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Exponential stabilization of periodic solutions of a system of KdV equations*

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Abstract. We consider a coupled nonlinear dispersive system of Korteweg-de Vries type in the presence of a dissipative mechanism. First we prove that the Cauchy problem is globally well posed in a suitable periodic Sobolev space and our main result says that the L^2 and L^∞ norms of the solutions decay exponentially fast as $t \to +\infty$.

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1. Introduction

We consider a coupled dispersive system of equations of Korteweg-de Vries type under the effect of dissipative mechanisms

$$\begin{aligned} u_t - (Hu)_x - a_3(Hv)_x + uu_x + a_1 vv_x + a_2(uv)_x + \varepsilon Lu = 0, \\ v_t - (Hv)_x - a_3(Hu)_x + vv_x + a_2 uu_x + a_1(uv)_x + \varepsilon Lv = 0, \end{aligned} \tag{1.1}$$

with initial conditions

$$u(x,0) = \varphi_1(x), \qquad v(x,0) = \varphi_2(x) \tag{1.2}$$

and periodic boundary conditions. In (1.1), a_1, a_2, a_3 and ε are real constants with $\varepsilon > 0$, $u = u(x,t)$, $v = v(x,t)$ are real-valued functions, $0 < x < 1$, $t > 0$, and H and L are pseudo-differential operators of orders $\mu \geq 0$ and $\eta \geq 0$, respectively, whose symbols $h(k)$ and $l(k)$ satisfy appropriate conditions stated below. A distinguished special case included in (1.1) (when $H = L = -\frac{\partial^2}{\partial x^2}$) is the following system

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$$\begin{aligned} u_t + u_{xxx} + a_3 v_{xxx} + u u_x + a_1 v v_x + a_2 (uv)_x - \varepsilon u_{xx} = 0, \\ v_t + v_{xxx} + a_3 u_{xxx} + v v_x + a_2 u u_x + a_1 (uv)_x - \varepsilon v_{xx} = 0. \end{aligned} \tag{1.3}$$

J. A. Gear and R. Grimshaw [10] derived model (1.3) with $\varepsilon = 0$ to describe strong interactions of two long waves in a stratified fluid. System (1.3) has been intensively studied in recent years. The Cauchy problem for (1.3) with $\varepsilon = 0$ was studied by J. Bona et al. [8], J. Marshall Ash et al. [2] and F. Linares and M. Panthee [13] (see also the references therein). In [5], E. Bisognin et al. studied the following generalization of system (1.3),

$$\begin{aligned} u_t + u_{xxx} + a_3 v_{xxx} + u^p u_x + a_1 v^p v_x + a_2 (u^p v)_x - \varepsilon u_{xx} = 0, \\ v_t + v_{xxx} + a_3 u_{xxx} + v^p v_x + a_2 u^p u_x + a_1 (u v^p)_x - \varepsilon v_{xx} = 0, \end{aligned} \tag{1.4}$$

where $p \geq 1$ is any integer, with $-\infty < x < \infty$ and $\varepsilon > 0$. One of the results given in [5] is that the solutions of (1.4) decay algebraically at the same rate enjoyed by the solutions of the generalized KdV–Burgers equation provided the initial data are sufficiently small, $|a_3| < 1$ and $p > 4$. Nevertheless, when the nonlinearity is as in (1.3), that is, $p = 1$, in [5] was only showed the asymptotic stability as $t \to +\infty$, without giving any specific rate of decay. Our main concern in this article is to give a satisfactory answer on the uniform stabilization for the solutions of system (1.1). Some other works on related dispersive models are [1], [3], [4], [6], [7], [14], [15] (and the references therein). Let $\Omega = \{x \in \mathbb{R} \mid 0 < x < 1\}$. For $1 \leq q \leq \infty$, $L^q(\Omega)$ denotes the Banach space of measurable functions defined on Ω which are q-th power Lebesgue integrable (essentially bounded in the case $q = \infty$). The usual norm of $L^q(\Omega)$ is denoted by $\|\cdot\|_{L^q}$. By $L^q_p(\Omega)$ we denote the space of real functions in $L^q(\Omega)$ which are periodic of period 1 equipped with the same norm of $L^q(\Omega)$. If $s \geq 0$ then we denote by $H^s_p(\Omega)$ the space of functions u in $L^2_p(\Omega)$ which satisfy

$$\|u\|^2_{H^s_p} = \sum_{k=-\infty}^{+\infty} (1 + |k|^2)^s |u_k|^2 < +\infty. \tag{1.5}$$

Here u_k are the Fourier coefficients of u with respect to the system $\{\exp(2k\pi i x) \mid k \in \mathbb{Z}\}$, and $H^s_p(\Omega)$ is a Hilbert space with respect to the inner product

$$(u, v)_{H^s_p} = \sum_{k=-\infty}^{+\infty} (1 + |k|^2)^s u_k \bar{v}_k,$$

whose norm (given by (1.5)) is equivalent to the one in the usual Sobolev space $H^s(\Omega)$ (see for instance R. Temam [17]). Notice that by Parseval's identity

$(u,v)_{H_p^0} = (u,v)_{L^2}$ for any u and v in $L_p^2(\Omega)$, where $(\ ,\ )_{L^2}$ denotes the usual inner product of $L^2(\Omega)$.

We denote by $\dot{L}_p^2(\Omega)$ (resp. $\dot{H}_p^s(\Omega)$) the space of functions $u \in L_p^2(\Omega)$ (resp. $H_p^s(\Omega)$) such that

$$u_0 = \int_\Omega u(x)\,dx = 0.$$

We recall that in $\dot{H}_p^1(\Omega)$ Poincaré's inequality holds, that is, there is a positive constant $c(\Omega)$ such that

$$\|u\|_{L^2} \le c(\Omega)\|u_x\|_{L^2},$$

for any $u \in \dot{H}_p^1(\Omega)$.

Given $\mu \ge 0$ and $\eta \ge 0$, we assume that H and L are pseudo-differential operators of order μ and η, respectively, defined by

$$Hu(x) = \sum_{k=-\infty}^{+\infty} h(k)u_k \exp(2k\pi i x), \qquad Lu(x) = \sum_{k=-\infty}^{+\infty} l(k)u_k \exp(2k\pi i x),$$

where the symbols $h(k)$ and $l(k)$ are even real-valued functions satisfying the following hypotheses:

There exist positive constants c_i, $i = 1, \dots, 4$ such that

$$c_1|k|^\mu \le h(k) \le c_2|k|^\mu, \qquad c_3|k|^\eta \le l(k) \le c_4|k|^\eta \tag{1.6}$$

for all $k \in \mathbb{Z}$.

Remark. Note that for system (1.3) hypotheses (1.6) are satisfied with $h(k) = \ell(k) = k^2$. Note also that we may consider in (1.3) more general dissipative terms of type $\varepsilon(-1)^m\partial_x^{2m}u$, $\varepsilon(-1)^m\partial_x^{2m}v$, which correspond to the symbols $\ell(k) = k^{2m}$, $m \in \{1, 2, \dots\}$.

The Cauchy problem (1.1)–(1.2) will be considered in the space $\dot{\mathscr{H}}_p^s(\Omega) = \dot{H}_p^s(\Omega) \times \dot{H}_p^s(\Omega)$ endowed with the inner product and the norm given by $(U,V)_s = (u,w)_{H_p^s} + (v,z)_{H_p^s}$ and $\|U\|_s = (U,V)_s^{1/2}$, where $U = (u,v)$, and $V = (w,z)$ are in $\dot{\mathscr{H}}_p^s(\Omega)$. To simplify notations we also denote by $\|\ \|_{L^q}$ the natural norm of $L^q(\Omega) \times L^q(\Omega)$ and by $(\ ,\ )_{L^2}$ the usual inner product of $L^2(\Omega) \times L^2(\Omega)$. We rewrite (1.1)–(1.2) as

$$\begin{aligned} U_t - (MU)_x + F(U)_x + \varepsilon BU &= 0 \\ U(x,0) &= \varphi(x), \end{aligned} \tag{1.7}$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix},$$
$$M = \begin{bmatrix} H & a_3 H \\ a_3 H & H \end{bmatrix}, \qquad B = \begin{bmatrix} L & 0 \\ 0 & L \end{bmatrix}, \tag{1.8}$$

and the components of $F(U)$ are given by $F(U) = \binom{F_1(U)}{F_2(U)}$ with

$$F_1(U) = \frac{u^2}{2} + a_1 \frac{v^2}{2} + a_2(uv),$$
$$F_2(U) = \frac{v^2}{2} + a_2 \frac{u^2}{2} + a_1(uv). \tag{1.9}$$

Now we can describe the content of the present paper. Under the hypotheses (1.6), we show in Section 2 that the Cauchy problem (1.7) is globally well posed in the space $\dot{\mathscr{H}}_p^s(\Omega)$, for $s \geq s_0 = \max\{\mu + 1, \eta\}$ and μ, η, a_3 satisfying suitable conditions (see Theorems 2.5 and 2.7). We first study the linear problem associated with (1.7) and prove the existence of a unique local solution for the Cauchy problem (1.7) by using a fixed point theorem and techniques from the theory of semigroups of linear operators. Then we use energy estimates to extend the local solution globally. In Section 3, we show that the energy of the global solution $U(\cdot, t)$ of (1.7) stabilizes exponentially. More precisely, we prove the following result: If $2 \leq q \leq \infty$, then there exist positive constants $C = C(q, \varphi)$ and γ such that

$$\|U(\cdot, t)\|_{L^q} \leq C \exp(-\gamma t) \qquad \text{for all } t \geq 0. \tag{1.10}$$

Our proof of (1.10) is based on some techniques developed in the work of C. Foias and J. C. Saut [9], adapted conveniently to model (1.7). The main point consists in proving that the function

$$\kappa(t) = \frac{\big(BU(\cdot, t), U(\cdot, t)\big)_{L^2}}{\|U(\cdot, t)\|_{L^2}^2}$$

is well defined for any $t > 0$ if $\varphi \not\equiv 0$, and has a finite positive limit as $t \to +\infty$. This is possible in our case because the system (1.7) has the backward uniqueness property (see Lemma 3.3).

Other notations used in this paper are as follows. $C(J; X)$ denotes the space of functions which are continuous in the real interval J and take values in the Banach space X. We denote by C a generic constant whose value may be different from a line or inequality to another. We also use the notation U^T to indicate the transpose of a vector $U = \binom{u}{v}$.

2. Global well-posedness

In this section we shall prove that the Cauchy problem (1.7) is globally well posed in the periodic Sobolev space $\dot{\mathscr{H}}_p^s(\Omega)$ for suitable values of s. First we study the linear problem associated with (1.7)

$$\begin{aligned} U_t - (MU)_x + \varepsilon BU = 0, \\ U(x,0) = \varphi(x), \end{aligned} \tag{2.1}$$

where the operators M and B are as in (1.8). We want to prove that problem (2.1) has a unique global solution using semigroup theory. We consider the initial data φ in $\dot{\mathscr{H}}_p^s(\Omega)$ with $s \geq s_0 = \max\{\mu + 1, \eta\}$, and study (2.1) as an evolution equation in $\dot{\mathscr{H}}_p^{s-s_0}(\Omega)$. Formally, the solution of (2.1) can be written as

$$U(x,t) = \sum_{k=-\infty}^{+\infty} e^{tA(k)} \varphi_k \exp(2k\pi i x),$$

where $\varphi_k = \binom{\varphi_{1_k}}{\varphi_{2_k}}$ and

$$A(k) = ikh(k)A - \varepsilon l(k)I \qquad \text{with } A = \begin{bmatrix} 1 & a_3 \\ a_3 & 1 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.2}$$

Lemma 2.1. *Assume that $|a_3| < 1$ and let λ_1, λ_2 be the eigenvalues of the matrix A. Then*

$$e^{tA(k)} = \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix} = D(k,t),$$

where

$$D_1 = D_4 = \tfrac{1}{2}\{\exp(ikh(k)\lambda_1 t) + \exp(ikl(k)\lambda_2 t)\}\exp(-\varepsilon l(k)t), \tag{2.3}$$

$$D_2 = D_3 = \tfrac{1}{2}\operatorname{sgn} a_3\{\exp(ikh(k)\lambda_1 t) - \exp(ikl(k)\lambda_2 t)\}\exp(-\varepsilon l(k)t). \tag{2.4}$$

Proof. This follows from a straightforward calculation using (2.2). □

Lemma 2.2. *Assume that* (1.6) *holds and let $|a_3| < 1$, $s \geq 0$, $\theta \geq 0$ and $\eta > 0$. Define*

$$E(t)\varphi(x) = \sum_{k=-\infty}^{+\infty} D(k,t)\varphi_k \exp(2k\pi i x), \qquad x \in \mathbb{R},\ t \geq 0. \tag{2.5}$$

Then there exists a positive constant $C = C(\theta, \eta, c_3) > 0$ *such that*

$$\|E(t)\varphi\|_{s+\theta} \le C[1 + (\varepsilon t)^{-2\theta/\eta}]^{1/2}\|\varphi\|_s \tag{2.6}$$

for all $\varphi \in \dot{\mathscr{H}}_p^s(\Omega)$ *and* $t > 0$.

Proof. From (1.6), (2.3) and (2.4) we obtain that

$$|D_j(k,t)| \le \exp\big(-\varepsilon l(k)t\big) \le \exp(-\varepsilon c_3|k|^\eta t), \qquad j = 1, \dots, 4.$$

Thus, by (2.5) we have

$$\begin{aligned} \|E(t)\varphi\|_{s+\theta}^2 &= \sum_{k=-\infty}^{+\infty} (1 + |k|^2)^{s+\theta}|D(k,t)\varphi_k|^2 \\ &\le \sum_{k=-\infty}^{+\infty} 4(1 + |k|^2)^{s+\theta}\exp(-2\varepsilon c_3|k|^\eta t)|\varphi_k|^2 \\ &\le 2^{\theta+2}\sup_{k\in\mathbb{Z}}[(1 + |k|^{2\theta})\exp(-2\varepsilon c_3|k|^\eta t)]\|\varphi\|_s^2 \end{aligned} \tag{2.7}$$

for all $t \ge 0$ whenever $\sup_{k\in\mathbb{Z}}[(1 + |k|^{2\theta})\exp(-2\varepsilon c_3|k|^\eta t)] < +\infty$. Clearly this is true if $\theta = 0$ and (2.6) follows from (2.7) (in fact we obtain that $\|E(t)\varphi\|_s \le 2\|\varphi\|_s$ for all $t \ge 0$). If $\theta > 0$, observe that

$$\begin{aligned} (1 + |k|^{2\theta})\exp(-2\varepsilon c_3|k|^\eta t) &\le 1 + \sup_{k\in\mathbb{Z}}[|k|^{2\theta}\exp(-2\varepsilon c_3|k|^\eta t)] \\ &\le 1 + \left(\frac{\theta}{c_3\eta}\right)^{2\theta/\eta}(\varepsilon t)^{-2\theta/\eta}\exp\left(-\frac{2\theta}{\eta}\right) \\ &\le \max\left\{1, \left(\frac{\theta}{c_3\eta}\right)^{2\theta/\eta}\right\}[1 + (\varepsilon t)^{-2\theta/\eta}] \end{aligned}$$

for all $k \in \mathbb{Z}$ and $t \ge 0$. Therefore, if $\theta > 0$, then (2.6) also follows from (2.7). □

Lemma 2.3. *Under the hypotheses of Lemma* 2.2, *let* $E(t)$ *be as defined in* (2.5), *for any* $\varphi \in \dot{\mathscr{H}}_p^s(\Omega)$. *Then* $\{E(t)\}_{t\ge 0}$ *is a* C_0 *semigroup in* $\dot{\mathscr{H}}_p^s(\Omega)$, *and the map* $t \in (0,\infty) \mapsto E(t)\varphi$ *is continuous with respect to the topology of* $\dot{\mathscr{H}}_p^{s+\theta}(\Omega)$ *for all* $\theta \ge 0$.

Proof. The proof is similar to the one given in Lemma 1.1 by R. J. Iorio [12]. □

As a consequence of Lemma 2.3 we obtain the following result.

Theorem 2.4. *Assume that* (1.6) *holds,* $|a_3| < 1$ *and* $s \geq s_0 = \max\{\mu + 1, \eta\}$ *with* $\eta > 0$. *If* $\varphi \in \dot{\mathscr{H}}_p^s(\Omega)$, *then the Cauchy problem* (2.1) *has a unique solution* $U(\cdot, t)$ *such that* $U \in C([0, \infty); \dot{\mathscr{H}}_p^s(\Omega))$ *and* $U_t \in C([0, \infty); \dot{\mathscr{H}}_p^{s-s_0}(\Omega))$.

Proof. Consider the linear operator $R_\varepsilon = -\varepsilon B + \partial_x M$ in $\dot{\mathscr{H}}_p^{s-s_0}(\Omega)$ with domain $\mathscr{D}(R_\varepsilon) = \dot{\mathscr{H}}_p^s(\Omega)$ and write (2.1) in $\dot{\mathscr{H}}_p^{s-s_0}(\Omega)$ as

$$U_t = R_\varepsilon U, \qquad U(\cdot, 0) = \varphi. \tag{2.8}$$

The above choice of $\mathscr{D}(R_\varepsilon)$ implies that $\mathscr{D}(R_\varepsilon) = \{\varphi \in \dot{\mathscr{H}}_p^{s-s_0}(\Omega) \mid R_\varepsilon\varphi \in \dot{\mathscr{H}}_p^{s-s_0}(\Omega)\}$. Denote by $\mathscr{L}$ the infinitesimal generator of the semigroup $\{E(t)\}_{t\geq 0}$ in $\dot{\mathscr{H}}_p^{s-s_0}(\Omega)$. Let us show that $\mathscr{L} = R_\varepsilon$. If $\varphi \in \mathscr{D}(R_\varepsilon)$, then $\varphi \in \dot{\mathscr{H}}_p^{s-s_0}(\Omega)$ and there exists $g \in \dot{\mathscr{H}}_p^{s-s_0}(\Omega)$ such that $\lim_{t\to 0^+} \left\| \frac{E(t)\varphi - \varphi}{t} - g \right\|_{s-s_0} = 0$. This implies that

$$\lim_{t\to 0^+} \left| \frac{\exp(tA(k))\varphi_k - \varphi_k}{t} - g_k \right|^2 = 0 \tag{2.9}$$

for any $k \in \mathbb{Z}$, where $\varphi_k = \binom{\varphi_{1_k}}{\varphi_{2_k}}$ and $g_k = \binom{g_{1_k}}{g_{2_k}}$. On the other hand, we have that

$$\begin{aligned} \lim_{t\to 0^+} \left| \frac{\exp(tA(k))\varphi_k - \varphi_k}{t} - g_k \right|^2 &= \lim_{t\to 0^+} \left| \frac{1}{t} \int_0^t [A(k)\exp(\sigma A(k))\varphi_k - g_k] \right|^2 d\sigma \\ &= |A(k)\varphi_k - g_k|^2 \end{aligned} \tag{2.10}$$

for any $k \in \mathbb{Z}$. From (2.9) and (2.10) we deduce that $g = R_\varepsilon\varphi$ in $\dot{\mathscr{H}}_p^{s-s_0}(\Omega)$ which together with $g \in \dot{\mathscr{H}}_p^{s-s_0}(\Omega)$ shows that $\mathscr{L} \subseteq R_\varepsilon$. Using similar arguments we can show that $\mathscr{L} \supseteq R_\varepsilon$. Since we know that $\{E(t)\}_{t\geq 0}$ is a C_0 semigroup of linear operators in $\dot{\mathscr{H}}_p^{s-s_0}(\Omega)$ by Lemma 2.3, it follows that $U(\cdot, t) = E(t)\varphi$ is the unique solution of (2.8) in the desired class. □

Now let us consider the nonlinear problem (1.7). As before, we assume that M and B are as in (1.8) and the components of $F(U)$ are given by (1.9).

Theorem 2.5 (Local existence and regularity). *Assume that* (1.6) *holds,* $|a_3| < 1$ *and* $s \geq s_0 = \max\{\mu + 1, \eta\}$ *with* $\mu \geq 0$, $\eta \geq 2$. *If* $\varphi \in \dot{\mathscr{H}}_p^s(\Omega)$, *then there exist* $T_0 > 0$ *and a unique solution* $U \in C([0, T_0]; \dot{\mathscr{H}}_p^s(\Omega))$ *of* (1.7) *such that* $U_t \in C([0, T_0]; \dot{\mathscr{H}}_p^{s-s_0}(\Omega))$. *Moreover,* $U \in C((0, T_0]; \dot{\mathscr{H}}_p^r(\Omega))$ *for all* $r \geq s$.

Proof. Let $T_0 > 0$, and consider the set of functions

$$Y_{s,T_0} = \left\{ U \in C([0, T]; \dot{\mathscr{H}}_p^s(\Omega)) \text{ such that } \sup_{0\leq t\leq T_0} \|U(\cdot, t) - E(t)\varphi\|_s \leq 1 \right\}, \tag{2.11}$$

endowed with the metric induced by the sup norm of $C\big([0,T_0];\dot{\mathscr{H}}_p^s(\Omega)\big)$. In the complete metric space Y_{s,T_0} we define the map $\mathscr{P}: Y_{s,T_0} \to C([0,T_0];\mathscr{H}^s)$ by

$$\mathscr{P}U(\cdot,t) = E(t)\varphi - \int_0^t E(t-\sigma)\partial_x F\big(U(\cdot,\sigma)\big)\,d\sigma$$

for $0 \le t \le T_0$. Using Lemma 2.2 with $\theta = 1$ and the inequality $\|uv\|_{H_p^s} \le C\|u\|_{H_p^s}\|v\|_{H_p^s}$, $u, v \in H_p^s(\Omega)$, $s > 1/2$ (see Lemma 1.1 in [16]) we can show that $\mathscr{P}(Y_{s,T_0}) \subset Y_{s,T_0}$ and $\mathscr{P}$ is a contraction in Y_{s,T_0}, if T_0 is chosen sufficiently small. In fact, if $U, V \in Y_{s,T_0}$, then

$$\begin{aligned}
\|\mathscr{P}U(\cdot,t) - E(t)\varphi\|_s &\le \int_0^t \big\|E(t-\sigma)\partial_x F\big(U(\cdot,\sigma)\big)\big\|_s\,d\sigma \\
&\le C\int_0^t [1+\varepsilon^{-2/\eta}(t-\sigma)^{-2/\eta}]^{1/2}\big\|\partial_x F\big(U(\cdot,\sigma)\big)\big\|_{s-1}\,d\sigma \\
&\le C(1+2\|\varphi\|_s)^2\int_0^t (1+\varepsilon^{-1/\eta}\sigma^{-1/\eta})\,d\sigma \\
&\le C(1+2\|\varphi\|_s)^2\left(T_0 + \varepsilon^{-1/\eta}\frac{\eta}{\eta-1}T_0^{(\eta-1)/\eta}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\|\mathscr{P}U(\cdot,t) - \mathscr{P}V(\cdot,t)\|_s \\
&\quad\le C\int_0^t [1+\varepsilon^{-2/\eta}(t-\sigma)^{-2/\eta}]^{1/2}\big\|\partial_x\big[F\big(U(\cdot,\sigma)\big) - F\big(V(\cdot,\sigma)\big)\big]\big\|_{s-1}\,d\sigma \\
&\quad\le 2C(1+2\|\varphi\|_s)^2\left(T_0 + \varepsilon^{-1/\eta}\frac{\eta}{\eta-1}T_0^{(\eta-1)/\eta}\right)\sup_{0\le t\le T_0}\|U-V\|_s,
\end{aligned}$$

where C is a positive constant that depends on η, c_3, $|a_1|$, $|a_2|$ and s. Choosing $T_0 > 0$ sufficient small, we can see that $\|\mathscr{P}U(\cdot,t) - E(t)\varphi\|_s \le 1$ and $\|\mathscr{P}U(\cdot,t) - \mathscr{P}V(\cdot,t)\|_s \le \alpha\sup_{0\le t\le T_0}\|U-V\|_s$, with $0<\alpha<1$. By the Fixed Point Theorem it follows that there exists a unique $U \in Y_{s,T_0}$ such that $\mathscr{P}U = U$. This gives a unique solution of the integral equation

$$U(\cdot,t) = E(t)\varphi - \int_0^t E(t-\sigma)\partial_x F\big(U(\cdot,\sigma)\big)\,d\sigma \tag{2.12}$$

for any $t \in [0,T_0]$. Since $U \in C\big([0,T_0];\mathscr{H}_p^s(\Omega)\big)$ (recall that $\mathscr{D}(R_\varepsilon) = \mathscr{H}_p^s(\Omega)$) we can differentiate (2.12) with respect to t to show that $U(\cdot,t)$ solves (1.7) and $U_t \in C\big([0,T_0];\mathscr{H}_p^{s-s_0}(\Omega)\big)$. The regularity result now follows from a bootstrapping argument. In fact, from (2.12) and (2.6) it is sufficient to show that $w \in C\big((0,T_0];\mathscr{H}_p^{s+\tau}(\Omega)\big)$ for all $\tau \geq 0$, where

$$w(t) = -\int_0^t E(t-\sigma)\partial_x F\big(U(\cdot,\sigma)\big)\,d\sigma \qquad \text{for all } t \in [0,T_0].$$

Assume, without loss of generality, that $t \in (0,T_0)$ and $t' > 0$ are such that $t+t' \in (0,T_0)$. Choosing $\theta = \tau + 1$ in (2.6) with $\tau \in [0,1)$ and proceeding as before we obtain that

$$\begin{aligned}
\|w(t+t') - w(t)\|_{s+\tau} &\leq C\int_t^{t+t'} \|E(t+t'-\sigma)\partial_x F(U)\|_{s+\tau}\,d\sigma \\
&\quad + \int_0^t \big\|\big(E(t+t'-\sigma) - E(t-\sigma)\big)\partial_x F(U)\big\|_{s+\tau}\,d\sigma \\
&\leq C(1+2\|\varphi\|_s)^2 \int_t^{t+t'} \big[1 + \big(\varepsilon(t+t'-\sigma)\big)^{-(2/\eta)(\tau+1)}\big]^{1/2}\,d\sigma \\
&\quad + \int_0^t \big\|\big(E(t+t'-\sigma) - E(t-\sigma)\big)\partial_x F(U)\big\|_{s+\tau}\,d\sigma.
\end{aligned}$$

Note that the first integral in the last inequality above tends to zero as $t' \to 0$ because $\eta \geq 2$, and applying the dominated convergence theorem we may show that the second term goes to zero too. Therefore, $U \in C\big((0,T_0];\dot{\mathscr{H}}_p^{s+\tau}(\Omega)\big)$ for all $0 \leq \tau < 1$. A repetition of this argument shows that $U \in C\big((0,T_0];\dot{\mathscr{H}}_p^{r+2\tau}(\Omega)\big)$. Finally, by induction, it follows that $U \in C\big((0,T_0];\dot{\mathscr{H}}_p^{s+n\tau}(\Omega)\big)$ for all $n \in \mathbb{N}$, which concludes the proof of Theorem 2.5. □

Next we prove some a priori estimates needed to extend the local solution $U(\cdot,t)$ of (1.7) for all $t \in [0,\infty)$.

Lemma 2.6. (i) *Assume the hypotheses of Theorem* 2.5 *and let* $U(\cdot,t)$ *be a solution of* (1.7) *such that* $U \in C\big([0,T^*);\dot{\mathscr{H}}_p^{s}(\Omega)\big)$ *and* $U_t \in C\big([0,T^*);\dot{\mathscr{H}}_p^{s-s_0}(\Omega)\big)$. *Then*

$$\|U(\cdot,t)\|_{L^2} \leq \|\varphi\|_{L^2} \qquad \textit{for all } 0 \leq t < T^*. \tag{2.13}$$

(ii) *Assume the hypotheses of Theorem* 2.5 *with* $\eta \geq \mu > 1$ *and* $\eta \geq 2$. *Then there exists a positive constant* $C_0 = C_0(a_1,a_2,a_3,\mu,\eta,T^*,\|\varphi\|_{L^2})$ *such that*

$$\|U(\cdot,t)\|_{\mu/2} \leq C_0, \qquad \textit{for all } 0 \leq t < T^*. \tag{2.14}$$

Proof. First we multiply the equation in (1.7) by U^T and integrate over Ω to obtain

$$\frac{d}{dt}\|U\|_{L^2}^2 + 2\varepsilon\int_\Omega U^T BU\,dx = 0. \tag{2.15}$$

Integrating (2.15) in t we get

$$\|U\|_{L^2}^2 + 2\varepsilon\int_0^t\int_\Omega U^T BU\,dx\,d\sigma = \|\varphi\|_{L^2}^2. \tag{2.16}$$

Note that by (1.6) and Parseval's identity

$$\int_\Omega U^T BU\,dx\,d\sigma = \sum_{k=-\infty}^{+\infty}\ell(k)(|u_k|^2 + |v_k|^2) \geq 0.$$

Thus (2.13) follows from (2.16).

Next we multiply the equation in (1.7) by $U^T F'(U) - 2(MU)^T$ and integrate over Ω to obtain

$$\begin{aligned}\int_\Omega \big(&U^T F'(U)U_t - 2(MU)^T U_t - U^T F'(U)(MU)_x + 2(MU)^T(MU)_x\\ &+ U^T F'(U)F(U)_x - 2(MU)^T F(U)_x + \varepsilon U^T F'(U)BU\\ &- 2\varepsilon(MU)^T BU\big)\,dx = 0.\end{aligned} \tag{2.17}$$

From (2.17), after some calculations, we find

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\left(\frac{1}{3}U^T F'(U)U - U^T MU\right)dx + \varepsilon\int_\Omega U^T F'(U)BU\,dx - 2\varepsilon\int_\Omega (MU)^T BU\,dx\\ &\quad + \int_\Omega \partial_x\left(\frac{1}{4}U^T F'(U)^2U + (MU)^T MU - U^T F'(U)MU\right)dx = 0.\end{aligned} \tag{2.18}$$

Observe that the last term in (2.18) vanishes due to the periodicity of U. Thus, an integration of (2.18) in t yields

$$\begin{aligned}&\int_\Omega\left(U^T MU - \frac{1}{3}U^T F'(U)U\right)dx + 2\varepsilon\int_0^t\int_\Omega \big(2(MU)^T BU - U^T F'(U)BU\big)\,dx\,d\sigma\\ &\quad = \int_\Omega\left(\varphi^T M\varphi - \frac{1}{3}\varphi^T F'(\varphi)\varphi\right)dx.\end{aligned} \tag{2.19}$$

Now, by hypotheses (1.6) we have

$$c_1(1-|a_3|)\|U\|_{\mu/2}^2 \leq \int_\Omega U^T MU\,dx \leq 2^{\mu/2}c_2(1+|a_3|)\|U\|_{\mu/2}^2. \tag{2.20}$$

Moreover, using the additional hypothesis $\eta \geq \mu > 1$ and part (i) we also have

$$\left| \int_\Omega U^T F'(U) U \, dx \right| \leq \|U\|_{L^2} \|F'(U)U\|_{L^2} \leq C_1 \|U\|_{L^2}^2 \|U\|_{L^\infty} \leq C_1 \|\varphi\|_{L^2}^2 \|U\|_{\mu/2}, \tag{2.21}$$

$$\left| \int_\Omega U^T F'(U) BU \, dx \right| \leq \|U\|_{L^2} \|U\|_{L^\infty} \|BU\|_{L^2} \leq 2^{\eta/2} C_2 \|\varphi\|_{L^2} \|U\|_{\mu/2}^2, \tag{2.22}$$

and

$$\left| \int_\Omega (MU)^T BU \, dx \right| \leq 2^{(\mu+\eta)/2} c_2 c_4 (1 + |a_3|) \|U\|_{\mu/2}^2, \tag{2.23}$$

for some positive constants C_1 and C_2. Then from (2.20)–(2.23) and (2.19) we deduce that

$$\|U\|_{\mu/2}^2 \leq \alpha + \beta \int_0^t \|U\|_{\mu/2}^2 \, ds \qquad \text{for all } 0 \leq t < T^* \tag{2.24}$$

for some positive constants α and β. Therefore, (2.14) follows from (2.24) and Gronwall's inequality. This completes the proof of Lemma 2.6. □

Theorem 2.7 (Global existence). *Assume that* (1.6) *holds,* $|a_3| < 1$ *and* $s \geq s_0 = \max\{\mu + 1, \eta\}$ *with* $\eta \geq \mu > 1$ *and* $\eta \geq 2$. *If* $\varphi \in \dot{\mathscr{H}}_p^s(\Omega)$, *then the Cauchy problem* (1.7) *has a unique solution* $U \in C([0,\infty); \dot{\mathscr{H}}_p^s(\Omega))$ *such that* $U_t \in C([0,\infty); \dot{\mathscr{H}}_p^{s-s_0}(\Omega))$.

Proof. First observe that by the construction of T_0 in Theorem 2.5 and a well-known technique (see [11] for example), we can extend the local solution U of (1.7) to a maximal interval of existence $[0, T^*)$ such that $U \in C([0, T^*); \dot{\mathscr{H}}_p^s(\Omega))$, $U_t \in C([0, T^*); \dot{\mathscr{H}}_p^{s-\eta}(\Omega))$, and $U \in C((0, T^*); \dot{\mathscr{H}}_p^r(\Omega))$ for all $r \geq s$. Moreover, either $T^* = +\infty$, or if $T^* < +\infty$, then $\lim_{t \to T^*} \|U(\cdot, t)\|_s = +\infty$. Thus, to prove Theorem 2.7 it is sufficient to show that $\|U(\cdot, t)\|_s$ is bounded on $[0, T^*)$ if $T^* < +\infty$. From (1.7), using the regularity of $U(\cdot, t)$ on $(0, T^*)$, we obtain that

$$\frac{1}{2} \frac{d}{dt} \|U\|_s^2 = (U, U_t)_s = (U_x, F(U))_s - (U, BU)_s \tag{2.25}$$

for $0 < t < T^*$. Since (1.6) holds and $\eta \geq 2$, then from (2.25) we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|U\|_s^2 &\le \|U_x\|_s\|F(U)\|_s - \varepsilon(U,BU)_s\\ &\le c_3^{-1/2}(U,BU)_s^{1/2}\|F(U)\|_s - \varepsilon(U,BU)_s\\ &\le \frac{1}{c_3\varepsilon}\|F(U)\|_s^2. \end{aligned}\tag{2.26}$$

Using (1.9) and the inequality $\|uv\|_{H_p^s} \le C\|u\|_{H_p^s}\|v\|_{H_p^s}$ for $u, v \in H_p^s(\Omega)$ and $s > 1/2$, we estimate $\|F(U)\|_s^2$ as follows:

$$\|F(U)\|_s^2 \le C(\|u^2\|_{H_p^s}^2 + \|uv\|_{H_p^s}^2 + \|v^2\|_{H_p^s}^2) \le C\|U\|_{\mu/2}^2\|U\|_s^2. \tag{2.27}$$

Therefore, from (2.26), (2.27) and Lemma 2.6 (ii) it follows that

$$\frac{d}{dt}\|U\|_s^2 \le C\|U\|_s^2 \quad \text{for all } 0 < t < T^*. \tag{2.28}$$

Now, integrating the inequality (2.28) over $[\delta, t]$ with $0 < \delta < t < T^*$ and then letting $\delta \to 0$, we deduce that

$$\sup_{0\le t<T^*}\|U(\cdot,t)\|_s \le C$$

for some positive constant C, which depends on s, T^* and $\|\varphi\|_s$. This completes the proof of Theorem 2.7. □

Theorem 2.8 (Continuous dependence). *Assume the hypotheses of Theorem* 2.7. *Then, for each* $T > 0$, *the map* $\mathscr{U} : \dot{\mathscr{H}}_p^s(\Omega) \to C([0,T]; \dot{\mathscr{H}}_p^s(\Omega))$, *defined by* $\mathscr{U}(\varphi) = U$ *where* $U = U(\cdot, t)$ *is the global solution of* (1.7), *is continuous.*

Proof. Let U and V denote the solutions of (1.7) with initial data $U(\cdot,0) = \varphi$ and $V(\cdot,0) = \psi$, respectively, and let $W = U - V$. Then W satisfies the initial value problem

$$\begin{aligned} W_t - (MW)_x + [F(U) - F(V)]_x + \varepsilon BW = 0.\\ W(\cdot,0) = \varphi - \psi. \end{aligned}$$

Proceeding as in the proof of Theorem 2.7, we obtain that

$$\frac{d}{dt}\|W\|_s^2 \le C(\varepsilon)\|F(U) - F(V)\|_s^2, \quad 0 < t \le T. \tag{2.29}$$

Estimating the right-hand side of (2.29) using (1.9) with $U = \binom{u}{v}$ and $V = \binom{w}{z}$ we find that

$$\|F(U) - F(V)\|_s^2 \le C(\varepsilon, |a_1|, |a_2|, s)(\|U\|_s + \|V\|_s)^2 \|W\|_s^2. \tag{2.30}$$

Since U and V satisfy (2.28) for any $t \in [0, T]$, it follows from (2.29) and (2.30) that

$$\frac{d}{dt}\|W(\cdot, t)\|_s^2 \le \tilde{C}\|W(\cdot, t)\|_s^2 \qquad \text{for all } 0 < t \le T,$$

where $\tilde{C}$ is a positive constant depending on s, $|a_1|$, $|a_2|$, T, $\|\varphi\|_s$, and $\|\psi\|_s$. Now, repeating the same argument used after (2.28), we obtain the inequality

$$\|W(\cdot, t)\|_s^2 \le \|\varphi - \psi\|_s^2 \exp(\tilde{C}T), \qquad 0 \le t \le T,$$

which implies the continuity of $\mathscr{U}$. □

3. Asymptotic behavior

Let $U = U(\cdot, t)$ denote the global solution of (1.7) obtained in Theorem 2.7. In this section we study the asymptotic behavior of $U(\cdot, t)$ as $t \to +\infty$. We begin with the following results.

Proposition 3.1. *Under all assumptions of Theorem* 2.7 *we have:*

(a) $\lim_{t\to+\infty} \big(BU(\cdot, t), U(\cdot, t)\big)_{L^2} = 0.$

(b) $\lim_{t\to+\infty} \|U(\cdot, t)\|_{L^2} = 0.$

(c) $\lim_{t\to+\infty} \|U(\cdot, t)\|_{L^\infty} = 0.$

Proof. From (2.16) we have

$$\int_0^\infty (BU, U)_{L^2}\, d\sigma = \int_0^\infty \int_\Omega U^T BU\, dx\, d\sigma \le \frac{1}{2\varepsilon}\|\varphi\|_{L^2}^2 < +\infty. \tag{3.1}$$

Multiplying the equation in (1.7) by $(BU)^T$ and integrating over Ω we obtain

$$\frac{d}{dt}(BU, U)_{L^2} + 2\varepsilon\|BU\|_{L^2}^2 = -2\int_\Omega (BU)^T F'(U) U_x\, dx, \tag{3.2}$$

because the term $\int_\Omega (BU)^T (MU)_x\, dx$ is equal to zero due to periodicity. Let us estimate the right-hand side of (3.2). By Lemma 2.6 (ii) and the embedding $\dot{H}_p^{\eta/2}(\Omega) \hookrightarrow \dot{L}_p^\infty(\Omega)$, $\eta \ge 2$, we know that $\|U(\cdot, t)\|_{L^\infty} \le C$ for any $t \ge 0$. Thus

$$\left|-2\int_\Omega (BU)^T F'(U) U_x\, dx\right| \le C\|BU\|_{L^2}\|U\|_{L^\infty}\|U_x\|_{L^2} \le C\|BU\|_{L^2}\|U_x\|_{L^2}. \tag{3.3}$$

But, using Parseval's identity and (1.6) we also know that

$$\|U_x(\cdot,t)\|_{L^2}^2 \le C\big(BU(\cdot,t),U(\cdot,t)\big)_{L^2} \qquad \text{for all } t \ge 0. \tag{3.4}$$

Therefore, from (3.3) and (3.4) we obtain the estimate

$$\left|-2\int_\Omega (BU)^T F'(U) U_x\,dx\right| \le \varepsilon\|BU\|_{L^2}^2 + C(\varepsilon)(BU,U)_{L^2}. \tag{3.5}$$

Now, integrating (3.2) over $[0,t]$ and using (3.5) and (3.1), we deduce that

$$(BU,U)_{L^2} + \varepsilon\int_0^t \|BU\|_{L^2}^2\,d\sigma \le (B\varphi,\varphi)_{L^2} + C(\varepsilon)\int_0^\infty (BU,U)_{L^2}\,d\sigma.$$

This implies that $\int_0^\infty \|BU(\cdot,\sigma)\|_{L^2}^2\,d\sigma < +\infty$. Consequently, from (3.2), (3.5) and (3.1) we conclude that

$$\int_0^\infty \left|\frac{d}{dt}(BU,U)_{L^2}\right| d\sigma < +\infty,$$

which together with (3.1) implies (a).

By Poincaré's inequality, (3.4) and part (a) we obtain (b). Finally, using the embedding $\dot H_p^1(\Omega) \hookrightarrow \dot L_p^\infty(\Omega)$, (3.4) and part (a) we also conclude (c). □

Next we shall show that, in fact, $(BU,U)_{L^2}$, $\|U\|_{L^2}$, and $\|U\|_{L^\infty}$ decay exponentially to zero as $t \to +\infty$. To do this we first prove some auxiliary lemmas.

Lemma 3.2. *Under all assumptions of Theorem* 2.7 *there exists a positive constant* C *such that*

$$\|F(U)_x\|_{L^2}^2 \le C(BU,U)_{L^2}^2. \tag{3.6}$$

Proof. Using (1.9), the embedding $\dot H_p^1(\Omega) \hookrightarrow \dot L_p^\infty(\Omega)$ and Poincaré's inequality, we have

$$\|F(U)_x\|_{L^2}^2 \le C\|U\|_{L^\infty}^2\|U_x\|_{L^2}^2 \le C\|U_x\|_{L^2}^4. \tag{3.7}$$

Then (3.6) follows from (3.7), since $\eta \ge 2$. □

The next lemma shows that the system (1.8) has the backward uniqueness property.

Lemma 3.3. *Under all assumptions of Theorem* 2.7, *if* $U(\cdot,t_0) = 0$ *for some* $t_0 > 0$, *then* $U(\cdot,t) = 0$ *for all* $0 \le t \le t_0$.

Proof. Assume that $U(\cdot, t_0) = 0$ for some $t_0 > 0$ and define $t_1 = \inf\{t \in [0, t_0] \mid U(\cdot, t) = 0\}$. Then, either (i) $t_1 = 0$ or (ii) $0 < t_1 \le t_0$. Let us show that (ii) does not occur. In fact, if (ii) holds then $U(\cdot, t) \neq 0$ for all $0 \le t < t_1$ and $U(\cdot, t_1) = 0$. Consider the function

$$\kappa(t) = \frac{\big(BU(\cdot, t), U(\cdot, t)\big)_{L^2}}{\|U(\cdot, t)\|_{L^2}^2}, \qquad 0 \le t < t_1. \tag{3.8}$$

A direct calculation gives us the identity

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\kappa(t) &= \|U\|_{L^2}^{-2}[(BU, U_t)_{L^2} - \kappa(t)(U, U_t)_{L^2}] \\ &= \|U\|_{L^2}^{-2}\big(BU - \kappa(t)U, U_t\big)_{L^2} \\ &= \|U\|_{L^2}^{-2}\big(BU - \kappa(t)U, (MU)_x - F(U)_x - \varepsilon BU\big)_{L^2}.\end{aligned} \tag{3.9}$$

Since $(\kappa U, BU - \kappa U)_{L^2} = 0$, it follows that

$$\begin{aligned}(BU - \kappa U, -\varepsilon BU)_{L^2} &= (BU - \kappa U, -\varepsilon BU)_{L^2} + \varepsilon(\kappa U, BU - \kappa U)_{L^2} \\ &= -\varepsilon\|BU - \kappa U\|_{L^2}^2.\end{aligned} \tag{3.10}$$

We also observe that $\big(BU - \kappa U, (MU)_x\big)_{L^2} = 0$. Thus, from (3.9) and (3.10) it follows that

$$\begin{aligned}\frac{d}{dt}\kappa(t) + \frac{2\varepsilon}{\|U\|_{L^2}^2}\|BU - \kappa U\|_{L^2}^2 &= 2\|U\|_{L^2}^{-2}\big(BU - \kappa U, -F(U)_x\big)_{L^2} \\ &\le 2\|U\|_{L^2}^{-2}\|BU - \kappa U\|_{L^2}\|F(U)_x\|_{L^2} \\ &\le \frac{\varepsilon}{\|U\|_{L^2}^2}\|BU - \kappa U\|_{L^2}^2 + \frac{1}{\varepsilon\|U\|_{L^2}^2}\|F(U)_x\|_{L^2}^2.\end{aligned}$$

Consequently,

$$\frac{d}{dt}\kappa(t) + \frac{\varepsilon}{\|U\|_{L^2}^2}\|BU - \kappa U\|_{L^2}^2 \le \frac{1}{\varepsilon\|U\|_{L^2}^2}\|F(U)_x\|_{L^2}^2.$$

The above inequality and Lemma 3.2 imply that

$$\frac{d}{dt}\kappa(t) + \frac{\varepsilon}{\|U\|_{L^2}^2}\|BU - \kappa U\|_{L^2}^2 \le C\frac{(BU, U)_{L^2}^2}{\|U\|_{L^2}^2} = C\kappa(t)(BU, U)_{L^2} \tag{3.11}$$

for $0 \le t < t_1$, where C is a positive constant. From (3.11), using Gronwall's inequality, we obtain that

$$\kappa(t) \le \kappa(0) \exp\Big(C \int_0^\infty \big(BU(\cdot,\sigma), U(\cdot,\sigma)\big)_{L^2}\, d\sigma \Big), \qquad 0 \le t < t_1. \tag{3.12}$$

On the other hand, observe that for $0 \le t < t_1$,

$$\frac{d}{dt} \log\|U(\cdot,t)\|_{L^2}^2 = \frac{2(U,U_t)_{L^2}}{\|U\|_{L^2}^2} = -2\varepsilon \frac{(BU,U)_{L^2}}{\|U\|_{L^2}^2},$$

that is,

$$-\frac{d}{dt} \log\|U(\cdot,t)\|_{L^2}^2 = 2\varepsilon\kappa(t), \qquad 0 \le t < t_1.$$

Integrating this equation in t and using (3.12) we obtain that

$$-\log\|U(\cdot,t)\|_{L^2}^2 \le 2\varepsilon C_1 t_1 + |\log\|\varphi\|_{L^2}^2|, \qquad 0 \le t < t_1,$$

where C_1 is a positive constant. This contradicts the fact that $U(\cdot,t_1) = 0$. Therefore, (ii) does not occur. Now, since $\|U(\cdot,t_1)\|_{L^2} = 0$, the equation (2.16) implies that $U(\cdot,t) = 0$ for all $0 \le t \le t_0$. □

Lemma 3.3 shows that the function $\kappa(t)$ introduced in (3.8) is well defined for all $t \ge 0$ if $\varphi \not\equiv 0$. Next we study the asymptotic behavior of $\kappa(t)$ as $t \to +\infty$.

Lemma 3.4. *Assume all assumptions of Theorem* 2.7 *and that* $\varphi \not\equiv 0$. *Then the limit* $\lim_{t\to+\infty} \kappa(t) = \lambda$ *exists and is positive.*

Proof. Since $\varphi \not\equiv 0$ then (3.11) holds for any $t \ge 0$. Let $W(t) = \frac{U(\cdot,t)}{\|U(\cdot,t)\|_{L^2}}$, so that $(BW,W)_{L^2} = \kappa(t)$. Then from (3.11) we have

$$\frac{d}{dt}\kappa(t) + \varepsilon\|BW - \kappa W\|_{L^2}^2 \le C(BW,W)_{L^2}^2 = C\kappa(t)(BU,U)_{L^2}. \tag{3.13}$$

Fix $t_0 > 0$. Integrating (3.13) over the interval $t_0 \le \sigma \le T$ we obtain that

$$\kappa(T) + \varepsilon \int_{t_0}^T \|BW - \kappa W\|_{L^2}^2\, d\sigma \le \kappa(t_0) + C \int_{t_0}^T \kappa(\sigma)(BU,U)_{L^2}\, d\sigma. \tag{3.14}$$

From (3.14) and Gronwall's inequality we deduce that

$$\kappa(T) \le \kappa(t_0) \exp\Big(C \int_{t_0}^T (BU,U)_{L^2}\, d\sigma \Big).$$

Using (3.1) and the above inequality we conclude that

$$\overline{\lim_{T\to+\infty}} \kappa(T) \le \kappa(t_0)\exp\Big(C\int_{t_0}^{\infty}(BU,U)_{L^2}\,d\sigma\Big) < +\infty. \tag{3.15}$$

Observe that (3.15) also implies $\overline{\lim}_{T\to+\infty}\kappa(T) \le \underline{\lim}_{t_0\to+\infty}\kappa(t_0)$. Consequently, $\lambda = \lim_{t\to+\infty}\kappa(t)$ exists. To see that it is positive observe that by Poincaré's inequality we have

$$\kappa(t) = \frac{(BU,U)_{L^2}}{\|U\|_{L^2}^2} \ge \frac{1}{C}\frac{(BU,U)_{L^2}}{(BU,U)_{L^2}} = \frac{1}{C} > 0.$$

Therefore $\lambda > 0$. This completes the proof of Lemma 3.4. □

Now we can state and prove our main result in this section.

Theorem 3.5. *Assume all assumptions of Theorem* 2.7 *and that* $\varphi \not\equiv 0$. *Then there exist positive constants* $C = C(\|\varphi\|_{L^2})$ *and* γ *such that*

(a) $\|U(\cdot,t)\|_{L^2} \le C\exp(-\gamma t)$ *for all* $t \ge 0$,

(b) $\big(BU(\cdot,t),U(\cdot,t)\big)_{L^2} \le C\exp(-\gamma t)$ *for all* $t \ge 0$,

(c) $\|U(\cdot,t)\|_{L^\infty} \le C\exp(-\gamma t)$ *for all* $t \ge 0$.

Proof. Since $\lambda = \lim_{t\to+\infty}\kappa(t)$ is finite by Lemma 3.4, then from (3.15) we deduce that

$$\kappa(t) \ge \lambda\exp\Big(-C\int_t^{\infty}\big(B(\cdot,\sigma),U(\cdot,\sigma)\big)_{L^2}\,d\sigma\Big), \qquad t > 0. \tag{3.16}$$

From equation (2.15) we know that

$$\frac{d}{dt}\|U\|_{L^2}^2 + 2\varepsilon\kappa(t)\|U\|_{L^2}^2 = 0. \tag{3.17}$$

Combining (3.16) and (3.17) we obtain that

$$\frac{d}{dt}\|U(\cdot,t)\|_{L^2}^2 + 2\varepsilon\lambda\exp\Big(-C\int_t^{\infty}\big(BU(\cdot,\sigma),U(\cdot,\sigma)\big)_{L^2}\,d\sigma\Big)\|U(\cdot,t)\|_{L^2}^2 \le 0.$$

Consequently

$$\|U(\cdot,t)\|_{L^2}^2 \le \|\varphi\|_{L^2}^2\exp\Big(-2\lambda\varepsilon\int_0^t\omega(r)\,dr\Big), \tag{3.18}$$

where

$$\omega(r) = \exp\Big(-C \int_r^\infty \big(BU(\cdot,\sigma), U(\cdot,\sigma)\big)_{L^2}\, d\sigma\Big).$$

On the other hand, since $\eta \geq 2$, using equation (2.15), Proposition 3.1 (b) and Poincaré's inequality we deduce that

$$\begin{aligned} 2\varepsilon \int_t^\infty \big(BU(\cdot,\sigma), U(\cdot,\sigma)\big)_{L^2}\, d\sigma &\leq \|U(\cdot,t)\|_{L^2}^2 \\ &\leq C\big(BU(\cdot,t), U(\cdot,t)\big)_{L^2} \qquad \text{for all } t \geq 0, \end{aligned}$$

which together with (3.1) implies that

$$\begin{aligned} \int_0^t \omega(r)\, dr &= t + \int_0^t \big(\omega(r) - 1\big)\, dr \\ &\geq t - C \int_0^t \int_r^\infty \big(BU(\cdot,\sigma), U(\cdot,\sigma)\big)_{L^2}\, d\sigma\, dr \geq t - \tilde{C}, \end{aligned} \tag{3.19}$$

where $\tilde{C}$ is a positive constant depending on ε and $\|\varphi\|_{L^2}$.

Now, substituting (3.19) into (3.18), we find that

$$\|U(\cdot,t)\|_{L^2}^2 \leq \|\varphi\|_{L^2}^2 \exp\big(-2\lambda\varepsilon(t - \tilde{C})\big) \leq C \exp(-2\lambda\varepsilon t) \qquad \text{for all } t \geq 0,$$

which proves (a) with $\gamma = \lambda\varepsilon$.

Since $\kappa(t)$ is bounded, (b) follows from (a). Finally, using the embedding $\dot{H}_p^1(\Omega) \hookrightarrow \dot{L}_p^\infty(\Omega)$, (3.4) and part (a) we obtain (c). □

Now our claim (1.10) in the introduction follows from Theorem 3.5 and interpolation.

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References

[1] J. Albert and F. Linares, Stability and symmetry of solitary-wave solutions to systems modeling interactions of long waves. *J. Math. Pures Appl.* (9) **79** (2000), 195–226. Zbl 0965.35128 MR 1750923

[2] J. M. Ash, J. Cohen, and G. Wang, On strongly interacting internal solitary waves. *J. Fourier Anal. Appl.* **2** (1996), 507–517. Zbl 0990.35120 MR 1412066

[3] P. Biler, Large-time behaviour of periodic solutions to dissipative equations of Kortweg-de Vries-Burgers type. *Bull. Polish Acad. Sci. Math.* **32** (1984), 401–405. Zbl 0561.35065 MR 782755

[4] V. Bisognin, On the asymptotic behaviour of the solutions of a nonlinear dispersive system of Benjamin-Bona-Mahony's type. *Boll. Un. Mat. Ital. B* (7) **10** (1996), 99–128. Zbl 0859.35108 MR 1385056

[5] E. Bisognin, V. Bisognin, and G. Perla Menzala, Asymptotic behavior in time of the solutions of a coupled system of KdV equations. *Funkcial. Ekvac.* **40** (1997), 353–370. Zbl 0899.35097 MR 1610697

[6] E. Bisognin, V. Bisognin, and G. Perla Menzala, Uniform stabilization and space-periodic solutions of a nonlinear dispersive system. *Dynam. Contin. Discrete Impuls. Systems* **7** (2000), 463–488. Zbl 0978.35044 MR 1795812

[7] J. L. Bona, M. Chen, and J.-C. Saut, Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media: II. The nonlinear theory. *Nonlinearity* **17** (2004), 925–952. Zbl 1059.35103 MR 2057134

[8] J. L. Bona, G. Ponce, J.-C. Saut, and M. M. Tom, A model system for strong interaction between internal solitary waves. *Comm. Math. Phys.* **143** (1992), 287–313. Zbl 0752.35056 MR 1145797

[9] C. Foias and J.-C. Saut, Asymptotic behavior, as $t \to +\infty$, of solutions of Navier-Stokes equations and nonlinear spectral manifolds. *Indiana Univ. Math. J.* **33** (1984), 459–477. Zbl 0565.35087 MR 740960

[10] J. A. Gear and R. Grimshaw, Weak and strong interactions between internal solitary waves. *Stud. Appl. Math.* **70** (1984), 235–258. Zbl 0548.76020 MR 742590

[11] J. A. Goldstein, *Semigroups of linear operators and applications*. The Clarendon Press, Oxford 1985. Zbl 0592.47034 MR 0790497

[12] R. J. Iório, Jr., KdV, BO and friends in weighted Sobolev spaces. In *Functional-analytic methods for partial differential equations* (Tokyo, 1989), Lecture Notes in Math. 1450, Springer, Berlin 1990, 104–121. Zbl 0717.35076 MR 1084604

[13] F. Linares and M. Panthee, On the Cauchy problem for a coupled system of KdV equations. *Commun. Pure Appl. Anal.* **3** (2004), 417–431. Zbl 1069.35074 MR 2098292

[14] C. P. Massarolo, G. P. Menzala, and A. F. Pazoto, A coupled system of Korteweg de Vries equations as singular limit of the Kuramoto-Sivashinsky equations. *Adv. Differential Equations* **12** (2007), 541–572. Zbl 1148.35078 MR 2321565

[15] J. M. Pereira, Stability and instability of solitary waves for a system of coupled BBM equations. *Appl. Anal.* **84** (2005), 807–819. Zbl 1081.35099 MR 2169878

[16] J. C. Saut and R. Temam, Remarks on the Korteweg-de Vries equation. *Israel J. Math.* **24** (1976), 78–87. Zbl 0334.35062 MR 0454425

[17] R. Temam, *Infinite-dimensional dynamical systems in mechanics and physics*. Appl. Math. Sci. 68, Springer-Verlag, New York 1988. Zbl 0662.35001 MR 0953967

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