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An existence theorem for a perturbed singular elliptic problem

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Abstract. We prove the existence of a positive solution for singular elliptic problems of the type $-\Delta u = \alpha(x)u^{-\beta} + \lambda f(x,u)$ in Ω, $u_{|\partial\Omega} = 0$, where β is any positive real number.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a non-empty bounded open set with boundary $\partial\Omega$ of class C^2. Let $\alpha : \Omega \to \mathbb{R}$ and $f : \bar{\Omega} \times [0, +\infty[\to \mathbb{R}$ be two given real functions.

The purpose of this article is to establish an existence theorem of positive solutions for the perturbed elliptic singular boundary value problem

$$\left.\begin{aligned} -\Delta u &= \alpha(x)u^{-\beta} + \lambda f(x,u) \quad \text{in } \Omega, \\ u_{|\partial\Omega} &= 0, \end{aligned}\right\} \tag{P_λ}$$

where $\lambda \in \mathbb{R}$ is nonnegative and sufficiently small. Here β is a positive real exponent.

Since the paper [2], problems like (P_λ) have been widely studied (for the most recent articles see, for instance, [1], [4], [6], [7]). A rather recurrent assumption to get the existence of solutions for problem (P_λ) is to impose that $\beta < 1$. Under this assumption, several results concerning existence as well as multiplicity and uniqueness for problem (P_λ) are available. On the contrary, it seems that relatively less articles deal with problem (P_λ) assuming no restriction on β. In this case, for $\lambda = 0$ we refer to [3], and for $\lambda = 1$ we refer to [1] where a variational approach is presented. Recently problem (P_λ) was studied in [4] where the p-Laplacian operator ($p > 1$) was considered. In [4], f is supposed to be a Carathéodory function and bounded on every set of the type $\Omega \times [0, s]$, with $s > 0$. In particular, in

[4] it is shown that there exists λ_0 such that, for all $\lambda \in [0, \lambda_0]$, problem (P_λ) admits a positive weak solution in $W_0^{1,p}(\Omega)$, provided that either there exists a nonnegative function $\phi \in C_0^1(\bar{\Omega})$ and $q > N$ such that $\alpha\phi \in L^q(\Omega)$ and f is nonnegative (Theorem 1.1 of [4]) or $\alpha \in L^\infty(\Omega)$ with $\operatorname{essinf}_\Omega \alpha > 0$ and $\beta < \frac{1}{N}$ (Theorem 1.3 of [4]). The proofs of these results are based on a super and sub-solution argument which does not work any longer when the assumption $\beta < \frac{1}{N}$ is removed in Theorem 1.3 of [4]. Here we study problem (P_λ) without assuming any restriction on β and keeping the perturbation term $\lambda f(x,t)$ in this general form, allowing f to change sign as well. In particular our result will be directly comparable to Theorem 1.3 of [4] in the case $p = 2$. Indeed, we will assume that the infimum of α on Ω is positive (note that this condition does not meet the one imposed on α in Theorem 1.1 of [4]). We observe that when $\beta > 3$, this assumption implies that no weak solution can exist. This fact can be deduced by adapting to our case the proof of Theorem 2 of [3]. So we are led to look for the existence of a classical solution for problem (P_λ), that is, a function $u_\lambda \in C(\bar{\Omega}) \cap C^2(\Omega)$ satisfying the equation $-\Delta u = \alpha(x)u^{-\beta} + \lambda f(x,u)$ and the boundary condition $u_{|\partial\Omega} \equiv 0$ pointwise in Ω and $\partial\Omega$, respectively.

In order to get a classical solution we have to impose standard assumptions on the nonlinearity. Precisely, we assume that

(H) there exists $\gamma \in]0,1[$ such that α and f are locally Hölderian with exponent γ in Ω and in $\Omega \times [0,+\infty[$, respectively.

In the main result of this paper we will establish the existence of a positive classical solution for small nonnegative λ under a further suitable local condition on f. The idea of the proof is suggested by the arguments presented in [1].

2. The result

We start stating the following well-known comparison lemma (for the proof, see for instance [5] where the reader can find an even more general version)

Lemma 1. *Let $g : \Omega \times]0,+\infty[\to \mathbb{R}$ be a continuous function such that $t \to t^{-1}g(x,t)$ is strictly decreasing in $]0,+\infty[$ for all $x \in \Omega$. Let $u, v \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfying*

$$\begin{cases} -\Delta u \le g(x,u), \ -\Delta v \ge g(x,v) & \text{in } \Omega, \\ u \le v & \text{in } \partial\Omega. \end{cases}$$

Then $u \le v$ in Ω.

Now we are able to prove the main result. From now on, we will make use of the following notations:

$$L_s = \sup_{x \in \bar{\Omega}} \sup_{\substack{t,\tau \in [0,s] \\ t > \tau}} \frac{f(x,t) - f(x,\tau)}{t - \tau};$$

if $u : \Omega \to \mathbb{R}$ is a given function, we put $u_+(x) = \max\{u(x), 0\}$ for each $x \in \Omega$.

Theorem 1. *Assume condition* (H) *and that $L_s < +\infty$ for all $s > 0$. Moreover, assume $\sup_{\bar{\Omega}\times[0,T]}|f| < +\infty$ for all $T > 0$ and α continuous and positive on $\bar{\Omega}$. Then, for every $\beta > 0$, there exists λ_0 such that, for each $\lambda \in [0, \lambda_0]$, problem (P_λ) admits a positive classical solution $u \in C(\bar{\Omega}) \cap C^{2+\gamma}(\Omega)$.*

Proof. Let $\beta > 0$. Let $u_1 \in C(\bar{\Omega}) \cap C^{2+\gamma}(\Omega)$ be the unique positive solution of the problem $-\Delta u = \alpha(x)$ in Ω, $u_{|\partial\Omega} = 0$, and put

$$\lambda_1 = \frac{(\max\{2^{1/\beta}, 2(\beta+1)\}\|u_1\|_\infty)^{-\beta/(\beta+1)}}{\sup_{\Omega\times[0,T]}|f|} \cdot \inf_{\Omega} \alpha,$$

where

$$T = \left(2(\beta+1)\|u_1\|_\infty\right)^{1/(\beta+1)}.$$

Moreover, define

$$h_n(t) = \begin{cases} \min\{(\min\{t, T\})^{-\beta}, n\} & \text{if } t > 0, \\ n & \text{if } t \le 0, \end{cases}$$

for each $n \in \mathbb{N}$, and

$$f_T(x,t) = \begin{cases} f(x,t) & \text{if } (x,t) \in \Omega \times [0,T], \\ f(x,T) & \text{if } (x,t) \in \Omega \times [T, \infty[. \end{cases}$$

Now consider the following problem

$$\left.\begin{aligned} -\Delta u &= \alpha(x)h_n(u) + \lambda f_T(x,u) \quad \text{in } \Omega, \\ u_{|\partial\Omega} &= 0. \end{aligned}\right\} \qquad (P_{\lambda,n})$$

Let us show that for all $\lambda \in [0, \lambda_1]$ the functions

$$\bar{u} = \left(2(\beta+1)u_1\right)^{1/(\beta+1)} \quad \text{and} \quad \underline{u} = 2^{-1/(\beta+1)}\|u_1\|_\infty^{-\beta/(\beta+1)} u_1$$

are, respectively, a supersolution and, if $n > (\|u_1\|_\infty)^{-\beta/(\beta+1)}$, a subsolution of problem $(P_{\lambda,n})$. Indeed, for $\lambda \in [0, \lambda_1]$ and $n \in \mathbb{N}$, one has

$$\begin{aligned} -\Delta\bar{u} &= 4\beta\big(2(\beta+1)u_1\big)^{(-2\beta-1)/(\beta+1)}(\nabla u_1)^2 - 2\big(2(\beta+1)u_1\big)^{-\beta/(\beta+1)}\Delta u_1 \\ &\geq 2\alpha(x)(\bar{u})^{-\beta} \geq 2\alpha(x)h_n(\bar{u}) \end{aligned}$$

and consequently

$$-\Delta\bar{u} \geq \alpha(x)h_n(\bar{u}) + \lambda f(x,\bar{u}).$$

Hence $\bar{u}$ is a supersolution of $(P_{\lambda,n})$. Now let $n > (\|u_1\|_\infty)^{-\beta/(\beta+1)}$ and, again, $\lambda \in [0, \lambda_1]$. Then, one has

$$-\Delta\underline{u} + \lambda \sup_{\bar{\Omega}\times[0,T]} |f| = \frac{1}{2}\alpha(x)\left\|\frac{u_1}{2}\right\|_\infty^{-\beta/(\beta+1)} + \lambda \sup_{\bar{\Omega}\times[0,T]} |f| \leq \alpha(x)n \tag{1}$$

and

$$\begin{aligned} -\Delta\underline{u} &= \frac{1}{2}\alpha(x)\left\|\frac{u_1}{2}\right\|_\infty^{-\beta/(\beta+1)} \\ &\leq \frac{1}{2}\alpha(x)\left\|\frac{u_1}{2}\right\|_\infty^{-\beta/(\beta+1)}\left(\frac{u_1}{\|u_1\|_\infty}\right)^{-\beta} \\ &= \frac{1}{2}\alpha(x)2^{\beta/(\beta+1)}\|u_1\|_\infty^{\beta^2/(\beta+1)}(u_1)^{-\beta} = \frac{1}{2}\alpha(x)(\underline{u})^{-\beta}, \end{aligned}$$

which implies that

$$-\Delta\underline{u} + \lambda \sup_{\bar{\Omega}\times[0,T]} |f| \leq \alpha(x)(\underline{u})^{-\beta}. \tag{2}$$

From (1) and (2) it follows that

$$-\Delta\underline{u} \leq \alpha(x)h_n(\underline{u}) + \lambda f(x,\underline{u})$$

and so $\underline{u}$ is a subsolution of $(P_{\lambda,n})$.

Therefore, by a standard argument, we infer that for every $\lambda \in [0, \lambda_1]$ and every $n > (\|u_1\|_\infty)^{-\beta/(\beta+1)}$, problem $(P_{\lambda,n})$ admits a classical solution

$$u_{\lambda,n} \in \{u \in C^{2+\gamma}(\bar{\Omega}) \mid \underline{u}(x) \leq u(x) \leq \bar{u}(x) \text{ for each } x \in \Omega\}.$$

Now put

$$\lambda_2 = \frac{\min\{1, T^{-\beta}\} \cdot \min_{\bar{\Omega}} \alpha}{M + T \cdot |L_T|}$$

where $M = \sup_{\bar{\Omega}\times[0,T]}|f|$. Then it is easy to show that for all $\lambda \in [0, \lambda_2[$, $n \in \mathbb{N}$, and $x \in \Omega$ the function $t \to t^{-1}\big(\alpha(x)h_n(t) + \lambda f_T(x,t)\big)$ is strictly decreasing in $[0, +\infty[$. To this aim, if λ is as above and if $t_1 > t_2 > 0$, it is enough to show that

$$\alpha(x)\big(t_2 h_n(t_1) - t_1 h_n(t_2)\big) < \lambda\big(t_1 f_T(x, t_2) - t_2 f_T(x, t_1)\big). \tag{3}$$

To prove (3), observe that, on the one hand, we have

$$t_2 h_n(t_1) - t_1 h_n(t_2) \leq n(t_2 - t_1) \qquad \text{if } n \leq (\min\{t_2, T\})^{-\beta}$$

and

$$t_2 h_n(t_1) - t_1 h_n(t_2) \leq T^{-\beta}(t_2 - t_1) \qquad \text{if } n \geq (\min\{t_2, T\})^{-\beta},$$

hence

$$t_2 h_n(t_1) - t_1 h_n(t_2) \leq \min\{1, T^{-\beta}\}(t_2 - t_1). \tag{4}$$

On the other hand, for $x \in \Omega$, we have

$$\begin{aligned} t_1 f_T(x, t_2) - t_2 f_T(x, t_1) &= (t_1 - t_2) f_T(x, t_2) - t_2\big(f_T(x, t_1) - f_T(x, t_2)\big) \\ &\geq (t_2 - t_1)M + T \cdot |L_T|(t_2 - t_1) \\ &= (t_2 - t_1)(M + T \cdot |L_T|). \end{aligned} \tag{5}$$

Consequently, (3) follows easily from (4), (5) and the choice of λ.

Now observe that $h_{n+1}(t) \geq h_n(t)$ for every $t > 0$ and every $n \in \mathbb{N}$. Then, by Lemma 1, if we put $\lambda_3 = \min\{\lambda_1, \lambda_2\}$, for each $\lambda \in [0, \lambda_3]$ and $n > (\|u_1\|_\infty)^{-\beta/(\beta+1)}$ we have

$$u_{\lambda,n}(x) \leq u_{\lambda,n+1}(x) \tag{6}$$

for each $x \in \Omega$. Therefore, $u_{\lambda,n}(x)$ is definitively non-decreasing uniformly with respect to $x \in \Omega$. We claim that $u_{\lambda,n}(x)$ is a Cauchy sequence in $L^\infty(\Omega)$ for sufficiently small λ.

Indeed, fix $R > \mathrm{diam}(\Omega)$, and put $\lambda_0 = \min\left\{\lambda_3, \frac{1}{R^2 L_T}\right\}$ and $\delta(x) = \cos\left(\frac{x_1}{R}\right)$, where x_1 is the first coordinate of $x \in \Omega$. Now let $\varepsilon > 0$. Observe that $\delta(x) \geq$

$\cos 1 > 0$ for each $x \in \Omega$. Let $\lambda \in [0, \lambda_0]$. Then, for every $n, m \in \mathbb{N}$ with $n \geq m > \max\{(\|u_1\|_\infty)^{-\beta/(\beta+1)}, \varepsilon^{-\beta}\}$, we have

$$\begin{aligned}
-\Delta\big(u_{\lambda,m}(x) + \varepsilon\delta(x)\big) &= \alpha(x)h_m\big(u_{\lambda,m}(x)\big) \\
&\quad + \lambda\big(f_T\big(x, u_{\lambda,m}(x)\big) - f_T\big(x, u_{\lambda,m}(x) + \varepsilon\delta(x)\big)\big) \\
&\quad + \lambda f_T\big(x, u_{\lambda,m}(x) + \varepsilon\delta(x)\big) - \varepsilon\Delta\delta(x) \\
&\geq \alpha(x)h_n\big(u_{\lambda,m}(x) + \varepsilon\delta(x)\big) + \lambda f_T\big(x, u_{\lambda,m}(x)\big) \\
&\quad - \lambda|L_T|\varepsilon\delta(x) + \frac{\varepsilon}{R^2}\delta(x) \\
&= \alpha(x)h_n\big(u_{\lambda,m}(x) + \varepsilon\delta(x)\big) + \lambda f_T\big(x, u_{\lambda,m}(x)\big) \\
&\quad + \varepsilon(R^{-2} - \lambda|L_T|)\cos 1 \\
&\geq \alpha(x)h_n\big(u_{\lambda,m}(x) + \varepsilon\delta(x)\big) + \lambda f_T\big(x, u_{\lambda,m}(x)\big).
\end{aligned}$$

This shows that $u_{\lambda,m} + \varepsilon\delta$ is a supersolution of the problem

$$\begin{cases} -\Delta u = \alpha(x)h_n(u) + \lambda f_T(x, u) & \text{in } \Omega, \\ u_{|\partial\Omega} = 0. \end{cases}$$

Consequently, by Lemma 1, one has $u_{\lambda,n}(x) \leq u_{\lambda,m}(x) + \varepsilon\delta(x) \leq u_{\lambda,m}(x) + \varepsilon$ for each $x \in \Omega$. Hence, in view of (6), we easily infer that $u_{\lambda,n}(x)$ is a Cauchy sequence in $L^\infty(\Omega)$ and so it converges to some $u_\lambda \in L^\infty(\Omega)$. Clearly we have

$$\underline{u}(x) \leq u_\lambda(x) \leq \bar{u}(x) \tag{7}$$

for a.a. $x \in \Omega$. This implies that u_λ is a.e. positive in Ω and that $u_\lambda^{-\beta} \in L^\infty_{\mathrm{loc}}(\Omega)$.

Now let σ any real positive number. Since $u_{\lambda,n}$ is a classical solution of $(P_{\lambda,n})$, we easily deduce

$$\begin{aligned}
\|(u_{\lambda,n} - \sigma)_+\|^2 &= \int_\Omega \alpha(x)h_n\big(u_{\lambda,n}(x)\big)\big(u_{\lambda,n}(x) - \sigma\big)_+\,dx \\
&\quad + \lambda\int_\Omega f_T\big(x, u_{\lambda,n}(x)\big)\big(u_{\lambda,n}(x) - \sigma\big)_+\,dx \\
&\leq \Big(\max_\Omega \alpha\sigma^{-\beta} + \lambda \sup_{(x,t)\in\Omega\times[0,T]} |f|\Big)\int_\Omega (u_{\lambda,n} - \sigma)_+\,dx.
\end{aligned}$$

This implies that the sequence $(u_{\lambda,n} - \sigma)_+$ is bounded in $W_0^{1,2}(\Omega)$. Thus, up to a subsequence, it is weakly converging in $W_0^{1,2}(\Omega)$ and, by standard embedding theorems, strongly in $L^2(\Omega)$ to $(u_\lambda - \sigma)_+$. Therefore,

$$(u_\lambda - \sigma)_+ \in W_0^{1,2}(\Omega) \qquad \text{for each } \sigma > 0.$$

Now let $\varphi \in C_0^\infty(\Omega)$ and fix

$$\sigma \in \left]0, \inf_{x \in \overline{\operatorname{supp}\varphi}} \underline{u}(x)\right[.$$

Taking into account that $u_{\lambda,n}(x)$ is non-decreasing with respect to n and that $u_{\lambda,n}(x) \geq \underline{u}(x)$ for all $x \in \Omega$, by the choice of σ we have

$$\bigl(u_{\lambda,n}(x) - \sigma\bigr)_+ = u_{\lambda,n}(x) - \sigma \quad \text{and} \quad \bigl(u_\lambda(x) - \sigma\bigr)_+ = u_\lambda(x) - \sigma$$

in $\overline{\operatorname{supp}\varphi}$. It follows that $u_{\lambda,n}(x)$ converges weakly in $W_0^{1,2}(\Omega)$ and strongly in $L^2(\Omega)$ to u_λ on $\overline{\operatorname{supp}\varphi}$. Using again the fact that $u_{\lambda,n}$ is a classical solution of problem $(P_{\lambda,n})$, it turns out that

$$\int_\Omega \nabla u_{\lambda,n}(x) \nabla\varphi(x)\, dx = \int_\Omega \bigl(\alpha(x) h_n\bigl(u_{\lambda,n}(x)\bigr) + \lambda f_T\bigl(x, u_{\lambda,n}(x)\bigr)\bigr)\varphi(x)\, dx.$$

Hence, as $n \to \infty$, we get

$$\int_\Omega \nabla u_\lambda \nabla\varphi\, dx = \int_\Omega \bigl(\alpha(x) u_\lambda(x)^{-\beta} + \lambda f_T\bigl(x, u_\lambda(x)\bigr)\bigr)\varphi(x)\, dx.$$

By the arbitrariness of the function φ, we conclude that u_λ solves the equation

$$-\Delta u = \alpha(x) u^{-\beta} + \lambda f_T(x, u)$$

in distributional sense. Then, from the standard interior regularity theory, one has $u_\lambda \in C^{2+\gamma}(\Omega)$ and

$$-\Delta u_\lambda(x) = \alpha(x) u_\lambda(x)^{-\beta} + \lambda f_T\bigl(x, u_\lambda(x)\bigr)$$

for all $x \in \Omega$. Moreover, from (7) and the regularity of $\partial\Omega$ (see Theorem 5.1 of [1]), one has $u_{|\partial\Omega} \equiv 0$ and $u_\lambda \in C^{2+\gamma}(\Omega) \cap C(\overline{\Omega})$ as well. Finally, note that since $u_\lambda(x) \leq T$ for a.a. $x \in \Omega$, in the previous equation we can replace the function f_T with the function f. The proof is now complete. □

Remark 1. Condition $L_s < +\infty$ for all $s > 0$, imposed on f, is clearly weaker than the Lipschitz condition with respect to the second variable (uniformly with respect to the first one) on every interval $[0, s]$. For example, we have $L_s \leq 0$ whenever $f(x, \cdot)$ is non increasing for all $x \in \Omega$. Nevertheless, when the previous Lipschitz condition is fulfilled, we can improve the conclusion of Theorem 1 allowing λ to belong to a interval of the type $[-\lambda_1, \lambda_1]$, as it can be easily checked applying Theorem 1 to f and $-f$.

Remark 2. As we have just said in the introduction, Theorem 1 is comparable to Theorem 1.3 of [4] in the case $p = 2$. Here we are able to remove any condition on β (except its positivity), but we require some further condition on the nonlinearity of f. Clearly, in view of Theorem 1.3 of [4], it would be interesting to extend Theorem 1 to the quasilinear case by replacing the Laplacian with the p-Laplacian operator ($p > 1$). However, in this case the proof of Theorem 1 does not seem to work due to the nonlinear feature of the p-Laplacian operator.

References

[1] A. Canino and M. Degiovanni, A variational approach to a class of singular semilinear elliptic equations. *J. Convex Anal.* **11** (2004), 147–162. Zbl 1073.35092 MR 2159469

[2] M. G. Crandall, P. H. Rabinowitz, and L. Tartar, On a Dirichlet problem with a singular nonlinearity. *Comm. Partial Differential Equations* **2** (1977), 193–222. Zbl 0362.35031 MR 0427826

[3] A. C. Lazer and P. J. McKenna, On a singular nonlinear elliptic boundary-value problem. *Proc. Amer. Math. Soc.* **111** (1991), 721–730. Zbl 0727.35057 MR 1037213

[4] K. Perera and Z. Zhang, Multiple positive solutions of singular p-Laplacian problems by variational methods. *Bound. Value Probl.* (2005), 377–382. Zbl 05050326 MR 2202224

[5] J. Shi and M. Yao, Positive solutions for elliptic equations with singular nonlinearity. *Electron. J. Differential Equations* (2005), no. 4, 11 pp. Zbl 1129.35344 MR 2119054

[6] Y. Sun, S. Wu, and Y. Long, Combined effects of singular and superlinear nonlinearities in some singular boundary value problems. *J. Differential Equations* **176** (2001), 511–531. Zbl 1109.35344 MR 1866285

[7] Z. Zhang, Critical points and positive solutions of singular elliptic boundary value problems. *J. Math. Anal. Appl.* **302** (2005), 476–483. Zbl 02136662 MR 2107848

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