Portugal. Math. (N.S.)
Vol. 66, Fasc. 2, 2009, 211–223

Portugaliae Mathematica

A note on the global regularity of steady flows of generalized Newtonian fluids

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(Communicated by Hugo Beirão da Veiga)

Abstract. We establish regularity results for solutions of a generalized Newtonian model in a cubic domain. We prove regularity results in the L^2-space for the second derivatives of the velocity and the first derivatives of the pressure. Further, we show that the gradient of weak solutions is integrable up to the boundary with any finite exponent.

Mathematics Subject Classification (2000). Primary 35Q30; Secondary 76D03, 35J65.
Keywords. Generalized Newtonian fluids, shear thinning fluids, global regularity.

1. Introduction

In this article we are concerned with the following boundary value problem

$$\left.\begin{aligned} -\nu_0\Delta u - \nu_1\nabla\cdot[(\mu+|\mathscr{D}u|)^{p-2}\mathscr{D}u] + (u\cdot\nabla)u + \nabla\pi = f, &\quad \text{in } \Omega,\\ \nabla\cdot u = 0, &\quad \text{in } \Omega, \end{aligned}\right\} \tag{1.1}$$

where

$$\mathscr{D}u = \frac{1}{2}(\nabla u + \nabla u^T),$$

$(u\cdot\nabla)u = u_k\partial_k u$, ν_0, ν_1 and μ are positive constants and $p \in (1,2)$. Below we show L^2-regularity, up to the boundary, for the second derivatives of the velocity field and for the first derivatives of the pressure field. The assumption $\nu_0 > 0$ is crucial in our proof. However we observe that even if such a case is easier to handle from a mathematical point of view, it is physically interesting and more realistic.

The up to the boundary regularity problem has been studied by few authors. In the case $p < 2$, considered here, the most significant result has been obtained in [4], in the more difficult case $\nu_0 = 0$ and with a non-flat boundary. For results with $\nu_0 = 0$ and a flat boundary see the articles [7], [2] and [9]. Further, cylindrical domains were considered by the author in the previous articles [12], [13].

Just for completeness, we recall that for $p > 2$ (the so-called shear thickening case) regularity results up to the boundary were obtained in the half-space case $\mathbb{R}^n_+$ in [1], in the "cubic domain"-case (see below) in [6], [5], and in suitable smooth domains in [3].

The general outline of our proof follows the one introduced in the pioneering article [1]. Actually, here we work in the simplified framework introduced in [6], namely, a three-dimensional cubic domain $\Omega = (]0,1[)^3$ instead of the half space $\mathbb{R}^n_+$.

In [4] it is proved that

$$u \in W^{1,\bar{q}}(\Omega) \cap W^{2,l}(\Omega), \qquad \nabla\pi \in L^l(\Omega),$$

where

$$\bar{q} = 4p - 2, \qquad l = \frac{\bar{q}}{p+1}.$$

Hence, if $p < 2$ and $\nu_0 = 0$, the integrability exponent l remains strictly less than 2. In the sequel we show that $u \in W^{2,2}(\Omega)$ provided that $\nu_0 > 0$.

The second question which arises is the following one: is it possible to prove the $W^{1,q}$-regularity, up to the boundary, of solutions for any finite power q? This, in particular, implies that the solution belongs to $C^{0,\alpha}(\overline{\Omega})$, for any $\alpha < 1$. This result, under the assumption $\nu_0 > 0$, was proved in [8] for a very large class of problems. In particular, in this last reference, the boundary condition is non-homogeneous, there are no convexity assumptions and the power $p = p(x) < 2$ may depend on x. We have shown the same result (independently from [8]) in the simpler case (1.1). Since our proof is very short, we present it to the reader. We observe that, for the time being, we were not able to prove that the full gradient belongs to $L^\infty(\Omega)$; see [8] for some considerations on this point. However, even for $\nu_0 = 0$ it seems possible to show better results, namely that $u \in C^{1,\alpha}(\overline{\Omega})$ for sufficiently small forces.

2. Notations and statement of the main results

Throughout the article Ω denotes a three dimensional cube $\Omega = (]0,1[)^3$. We denote by Γ two opposite faces in the x_3 direction of Ω, i.e.,

$$\Gamma = \{x : |x_1| < 1, |x_2| < 1, x_3 = 0\} \cup \{x : |x_1| < 1, |x_2| < 1, x_3 = 1\}.$$

We set $x' = (x_1, x_2)$ and say that a function is x'-periodic if it is periodic in both directions x_1 and x_2. We impose Dirichlet boundary conditions on Γ and periodicity in the other two directions. Therefore, we can write the boundary conditions as

$$u_{|\Gamma} = 0, \qquad u \text{ is } x'\text{-periodic.} \tag{2.1}$$

By $L^p(\Omega)$, $p \in [1, +\infty]$, we denote the usual Lebesgue space with norm $\|\cdot\|_p$. Further, we set $\|\cdot\| = \|\cdot\|_2$. By $W^{m,p}(\Omega)$, m a non-negative integer and $p \in (1, +\infty)$, we denote the usual Sobolev space with norm $\|\cdot\|_{m,p}$. We denote by $W_0^{1,p}(\Omega)$ the closure in $W^{1,p}(\Omega)$ of $C_0^\infty(\Omega)$ and by $W^{-1,p'}(\Omega)$, $p' = p/(p-1)$, the strong dual of $W_0^{1,p}(\Omega)$ with norm $\|\cdot\|_{-1,p'}$. In notation concerning duality pairings, norms and functional spaces, we will not distinguish between scalar and vector fields.

We set

$$\mathscr{V} = \{v \in C_0^\infty(\Omega) : \nabla \cdot v = 0\}$$

and

$$V_q = \{v \in W^{1,q}(\Omega) : \nabla \cdot v = 0, v_{|\Gamma} = 0, v \text{ is } x'\text{-periodic}\}.$$

By V_q' we denote the dual space of V_q. Recall that, by appealing to inequalities of Korn type, one gets the following result (see [17] Proposition 1.1).

Lemma 2.1. *There exists a constant c such that*

$$\|v\|_q + \|\nabla v\|_q \le c\|\mathscr{D}v\|_q \qquad \textit{for each } v \in V_q.$$

This result implies that the two sides of the above inequality give equivalent norms in V_q.

We denote by D^2u the set of all the second partial derivatives of u. The symbol D_*^2u may denote any second-order partial derivative $\partial_{ik}^2 u_j$ (with the obvious meaning $\partial_{ik}^2 u_j = \frac{\partial^2 u_j}{\partial x_i \partial x_k}$) except for the derivatives $\partial_{33}^2 u_j$, $j = 1, 2$. Moreover we set

$$|D_*^2 u|^2 := |\partial_{33}^2 u_3|^2 + \sum_{\substack{i,j,k=1\\(i,k)\neq(3,3)}}^{3} |\partial_{ik}^2 u_j|^2.$$

By the symbol $\nabla_* \pi$ we denote the second and the third components of the gradient of π.

We denote by c positive constants that may have different values even in the same equation.

Definition 2.2. Assume that $f \in V_2'$. We say that u is a *weak solution* of problem (1.1)–(2.1) if $u \in V_2$ and satisfies

$$\nu_0 \int_\Omega \nabla u \cdot \nabla \varphi \, dx + \nu_1 \int_\Omega (\mu + |\mathscr{D}u|)^{p-2} \mathscr{D}u \cdot \mathscr{D}\varphi \, dx - \int_\Omega (u \cdot \nabla)\varphi \cdot u \, dx = \int_\Omega f \cdot \varphi \, dx \tag{2.2}$$

for all $\varphi \in V_2$.

The existence of a weak solution can be obtained using arguments of the theory of monotone operators, following J.-L. Lions [16]. For a proof of the existence result we refer to [15].

Note that we can replace φ by u in (2.2). Then, using Lemma 2.1, it is easy to get the estimates

$$\begin{aligned} \|\nabla u\| &\le c\|f\|_{-1,2}, \\ \|\mathscr{D}u\|_p &\le c(\|f\|_{-1,p'}^{1/(p-1)} + 1). \end{aligned} \tag{2.3}$$

By restriction of (2.2) to divergence-free test functions with compact support and by de Rham's Theorem, one can associate the pressure field π, determined up to a constant.

Our aim is to prove the following regularity theorems.

Theorem 2.3. *Let be* $p \in \left[\frac{3}{2}, 2\right)$, $f \in L^2(\Omega)$, *u a weak solution of problem* (1.1)–(2.1), *and π the corresponding pressure field. Then* $u \in W^{2,2}(\Omega)$, $\nabla\pi \in L^2(\Omega)$ *and*

$$\|D^2 u\| + \|\nabla\pi\| \le c\|f\|(1 + \|f\|^2). \tag{2.4}$$

Theorem 2.4. *Let* $p \in \left[\frac{3}{2}, 2\right)$, $f \in L^3(\Omega)$, *and u, π as in Theorem* 2.3. *Then, besides the regularity stated in Theorem* 2.3, *there holds*

$$u \in W^{1,q}(\Omega) \quad \textit{for all } q \in (1, +\infty).$$

The above assumption on f may be replaced by the condition

$$f \in W^{-1,q}(\Omega) \quad \text{for all } q < +\infty.$$

In the remaining part of this section we recall some preliminary results. The first one is a well known regularity result of solutions of the Stokes system, due to Cattabriga [11]. Let us consider the following Stokes problem:

$$\left.\begin{aligned} \Delta W = \nabla \Pi + G \quad &\text{in } \Omega, \\ \nabla \cdot W = 0 \quad &\text{in } \Omega, \\ W_{|\Gamma} = 0,\ W \quad &x'\text{-periodic.} \end{aligned}\right\} \tag{2.5}$$

A field W is called a q-weak solution of (2.5) if $W \in V_q$ for some $q \in (1, +\infty)$, and W satisfies the identity

$$\int_\Omega \nabla W \cdot \nabla \varphi \, dx = \int_\Omega G \cdot \varphi \, dx \tag{2.6}$$

for all $\varphi \in V_{q'}$.

Lemma 2.5. *For every $G \in W^{-1,q}(\Omega)$, $1 < q < +\infty$, there exists one and only one q-weak solution W of the Stokes problem* (2.5)*. Moreover, the solution satisfies the estimate*

$$\|W\|_{1,q} + \|\Pi\|_q \leq C_q \|G\|_{-1,q},$$

where C_q is a positive constant and Π is the pressure field associated to W by de Rham's Theorem.

For the proof of the above result we refer to [11], [14].

Lemma 2.6. *If $\nabla g = \nabla \cdot G$ for some $G \in L^q(\Omega)$, then $g \in L^q(\Omega)$ and*

$$\|g - \bar{g}\|_q \leq c \|G\|_q,$$

where $\bar{g}$ is the mean value of g in Ω.

For the proof we refer, for instance, to [10].

3. Proof of Theorem 2.3

As in previous articles (see, for instance, [6], [7], [12]), we replace the use of the differential quotients method in the tangential directions by formal differentiation in the same directions.

Let us define the second order tensor S as

$$S = (\mu + |D|)^{p-2} D,$$

where D is an arbitrary second order tensor. It is easy to verify that

$$\frac{\partial S_{ij}}{\partial D_{kl}} C_{ij} C_{kl} \geq (p-1)(\mu + |D|)^{p-2} |C|^2 \tag{3.1}$$

for any tensors C. Moreover,

$$\left| \frac{\partial S_{ij}}{\partial D_{kl}} \right| \leq (3-p)(\mu + |D|)^{p-2}. \tag{3.2}$$

Define, for $s = 1, 2$,

$$J_s(u) = \int_\Omega \nabla \cdot [(\mu + |\mathscr{D}u|)^{p-2}\mathscr{D}u] \cdot \partial_{ss}u \, dx$$

and

$$I_s(u) = \int_\Omega (\mu + |\mathscr{D}u|)^{p-2}|\partial_s \mathscr{D}u|^2 \, dx. \tag{3.3}$$

By integrating twice by parts and taking into account (3.1) it is easily seen that (see [7] for details)

$$J_s(u) \geq (p-1)I_s(u). \tag{3.4}$$

Let us consider the following generalized Stokes system:

$$\left.\begin{aligned} -\nu_0\Delta u - \nu_1\nabla \cdot [(\mu + |\mathscr{D}u|)^{p-2}\mathscr{D}u] + \nabla\pi = f \quad \text{in } \Omega, \\ \nabla \cdot u = 0 \quad \text{in } \Omega, \end{aligned}\right\} \tag{3.5}$$

with the boundary conditions (2.1). For such a system we prove the following result

Proposition 3.1. *Let* $p \in \left[\frac{3}{2}, 2\right)$, $f \in L^2(\Omega)$, *and* (u, π) *be a weak solution of problem* (3.5)–(2.1). *Then* $u \in W^{2,2}(\Omega)$, $\nabla\pi \in L^2(\Omega)$ *and*

$$\|D^2u\| + \|\nabla\pi\| \leq c\|f\|. \tag{3.6}$$

The proof of this result is split into three fundamental steps: the first step consists of estimating the tangential derivatives of the velocity and pressure fields; the second step consists of estimating the normal derivatives of the velocity field; the last step, which is a direct consequence of the previous ones, consists of estimating the normal derivative of the pressure field. In order to make the reading easier, we prefer to present each step in a separate lemma.

Lemma 3.2. *Let* $p \in (1, 2)$, $f \in L^2(\Omega)$, *and let* (u, π) *be a weak solution of problem* (3.5)–(2.1). *Then* $D_*^2 u, \nabla_*\pi \in Ł^2(\Omega)$ *and, for* $s = 1, 2$,

$$\|D_*^2u\|^2 + I_s(u) + \|\nabla_*\pi\|^2 \leq c\|f\|^2. \tag{3.7}$$

Proof. Multiplying equation $(3.5)_1$ by $\partial_{ss}^2 u$, $s = 1, 2$, integrating twice by parts and, finally, using (3.4) one has

$$\nu_0 \int_\Omega |\nabla_* \nabla u|^2 \, dx + \nu_1 (p-1) \int_\Omega (\mu + |\mathscr{D}u|)^{p-2} |\partial_s \mathscr{D}u|^2 \, dx = - \int_\Omega f \cdot \partial^2_{ss} u \, dx. \quad (3.8)$$

By applying the Hölder and Cauchy–Schwarz inequalities, one obtains that

$$\|\nabla_* \nabla u\| \le c\|f\|$$

and

$$I_s(u) \le c\|f\|^2.$$

As far as the pressure term is concerned, let us differentiate the first equation $(3.5)_1$ with respect to x_s, $s = 1, 2$:

$$\nabla \partial_s \pi = \nu_0 \nabla \cdot \partial_s \nabla u + \nu_1 \nabla \cdot \partial_s [(\mu + |\mathscr{D}u|)^{p-2} \mathscr{D}u] + \partial_s f.$$

From Lemma 2.6, we only have to estimate the term $\partial_s[(\mu + |\mathscr{D}u|)^{p-2}\mathscr{D}u]$. Since

$$\begin{aligned} &\partial_s[(\mu + |\mathscr{D}u|)^{p-2}\mathscr{D}u] \\ &\qquad = (\mu + |\mathscr{D}u|)^{p-2} \partial_s \mathscr{D}u + (p-2)(\mu + |\mathscr{D}u|)^{p-3} |\mathscr{D}u|^{-1} (\mathscr{D}u \cdot \partial_s \mathscr{D}u) \mathscr{D}u, \end{aligned} \quad (3.9)$$

we have

$$|\partial_s[(\mu + |\mathscr{D}u|)^{p-2}\mathscr{D}u]| \le (3-p)(\mu + |\mathscr{D}u|)^{p-2} |\partial_s \mathscr{D}u|$$

almost everywhere in Ω. Hence $\partial_s[(\mu + |\mathscr{D}u|)^{p-2}\mathscr{D}u]$ belongs to $L^2(\Omega)$ and

$$\int_\Omega |\partial_s[(\mu + |\mathscr{D}u|)^{p-2}\mathscr{D}u]|^2 \, dx \le c\mu^{p-2} I_s(u).$$

By applying Lemma 2.6 we have

$$\|\partial_s \pi\|^2 \le c\mu^{p-2} I_s(u) + \|\partial_s \nabla u\|^2 + c\|f\|^2, \qquad s = 1, 2,$$

from which, by the above estimates on $|\nabla_* \nabla u|$ and $I_s(u)$, one obtains (3.7). □

Lemma 3.3. *Let* $p \in \left[\frac{3}{2}, 2\right)$, $f \in L^2(\Omega)$, *and let* (u, π) *be a weak solution of problem* (3.5)–(2.1). *Then*

$$\sum_{l=1}^{2} \|\partial^2_{33} u_l\| \le c\|f\|. \quad (3.10)$$

Proof. By using (3.9), the j-th equation $(3.5)_1$, $j = 1, 2$, takes the form

$$2\nu_0 \partial^2_{33} u_j + \nu_1 (\mu + |\mathscr{D}u|)^{p-2} \partial^2_{33} u_j \\ + 2\nu_1 (p-2)(\mu + |\mathscr{D}u|)^{p-3} |\mathscr{D}u|^{-1} \mathscr{D}_{j3} \sum_{l=1}^{2} \mathscr{D}_{l3} \partial^2_{33} u_l = F_j - 2f_j + 2\partial_j \pi, \qquad (3.11)$$

where

$$F_j(x) = -[2\nu_0 + \nu_1(\mu + |\mathscr{D}u|)^{p-2}] \sum_{k=1}^{2} \partial^2_{kk} u_j \\ - 2\nu_1 (p-2)(\mu + |\mathscr{D}u|)^{p-3} |\mathscr{D}u|^{-1} \left[\partial^2_{33} u_3 \mathscr{D}_{33} \mathscr{D}_{j3} + \sum_{\substack{l,m,k=1 \\ (k,m) \neq (3,3)}}^{3} \partial^2_{km} u_l \mathscr{D}_{jk} \mathscr{D}_{lm} \right].$$

Equations (3.11), $j = 1, 2$, can be treated as a 2×2 linear system in the unknowns $\partial^2_{33} u_j$, $j = 1, 2$. We denote the elements of the matrix $A = A(x)$ associated with such a system as a_{jl}, where $j, l = 1, 2$. Then we can rewrite the system as

$$\sum_{l=1}^{2} a_{jl} \partial^2_{33} u_l = G_j, \qquad (3.12)$$

where the elements of the matrix of the system are given by

$$a_{jl} = [2\nu_0 + \nu_1(\mu + |\mathscr{D}u|)^{p-2}] \delta_{jl} + 2\nu_1 (p-2)(\mu + |\mathscr{D}u|)^{p-3} |\mathscr{D}u|^{-1} \mathscr{D}_{j3} \mathscr{D}_{l3}$$

and

$$G_j = F_j - 2f_j + 2\partial_j \pi.$$

Note that $a_{jl} = a_{lj}$; moreover, if $\xi = (\xi_1, \xi_2, 0)$ then

$$\sum_{j,l=1}^{2} a_{jl} \xi_j \xi_l = [2\nu_0 + \nu_1(\mu + |\mathscr{D}u|)^{p-2}] |\xi|^2 \\ + 2\nu_1 (p-2)(\mu + |\mathscr{D}u|)^{p-3} |\mathscr{D}u|^{-1} [(\mathscr{D}u)\xi]^2_3,$$

where $(\mathscr{D}u)\xi = (\mathscr{D}u)_{ij} \xi_j$; hence

$$\sum_{j,l=1}^{2} a_{jl} \xi_j \xi_l \geq \left[2\nu_0 + 2\nu_1 \left(p - \frac{3}{2} \right) (\mu + |\mathscr{D}u|)^{p-2} \right] |\xi|^2. \qquad (3.13)$$

This means that, in the hypothesis $p \geq \frac{3}{2}$, the matrix $A = (a_{jl})$ is definite positive a.e. in $x \in \Omega$. From (3.12) and (3.13) with $\xi_j = \partial_{33}^2 u_j$, $j = 1, 2$, we get

$$\left[2\nu_0 + 2\nu_1\left(p - \frac{3}{2}\right)(\mu + |\mathscr{D}u|)^{p-2}\right]\sum_{l=1}^{2}|\partial_{33}^2 u_l| \leq \left[\sum_{l=1}^{2}|G_l|^2\right]^{1/2} \text{ a.e. in } x \in \Omega. \tag{3.14}$$

Straightforward calculations show that, for $j = 1, 2$,

$$|G_j| \leq c\left[2\nu_0 + 2\nu_1\left(p - \tfrac{3}{2}\right)(\mu + |\mathscr{D}u|)^{p-2}\right]|D_*^2 u| + 2|\partial_j \pi| + 2|f_j|$$

almost everywhere in Ω. By using the above estimate in (3.14) and dividing both sides by $\left[2\nu_0 + 2\nu_1\left(p - \frac{3}{2}\right)(\mu + |\mathscr{D}u|)^{p-2}\right]$ we get

$$\sum_{l=1}^{2}|\partial_{33}^2 u_l| \leq c|D_*^2 u| + \frac{c}{2\nu_0}(|\nabla_* \pi| + |f|), \tag{3.15}$$

almost everywhere in Ω. From the hypothesis and the previous lemma, all the terms on the right-hand side belong to $L^2(\Omega)$. The lemma is proven. □

From the above two lemmas we have obtained $D^2 u \in L^2(\Omega)$ and

$$\|D^2 u\| \leq c\|f\|.$$

Lemma 3.4. *Let* $p \in \left[\frac{3}{2}, 2\right)$, $f \in L^2(\Omega)$, *and let* (u, π) *be a weak solution of problem* (3.5)–(2.1). *Then*

$$\|\partial_3 \pi\| \leq c\|f\|. \tag{3.16}$$

Proof. From the third of the three equations $(3.5)_1$, one can estimate $\partial_3 \pi$ in terms of quantities already estimated. Since

$$|\partial_3 \pi| \leq c[\nu_0 + 2\nu_1(p-2)(\mu + |\mathscr{D}u|)^{p-2}]|D^2 u| + |f_3|,$$

almost everywhere in Ω, straightforward calculations together with Lemmas 3.2–3.3 lead to (3.16). □

Therefore, we have also obtained

$$\|\nabla \pi\| \leq c\|f\|$$

and the proof of Proposition 3.1 is complete.

Proof of Theorem 2.3. For the proof we follow the usual way of treating the convective term $(u \cdot \nabla)u$ as a right-hand side and deriving a priori estimates. First of all we observe that the validity of the identity

$$\int_\Omega (u \cdot \nabla)u \cdot u \, dx = 0$$

implies that estimates (2.3) still hold for weak solutions of the complete system (1.1). Hence the solution belongs to $W^{1,2}(\Omega)$ and

$$\|u\|_{1,2} \le c\|f\|.$$

Set

$$F = f - (u \cdot \nabla)u.$$

By Proposition 3.1 it follows that $F \in L^2(\Omega)$ implies

$$\|u\|_{2,2} \le c\|F\|.$$

Let us estimate the L^2-norm of F. By applying Hölder's inequality, then Sobolev's inequality and Gagliardo–Nirenberg's inequality, there holds

$$\|(u \cdot \nabla)u\| \le \|u\|_6 \|\nabla u\|_3 \le c\|\nabla u\|^{3/2} \|D^2 u\|^{1/2} + c\|\nabla u\|^2.$$

Then by using the Cauchy–Schwarz inequality, we get

$$\|(u \cdot \nabla)u\| \le c\|\nabla u\|^2 + c\|\nabla u\|^3 + c\|D^2 u\|.$$

Therefore

$$\|F\| \le \|f\| + \|(u \cdot \nabla)u\| \le \|f\| + c\|f\|^3 + c\|D^2 u\|.$$

This enables us to obtain the desired estimate on the $W^{2,2}$-norm of u. □

4. Proof of Theorem 2.4

For simplicity, from now on, we assume that the force field f is in divergence form. Indeed it is well known that for any $f \in L^3(\Omega)$, there exists a tensor field $F \in W^{1,3}(\Omega)$ such that $f = \nabla \cdot F$.

Let (u, π) be a weak solution of problem (1.1)–(2.1) and let us consider the following Stokes problem:

$$\left.\begin{aligned} -\nu_0 \Delta v = -\nabla\sigma + \nu_1 \nabla \cdot S(\mathscr{D}u) + \nabla \cdot F - (u \cdot \nabla)u \quad &\text{in } \Omega, \\ \nabla \cdot v = 0 \quad &\text{in } \Omega, \\ v_{|\Gamma} = 0,\ v \quad x'\text{-periodic}, \end{aligned}\right\} \tag{4.1}$$

with $S(\mathscr{D}u) = (\mu + |\mathscr{D}u|)^{p-2}\mathscr{D}u$. We set

$$G(x) = \nu_1 \nabla \cdot S(\mathscr{D}u) + \nabla \cdot F - (u \cdot \nabla)u. \tag{4.2}$$

Lemma 4.1. *Let be $u \in W^{1,r}(\Omega)$ for some $r \in [6, +\infty)$. Then the corresponding Stokes problem* (4.1) *admits a unique weak solution v. Further, the solution v belongs to $W^{1,r/(p-1)}(\Omega)$ and the associated pressure field σ belongs to $L^{r/(p-1)}(\Omega)$.*

Proof. By the embedding of $W^{1,3}(\Omega)$ in $L^q(\Omega)$, for any $q \in (1, +\infty)$, we have $F \in L^q(\Omega)$; thus $\nabla \cdot F \in W^{-1,q}(\Omega)$ for any $q \in (1, +\infty)$. Further, let us note that $u \in W^{1,r}(\Omega)$ implies $S(\mathscr{D}u) \in L^{r/(p-1)}(\Omega)$; thus, as before, $\nabla \cdot S(\mathscr{D}u) \in W^{-1,r/(p-1)}(\Omega)$. Finally $(u \cdot \nabla)u$ belongs to $W^{-1,r/(p-1)}(\Omega)$, too. Indeed, for any $\varphi \in W_0^{1,r/(r-p+1)}(\Omega)$, where $\frac{r}{r-p+1}$ is the dual exponent of $\frac{r}{p-1}$, we have

$$\left|\int_\Omega (u \cdot \nabla)u \cdot \varphi \, dx\right| = \left|\int_\Omega (u \cdot \nabla)\varphi \cdot u \, dx\right| \le \|\nabla\varphi\|_{r/(r-p+1)} \|u^2\|_{r/(p-1)} < +\infty,$$

since $u \in W^{1,r}(\Omega)$ implies that $u \in L^q(\Omega)$ for any q. Hence $G(x) \in W^{-1,r/(p-1)}(\Omega)$, where $G(x)$ is defined by (4.2). These arguments allow us to apply Lemma 2.5, which leads to the desired result. □

Lemma 4.2. *Let u be a weak solution of* (1.1)–(2.1) *and let v be the corresponding weak solution of* (4.1). *Then $u = v$.*

Proof. Let us take the difference $u - v$. By Definition 2.2 and the definition (2.6) of weak solution for the Stokes problem and Lemma 2.5 we have that u and v belong to V_2 and

$$\nu_0 \int_\Omega \nabla(u - v) \cdot \nabla\varphi = 0 \qquad \text{for any } \varphi \in \mathscr{V}(\Omega).$$

By standard arguments this implies that $\nabla(u - v) = 0$ a.e. and therefore, employing the boundary conditions, we get $u = v$. □

Proof of Theorem 2.4. Set

$$q_0 = 6, \qquad q_{n+1} = \frac{q_n}{p-1} \tag{4.3}$$

for each non-negative integer n. By Theorem 2.3 we know that $f \in L^3(\Omega)$ implies (at least) that the velocity field u belongs to $W^{2,2}(\Omega)$, hence to $W^{1,q_0}(\Omega)$, $q_0 = 6$, by standard embedding. Assume that $u \in W^{1,q_n}(\Omega)$ for some $q_n \geq 6$. By Lemma 4.1 the solution v of system (4.1) belongs to $W^{1,q_{n+1}}(\Omega)$, with q_{n+1} given by (4.3). Furthermore, by Lemma 4.2, this implies that $u \in W^{1,q_{n+1}}(\Omega)$. Observing that the above sequence monotonically increases and diverges to infinity as n goes to $+\infty$, we obtain that $u \in W^{1,q}(\Omega)$ for any $q < +\infty$. □

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Received March 18, 2008; revised July 22, 2008

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