{F}$ is a simple finite-dimensional Filippov superalgebra over a field of characteristic 0, then $L = L(\mathscr{F})$ is a semisimple Lie superalgebra.*

Given an n-ary superalgebra A with a multiplication $(\cdot, \dots, \cdot)$, we have $\mathrm{End}(A) = \mathrm{End}_{\bar{0}} A \oplus \mathrm{End}_{\bar{1}} A$. The element $D \in \mathrm{End}_{\bar{s}} A$ is called a *derivation* of degree s of A if, for every $a_1, \dots, a_n \in A$, $p(a_i) = p_i$, the following equality holds:

$$D(a_1, \dots, a_n) = \sum_{i=1}^{n} (-1)^{sq_i}(a_1, \dots, Da_i, \dots, a_n),$$

where $q_i = \sum_{j=1}^{i-1} p_j$. We denote by $\mathrm{Der}_{\bar{s}} A \subset \mathrm{End}_{\bar{s}} A$ the subspace of all derivations of degree s and set $\mathrm{Der}(A) = \mathrm{Der}_{\bar{0}} A \oplus \mathrm{Der}_{\bar{1}} A$. The subspace $\mathrm{Der}(A) \subseteq \mathrm{End}(A)$ is easily seen to be closed under the bracket

$$[a, b] = ab - (-1)^{\deg(a)\deg(b)} ba,$$

(known as the *supercommutator*) and is called the *superalgebra of derivations* of A.

Fix elements $x_1, \dots, x_{n-1} \in A$, $i \in \{1, \dots, n\}$, and define a transformation $\mathrm{ad}_i(x_1, \dots, x_{n-1}) \in \mathrm{End}(A)$ by the rule

$$\mathrm{ad}_i(x_1, \dots, x_{n-1})x = (-1)^{pq_i}(x_1, \dots, x_{i-1}, x, x_i, \dots, x_{n-1}), \tag{5}$$

where $p = p(x)$, $p_i = p(x_i)$, $q_i = \sum_{j=i+1}^{n-1} p_j$.

If the transformations $\mathrm{ad}_i(x_1, \dots, x_{n-1}) \in \mathrm{End}(A)$ are derivations of A for all $i = 1, \dots, n$ and $x_1, \dots, x_{n-1} \in A$, then we call them *strictly inner derivations* and A an *inner-derivation superalgebra* ($\mathscr{ID}$*-superalgebra*). Notice that the n-ary Filippov superalgebras and the n-ary commutative Leibniz algebras are examples of $\mathscr{ID}$-superalgebras.

Now let us denote by $\mathrm{Inder}(A)$ the linear space spanned by the strictly inner derivations of A. If A is an n-ary $\mathscr{ID}$-superalgebra, then it is easy to see that $\mathrm{Inder}(A)$ is an ideal of $\mathrm{Der}(A)$.

Lemma 2.2. *Given a simple $\mathscr{ID}$-superalgebra A over k, the Lie superalgebra* $\mathrm{Inder}(A)$ *acts faithfully and irreducibly on A.*

Let $\mathscr{F}$ be an n-ary Filippov superalgebra over k. We point out that the map $\mathrm{ad} := \mathrm{ad}_n : \bigotimes^{n-1} \mathscr{F} \mapsto \mathrm{Inder}(\mathscr{F})$ satisfies

$$[D, \mathrm{ad}(x_1, \dots, x_{n-1})] = \sum_{i=1}^{n-1} (-1)^{pq_i} \mathrm{ad}(x_1, \dots, x_{i-1}, Dx_i, x_{i+1}, \dots, x_{n-1})$$

for all $D \in \mathrm{Inder}(\mathscr{F})$, and the associated map

$$(x_1, \dots, x_n) \mapsto \mathrm{ad}(x_1, \dots, x_{n-1})x_n,$$

from $\bigotimes^n \mathscr{F}$ to $\mathscr{F}$ is $\mathbb{Z}_2$-skew-symmetric. If we consider $\mathscr{F}$ as an $\mathrm{Inder}(\mathscr{F})$-module then ad induces an $\mathrm{Inder}(\mathscr{F})$-module morphism from the $(n-1)$-th exterior power $\bigwedge^{n-1} \mathscr{F}$ to $\mathrm{Inder}(\mathscr{F})$ (which we also denote by ad) such that the map $(x_1, \dots, x_n) \mapsto \mathrm{ad}(x_1, \dots, x_{n-1})x_n$ is $\mathbb{Z}_2$-skew-symmetric. (Note that in $\bigwedge^{n-1} \mathscr{F}$ we have: $x_1 \wedge \cdots \wedge x_i \wedge x_{i+1} \wedge \cdots \wedge x_{n-1} = -(-1)^{p_i p_{i+1}} x_1 \wedge \cdots \wedge x_{i+1} \wedge x_i \wedge \cdots \wedge x_{n-1}$.) Conversely, if L is a Lie superalgebra, V is an L-module, and ad is an L-module morphism from $\bigwedge^{n-1} V \mapsto L$ such that the map $(v_1, \dots, v_n) \mapsto \mathrm{ad}(v_1 \wedge \cdots \wedge v_{n-1})v_n$ from $\bigotimes^n V$ to V is $\mathbb{Z}_2$-skew-symmetric (we call the homomorphisms of this type *skew-symmetric*), then V becomes an n-ary Filippov superalgebra by putting

$$[v_1, \dots, v_n] = \mathrm{ad}(v_1 \wedge \cdots \wedge v_{n-1})v_n.$$

Therefore, we have a correspondence between the set of n-ary Filippov superalgebras and the set of the triples (L, V, ad), satisfying the conditions above.

We shall assume that all vector spaces appearing in the following section are finite-dimensional over F.

If $\mathscr{F}$ is a simple n-ary Filippov superalgebra, then Theorem 2.1 establishes that the Lie superalgebra $\mathrm{Inder}(\mathscr{F})$ is semisimple, and $\mathscr{F}$ is a faithful and irreducible $\mathrm{Inder}(\mathscr{F})$-module. Moreover, the $\mathrm{Inder}(\mathscr{F})$-module morphism $\mathrm{ad} : \bigwedge^{n-1} \mathscr{F} \mapsto \mathrm{Inder}(\mathscr{F})$ is surjective.

Conversely, if (L, V, ad) is a triple such that L is a semisimple Lie superalgebra over F, V is a faithful irreducible L-module, ad is a surjective L-module morphism from $\bigwedge^{n-1} V$ onto the adjoint module L and the map $(v_1, \dots, v_n) \mapsto \mathrm{ad}(v_1 \wedge \cdots \wedge v_{n-1})v_n$ from $\bigotimes^n V$ to V is $\mathbb{Z}_2$-skew-symmetric, then the corresponding n-ary Filippov superalgebra is simple. A triple with these conditions will be called a *good triple*. Thus, the problem of determining the simple n-ary Filippov superalgebras over F can be translated to that of finding the good triples.

3. Lie superalgebra $B(0,n)$

In this section, we recall some notations and results from [5], [6] on the Lie superalgebra $B(0,n)$ (and its irreducible faithful finite-dimensional representations) and give some explicit constructions which shall be used later on. Then we apply these results to the study of the simple n-ary Filippov superalgebras of type $B(0,n)$. Let us start recalling the definition of an induced module.

Let $\mathscr{L}$ be a Lie superalgebra, $U(\mathscr{L})$ its universal enveloping superalgebra [5], H a subalgebra of $\mathscr{L}$, and V an H-module. The module V can be extended to $U(H)$-module. We consider the $\mathbb{Z}_2$-graded space $U(\mathscr{L}) \otimes_{U(H)} V$, the quotient space of $U(\mathscr{L}) \otimes V$ by the linear span of the elements of the form $gh \otimes v - g \otimes h(v)$, $g \in U(\mathscr{L})$, $h \in U(H)$. This space can be endowed with a structure of a $\mathscr{L}$-module as follows: $g(u \otimes v) = gu \otimes v$, $g \in \mathscr{L}$, $u \in U(\mathscr{L})$, $v \in V$. The so constructed $\mathscr{L}$-module is said to be *induced from the H-module V* and is denoted be $\operatorname{Ind}_H^{\mathscr{L}} V$.

From now on we denote by G a contragredient Lie superalgebra over Φ and consider it with the "standard" $\mathbb{Z}$-grading (cf. [5], Sections 5.2.3 and 2.5.7).

Let $G = \bigoplus_{i \geq -d} G_i$. Set $H = (G_0)_{\bar{0}} = \langle h_1, \dots, h_r \rangle$, $N^+ = \bigoplus_{i>0} G_i$ and $B = H \oplus N^+$. Let $\Lambda \in H^*$, $\Lambda(h_i) = a_i \in \Phi$, and let $\langle v_\Lambda \rangle$ be an one-dimensional B-module such that $N^+(v_\Lambda) = 0$, $h_i(v_\Lambda) = a_i v_\Lambda$. Let $V_\Lambda = \operatorname{Ind}_B^G \langle v_\Lambda \rangle / I_\Lambda$, where I_Λ is the (unique) maximal submodule of the G-module $\operatorname{Ind}_B^G \langle v_\Lambda \rangle$. Then Λ is called the *highest weight* of the G-module V_Λ. The numbers a_i are called the *numerical marks* of Λ. By [5], every faithful irreducible finite-dimensional G-module may be obtained this way. Note that now we suppose that $1 \otimes v \in V_{\bar{1}}$, which provides a $\mathbb{Z}_2$-graded structure of V.

Lemma 3.1. *Let V be a module over a Lie superalgebra G, let $V = \bigoplus V_{\gamma_i}$ be its weight decomposition, and let ϕ be a homomorphism from $\bigwedge^m V$ into G. Then, for every $v_i \in V_{\gamma_i}$,*

$$\begin{aligned} &\phi(v_1, \dots, v_m) \in G_{\gamma_1 + \dots + \gamma_m}, \qquad \textit{if } \gamma_1 + \dots + \gamma_m \textit{ is a root of } G, \\ &\phi(v_1, \dots, v_m) = 0, \qquad \textit{otherwise.} \end{aligned}$$

Proof. We only have to consider the action of an element h of a Cartan subalgebra of G on $\phi(v_1, \dots, v_m)$. □

Consider the algebra $G = B(0,1)$. It consists of the matrices of type

$$\left(\begin{array}{c|cc} 0 & x & y \\ \hline y & z & u \\ -x & v & -z \end{array} \right).$$

Choose the classical basis of $G_{\bar{0}} : \{h = e_{22} - e_{33}, g_{-2\delta} = e_{32}, g_{2\delta} = e_{23}\}$, and of $G_{\bar{1}} : \{g_{-\delta} = e_{12} - e_{31}, g_\delta = e_{13} + e_{21}\}$. Here $H = \langle h \rangle$ is a Cartan subalgebra of G, and $\delta \in H^*$ is such that $\delta(h) = 1$. We have

$$G = \langle g_{-2\delta} \rangle \oplus \langle g_{-\delta} \rangle \oplus \langle h \rangle \oplus \langle g_\delta \rangle \oplus \langle g_{2\delta} \rangle = \sum_{i=-2}^{2} G_i.$$

This gives the canonical $\mathbb{Z}$-grading of G. Therefore,

$$\begin{aligned} B &= \langle h, g_\delta, g_{2\delta} \rangle, \\ U(B) &= \langle h^{k_1} g_{2\delta}^{k_2} g_\delta^{\varepsilon}; k_i \in \mathbb{N}_0, \varepsilon \in \{0,1\} \rangle, \\ U(G) &= \langle h^{k_1} g_{2\delta}^{k_2} g_{-2\delta}^{k_3} g_\delta^{\varepsilon_1} g_{-\delta}^{\varepsilon_2}; k_i \in \mathbb{N}_0, \varepsilon_i \in \{0,1\} \rangle. \end{aligned}$$

Note some relations in the universal enveloping algebra $U(G)$:

$$g_{2\delta}g_{-2\delta} = g_{-2\delta}g_{2\delta} + h, \qquad g_{\delta}g_{-2\delta} = g_{-2\delta}g_{\delta} + g_{-\delta}, \qquad g_{-\delta}g_{-2\delta} = g_{-2\delta}g_{-\delta},$$
$$g_{2\delta}g_{-\delta} = g_{-\delta}g_{2\delta} - g_{\delta}, \qquad g_{\delta}g_{-\delta} + g_{-\delta}g_{\delta} = h, \qquad g_{-\delta}g_{-\delta} = -g_{-2\delta}.$$

Let $\Lambda(h) = a \in \Phi$ and $U_\Lambda = \mathrm{Ind}_B^G\langle v_\Lambda\rangle$. Set $v = v_\Lambda$. It is clear that U_Λ has the following basis: $\{v_k = g_{-2\delta}^k \otimes v, w_m = g_{-2\delta}^m g_{-\delta} \otimes v; k, m \in \mathbb{N}_0\}$. Using the relations in $U(G)$, we obtain the following action of the basis elements of G on U_Λ:

$$\begin{aligned} hv_k &= (a-2k)v_k, & hw_k &= (a-2k-1)w_k, \\ g_{2\delta}v_k &= k(a-k+1)v_{k-1}, & g_{2\delta}w_k &= k(a-k)w_{k-1}, \\ g_{-2\delta}v_k &= v_{k+1}, & g_{-2\delta}w_k &= w_{k+1}, \\ g_{\delta}v_k &= kw_{k-1}, & g_{\delta}w_k &= (a-k)v_k, \\ g_{-\delta}v_k &= w_k, & g_{-\delta}w_k &= -v_{k+1}. \end{aligned}$$

One can see that U_Λ has a finite-dimensional quotient module if and only if $a = k-1$ for some $k \in \mathbb{N}$. In this case, $I_\Lambda = \{v_j, w_i; j \geq k, i \geq k-1\}$ and $\dim V_\Lambda = U_\Lambda / I_\Lambda = 2k-1$.

Definition 3.1. Given a Lie superalgebra G, we say that a Filippov superalgebra $\mathscr{F}$ has *type G* if $\mathrm{Inder}(\mathscr{F}) \cong G$.

Lemma 3.2. *There are no simple finite-dimensional Filippov superalgebras of type* $B(0,1)$ *over* Φ.

Proof. Assume the contrary. Let $\mathscr{F}$ be a simple $(n+1)$-ary finite-dimensional Filippov superalgebra of type $B(0,1)$ over Φ. Let $G = B(0,1)$ and $V = V_\Lambda = V(k)$ be a faithful irreducible G-module with the highest weight Λ, $\Lambda(h) = a$, $a = k-1 \in \mathbb{N}_0$. Then $k \neq 1$ (i.e., $a \neq 0$), since otherwise $\dim V = 1$ and $\mathscr{F}$ is either a Filippov algebra or an n-ary Leibniz algebra. Since ϕ is surjective, there are $u_i \in V_{\gamma_i}$ such that $\phi(u_1 \wedge \cdots \wedge u_n) = h$ (in what follows, we denote $\phi(u_1 \wedge \cdots \wedge u_n)$ by $\phi(u_1, \ldots, u_n)$). Then

$$\phi(u_1, \ldots, u_n)v_0 = hv_0 = av_0.$$

Since ϕ is skew-symmetric, we have $|-\gamma_i + a| \leq 2$ for every i, i.e., $|-\gamma_i + k - 1| \leq 2$. Therefore, we have either $k = 2$ or $k = 3$.

If $k = 2$ then $a = 1$ and $V = \langle v_0\rangle \oplus \langle w_0\rangle \oplus \langle v_1\rangle = V_1 \oplus V_0 \oplus V_{-1}$. Then there are $u_i \in V_{\gamma_i}$ such that $\phi(u_1, \ldots, u_n) = g_\delta$. By [9], we may assume that $1 \otimes v$ is odd. Since the action of g_δ on $g_{-\delta} \otimes v$ provides a nonzero element and $g_{-\delta} \otimes v$ is even, it follows that $u_i \neq g_{-\delta} \otimes v$ for $i = 1, \ldots, n$. Thus we have $n = 2k+1$, $k \geq 1$, and

$$A := \phi(1 \otimes v, \underline{1 \otimes v, g_{-2\delta} \otimes v}_k) = \alpha g_\delta$$

for some $0 \neq \alpha \in \Phi$ (where $\underline{u, v}_k$ means that the elements u and v are k-times repeating: $\underbrace{u, v, \dots, u, v}_{2k}$, and we omit the index k when its value is clear from the context).

Multiplying the latter equality by $g_{-\delta}$, we have

$$(k+1)\phi(g_{-\delta} \otimes v, \underline{1 \otimes v, g_{-2\delta} \otimes v}_k) = \alpha h.$$

Repeating this procedure with g_δ, we come to $(k+1)A = -\alpha g_\delta$ and $A = 0$, which is a contradiction.

If $k = 3$ then $a = 2$, $\gamma_i = 0$ for all i, and

$$V = \langle v_0 \rangle \oplus \langle w_0 \rangle \oplus \langle v_1 \rangle \oplus \langle w_1 \rangle \oplus \langle v_2 \rangle = V_2 \oplus V_1 \oplus V_0 \oplus V_{-1} \oplus V_{-2}.$$

Therefore, $u_i = v_1$ and $\phi(v_1, \dots, v_1) = \alpha h$ for some $0 \neq \alpha \in \Phi$. Multiplying this equality twice by g_δ, we obtain that $n\phi(w_0, v_1, \dots, v_1) = -\alpha g_\delta$ and $n\phi(v_0, v_1, \dots, v_1) = -\alpha g_{2\delta}$. Acting with both sides of $\phi(v_1, \dots, v_1) = \alpha h$ on v_0 and of $n\phi(v_0, v_1, \dots, v_1) = -\alpha g_{2\delta}$ on v_1, we come to

$$[v_1, \dots, v_1, v_0] = 2\alpha v_0 \quad \text{and} \quad n[v_0, v_1, \dots, v_1] = -2\alpha v_0.$$

Therefore, $n = -1$, which gives again a contradiction. □

Let G be a contragredient Lie superalgebra of rank n, $U = \mathrm{Ind}_B^G \langle v_\Lambda \rangle$, and $V = V_\Lambda = U/N$ be a finite-dimensional representation of G, where $N = I_\Lambda$ is a maximal proper submodule of the G-module U. Let $G = \bigoplus_\alpha G_\alpha$ be a root decomposition of G relative to a Cartan subalgebra H. Denote by $\mathscr{A}$ the following set of roots: $\mathscr{A} = \{\alpha; g_\alpha \notin B\}$.

Lemma 3.3. *Let $g_\alpha \in G_\alpha$ and $g_\alpha \otimes v \neq 0$ $(v = v_\Lambda)$. Then*

$$g_\alpha^j \otimes v \in U_{\sum_{i=1}^n (j\alpha(h_i) + \Lambda(h_i))\delta_i}$$

for all $j \in \mathbb{N}$, and there exists a minimal positive integer $k \in \mathbb{N}$ such that $g_\alpha^k \otimes v \in N$ and the set $\mathscr{E}_{\alpha,k} = \{1 \otimes v, g_\alpha \otimes v, \dots, g_\alpha^{k-1} \otimes v\}$ is linearly independent in V. Moreover, setting $h = [g_{-\alpha}, g_\alpha]$, we have:

1) $\Lambda(h) = -\frac{(k-1)\alpha(h)}{2}$ *if either $g_\alpha \in G_{\bar{0}}$ or k odd;*
2) $\alpha(h) = 0$ *if $g_\alpha \in G_{\bar{1}}$ and k even.*

Proof. Using induction, the first inclusion is clear. Suppose that there is no $k \in \mathbb{N}$ with these properties. Construct a basis of V starting with the elements $1 \otimes v$, $g_\alpha \otimes v$, $g_\alpha^2 \otimes v, \dots$. Since $\dim V < \infty$, there is a minimal number k such that

$u = \sum_{i=0}^{k} \beta_i g_\alpha^i \otimes v \in N$ and $\beta_k \neq 0$. Choose $h \in H$ such that $\alpha(h) \neq 0$. We have $hu = \sum_{i=0}^{k} \beta_i \gamma_i g_\alpha^i \otimes v \in N$, where $\gamma_i = i\alpha(h) + \Lambda(h)$. If $\gamma_k = 0$ then $\gamma_i = 0$ for some $i < k$, which is impossible. Therefore, $u - \frac{1}{\gamma_k} hu \in N$ and $\gamma_i = \gamma_k$, which is again impossible. Thus, there exists $k \in \mathbb{N}$ such that $\mathscr{E}_{\alpha,k}$ is linearly independent in V and $g_\alpha^{k+i} \otimes v \in N$ for every $i \in \mathbb{N} \cup \{0\}$. Moreover, since $g_\alpha^k \otimes v \in N$, in the case $g_\alpha \in G_{\bar{0}}$ we have

$$g_{-\alpha} g_\alpha^k \otimes v = k\big(\alpha(h)(k-1)/2 + \Lambda(h)\big) g_\alpha^{k-1} \otimes v \in N,$$

where $h = [g_{-\alpha}, g_\alpha]$. Therefore, $\Lambda(h) = -\frac{(k-1)\alpha(h)}{2}$. The remaining cases may be considered analogously. Namely, if $k = 2s$ and $g_\alpha \in G_{\bar{1}}$, then $g_{-\alpha} g_\alpha^k \otimes v = s\alpha(h) g_\alpha^{k-1} \otimes v$ and $\alpha(h) = 0$. If $k = 2s+1$ and $g_\alpha \in G_{\bar{1}}$, then $g_{-\alpha} g_\alpha^{2s+1} \otimes v = \big(\Lambda(h) + s\alpha(h)\big) g_\alpha^{2s} \otimes$ and $\Lambda(h) = -\frac{(k-1)\alpha(h)}{2}$. □

Remark 3.1. Note that if we start with a root β, then there exists $s \in \mathbb{N}$ such that $\mathscr{E}_{\beta,s}$ is linearly independent, but $\mathscr{E}_{\alpha,k} \cup \mathscr{E}_{\beta,s}$ may not be linearly independent.

Recall that a set $\mathscr{E}$ is called a *pre-basis* of a vector space W if $\langle \mathscr{E} \rangle = W$.

Let $\{g_{\alpha_1}^{k_1} \dots g_{\alpha_s}^{k_s}; k_i \in \mathbb{N}_0, \alpha_i \in \mathscr{A}\}$ be a basis of V. As we have seen above, for every $i = 1 \dots, s$, there exists a minimal number $p_i \in \mathbb{N}$ such that $g_{\alpha_i}^{p_i} \in N$. Using the induction on the word length, it is easy to show that $\{g_{\alpha_1}^{k_1} \dots g_{\alpha_s}^{k_s}; k_i \in \mathbb{N}_0, k_i < p_i, \alpha_i \in \mathscr{A}\}$ is a pre-basis of V/N.

Consider the algebra $B(0,n)$. It consists of the matrices of type

$$\begin{pmatrix} 0 & x & y \\ y^\top & A & B \\ -x^\top & C & -A^\top \end{pmatrix},$$

where A is a $(n \times n)$-matrix, B and C are some symmetric $(n \times n)$-matrices, and x, y are some $(n \times 1)$-matrices.

Choose the following generators of $G = B(0,n)$ [4]:

$$\left.\begin{aligned} h_i &= e_{i+1,i+1} - e_{i+n+1,i+n+1}, \\ h_n &= e_{n+1,n+1} - e_{2n+1,2n+1}, \\ g_{\delta_{i+1}-\delta_i} &= e_{i+2,i+1} - e_{i+n+1,i+n+2}, \\ g_{\delta_i-\delta_{i+1}} &= e_{i+1,i+2} - e_{i+n+2,i+n+1} \end{aligned}\right\} \in B(0,n)_{\bar{0}}$$

$(i = 1, \dots, n-1)$, and

$$\left.\begin{aligned} g_{-\delta_n} &= e_{1,n+1} - e_{2n+1,1}, \\ g_{\delta_n} &= e_{n+1,1} + e_{1,2n+1}, \end{aligned}\right\} \in B(0,n)_{\bar{1}}.$$

We write out also some elements and multiplications that will be needed in the following:

$$
\begin{array}{ll}
g_{\delta_i} = e_{i+1,1} + e_{1,n+i+1}, & g_{-\delta_i} = e_{1,i+1} - e_{n+i+1,1}, \\
g_{2\delta_i} = e_{i+1,n+i+1}, & g_{-2\delta_i} = e_{n+i+1,i+1}, \\
[g_{2\delta_i}, g_{-2\delta_i}] = [g_{\delta_i}, g_{-\delta_i}] = h_i, & [g_{\delta_i}, g_{-2\delta_i}] = g_{-\delta_i}, \\
[g_{2\delta_i}, g_{-\delta_i}] = -g_{\delta_i} & \\
g_{\delta_i-\delta_j} = e_{i+1,j+1} - e_{j+n+1,n+i+1} & g_{-\delta_i-\delta_j} = e_{n+i+1,j+1} + e_{n+j+1,i+1} \\
g_{\delta_i+\delta_j} = e_{j+1,n+i+1} + e_{i+1,n+j+1} & [g_{\delta_i+\delta_j}, g_{-\delta_i-\delta_j}] = h_i + h_j \\
[g_{\delta_j-\delta_i}, g_{\delta_i-\delta_j}] = h_j - h_i & [g_{2\delta_i}, g_{-\delta_i-\delta_j}] = g_{\delta_i-\delta_j} \\
[g_{-\delta_i}, g_{-\delta_j}] = -g_{-\delta_i-\delta_j} & [g_{\delta_i}, g_{\delta_j}] = g_{\delta_i+\delta_j} \\
[g_{-\delta_j+\delta_i}, g_{-2\delta_i}] = -g_{-\delta_i-\delta_j} & [g_{\delta_i}, g_{-\delta_i-\delta_j}] = g_{-\delta_j} \\
[g_{\delta_k-\delta_i}, g_{-\delta_k-\delta_j}] = -g_{-\delta_i-\delta_j} & [g_{\delta_j+\delta_i}, g_{-2\delta_i}] = g_{-\delta_i+\delta_j}
\end{array}
$$

The space $H = \langle h_i; i = 1, \ldots, n\rangle$ is a Cartan subalgebra of $B(0,n)$, and δ_i, $i = 1, \ldots, n$, are the linear functions on H such that $\delta_i(h_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta. Then $\Delta = \Delta_0 \cup \Delta_1$ is a root system for $B(0,n)$, where $\Delta_0 = \{0, \pm\delta_i \pm \delta_j\}$ and $\Delta_1 = \{\pm\delta_i\}$, $i, j = 1, \ldots, n$. The roots $\{\delta_i - \delta_{i+1}$, $i = 1, \ldots, n-1, \delta_n\}$ are simple. The conditions $G_{\delta_k} \subseteq G_{n-k+1}$, $H \subseteq G_0$ and $G_{-\delta_k} \subseteq G_{-n+k-1}$ provide the standard grading of $B(0,n)$ [5], Section 5.2.3. The negative part of this grading is $G_{-\delta_i-\delta_j}$ for every i, j; $G_{\delta_i-\delta_j}$ for $i > j$, and $G_{-\delta_i}$ for every i. Henceforth, the set

$$
\begin{aligned}
\mathscr{E} = \{ & g^{k_1}_{\delta_n-\delta_{n-1}} \cdots g^{k_{n-1}}_{\delta_n-\delta_1} g^{k_n}_{\delta_{n-1}-\delta_{n-2}} \cdots g^{k_s}_{\delta_2-\delta_1} g^{k_{s+1}}_{-2\delta_n} g^{k_{s+2}}_{-\delta_n-\delta_{n-1}} \\
& \cdots g^{k_r}_{-2\delta_1} g^{\varepsilon_n}_{-\delta_n} \cdots g^{\varepsilon_1}_{-\delta_1} \otimes v; k_i \in \mathbb{N}, \varepsilon_i \in \mathbb{Z}_2\}
\end{aligned}
\tag{6}
$$

is a basis of the induced module $M = \mathrm{Ind}^G_B\langle v_\Lambda\rangle$ $(v = v_\Lambda)$.

For $\alpha \in \Delta$ and $w \in \mathscr{E}$, we denote by $\theta(\alpha, w)$ the degree of the element g_α in w. For example, $\theta(-2\delta_1, w) = k_r$, where w from (6). By Lemmas 3.1 and 3.3, it is easy to obtain the following

Lemma 3.4. *Given $w \in \mathscr{E}$, $\gamma(w) = \sum_{i=1}^n \gamma_i(w)\delta_i$ is a weight of M, where*

$$
\begin{aligned}
\gamma_i(w) = & \sum_{j<i} \theta(\delta_i - \delta_j, w) - \sum_{j>i} \theta(\delta_j - \delta_i, w) - \sum_{j\neq i} \theta(-\delta_j - \delta_i, w) \\
& - \theta(-\delta_i, w) - 2\theta(-2\delta_i, w) + \Lambda(h_i).
\end{aligned}
\tag{7}
$$

Let V be an irreducible module over $G = B(0,n)$ with the highest weight Λ, $\Lambda(H_i) = b_i$. (Here H_i are the elements of the standard basis of H, cf. [5], $H_i = h_i - h_{i+1}$, $H_n = 2h_n$.) By [5], $b_i \in \mathbb{N}$, $b_n \in 2\mathbb{N}$. It is possible to check that $a_i := \Lambda(h_i) = (\sum_{j=i}^{n-1} b_j) + b_n/2 \geq 0$, $i = 1, \dots, n-2$, $a_{n-1} := \Lambda(h_{n-1}) = b_{n-1} + b_n/2 \geq 0$, $a_n := \Lambda(h_n) = b_n/2 \geq 0$, and $a_1 \geq \cdots \geq a_n \geq 0$. We see that the weight Λ can be defined by means of the n-tuple $(a_1, \dots, a_n)$, with $a_i \in \mathbb{N}_0$, $i = 1, \dots, n$, such that $a_1 \geq \cdots \geq a_n \geq 0$ and $\Lambda(h_i) = a_i$. Denote $\Lambda = (a_1, \dots, a_n)$.

Before proving the main theorem, we present some technical lemmas on irreducible modules of a special type $(a_1 = 1)$ over $B(0,n)$.

Lemma 3.5. *Let $V = V_\Lambda$ be an irreducible module over $B(0,n)$ with $\Lambda = (1, a_2, \dots, a_n)$. Then we have the following:*

1) $g^2_{-2\delta_1} \otimes v = 0$;
2) $g_{-2\delta_1} g_{-\delta_1} \otimes v = 0$;
3) $g^3_{-\delta_1} \otimes v = 0$ $(g^2_{-\delta_1} \otimes v \neq 0)$;
4) $g_{-2\delta_1} g_{\delta_j - \delta_1} \otimes v = 0$;
5) $g_{-2\delta_1} g_{-\delta_j - \delta_1} \otimes v = 0$;
6) $g_{-\delta_1} g_{\delta_j - \delta_1} \otimes v = 0$;
7) $g_{\delta_i - \delta_1} g_{\delta_j - \delta_1} \otimes v = 0$;
8) $g_{-\delta_i - \delta_1} g_{-\delta_j - \delta_1} \otimes v = -g_{-2\delta_1} g_{-\delta_j - \delta_i} \otimes v$;
9) $g^2_{-\delta_1} \otimes v = -g_{-2\delta_1} \otimes v$;
10) $g_{\delta_i - \delta_1} g_{-\delta_j - \delta_1} \otimes v = -g_{-2\delta_1} g_{\delta_i - \delta_j} \otimes v$, $i \neq j$;
11) $g_{\delta_i - \delta_1} g_{-\delta_i - \delta_1} \otimes v = -(1 + a_i) g_{-2\delta_1} \otimes v$;
12) $g_{-\delta_i - \delta_1} g_{-\delta_2} \otimes v \neq 0$ (*if* $a_2 = 1$);
13) $g_{\delta_2 - \delta_1} g_{-\delta_2} \otimes v \neq 0$ (*if* $a_2 = 1$);
14) $g_{\delta_i - \delta_1} g_{-\delta_2} \otimes v \neq 0$ (*if* $a_2 = 1$, $a_i = 0$).

Proof. 1) By Lemma 3.3, if $\alpha = -2\delta_1$, then $h = [g_{2\delta_1}, g_{-2\delta_1}] = h_1$, $1 = \Lambda(h) = k - 1$ and $k = 2$.

2) By 1), $g_{\delta_1} g^2_{-2\delta_1} \otimes v = 0$. Since $[g_{\delta_1}, g_{-2\delta_1}] = g_{-\delta_1}$, we have

$$(g_{-2\delta_1} g_{\delta_1} + g_{-\delta_1}) g_{-2\delta_1} \otimes v = g^2_{-2\delta_1} g_{\delta_1} \otimes v + g_{-2\delta_1} g_{-\delta_1} \otimes v + g_{-\delta_1} g_{-2\delta_1} \otimes v$$
$$= 2 g_{-2\delta_1} g_{-\delta_1} \otimes v = 0.$$

3) It is easy to see that $g_{-\delta_1} \otimes v \neq 0$, $h = [g_{\delta_1}, g_{-\delta_1}] = h_1$, $-\delta_1(h_1) \neq 0$. Therefore, by Lemma 3.3, k is odd and $1 = \Lambda(h_1) = -\frac{(k-1)}{2}(-1)$, $k = 3$.

4) We have $[g_{\delta_1 + \delta_j}, g_{-2\delta_1}] = g_{\delta_j - \delta_1}$ and $g_{\delta_1 + \delta_j} g^2_{-2\delta_1} \otimes v$. Hence, $(g_{-2\delta_1} g_{\delta_1 + \delta_j} + g_{-\delta_1 + \delta_j}) g_{-2\delta_1} \otimes v = g_{-2\delta_1} g_{-\delta_1 + \delta_j} \otimes v + g_{-\delta_1 + \delta_j} g_{-2\delta_1} \otimes v = 0$.

5) Since $[g_{\delta_1 - \delta_j}, g_{-2\delta_1}] = -g_{-\delta_j - \delta_1}$ and $g_{\delta_1 - \delta_j} g^2_{-2\delta_1} \otimes v = 0$, we have $(g_{-2\delta_1} g_{\delta_1 - \delta_j} - g_{-\delta_j - \delta_1}) g_{-2\delta_1} \otimes v = -2 g_{-2\delta_1} g_{-\delta_j - \delta_1} \otimes v = 0$.

6) $g_{\delta_1 + \delta_i} g_{-2\delta_1} g_{-\delta_1} \otimes v = 0 \implies g_{-\delta_1 + \delta_i} g_{-\delta_1} \otimes v = 0$.

7) $g_{\delta_i} g_{-\delta_1} g_{\delta_j - \delta_1} \otimes v = g_{\delta_i - \delta_1} g_{\delta_j - \delta_1} \otimes v = 0$.

8) By 5), $g_{\delta_1-\delta_i}g_{-2\delta_1}g_{-\delta_j-\delta_1}=0$. Since $[g_{\delta_1-\delta_i},g_{-2\delta_1}]=-g_{-\delta_i-\delta_1}$, $(g_{-2\delta_1}g_{\delta_1-\delta_i}-g_{-\delta_1-\delta_i})g_{-\delta_1-\delta_j}\otimes v=0$. Since

$$[g_{\delta_1-\delta_i},g_{-\delta_1-\delta_j}]=-g_{-\delta_i-\delta_j},\qquad (-g_{-2\delta_1}g_{-\delta_i-\delta_j}-g_{-\delta_1-\delta_i}g_{-\delta_1-\delta_j})\otimes v=0.$$

9) Since $[g_{\delta_1},g_{-2\delta_1}]=g_{-\delta_1}$ and $[g_{\delta_1},g_{-\delta_1}]=h_1$, we have $0=g_{\delta_1}g_{-\delta_1}g_{-2\delta_1}\otimes v=(-g_{-\delta_1}g_{\delta_1}+h_1)g_{-2\delta_1}\otimes v=-g^2_{-\delta_1}\otimes v-g_{-2\delta_1}\otimes v$.

10) We have to apply $g_{\delta_1+\delta_i}$ to 5) and use $[g_{\delta_1+\delta_i},g_{-2\delta_1}]=g_{-\delta_1+\delta_i}$, $[g_{\delta_1+\delta_i},g_{-\delta_1-\delta_j}]=g_{\delta_i-\delta_j}$.

11) In 10) we have to use $[g_{\delta_1+\delta_i},g_{-\delta_1-\delta_i}]=h_1+h_i$ instead of the last equality.

12) If $i\neq 2$ and we suppose that $g_{-\delta_1-\delta_i}g_{-\delta_2}\otimes v=0$, then the action with g_{δ_2} gives $g_{-\delta_1-\delta_i}\otimes v=0$, which is a contradiction. If $g_{-\delta_1-\delta_2}g_{-\delta_2}\otimes v=0$, then the action with g_{δ_1} leads to $g^2_{-\delta_2}\otimes v=0$, again a contradiction.

13) If $g_{-\delta_1+\delta_2}g_{-\delta_2}\otimes v=0$, then $0=g_{-\delta_1+\delta_2}g_{-\delta_2}\otimes v=g_{-\delta_2}g_{-\delta_1+\delta_2}\otimes v-g_{-\delta_1}\otimes v$, which is a contradiction.

14) If $a_i=0$ and $g_{-\delta_1+\delta_i}g_{-\delta_2}\otimes v=0$, then the action with g_{δ_2} gives $g_{-\delta_1+\delta_i}\otimes v=0$, which is a contradiction. □

Corollary 3.1. *Under the assumptions of Lemma* 3.5 *and* $a_2=0$,

$$\{g_{-2\delta_1}\otimes v, g_{-\delta_1\pm\delta_i}\otimes v, g_{-\delta_1}\otimes v, 1\otimes v\}$$

is a pre-basis of V.

Proof. Note that in this case we have $g_{-\delta_i+\alpha}\otimes v=0$, for $\alpha\in\{0,\pm\delta_j\}\backslash\{\delta_i\}$. □

Lemma 3.6. *Under the assumptions of Lemma* 3.5,

$$\dim V_{-k\delta_1+\sum_{i=2}^{n}\alpha_i\delta_i}=0,$$

when $k\geq 2$, $\alpha_i\in\Phi$.

Proof. By Lemma 3.5, $g^s_{-2\delta_1}$ appears in the expression (6) for a nonzero element of V only if $s=1$, and in this case we cannot find the element of the types $g_{-\delta_1+\delta_i}$, $g_{-\delta_1-\delta_i}$, $g_{-\delta_1}$ in this expression. By the same reason, in such expression (6), we may find $g_{-\delta_1}$ only in degree 1, and it is not possible to find two elements of the type $g_{-\delta_1-\delta_i}$ (or $g_{-\delta_1+\delta_i}$). The lemma follows. □

Now we are in a condition to state and prove the main result of this article.

Theorem 3.1. *There are no simple finite-dimensional Filippov superalgebras of type* $B(0,n)$ *over* Φ.

Proof. Let $G = B(0,n)$, let V be a finite-dimensional irreducible module over G with the highest weight $\Lambda = (a_1, \dots, a_n)$, and let ϕ be a surjective skew-symmetric homomorphism from $\bigwedge^m V$ on G. Then there exist $u_i \in V_{\gamma_i}$ such that

$$\phi(u_1, \dots, u_m) = g_{-2\delta_1}. \tag{8}$$

If $u \in V_\gamma$ (or G_γ) and $\gamma = \sum \alpha_i \delta_i$, then we denote by $\delta_i(u)$ the element α_i, and we denote by $\delta(u)$ the element α_1. By Lemma 3.4, $\delta(u_i) = a_1 - k_i$ for some $k_i \in \mathbb{N}_0$. By Lemma 3.1, $ma_1 - \sum_{i=1}^m k_i = -2$. Since $g_{-2\delta_1}(1 \otimes v) \neq 0$ and ϕ is a skew-symmetric homomorphism, $\phi(u_1, \dots, u_m)(1 \otimes v) = g_{-2\delta_1}(1 \otimes v) \neq 0$ and $\phi(u_1, \dots, u_{i-1}, 1 \otimes v, u_{i+1}, \dots, u_m) \neq 0$. Since $\delta(1 \otimes v) = a_1$, the inequality $|k_i - 2| \leq 2$ follows. Let $a_1 \geq 2$. By Lemma 3.3, we have

$$\phi(u_1, \dots, u_m)(g_{-2\delta_1}^{a_1 - 1} \otimes v) = g_{-2\delta_1}(g_{-2\delta_1}^{a_1 - 1} \otimes v) \neq 0$$

and, analogously, $|k_i - 2a_1| \leq 2$. From these inequalities we see that the required skew-symmetric homomorphism does not exist if $a_1 \geq 4$, and, in the case $a_1 = 3$, we have the condition $k_i = 4$ for all i.

Consider the case $a_1 = 3$. Then, by (8), we have $\phi(u_1, u_2) = g_{-2\delta_1}$, where $\delta(u_1) = \delta(u_2) = -1$. Since $\delta(1 \otimes v) = 3$ and $g_{-2\delta_1}(1 \otimes v) \neq 0$, we have $\phi(1 \otimes v, u_2) \doteq g_{2\delta_1}$. (In what follows, the symbol $\doteq$ denotes an equality up to a nonzero coefficient.) Since $g_{2\delta_1}(g_{-2\delta_1} \otimes v) \neq 0$, we have $\phi(1 \otimes v, g_{-2\delta_1} \otimes v) \neq 0$, $\delta(1 \otimes v) = 3$ and $\delta(g_{-2\delta_1} \otimes v) = 1$, which is a contradiction.

Consider the case $a_1 = 2$. By [9], we may assume that $1 \otimes v$ is odd. Let $\phi(u_1, \dots, u_m) = g_{-\delta_1}$, $u_i \in V_{\gamma_i}$. Then $\sum_{i=1}^m \delta(u_i) = -1$. Since

$$\phi(u_1, \dots, u_m)(1 \otimes v) = g_{-\delta_1} \otimes v \neq 0, \tag{9}$$

we have $|\sum_{i=1}^m \delta(u_i) - \delta(u_j) + 2| \leq 2$ for every $j = 1, \dots, m$, and $|1 - \delta(u_j)| \leq 2$. On the other hand, since $\phi(u_1, \dots, u_m)(g_{-\delta_1}^3 \otimes v) = g_{-\delta}^4 \otimes v \neq 0$ and $\delta(g_{-\delta_1}^3 \otimes v) = -1$, we have $|2 + \delta(u_j)| \leq 2$. Therefore, $\delta(u_j) = 0, -1$ and we may assume that $\delta(u_1) = -1$, $\delta(u_i) = 0$, $i \geq 2$. By (9), $\phi(1 \otimes v, u_2, \dots, u_m) \doteq g_{2\delta_1}$, and

$$\phi(1 \otimes v, u_2, \dots, u_m)(g_{-2\delta_1} \otimes v) \doteq g_{2\delta_1}(g_{-2\delta_1} \otimes v) = 2(1 \otimes v).$$

Thus, we may interchange, for example, the elements u_2 and $g_{-2\delta_1} \otimes v$. Repeating this process, we obtain that

$$\phi(1 \otimes v, \underline{g_{-2\delta_1} \otimes v}) \doteq g_{2\delta_1}.$$

Multiplying by $g_{-\delta_1}$, we come to the following:

$$\phi(g_{-\delta_1} \otimes v, \underline{g_{-2\delta_1} \otimes v}) - (m-1)\phi(1 \otimes v, g_{-\delta_1} g_{-2\delta_1} \otimes v, \underline{g_{-2\delta_1} \otimes v}) \doteq g_{\delta_1}.$$

Acting with the both sides of the last equality on $g_{-\delta_1} \otimes v$, we arrive at

$$\phi(1 \otimes v, g_{-\delta_1} g_{-2\delta_1} \otimes v, \underline{g_{-2\delta_1} \otimes v})(g_{-\delta_1} \otimes v) \neq 0$$

and

$$A := \phi(1 \otimes v, g_{-\delta_1} \otimes v, \underline{g_{-2\delta_1} \otimes v}) \neq 0.$$

It remains to notice that $\delta(A) = 3$, which is a contradiction.

Lemma 3.7. *There are no good triples of the type (G, V, ϕ), where $G = B(0, n)$, $V = V_\Lambda$, $\Lambda = (1, 1, a_3, \dots, a_n)$.*

Proof. By above, there are the elements $u_i \in V_{\gamma_i}$ such that $\phi(u_1, \dots, u_m) = g_{-2\delta_1}$, where $-3 \leq \delta(u_i) \leq 1$. By Lemma 3.6, $-1 \leq \delta(u_i) \leq 1$. If $\delta(u_i) = 1$ for some i, then the action by the last equality on $g_{-\delta_2} \otimes v$ twice gives a contradiction (note that $g_{-\delta_2} \otimes v$ is an even element). Therefore, we come to the case $\phi(u_1, \dots, u_m) = g_{-2\delta_1}$, where $\delta(u_1) = \delta(u_2) = -1$, $\delta(u_i) = 0$, $i > 2$. The action on $g_{-\delta_2} \otimes v$ gives $w \doteq \phi(u_1, u_2, g_{-\delta_2} \otimes v, u_4, \dots, u_m) \in \{g_{-\delta_1}, g_{-\delta_1-\delta_i}, g_{-\delta_1+\delta_i}\}$. If $w \in \{g_{-\delta_1}, g_{-\delta_1-\delta_i}\}$ then the action on $g_{-\delta_2} \otimes v$ leads to a contradiction, by Lemma 3.5. Thus, $w \doteq g_{-\delta_1+\delta_i}$, $i \neq 2$, $a_i = 1$, using Lemma 3.5 and the action on $g_{-\delta_2} \otimes v$. We have proved that if $w \doteq \phi(u_1, u_2, g_{-\delta_2} \otimes v, u_4, \dots, u_m) \neq 0$, where $\delta(u_1) = \delta(u_2) = -1$, $\delta(u_i) = 0$, $i > 3$, then $w \doteq g_{-\delta_1+\delta_i}$, $i \neq 2$, $a_i = 1$. Applying the equality $w \doteq g_{-\delta_1+\delta_i}$ to $g_{-\delta_1-\delta_i} \otimes v$, we have $w \doteq \phi(u_1, u_2, g_{-\delta_2} \otimes v, g_{-\delta_1-\delta_i} \otimes v, u_5, \dots, u_m) \neq 0$ by Lemma 3.5. By the above, $w \doteq g_{-\delta_1+\delta_j}$, $j \neq 2$, $a_j = 1$. Repeating this process, we arrive at

$$v_0 = \phi(u_1, u_2, g_{-\delta_2} \otimes v, g_{-\delta_1-\delta_{i_3}} \otimes v, \dots, g_{-\delta_1-\delta_{i_m}} \otimes v) \doteq g_{-\delta_1+\delta_{i_2}}, \tag{10}$$

where $i_j \neq 2$ and $a_{i_j} = 1$. If $m \geq 4$ then applying (10) to $g_{-\delta_1-\delta_{i_2}} \otimes v$ we obtain

$$v_1 = \phi(u_1, u_2, g_{-\delta_1-\delta_{i_2}} \otimes v, \dots, g_{-\delta_1-\delta_{i_m}} \otimes v) \doteq g_{-2\delta_1}$$

and $\delta_2(v_1) = m - 2 + \delta_2(u_1) + \delta_2(u_2) = 0$, $\delta_2(v_0) = 2 - m + m - 3 = -1$, which is a contradiction. If $m = 3$ then $v_2 = \phi(u_1, u_2, g_{-\delta_2} \otimes v) = g_{-\delta_1+\delta_i}$, $i \neq 2$. Replacing $u = g_{-\delta_2} \otimes v$ by $g_{-\delta_1-\delta_i} \otimes v = u'$, we obtain that $\delta_2(u_1) + \delta_2(u_2) = -1$ (note that $\delta_2(u) = 1$). Therefore, $\delta_2(v_2) = -1$, which leads to a contradiction.

Consider now the case $m = 2$. In this case, we have $\phi(u_1, u_2) = g_{-2\delta_1}$. We may assume that $\delta_2(u_2) \geq 0$. We have $\phi(u_1, u_2)(1 \otimes v) \neq 0$. It follows that

$$w = \phi(1 \otimes v, u_2) \in \{g_{2\delta_2}, g_{\delta_2}, g_{\delta_2+\delta_i}, g_{\delta_2-\delta_i}\}.$$

If $w \in \{g_{2\delta_2}, g_{\delta_2}, g_{\delta_2-\delta_i}\}$, then $wg_{-2\delta_2} \otimes v \neq 0$ and we have

$$w_1 = \phi(1 \otimes v, g_{-2\delta_2} \otimes v) \doteq g_{2\delta_1}.$$

Therefore, $(g_{-\delta_1} w_1) g_{-\delta_1} \otimes v \neq 0$ and $w_2 = \phi(1 \otimes v, g_{-\delta_1} \otimes v) \neq 0$, $\delta(w_2) = 1$, $\delta_2(w_2) = 2$, which is a contradiction. Thus, $w \doteq g_{\delta_2+\delta_i}$. In this case, $w g_{-\delta_2-\delta_i} \otimes v \neq 0$ and $u = \phi(1 \otimes v, g_{-\delta_2-\delta_i} \otimes v) \neq 0$. Hence, $i = 1$, $a_3 = \cdots = a_n = 0$ and $u \doteq g_{\delta_1+\delta_2}$. Furthermore,

$$g_{-\delta_1} u = \phi(g_{-\delta_1} \otimes v, g_{-\delta_2-\delta_1} \otimes v) - \phi(1 \otimes v, g_{-\delta_1} g_{-\delta_2-\delta_1} \otimes v) := u_1 - u_2 \doteq g_{\delta_2}$$

and $(g_{-\delta_1} u)(g_{-\delta_2} \otimes v) \neq 0$ (observe that if $u_2 g_{-\delta_2} \otimes v \neq 0$, then $u'' = \phi(1 \otimes v, g_{-\delta_2} \otimes v) \neq 0$ and $\delta(u'') = 2$, $\delta_2(u'') = 1$). Therefore,

$$u' = \phi(g_{-\delta_1} \otimes v, g_{-\delta_2} \otimes v) = g_{\delta_1+\delta_2}.$$

We have

$$(g_{-\delta_2} u') g_{-\delta_1} \otimes v \neq 0, \qquad u_3 = \phi(g_{-\delta_2} g_{-\delta_1} \otimes v, g_{-\delta_2} \otimes v) \doteq g_{\delta_1},$$
$$u_3 g_{-2\delta_1} \otimes v \neq 0, \qquad u_4 = \phi(g_{-2\delta_1} \otimes v, g_{-\delta_2} \otimes v) \neq 0, \qquad u_4 \doteq g_{\delta_2}$$

and $u_4 g_{-\delta_2} \otimes v \neq 0$, which is again a contradiction. □

Thus, we have come to the case $\Lambda = (1, 0, \ldots, 0)$. In this case, there are some weight vectors $u_i \in V$ such that

$$\phi(u_1, \ldots, u_m) = h_1 + \sum_{i=2}^{n} \alpha_i h_i. \tag{11}$$

Notice that we may assume that $u_i \neq g_{-\delta_1} \otimes v$. Act on (11) with g_{δ_1} and use Corollary 3.1. If $\delta(u_i) = 1$, then $u_i \doteq 1 \otimes v$ and $g_{\delta_1} u_i = 0$. If $\delta(u_i) = 0$, then $u_i \in \langle g_{-\delta_1 \pm \delta_i} \otimes v; i \neq 1 \rangle$ and $g_{\delta_1} u_i = 0$. If $\delta(u_i) = -1$, then $u_i \doteq g_{-2\delta_1} \otimes v$ and $g_{\delta_1} u_i \doteq g_{-\delta_1} \otimes v$. Finally, considering the action on $g_{-\delta_1} \otimes v$, we come to a contradiction. □

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