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Nonlinear stability properties of periodic travelling wave solutions of the classical Korteweg–de Vries and Boussinesq equations

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Abstract. This article is concerned with nonlinear stability properties of periodic travelling wave solutions of the classical Korteweg–de Vries and Boussinesq equations. Periodic travelling wave solutions with a fixed fundamental period L will be constructed by using Jacobi's elliptic functions. It will be shown that these solutions, called *cnoidal waves*, are nonlinearly stable in the respective energy space by periodic disturbances with period L.

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1. Introduction

The original Korteweg–de Vries and Boussinesq equations were among the first models for nonlinear, dispersive wave propagation [11], [12], [25]. These equations possess special travelling wave solutions known as Scott Russel's solitary waves or *solitons* [1], [10], [11], [12], [28] (the latter name, which suggests an analogy with particles, is appropriate since the solitary waves retain their form even after joint interactions), and *cnoidal waves* (Korteweg and de Vries generalization of the sinusoidal wave—cf. Korteweg–de Vries [25], Benjamin [7], Boussinesq [11], [12] and Lamb [26]). The cnoidal wave solutions are periodic travelling waves written in terms of the Jacobian elliptic functions (see Section 2 below).

Stability results for solitary waves of the Korteweg–de Vries and Boussinesq equations go back to the works of Benjamin [6], Bona [8], Bona et al. [9], Bona and Sachs [10], Souganidis and Strauss [32] and Weinstein [34], [35].

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The aim of this article is to investigate the nonlinear stability of periodic travelling wave solutions $\phi(x - ct)$ of the Korteweg–de Vries equation

$$u_t + uu_x + u_{xxx} = 0, \tag{1.1}$$

and of the Boussinesq equation

$$u_{tt} - u_{xx} + \left(\frac{u^2}{2} + u_{xx}\right)_{xx} = 0. \tag{1.2}$$

The latter equation has the following equivalent form

$$\begin{cases} u_t = v_x, \\ v_t = \left(u - u_{xx} - \frac{u^2}{2}\right)_x \end{cases} \tag{1.3}$$

for $x \in \mathbb{R}$, $t > 0$. Here subscripts t and x denote partial differentiation with respect to t and x.

The above equations have important conservation laws. In fact, if $u = u(x, t)$ is an appropriately smooth solution of (1.1), then the integrals

$$E(u) = \frac{1}{2}\int_0^L \left[u_x^2 - \frac{1}{3}u^3\right] dx \quad \text{and} \quad F(u) = \frac{1}{2}\int_0^L u^2\, dx \tag{1.4}$$

are independent of the temporal variable t. Also, if $U(x, t) = \big(u(x, t), v(x, t)\big)$ is a solution of system (1.3), we have the following invariants of motion, which are integrals over x of the densities:

$$u(x,t), \qquad v(x,t), \qquad u(x,t)v(x,t), \qquad \tfrac{1}{2}\big(u^2(x,t) + v^2(x,t) + u_x^2(x,t) - \tfrac{1}{3}u^3(x,t)\big).$$

The last two, which we denote by I and H respectively, are impulse and energy integrals, while the first two are Casimirs with no dynamical significance. These invariants turn out to be relevant quantities in the investigation of stability properties of travelling waves. In fact, we prove here stability for cnoidal wave solutions of the Boussinesq system (1.3) in the space $H^1_{\text{per}}([0, L]) \times L^2_{\text{per}}([0, L])$ associated with the conserved quantities I and H above. Note that

$$I = \int_0^L uv\, dx \quad \text{and} \quad H = \frac{1}{2}\int_0^L \left(u^2(x,t) + v^2(x,t) + u_x^2(x,t) - \frac{1}{3}u^3(x,t)\right) dx \tag{1.5}$$

are smooth functionals on $H^1_{\text{per}}([0, L]) \times L^2_{\text{per}}([0, L])$.

The periodic travelling wave of speed c and period L to equations (1.1) and (1.2), if it exists, is a solution of (1.1) and (1.3), respectively, depending only on $\xi = x - ct$.

Inserting the L-periodic travelling wave $u(x,t) = \phi_c(x - ct)$ in (1.1), we see that ϕ_c must satisfy

$$\phi_c'' + \frac{1}{2}\phi_c^2 - c\phi_c = A_{\phi_c}, \tag{1.6}$$

where A_{ϕ_c} is an integration constant, which will be considered equal to zero here (actually, one may always perform the change of unknown $\tilde{\phi}_c = \phi_c + \sqrt{c^2 + 2A_{\phi_c}} - c$). So the travelling wave equation (1.6) takes the form

$$E'(\phi_c) + cF'(\phi_c) = 0. \tag{1.7}$$

The characterization of the periodic travelling wave ϕ_c of speed c as a critical point of the functional $E + cF$ is crucial to the stability argument.

Nonlinear stability of L-periodic travelling wave solutions associated to the generalized KdV and Boussinesq-type equations was studied by Benjamin in [7], by Angulo, Bona and Scialom in [4], by Angulo and Quintero in [5], by Angulo [3], and by Hakkaev, Iliev and Kirchev in [19]. In [7], Benjamin proved that the trivial nonzero solution of the problem, namely the constant $\phi_0 = 2c$, is stable if $c < \frac{4\pi^2}{L^2}$. Angulo, Bona and Scialom considered the constant of integration A_ϕ in (1.6) different from zero and obtained a nonlinear stability result for cnoidal wave solutions of the KdV equation, which are defined in an a priori fundamental interval $[0, L]$ and have mean zero on it (see [4]). In [5], the authors showed that special periodic travelling wave solutions with an arbitrary fundamental period L of a one-dimensional Boussinesq-type equation are orbitally stable in the space $\{u \in H^1_{\mathrm{per}}([0, L]) \mid \int_0^L u\,dx = 0\}$ for a range of their speeds of propagation and periods. In [3], the author proves nonlinear stability of dnoidal waves associated to the Schrödinger and modified Korteweg–de Vries equations. In [19], the authors prove stability of periodic travelling shallow-water waves determined by Newton's equation.

Korteweg–de Vries and Boussinesq equations are strongly related. In fact, substituting the L-periodic travelling wave solution $\vec{\phi} = \big(\phi_c(x - ct), \psi_c(x - ct)\big)$ in (1.3) leads to the system

$$\begin{cases} -c\phi_c'(\xi) = \psi_c'(\xi), \\ -c\psi_c'(\xi) = \Big(\phi_c - \phi_c'' - \frac{\phi_c^2}{2}\Big)'(\xi), \end{cases} \tag{1.8}$$

where $'$ denotes $\frac{d}{d\xi}$ and $\xi = x - ct$. Integrating (1.8), we obtain the nonlinear system

$$\begin{cases} -c\phi_c(\xi) = \psi_c(\xi) + K_1, \\ -c\psi_c(\xi) = \phi_c(\xi) - \phi_c''(\xi) - \frac{\phi_c^2}{2}(\xi) + K_2, \end{cases} \tag{1.9}$$

where K_1, K_2 are integration constants that will be considered equal to zero here. Then ϕ_c must satisfy

$$\phi_c'' - (1 - c^2)\phi_c + \frac{\phi_c^2}{2} = 0, \tag{1.10}$$

where $w = w(c) = 1 - c^2$ will be considered positive. So (1.10) takes the form of (1.6) and hence a KdV-type theory of existence and stability of cnoidal wave solutions can be established in the case of the Boussinesq equation (see Theorem 1.4 below).

In this article we first show the existence of a smooth curve $c \mapsto \phi_c$ of cnoidal wave solutions to equation (1.1), with a fixed period L. Then orbital stability of these solutions is established in $H^1_{\text{per}}([0, L])$ for a certain range of their speeds of propagation and periods by using the Lyapunov method [35]. By orbital stability we mean stability modulo spatial translation. More precisely, our first result is the following.

Theorem 1.1. *Let* $c \in \left(\frac{4\pi^2}{L^2}, +\infty\right)$. *Then the orbit* $\mathcal{O}_{\phi_c}$ *is* $H^1_{\text{per}}([0, L])$*-stable with regard to the flow of the Korteweg–de Vries equation.*

Here the set $\mathcal{O}_{\phi_c} = \{\phi_c(\cdot + s) \mid s \in \mathbb{R}\}$ is the orbit generated by the L-periodic cnoidal wave solution given by Theorem 2.1 below.

Remark 1.2. The proof of Theorem 1.1 is an adaptation to the periodic case $(p = 1)$ of Theorem 4.1 in [2].

In order to prove our theorem we construct the Lyapunov functional of the form $\mathscr{E}[u] = E(u) + cF(u)$, where E and F are the well-defined C^∞-mappings of $H^1_{\text{per}}([0, L])$ into $\mathbb{R}$ given by (1.4). We find that restricted to the manifold of functions, $u \in H^1_{\text{per}}([0, L])$ for which $F(u) = F(\phi_c)$, $\phi_c(\cdot)$ is a local minimum, provided that

$$\frac{d}{dc} F\big(\phi_c(\cdot)\big) > 0. \tag{1.11}$$

Condition (1.11) is arrived at through a spectral analysis of the operator $\mathscr{L}_{\text{cn}}$ (see (3.1), obtained by linearizing the travelling wave equation (1.6) about ϕ_c. Here it is seen that stability relies on a suitable lower bound on the second variation of the energy functional $\mathscr{E}$. This lower bound is obtained by using the analysis of a constrained variational problem for $\mathscr{L}_{\text{cn}}$.

Remark 1.3. Condition (1.11) is the analogue of the convexity condition obtained by Shatah [31] (see also [18]) for the stability of standing waves for the Klein–Gordon equations. In fact, relation (1.7) implies that condition (1.11) is equivalent to the convexity of the function

$$d(c) = E\big(\phi_c(\cdot)\big) + cF\big(\phi_c(\cdot)\big). \tag{1.12}$$

With respect to the system (1.3), as a consequence of the theory presented for KdV before, we conclude the existence of a smooth curve $c \mapsto \vec{\phi}_c = (\phi_c, \psi_c)$ of cnoidal wave solutions to system (1.3), with a fixed period L. Then orbital stability of these solutions is established in $H^1_{\text{per}}([0, L]) \times L^2_{\text{per}}([0, L])$ for a certain range of their speeds of propagation and periods. Specifically, our second result is the following.

Theorem 1.4. *Let $c \in (-1, 1)$ and $L > 2\pi$. Then the orbit $\mathcal{O}_{\vec{\phi}_c}$ is $H^1_{\text{per}}([0, L]) \times L^2_{\text{per}}([0, L])$-stable with regard to the flow of the Boussinesq equation, provided that $c^2 > \frac{1}{3}$ and $1 - c^2 > \frac{4\pi^2}{L^2}$.*

The outline of the proof is as follows. First, we prove local existence of smooth solutions for the initial value problem (1.2) with initial conditions

$$u(x, 0) = u_0(x) \quad \text{and} \quad u_t(x, 0) = u_1(x). \tag{1.13}$$

The nonlinear stability of the periodic travelling wave solutions of this equation follows the same general lines as those in the KdV case, which are based on ideas described in [6], [8], [10] and [35].

We remark that stability of $\vec{\phi}_c$ is established with respect to perturbations of periodic functions of the same period L in $H^1_{\text{per}}([0, L]) \times L^2_{\text{per}}([0, L])$.

Finally, local existence coupled with the stability result is shown to imply the conditions that lead to global existence, at least for initial data close to the stable cnoidal wave.

Remark 1.5. Also for Boussinesq's model, as we will see below, it is important to mention that stability relies on the convexity property of the so-called *moment of instability*

$$m(c) = H\big(\vec{\phi}_c(\cdot)\big) + cI\big(\vec{\phi}_c(\cdot)\big),$$

where H and I are the functionals defined by (1.5).

The following notation will be used:

$\langle f, g\rangle = \langle f, g\rangle_{L^2_{\text{per}([0,L])}} = \int_0^L fg\,dx,$

$\langle f, g\rangle_1 = \langle f, g\rangle_{H^1_{\text{per}([0,L])}} = \int_0^L fg\,dx + \int_0^L f'g'\,dx,$

$$
\begin{aligned}
&\|f\| = \|f\|_{L^2_{\mathrm{per}([0,\,L])}} = \left(\int_0^L f^2\,dx\right)^{1/2},\\
&\|f\|_1 = \|f\|_{H^1_{\mathrm{per}([0,\,L])}} = \left(\int_0^L f^2\,dx + \int_0^L f'^2\,dx\right)^{1/2},\\
&\langle (f,g),(u,v)\rangle = \langle (f,g),(u,v)\rangle_{L^2_{\mathrm{per}([0,\,L])}\times L^2_{\mathrm{per}([0,\,L])}} = \int_0^L fu\,dx + \int_0^L gv\,dx,\\
&\|(f,g)\| = \|(f,g)\|_{L^2_{\mathrm{per}([0,\,L])}\times L^2_{\mathrm{per}([0,\,L])}} = \left(\int_0^L f^2\,dx + \int_0^L g^2\,dx\right)^{1/2},\\
&\|(f,g)\|_X = \|(f,g)\|_{H^1_{\mathrm{per}([0,\,L])}\times L^2_{\mathrm{per}([0,\,L])}} = \left(\int_0^L f^2\,dx + \int_0^L f'^2\,dx + \int_0^L g^2\,dx\right)^{1/2}.
\end{aligned}
$$

2. Existence of a smooth curve of cnoidal wave solutions with a fixed period L for the equation (1.1) and system (1.3)

In this section we establish the existence of a family of even L-periodic travelling wave solutions $\phi = \phi_c(x - ct)$ for the equation

$$\phi'' - c\phi + \frac{\phi^2}{2} = 0 \tag{2.1}$$

such that the mapping $c \mapsto \phi_c$ is C^1.

Multiplying (2.1) by ϕ', a second integration is possible yielding the first-order equation

$$
\begin{aligned}
\left(\phi'(\xi)\right)^2 &= \frac{1}{3}[-\phi^3(\xi) + 3c\phi^2(\xi) + 6B_\phi] \equiv \frac{1}{3}p_\phi\left(\phi(\xi)\right)\\
&= \frac{1}{3}(\phi - \beta_1)(\phi - \beta_2)(\beta_3 - \phi),
\end{aligned} \tag{2.2}
$$

where B_{ϕ_c} is an integration constant and β_1, β_2, β_3 are the zeros of the polynomial $p_\phi(t) = -t^3 + 3ct^2 + 6B_\phi$ and so satisfy the relations

$$
\begin{cases}
3c = \beta_1 + \beta_2 + \beta_3,\\
0 = \beta_1\beta_2 + \beta_2\beta_3 + \beta_3\beta_1,\\
B_\phi = \frac{1}{6}\beta_1\beta_2\beta_3.
\end{cases}
$$

Moreover, we assume that $\beta_1 < \beta_2 < \beta_3$ and $\beta_3 > 0$, and we obtain from (2.2) that $\beta_2 \le \phi \le \beta_3$. By defining $\varphi = \phi/\beta_3$, (2.2) becomes $(\varphi')^2 = \frac{\beta_3}{3}(\varphi - \eta_1)(\varphi - \eta_2)\cdot(1 - \varphi)$, where $\eta_i = \beta_i/\beta_3$, $i = 1, 2$. We also impose the crest of the wave to be at $\xi = 0$, that is, $\varphi(0) = 1$. Now we define a further variable ψ via the relation $\varphi = 1 + (\eta_2 - 1)\sin^2\psi$ and thus obtain that

$$(\psi')^2 = \frac{\beta_3}{12}(1 - \eta_1)\left[1 - \left(\frac{1 - \eta_2}{1 - \eta_1}\right)\sin^2\psi\right]$$

and $\psi(0) = 0$. In order to write this in a standard form we define $k^2 = \frac{1-\eta_2}{1-\eta_1}$, $l = \frac{\beta_3}{12}(1 - \eta_1)$. It follows that $0 \le k^2 \le 1$ and $l > 0$ and we obtain $\int_0^\psi \frac{dt}{\sqrt{1-k^2\sin^2 t}} =$

$\sqrt{l}\xi$. Therefore, from the definition of the Jacobi elliptic function $y = sn(u;k)$ (see (5.1), we can write the last equality as $\sin\psi = \mathrm{sn}(\sqrt{l}\xi;k)$, and hence $\varphi = 1 + (\eta_2 - 1)\,\mathrm{sn}^2(\sqrt{l}\xi;k)$. Using the relation $\mathrm{sn}^2 + \mathrm{cn}^2 = 1$, we arrive finally to the conventional form

$$\phi(\xi) = \phi(\xi, \beta_1, \beta_2, \beta_3) = \beta_2 + (\beta_3 - \beta_2)\,\mathrm{cn}^2\left[\sqrt{\frac{\beta_3 - \beta_1}{12}}\,\xi; k\right], \tag{2.3}$$

where

$$k^2 = \frac{\beta_3 - \beta_2}{\beta_3 - \beta_1}, \qquad -\beta_1 = \beta_2 + \beta_3 - 3c = \frac{\beta_2\beta_3}{\beta_2 + \beta_3}, \qquad \beta_1 < \beta_2 < \beta_3. \tag{2.4}$$

From (2.4) we have that β_2, β_3 belong to the ellipse Σ given by

$$\beta_2^2 + \beta_3^2 + \beta_2\beta_3 - 3c(\beta_2 + \beta_3) = 0, \tag{2.5}$$

and since $\beta_2 < \beta_3$, it follows that $0 < \beta_2 < 2c < \beta_3 < 3c$.

Next, since cn^2 has fundamental period $2K(k)$, ϕ has fundamental period T_ϕ equal to

$$T_\phi \equiv \frac{4\sqrt{3}}{\sqrt{\beta_3 - \beta_1}} K(k).$$

Now we prove that $T_\phi > \frac{2\pi}{\sqrt{c}}$. Initially we express T_ϕ as a function of β_3 and c. In fact, following (2.5), every $\beta_3 \in (2c, 3c)$ defines a unique real value of $\beta_2 \in (0, 2c)$ such that (β_2, β_3) is in the interior of the ellipse Σ and

$$2\beta_2 = 3c - \beta_3 + \sqrt{9c^2 + 6c\beta_3 - 3\beta_3^2}. \tag{2.6}$$

So, by defining $\beta_1 \equiv 3c - \beta_2 - \beta_3$, we obtain for

$$k^2(\beta_3, c) = \frac{3\beta_3 - 3c - \sqrt{9c^2 + 6c\beta_3 - 3\beta_3^2}}{3\beta_3 - 3c + \sqrt{9c^2 + 6c\beta_3 - 3\beta_3^2}} \tag{2.7}$$

that

$$T_\phi(\beta_3, c) = \frac{4\sqrt{6}}{\sqrt{3\beta_3 - 3c + \sqrt{9c^2 + 6c\beta_3 - 3\beta_3^2}}} K\big(k(\beta_3), c\big).$$

Then by fixing $c > 0$, we have that $T_\phi(\beta_3, c) \to \infty$ as $\beta_3 \to 3c$ and $T_\phi(\beta_3, c) \to \frac{2\pi}{\sqrt{c}}$ as $\beta_3 \to 2c$. So, since the mapping $\beta_3 \in (2c, 3c) \to T_\phi(\beta_3, c)$ is strictly increasing (see proof of Theorem 2.1), it follows that $T_\phi > \frac{2\pi}{\sqrt{c}}$.

Now we obtain a cnoidal wave solution with period L. For $c_0 > \frac{4\pi^2}{L^2}$ there is a unique $\beta_{3,0} \in (2c_0, 3c_0)$ such that $T_\phi(\beta_{3,0}, c_0) = L$. So, for c_0 and $\beta_{3,0}$ such that $(\beta_{2,0}, \beta_{3,0}) \in \Sigma(c_0)$, we have that the cnoidal wave $\phi(\cdot) = \phi(\cdot; \beta_{1,0}, \beta_{2,0}, \beta_{3,0})$ with $\beta_{1,0} = 3c_0 - \beta_{2,0} - \beta_{3,0}$ has fundamental period L and satisfies (2.1) with $c = c_0$.

By the above analysis the cnoidal wave $\phi(\cdot; \beta_1, \beta_2, \beta_3)$ in (2.3) is completely determined by c and β_3 and will be denoted by $\phi_c(\cdot; \beta_3)$ or ϕ_c.

Next we show the existence of a smooth curve of cnoidal wave solutions for equation (2.1). In other words, we show that at least locally the choice of $\beta_{3,0}$ above depends smoothly of c_0.

Theorem 2.1. *Let $L > 0$ arbitrary but fixed. Consider $c_0 > \frac{4\pi^2}{L^2}$ and $\beta_{3,0} = \beta_3(c_0) \in (2c_0, 3c_0)$ such that $T_{\phi_{c_0}} = L$. Then the following holds:*

(1) *There exists an interval $\mathscr{I}(c_0)$ around c_0, an interval $J(\beta_{3,0})$ around $\beta_{3,0}(c_0)$, and a unique smooth function $\Lambda : \mathscr{I}(c_0) \to J(\beta_{3,0})$ such that $\Lambda(c_0) = \beta_{3,0}$ and*

$$\frac{4\sqrt{6}}{\sqrt{3\beta_3 - 3c + \sqrt{9c^2 + 6c\beta_3 - 3\beta_3^2}}} K(k) = L, \tag{2.8}$$

where $c \in \mathscr{I}(c_0)$, $\beta_3 = \Lambda(c)$, and $k^2 \equiv k^2(c) \in (0,1)$ is defined in (2.7).

(2) *The cnoidal wave solution given by* (2.3), *$\phi_c(\cdot; \beta_1, \beta_2, \beta_3)$, determined by $\beta_1 \equiv \beta_1(c)$, $\beta_2 \equiv \beta_2(c)$ and $\beta_3 \equiv \beta_3(c)$, has fundamental period L and satisfies the equation* (2.1). *Moreover, the mapping*

$$c \in \mathscr{I}(c_0) \to \phi_c \in H^1_{\text{per}}([0, L])$$

is a smooth function.

(3) *$\mathscr{I}(c_0)$ can be chosen as $\left(\frac{4\pi^2}{L^2}, +\infty\right)$.*

Proof. The idea of the proof is to apply the Implicit Function Theorem. We consider the open set $\Omega = \left\{(\beta, c) \mid c > \frac{4\pi^2}{L^2}, \beta \in (2c, 3c)\right\} \subseteq \mathbb{R}^2$ and define $\Psi : \Omega \to \mathbb{R}$ by

$$\Psi(\beta, c) = \frac{4\sqrt{6}}{\sqrt{3\beta - 3c + \sqrt{9c^2 + 6c\beta - 3\beta^2}}} K\big(k(\beta, c)\big),$$

where $k(\beta, c)$ is defined in (2.7), with $\beta_3 = \beta$. By the hypotheses, $\Psi(\beta_{3,0}, c_0) = L$.

Denoting $a \equiv a(\beta) = 3\beta - 3c$ and $b \equiv b(\beta) = 9c^2 + 6c\beta - 3\beta^2$, we have that

$$\frac{\partial \Psi}{\partial \beta} = \frac{\sqrt{a + \sqrt{b}}\left[4\sqrt{6}\frac{dK}{dk}\frac{dk}{d\beta}\right] - 4\sqrt{6}K\frac{1}{2}\left[a + \sqrt{b}\right]^{-1/2}\left[3 + \frac{1}{2}b^{-1/2}(6c - 6\beta)\right]}{a + \sqrt{b}}.$$

Now from (2.7) it follows that $\frac{dk^2}{d\beta} = \frac{2a^2+6b}{\sqrt{b}(a+\sqrt{b})^2}$. Since $\frac{dk^2}{d\beta} = 2k\frac{dk}{d\beta}$, we obtain that $\frac{dk}{d\beta} = \frac{1}{2k}\frac{2a^2+6b}{\sqrt{b}(a+\sqrt{b})^2} > 0$. Thus, $\frac{\partial\Psi}{\partial\beta} > 0$. In fact,

$$
\begin{aligned}
\frac{\partial\Psi}{\partial\beta} &= 4\sqrt{6}\frac{dK}{dk}\frac{1}{2k}\frac{(2a^2+6b)}{\sqrt{b}(a+\sqrt{b})^{5/2}} - 4\sqrt{6}\frac{1}{2}\frac{(3\sqrt{b}-a)}{\sqrt{b}(a+\sqrt{b})^{3/2}}K > 0 \\
&\Leftrightarrow \frac{dK}{dk}\frac{1}{k}\frac{(2a^2+6b)}{(a+\sqrt{b})} > (3\sqrt{b}-a)K \\
&\Leftrightarrow (2a^2+6b)(E-k'^2K) > (3\sqrt{b}-a)(a+\sqrt{b})k^2k'^2K \\
&\Leftrightarrow (2a^2+6b)E > (3\sqrt{b}-a)(a+\sqrt{b})k^2k'^2K + (2a^2+6b)k'^2K \\
&\Leftrightarrow (2a^2+6b)E > (2a\sqrt{b}+3b-a^2)k^2k'^2K + (2a^2+6b)k'^2K \\
&\Leftrightarrow (2a^2+6b)E > (2a\sqrt{b}+3b)k^2k'^2K + a^2k'^2K + a^2k'^4K + 6bk'^2K.
\end{aligned}
$$

Now $(2a^2+6b)E = (1+k'^2)a^2E + k^2a^2E + 6bE = k^2(1+k'^2)a^2E + k'^2(1+k'^2)a^2E + k^2a^2E + 6bE$. Since $a > \sqrt{b}$ and the fact that $k \to E(k) + K(k)$ is strictly increasing implies that $(1+k'^2)E > 2k'^2K$, we have that $k^2(1+k'^2)a^2E > 2k^2k'^2a^2K > 2a\sqrt{b}k^2k'^2K$. Moreover, $6bE = 6k^2bE + 6k'^2bE$ and $E - k'^2K > 0$ imply that $3k^2bE > 3bk^2k'^2K$. Also, by using the inequality $(1+k'^2)E > 2k'^2K$, we obtain that $3k^2bE + 6k'^2bE = 3bE + 3k'^2bE = 3(1+k'^2)bE > 6bk'^2K$.

Now we have to show that $k^2a^2E + k'^2(1+k'^2)a^2E - a^2k'^2K - a^2k'^4K > 0$. This follows from $k'^2(1+k'^2)a^2E > 2k'^4a^2K$ and the relation $k^2a^2E + k'^4a^2K - a^2k'^2K = k^2a^2E - k^2k'^2a^2K = k^2a^2(E-k'^2K) > 0$. Therefore, there exists a unique smooth function Λ, defined in a neighborhood $\mathscr{I}(c_0)$ of c_0, such that $\Psi(\Lambda(c), c) = L$ for every $c \in \mathscr{I}(c_0)$. So we obtain (2.8). Finally, since c_0 was chosen arbitrarily in the interval $\mathscr{I} = \left(\frac{4\pi^2}{L^2}, +\infty\right)$, it follows that Λ can be extended to $\mathscr{I}$. This completes the proof of theorem. □

Corollary 2.2. *Consider the mapping* $\Lambda : \mathscr{I}(c_0) \to J(\beta_{3,0})$ *determined by Theorem* 2.1. *Then* Λ *is a strictly increasing function in* $\mathscr{I}(c_0)$.

Proof. By Theorem 2.1 we have that $\Psi(\Lambda(c), c) = L$ for every $c \in \mathscr{I}(c_0)$ and so

$$\frac{d}{dc}\Lambda(c) = -\frac{\partial\Psi/\partial c}{\partial\Psi/\partial\beta}. \tag{2.9}$$

We will show that $\partial\Psi/\partial c < 0$. In order to do this, we denote again $a(c) = 3\beta - 3c$ and $b(c) = 9c^2 + 6c\beta - 3\beta^2$, and we note that

$$\frac{\partial\Psi}{\partial c}=\frac{4\sqrt{6}\sqrt{a(c)+\sqrt{b(c)}}\,\frac{dK}{dk}\,\frac{\partial k}{\partial c}}{a(c)+\sqrt{b(c)}}$$
$$+\frac{-4\sqrt{6}K\frac{1}{2}\left(a(c)+\sqrt{b(c)}\right)^{-1/2}\left(-3+\frac{1}{2}b(c)^{-1/2}(18c+6\beta)\right)}{a(c)+\sqrt{b(c)}},$$

where $k(\beta,c)$ is defined by (2.7), with $\beta_3=\beta$. Now

$$\frac{\partial k}{\partial c}=\frac{1}{2k}\,\frac{(3c-3\beta)b(c)^{-1/2}(18c+6\beta)-6\sqrt{b(c)}}{[a(c)+\sqrt{b(c)}]^2}<0$$

and $-3+\frac{1}{2}b(c)^{-1/2}(18c+6\beta)=\frac{-6\sqrt{b(c)}+18c+6\beta}{2\sqrt{b(c)}}>0$ because $-6\sqrt{b(c)}+18c+6\beta>0\Leftrightarrow 3c+\beta>\sqrt{b(c)}\Leftrightarrow 9c^2+6c\beta+\beta^2>9c^2+6c\beta-3\beta^2\Leftrightarrow 4\beta^2>0$. So it follows that $\frac{d\Psi}{dc}<0$, and the proof is completed. □

Now we prove that the modulus function $k(c)$ is strictly increasing.

Lemma 2.3. *Consider* $c\in\left(\frac{4\pi^2}{L^2},\infty\right)$, $\beta_3=\Lambda(c)$ *and the modulus function*

$$k(c)\equiv k\big(\Lambda(c),c\big)=\sqrt{\frac{3\beta_3(c)-3c-\sqrt{9c^2+6c\beta_3(c)-3\beta_3(c)^2}}{3\beta_3(c)-3c+\sqrt{9c^2+6c\beta_3(c)-3\beta_3(c)^2}}}.$$

Then $\frac{d}{dc}k(c)>0$.

Proof. Denoting $a(c)=3\beta_3(c)-3c$ and $b(c)=9c^2+6w\beta_3(c)-3\beta_3(c)^2$, we have that

$$\frac{dk}{dc}(c)=\frac{1}{2k}\left[\frac{2a^2+6b}{\sqrt{b}(a+\sqrt{b})^2}\right]\beta_3'(c)+\frac{1}{2k}\left[\frac{-ab^{-1/2}(18c+6\beta_3)-6\sqrt{b}}{(a+\sqrt{b})^2}\right].$$

Using that $\beta_3'(c)=\frac{d}{dc}\Lambda(c)$ by (2.9), we get that

$$\frac{dk}{dc}(c)>0\;\Leftrightarrow\;\frac{1}{2k}\left\{\frac{2a^2+6b}{\sqrt{b}(a+\sqrt{b})^2}\right\}$$
$$\left\{\frac{\frac{dK}{dk}\left[\frac{1}{2k}\,\frac{\left(ab^{-1/2}(18c+6\beta_3)+6\sqrt{b}\right)}{(a+\sqrt{b})^{3/2}}\right]+\frac{K(a+\sqrt{b})^{-1/2}\left(-3+\frac{1}{2}b^{-1/2}(18c+6\beta_3)\right)}{2}}{\frac{dK}{dk}\left[\frac{1}{2k}\,\frac{2a^2+6b}{\sqrt{b}(a+\sqrt{b})^{3/2}}\right]+\frac{K(a+\sqrt{b})^{-1/2}\left(-3+\frac{1}{2}b^{-1/2}(6\beta_3-6c)\right)}{2}}\right\}$$
$$+\frac{1}{2k}\left[\frac{-ab^{-1/2}(18c+6\beta_3)-6\sqrt{b}}{(a+\sqrt{b})^2}\right]>0.$$

Now the last inequality is true if and only if

$$\frac{1}{2k}\left\{\frac{2a^2+6b}{\sqrt{b}(a+\sqrt{b})^2}\right\}\frac{K}{2}(a+\sqrt{b})^{-1/2}\left(-3+\frac{1}{2}b^{-1/2}(18c+6\beta_3)\right)$$
$$>\frac{1}{2k}\left[\frac{ab^{-1/2}(18c+6\beta_3)+6\sqrt{b}}{(a+\sqrt{b})^2}\right]\frac{K}{2}(a+\sqrt{b})^{-1/2}\left(-3+\frac{1}{2}b^{-1/2}(6\beta_3-6c)\right),$$

and this happens if and only if

$$\begin{aligned}(2a^2+6b)\left[-3+\tfrac{1}{2}b^{-1/2}(18c+6\beta_3)\right]\\ >\left[a(18c+6\beta_3)+6b\right]\left[-3+\tfrac{1}{2}b^{-1/2}(6\beta_3-6c)\right].\end{aligned}\tag{2.10}$$

Now (2.10) is equivalent to $-6a^2+4\times 18c\sqrt{b}+3a(18c+6\beta_3)>0$, which is satisfied since $3a(3c+\beta_3)-a^2=a(9c+3\beta_3-3\beta_3+3c)=12ac>0$. This completes the proof. □

From the last results above we conclude the following existence theorem.

Theorem 2.4. *Let $L>2\pi$. Then there exists a smooth curve of cnoidal wave solutions for the system* (1.3) *in $H^n_{\text{per}}([0,L])\times H^m_{\text{per}}([0,L])$, $n,m\geq 0$, which satisfy the system* (1.9) *with integration constants $K_1=K_2=0$; this curve is given, for $w(c)=1-c^2$, by*

$$c\in\left(-\sqrt{1-\frac{4\pi^2}{L^2}},\sqrt{1-\frac{4\pi^2}{L^2}}\right)\to(\phi_{w(c)},\psi_{w(c)}).$$

Moreover,

$$\phi_{w(c)}(\xi)=\beta_2+(\beta_3-\beta_2)cn^2\left[\sqrt{\frac{\beta_3-\beta_1}{12}}\xi;k\right],$$

where the smooth function $\beta_3\equiv\beta_3(w(c))$ is given by Theorem 2.1, *$k=k(w(c))$ by* (2.7), *$\beta_2=\frac{3w-\beta_3+\sqrt{9w^2+6w\beta_3-3\beta_3^2}}{2}$, and $\beta_1=3w-\beta_2-\beta_3$.*

3. Stability of cnoidal waves for the KdV equation

In this section we shall show that the orbit $\mathcal{O}_{\phi_c}$ is stable in the $H^1_{\text{per}}([0,L])$-sense by the flow of the KdV equation, that is, for each $\varepsilon>0$ there is $\delta=\delta(\varepsilon)>0$ such that if $\inf_{s\in\mathbb{R}}\|u_0-\phi_c(\cdot+s)\|_1<\delta$ then the solution $u(t)$ of (1.1) with $u(0)=u_0$ satisfies

$$\inf_{s \in \mathbb{R}} \|u(t) - \phi_c(\cdot + s)\|_1 < \varepsilon$$

for all t for which $u(t)$ exists. Before we start to study stability we show the following result about the periodic initial value problem associated to the KdV equation.

Theorem 3.1. *Let $L > 0$ fixed. Then the periodic initial value problem associated to the KdV equation* (1.1) *is globally well posed in $H^s_{\text{per}}([0, L])$, $s \geq 1/2$.*

Proof. See Colliander, Keel, Staffilani, Takaoka and Tao in [14] (or [24]). □

3.1. Spectral analysis. In this section we study the spectral properties associated to the periodic eigenvalue problem considered on $[0, L]$

$$\begin{cases} \mathscr{L}_{\text{cn}} v := \left[-\frac{d^2}{dx^2} + c - \phi_c \right] v = \mu v, \\ v(0) = v(L),\ v'(0) = v'(L), \end{cases} \tag{3.1}$$

where $c > \frac{4\pi^2}{L^2}$ and ϕ_c is the L-periodic cnoidal wave (2.3) given by Theorem 2.1.

The theory of compact self-adjoint operators implies that the spectrum of $\mathscr{L}_{\text{cn}}$ is a countable infinite set of eigenvalues $(\mu_n)_{n \geq 0}$ with

$$\mu_0 \leqq \mu_1 \leqq \mu_2 \leqq \mu_3 \leqq \mu_4 \leqq \cdots, \tag{3.2}$$

counting twice double eigenvalues and $\mu_n \to \infty$ as $n \to \infty$. In fact, since $\mathscr{L}_{\text{cn}} = \left(-\frac{d^2}{dx^2} + c \right) + (-\phi_c) \equiv \mathscr{L} + \mathscr{M}$, with $\mathscr{L}$ having a discrete spectrum, it follows from Weyl's essential spectral theorem [33] that $\sigma_{\text{ess}}(\mathscr{L}_{\text{cn}}) = \sigma_{\text{ess}}(\mathscr{L}) = \emptyset$. Here we shall denote by v_n the corresponding eigenfunction to the eigenvalue μ_n. By the conditions $v(0) = v(L)$, $v'(0) = v'(L)$, v_n can be extended to the whole of $(-\infty, +\infty)$ as a continuously differentiable function with period L. Next, by using the classical Floquet theory (cf. Ince [20] and Magnus and Winkler [30]) we show that the first two eigenvalues μ_0 and μ_1 of $\mathscr{L}_{\text{cn}}$ are simple, $\mu_1 = 0$ and the corresponding eigenfunction is $\frac{d}{dx}\phi_c$.

From Floquet's theory, the periodic eigenvalue problem (3.1) is related to the following semi-periodic eigenvalue problem considered on $[0, L]$:

$$\begin{cases} \mathscr{L}_{\text{cn}} f = \nu f, \\ f(0) = -f(L),\ f'(0) = -f'(L). \end{cases}$$

This is also a self-adjoint problem and therefore determines a sequence of eigenvalues $(\nu_n)_{n \geq 0}$ with

$$\nu_0 \leqq \nu_1 \leqq \nu_2 \leqq \nu_3 \leqq \nu_4 \leqq \cdots, \tag{3.3}$$

counted with multiplicities and $\nu_n \to \infty$ as $n \to \infty$. We shall denote by f_n the corresponding eigenfunction to the eigenvalue ν_n. A function f such that $f(x+L) = -f(x)$ for all x is said to be *semi-periodic* with semi-period L. Evidently, such a function has period $2L$. So we have that the Hill equation

$$\mathscr{L}_{\mathrm{cn}} f = o f \tag{3.4}$$

has a solution of period L if and only if $o = \mu_n$, $n = 0, 1, 2, \ldots$, as well as it has a solution of period $2L$ if and only if $o = \nu_n$, $n = 0, 1, 2, \ldots$. If all eigenfunctions of (3.4) are bounded we say that they are *stable*; otherwise we say that they are *unstable*.

Next it follows from the oscillation theorem ([30], Theorem 2.1, p. 11) applied to the differential equation (3.4) that the sequences (3.2) and (3.3) satisfy the inequalities

$$\mu_0 < \nu_0 \leqq \mu_1 < \nu_1 \leqq \mu_2 < \nu_2 \leqq \nu_3 < \mu_3 \leqq \mu_4 \ldots. \tag{3.5}$$

The eigenfunctions of (3.4) are stable in the intervals $(\mu_0, \nu_0), (\mu_1, \nu_1), \ldots$. These intervals are called *intervals of stability*. At the endpoints of these intervals the solutions of (3.4) are, in general, unstable. This is always true for $o = \mu_0$ (μ_0 is always simple). The solutions of (3.4) are stable for $o = \mu_{2n+1}$ or $o = \mu_{2n+2}$ if and only if $\mu_{2n+1} = \mu_{2n+2}$, and they are stable for $o = \nu_{2n}$ or $o = \nu_{2n+1}$ if and only if $\nu_{2n} = \nu_{2n+1}$. The intervals $(-\infty, \mu_0), (\nu_0, \nu_1), (\mu_1, \mu_2), \ldots$, are called *intervals of instability*, omitting however any interval which is absent as a result of having a double eigenvalue. The interval of instability $(-\infty, \mu_0)$ will always be present. We note that the absence of an instability interval means that there is a value of o for which all solutions of (3.4) have either period L or semi-period L, that is, coexistence of solutions of (3.4) with period L or period $2L$ occurs for that value of o.

Before establishing our theorem, we note that the number of zeros of v_n and f_n is determined in the following form ([30], Theorem 2.14, p. 43):

$$\begin{aligned} &\text{(i)}\ \ v_0 \text{ has no zeros in } [0, L],\\ &\text{(ii)}\ \ v_{2n+1} \text{ and } v_{2n+2} \text{ have exactly } 2n+2 \text{ zeros in } [0, L),\\ &\text{(iii)}\ \ f_{2n} \text{ and } f_{2n+1} \text{ have exactly } 2n+1 \text{ zeros in } [0, L). \end{aligned} \tag{3.6}$$

Theorem 3.2. *Let $\mathscr{L}_{\mathrm{cn}}$ be the linear operator defined on $H^2_{\mathrm{per}}([0, L])$ by* (3.1). *Then the first two eigenvalues μ_0 and μ_1 of $\mathscr{L}_{\mathrm{cn}}$ are simple and satisfy $\mu_0 < \mu_1 = 0$; and ϕ_c' is the eigenfunction of μ_1.*

Proof. From (3.5) and (3.6), it follows that $0 = \mu_1 \le \mu_2$. In fact, since $\mathscr{L}_{\mathrm{cn}} \phi_c' = 0$ and ϕ_c' has two zeros in $[0, L)$, we have that 0 is either μ_1 or μ_2. We will show that

$0 = \mu_1$. Indeed, define the transformation $T_\eta v(x) := v(\eta x)$ for

$$\eta^2 = 12/(\beta_3 - \beta_1), \tag{3.7}$$

where the β_i, $i = 1, 2, 3$, are defined in (2.2) and also reappear in Theorem 2.1. Then using the explicit form (2.3) for ϕ_c, we see that problem (3.1) is equivalent to the eigenvalue problem

$$\begin{cases} \left[-\frac{d^2}{dx^2} + 12k^2 \operatorname{sn}^2(x)\right] y = \rho y, \\ y(0) = y(2K),\ y'(0) = y'(2K), \end{cases} \tag{3.8}$$

for $y \equiv T_\eta v$,

$$\rho = -12[c - \beta_3 - \mu]/(\beta_3 - \beta_1). \tag{3.9}$$

The first three eigenvalues ρ_0, ρ_1 and ρ_2 and their corresponding eigenfunctions are known explicitly. Since $\rho_1 = 4 + 4k^2$ is a simple eigenvalue of (3.8) with eigenfunction $y_1(x) = \operatorname{cn}(x) \operatorname{sn}(x) \operatorname{dn}(x) = CT_\eta \phi_c'$, it follows from (3.9) that $\mu = 0$ is a simple eigenvalue of problem (3.1) with corresponding eigenfunction ϕ_c'. Consider the Lamé polynomials [20] defined by

$$P_0(x) = \operatorname{dn}(x)[1 - (1 + 2k^2 - \sqrt{1 - k^2 + 4k^4}) \operatorname{sn}^2(x)]$$

and

$$P_2(x) = \operatorname{dn}(x)[1 - (1 + 2k^2 + \sqrt{1 - k^2 + 4k^4}) \operatorname{sn}^2(x)].$$

These functions have period $2K$ and are the corresponding eigenfunctions to the eigenvalues ρ_0 and ρ_2, respectively. In fact, the equation

$$\rho = k^2 + \frac{5k^2}{1 + \frac{9}{4}k^2 - \frac{1}{4}\rho}$$

has two roots, namely, $\rho_0 = 2 + 5k^2 - 2\sqrt{1 - k^2 + 4k^4}$ and $\rho_2 = 2 + 5k^2 + 2\sqrt{1 - k^2 + 4k^4}$. Since P_0 has no zeros in $[0, 2K]$ and P_2 has exactly 2 zeros in $[0, 2K)$ it must be the case that P_0 is the eigenfunction associated to ρ_0, the first eigenvalue of (3.8). On the other hand, since $\rho_0 < \rho_1$ for all $k^2 \in (0, 1)$, we obtain from (3.9) and the relation $-\beta_1(1 + k^2) = (2 - k^2)\beta_3 - 3c$ the inequality

$$\mu_0 = \frac{1}{4} \frac{\beta_3 - c}{k^2 + 1} \rho_0 + (c - \beta_3) < 0.$$

It follows that the first eigenvalue μ_0 of $\mathscr{L}_{\mathrm{cn}}$ is negative and has eigenfunction $v_0(x) = P_0\left(\frac{1}{\eta}x\right)$, with η given by (3.7). Moreover, as $\rho_1 < \rho_2$ for all $k \in (0, 1)$, it follows from (3.9) that

$$\mu_2 = \frac{1}{4}\frac{\beta_3 - c}{k^2+1}\rho_2 + (c - \beta_3) > 0,$$

and so μ_2 is the third eigenvalue of $\mathscr{L}_{\mathrm{cn}}$ with eigenfunction $v_2(x) = P_2\left(\frac{1}{\eta}x\right)$. □

Remark 3.3. It can be shown that the first three intervals of instability associated to $\mathscr{L}_{\mathrm{cn}}$ are $(-\infty, \mu_0)$, (ν_0, ν_1), (μ_1, μ_2) and that the last interval of instability of $\mathscr{L}_{\mathrm{cn}}$ is (ν_2, ν_3).

We now use the function $d(c)$ given by (1.12) to prove the following result which is the heart of Theorem 3.5 below.

Lemma 3.4 (Convexity of $d(c)$). *If* $c \in \left(\frac{4\pi^2}{L^2}, +\infty\right)$ *then* $d(c)$ *is a convex function.*

Proof. By (2.8), we have that $a + \sqrt{b} = \frac{48\times 2K^2}{L^2}$, and by (2.7) we have that $a - \sqrt{b} = \frac{48\times 2k^2K^2}{L^2}$. So we conclude that

$$a = \frac{48(1+k^2)K^2}{L^2} \quad \text{and} \quad \sqrt{b} = \frac{48(1-k^2)K^2}{L^2}.$$

Thus,

$$\begin{cases} 3\beta_3 - 3c = \frac{48(1+k^2)K^2}{L^2}, \\ 9c^2 + 6c\beta_3 - 3\beta_3^2 = \frac{48^2(1-k^2)^2K^4}{L^4}. \end{cases}$$

Solving the system above, we get

$$c = \frac{8K^2\sqrt{3k'^4 + (1+k^2)^2}}{L^2}$$

and

$$\beta_3 = \frac{8K^2[2(1+k^2) + \sqrt{3k'^4 + (1+k^2)^2}]}{L^2}.$$

Now by (2.1) we have that $\int_0^L \phi_c^2\,d\xi = 2c\int_0^L \phi_c\,d\xi$, from which, by using (1.7), we get

$$d'(c) = \left\langle E'(\phi_c) + F'(\phi_c), \frac{d}{dc}\phi_c \right\rangle + F(\phi_c) = F(\phi_c) = c\int_0^L \phi_c\,dx.$$

Using that $\int_0^L \mathrm{cn}^2\left[\sqrt{\frac{\beta_3-\beta_1}{12}}\xi; k\right] d\xi = \frac{L}{K}\left[\frac{E(k)-k'^2K(k)}{k^2}\right]$, we get

$$\begin{aligned}\frac{d}{dc}\left(\int_0^L \phi_c^2\, d\xi\right) &= 2\int_0^L \phi_c(\xi)\, d\xi + 2c\frac{d}{dc}\left(\int_0^L \phi_c\, d\xi\right)\\ &= 2\int_0^L \phi_c(\xi)\, d\xi + 2c\frac{d}{dc}\left[\beta_2 + \frac{(\beta_3-\beta_2)}{K}\,\frac{(E-k'^2K)}{k^2}\right]L\\ &= 2\int_0^L \phi_c(\xi)\, d\xi + 2c\beta_2'(c)L\\ &\quad + 2c\frac{d}{dk}\left[\frac{(\beta_3-\beta_2)}{K(k)}\,\frac{\left(E(k)-k'^2K(k)\right)}{k^2}\right]k'(c)L. \qquad (3.10)\end{aligned}$$

Then convexity of d follows, provided that β_2 is an increasing function of c, since $\phi_c > 0$ and the last term in (3.10) is also positive. In fact, from equation (2.6) we have that

$$\beta_2'(c) = \frac{6\sqrt{9c^2+6c\beta_3-3\beta_3^2}+18c+6\beta_3+2\beta_3'[3c-3\beta_3-\sqrt{9c^2+6c\beta_3-3\beta_3^2}]}{4\sqrt{9c^2+6c\beta_3-3\beta_3^2}}$$

is positive since $\beta_3' > 0$ (see Corollary 2.2) and $3c-3\beta_3-\sqrt{9c^2+6c\beta_3-3\beta_3^2} > 0$ if and only if $\beta_3 > 2c$. This shows that β_2 is an increasing function for $\beta_3 > 2c$. Also, the last term in (3.10) is positive, since $\frac{d}{dk}\left\{(\beta_3-\beta_2)\frac{1}{K}\left[\frac{E(k)-k'^2K(k)}{k^2}\right]\right\} = \frac{48}{L^2}kK(k)^2 > 0$. □

The next result has been proved in the real case in ([2], Lemma 4.6 and Lemma 4.7).

Theorem 3.5. *Let ϕ_c the cnoidal wave solution given by Theorem 2.1 and $c \in \left(\frac{4\pi^2}{L^2}, \infty\right)$. Define*

$$\mathscr{A} = \{\psi \in H^1_{\mathrm{per}}([0,L]) \mid \langle\psi,\phi_c\rangle = 0 \text{ and } \|\psi\|_{L^2_{\mathrm{per}}([0,L])} = 1\}$$

and

$$\mathscr{B} = \{\psi \in H^1_{\mathrm{per}}([0,L]) \mid \langle\psi,\phi_c\rangle = \langle\psi,\phi_c\phi_c'\rangle = 0 \text{ and } \|\psi\|_{L^2_{\mathrm{per}}([0,L])} = 1\}.$$

Then the linear operator $\mathscr{L}_{\mathrm{cn}}$ satisfies

(a) $\gamma := \min\{\langle\mathscr{L}_{\mathrm{cn}}\psi,\psi\rangle \mid \psi \in \mathscr{A}\} = 0,$

(b) $\zeta := \min\{\langle\mathscr{L}_{\mathrm{cn}}\psi,\psi\rangle \mid \psi \in \mathscr{B}\} > 0$ *and consequently* $\langle\mathscr{L}_{\mathrm{cn}}\psi,\psi\rangle \geq \zeta\|\psi\|$ *for all* ψ *with* $\langle\psi,\phi_c\rangle = \langle\psi,\phi_c\phi_c'\rangle = 0$.

Proof. (a) From the characterization of the eigenvalues of $\mathscr{L}_{\mathrm{cn}}$ (Theorem 3.2) it follows that γ is finite. Since $\langle \phi_c', \phi_c \rangle = 0$ and $\mathscr{L}_{\mathrm{cn}}(\phi_c') = 0$ it follows that $\gamma \le 0$.

Now we show that the minimum is attained. Let (ψ_j) be a minimizing sequence, i.e., $\psi_j \in \mathscr{A}$ for all j and $\langle \mathscr{L}_{\mathrm{cn}}\psi_j, \psi_j \rangle \to \gamma$ as $j \to \infty$. Then for any $\varepsilon > 0$ we can choose ψ_j so that

$$0 < c \le \int_0^L \psi_j'^2 \, dx + c\int_0^L \psi_j^2 \, dx \le \int_0^L \phi_c \psi_j^2 \, dx + \gamma + \varepsilon. \tag{3.11}$$

Since $\|\psi_j\|_{L^2_{\mathrm{per}}([0,L])} = 1$, (3.11) implies that $\|\psi_j\|_{H^1_{\mathrm{per}}([0,L])}$ is uniformly bounded as j varies. Thus there exists a subsequence, which we denote again by (ψ_j), such that $\langle \mathscr{L}_{\mathrm{cn}}\psi_j, \psi_j \rangle \to \gamma$ as $j \to \infty$ and $\psi_j \rightharpoonup \tilde{\psi}$ weakly in $H^1_{\mathrm{per}}([0,L])$. By weak convergence in $L^2_{\mathrm{per}}([0,L])$, $\langle \tilde{\psi}, \phi_c \rangle = 0$. We also have $\|\tilde{\psi}\| = 1$ and $\int_0^L \phi_c \psi_j^2 \, dx \to \int_0^L \phi_c \tilde{\psi}^2 \, dx$, since the embedding $H^1_{\mathrm{per}}([0,L]) \subset L^2_{\mathrm{per}}([0,L])$ is compact. Moreover,

$$\gamma \le \langle \mathscr{L}_{\mathrm{cn}}\tilde{\psi}, \tilde{\psi} \rangle \le \liminf_{j\to\infty} \langle \mathscr{L}_{\mathrm{cn}}\psi_j, \psi_j \rangle = \gamma.$$

Hence, the minimum is attained at $\tilde{\psi}$.

Now we want to show that $\gamma \ge 0$. In this case, we will apply Lemma E1 in Weinstein [34] in the case that $A = \mathscr{L}_{\mathrm{cn}}$ and $R = \phi_c$. In fact, from Theorem 3.2 we have that $\mathscr{L}_{\mathrm{cn}}$ has the necessary spectral properties required by Lemma E1. Then $\gamma \ge 0$ if $\langle \mathscr{L}_{\mathrm{cn}}^{-1}\phi_c, \phi_c \rangle \le 0$. Now from Theorem 2.1 we have that the mapping $c \in \left(\frac{4\pi^2}{L^2}, \infty\right) \to \phi_c$ is of class C^1, so by taking differentiation with regard to c in (2.1) we obtain that $f = -\frac{d}{dc}\phi_c$ satisfies

$$\mathscr{L}_{\mathrm{cn}}\left(-\frac{d}{dc}\phi_c\right) = \phi_c.$$

Therefore we get

$$\langle \mathscr{L}_{\mathrm{cn}}^{-1}\phi_c, \phi_c \rangle = -\frac{1}{2}\frac{d}{dc}\int_0^L \phi_c^2(\xi) \, d\xi \le 0 \iff \frac{1}{2}\frac{d}{dc}\int_0^L \phi_c^2(\xi) \, d\xi \ge 0.$$

But from Lemma 3.4 we have

$$\frac{d}{dc}\frac{1}{2}\int_0^L \phi_c^2(\xi) \, d\xi > 0 \tag{3.12}$$

for $c > \frac{4\pi^2}{L^2}$.

Thus $\gamma = 0$. This finishes the proof of (a).

(b) From part (a) it is inferred that $\zeta \ge 0$. We will prove that $\zeta > 0$ by showing that the assumption $\zeta = 0$ leads to a contradiction. We first show that $\zeta = 0$ implies that the minimum is attained in the admissible class. We then consider an associated Lagrange multiplier problem to deduce $\zeta > 0$.

If $\zeta = 0$, using the same analysis as that in proof of (a) above, it is easy to see that the minimum is attained at an admissible function $\tilde{\psi} \neq 0$. Then there exists $(\tilde{\psi}, \lambda_*, \alpha_*, \beta_*)$ among the critical points of the Lagrange multiplier problem

$$\mathscr{L}_{\mathrm{cn}}(\tilde{\psi}) = \lambda_* \tilde{\psi} + \alpha_* \phi_c + \beta_* \phi_c \phi_c'. \tag{3.13}$$

So taking the inner product of (3.13) with $\tilde{\psi}$ and ϕ_c', we get $\lambda_* = \beta_* = 0$. Therefore, $\mathscr{L}_{\mathrm{cn}} \tilde{\psi} = \alpha_* \phi_c$. Now, since $\mathscr{L}_{\mathrm{cn}} f = \phi_c$ with $f = -\frac{d}{dc}\phi_c$, it follows that $\mathscr{L}_{\mathrm{cn}}(\tilde{\psi} - \alpha_* f) = 0$. So there is an $\theta \in \mathbb{R}$ such that $\tilde{\psi} - \alpha_* f = \theta \phi_c'$ since the null space of $\mathscr{L}_{\mathrm{cn}}$ has dimension 1 and is spanned by ϕ_c'. From (3.12) we have that $\langle f, \phi_c \rangle \neq 0$, so $\alpha_* = 0$. Then $\tilde{\psi} = \theta \phi_c'$ and hence ϕ_c' is orthogonal to $\phi_c \phi_c'$, which is a contradiction. Therefore the minimum in (b) is positive and the proof of the theorem is completed. □

Remark 3.6. It follows from Poincaré's inequality that $\langle \mathscr{L}_{\mathrm{cn}} \psi, \psi \rangle \geq \zeta' \|\psi\|_1^2$ for all $\psi \in H^1_{\mathrm{per}}([0, L])$ with $\langle \psi, \phi_c \rangle = \langle \psi, \phi_c \phi_c' \rangle = 0$ for some $\zeta' > 0$, since $\phi_c > 0$.

3.2. Proof of Theorem 1.1. In this section we shall use the Lyapunov method for studying the nonlinear stability of solutions $u(x, t) = \phi_c(x - ct)$ with ϕ_c given by Theorem 2.1. The proof is based on ideas developed by Benjamin [6], Bona [8] and Weinstein [35].

We use the conserved quantity

$$\mathscr{E}_{\mathrm{KdV}}[u] = E(u) + cF(u)$$

as a Lyapunov function.

Initially we measure the deviation of the solution $u(t)$ from the orbit $\mathcal{O}_{\phi_c}$ using the metric

$$\rho_c(u(t), \mathcal{O}_{\phi_c}) := \sqrt{\inf \Omega_t(y)},$$

where

$$\Omega_t(y) = \|u'(\cdot + y, t) - \phi_c'\|^2_{L^2_{\mathrm{per}}([0,L])} + c\|u(\cdot + y, t) - \phi_c\|^2_{L^2_{\mathrm{per}}([0,L])}$$

and the infimum is taken over all $y \in [0, L]$. Since Ω_t is a continuous function of y, the minimum is attained in an interval of time $[0, T]$ (see [8]) and this defines $y = y(t)$. Hence, we have that

$$\rho_c(u(t), \mathcal{O}_{\phi_c}) = \sqrt{\Omega_t(y(t))} \tag{3.14}$$

for all $t \in [0, T]$.

Consider the perturbation

$$u(x+y,t)=\phi_c(x)+v(x,t), \tag{3.15}$$

for $t\in[0,T]$ and $y=y(t)$ determined by (3.14). The property of the minimum leads to the constraint

$$\int_0^L \phi_c(x)\phi_c'(x)v(x,t)\,dx=0. \tag{3.16}$$

By conservation of $\mathscr{E}_{\mathrm{KdV}}$, scale invariance, representation (3.15), the embedding $H^1_{\mathrm{per}}([0,L])\hookrightarrow L^r_{\mathrm{per}}([0,L])$ for all $r\geq 2$, and the fact that ϕ_c satisfies (2.1), we have the following variation for $\mathscr{E}_{\mathrm{KdV}}[u]=E(u)+cF(u)$:

$$\begin{aligned}\Delta\mathscr{E}_{\mathrm{KdV}}(t)&=\mathscr{E}_{\mathrm{KdV}}[u_0]-\mathscr{E}_{\mathrm{KdV}}[\phi_c]\\&=\mathscr{E}_{\mathrm{KdV}}\big(u(\cdot,t)\big)-\mathscr{E}_{\mathrm{KdV}}\big(\phi_c(\cdot)\big)\\&=\mathscr{E}_{\mathrm{KdV}}\big(u(\cdot+y,t)\big)-\mathscr{E}_{\mathrm{KdV}}\big(\phi_c(\cdot)\big)\\&=\mathscr{E}_{\mathrm{KdV}}\big(\phi_c+v(t)\big)-\mathscr{E}_{\mathrm{KdV}}(\phi_c)\\&\geq\frac{1}{2}\langle\mathscr{L}_{\mathrm{cn}}v,v\rangle-C_1\|v\|_1^3,\end{aligned} \tag{3.17}$$

where $C_1>0$ is a constant. Now we obtain a suitable lower bound on the quadratic form in (3.17). Initially we consider the normalization $\|u(t)\|^2=\|\phi_c\|^2$ for every $t\in[0,T]$. By (3.15),

$$\langle v,\phi_c\rangle=-\frac{1}{2}\|v\|^2. \tag{3.18}$$

Define $P_\| := \frac{\langle v,\phi_c\rangle}{\|\phi_c\|^2}\phi_c$ and $P_\perp := v-P_\|$. Without lost of generality, we may suppose that $\|\phi_c\|=1$. Then

$$\langle\mathscr{L}_{\mathrm{cn}}v,v\rangle=\langle\mathscr{L}_{\mathrm{cn}}P_\|,P_\|\rangle+2\langle\mathscr{L}_{\mathrm{cn}}P_\|,P_\perp\rangle+\langle\mathscr{L}_{\mathrm{cn}}P_\perp,P_\perp\rangle.$$

Moreover, by (3.16) we get $\langle P_\perp,\phi_c\rangle=\langle P_\perp,\phi_c\phi_c'\rangle=0$. Therefore, Theorem 3.5 (b) and (3.18) imply that

$$\langle\mathscr{L}_{\mathrm{cn}}P_\perp,P_\perp\rangle\geq\zeta\langle P_\perp,P_\perp\rangle=\zeta\big[\|v\|^2-\tfrac{1}{4}\|v\|^4\big]. \tag{3.19}$$

Also, by (3.18),

$$\langle\mathscr{L}_{\mathrm{cn}}P_\|,P_\|\rangle=\tfrac{1}{4}\langle\mathscr{L}_{\mathrm{cn}}\phi_c,\phi_c\rangle\|v\|^4. \tag{3.20}$$

Finally,

$$\langle \mathscr{L}_{\text{cn}} P_\perp, P_\parallel \rangle = -\tfrac{1}{2}\|v\|^2 \langle \mathscr{L}_{\text{cn}} P_\perp, \phi_c \rangle = -\tfrac{1}{2}\|v\|^2 \langle P_\perp, \mathscr{L}_{\text{cn}} \phi_c \rangle$$
$$= -\tfrac{1}{4}\|v\|^4 \|\phi\|_1^2. \tag{3.21}$$

Using (3.19), (3.20) and (3.21) and the fact that $\langle \mathscr{L}_{\text{cn}} \phi_c, \phi_c \rangle < 0$ we obtain from (3.17) that

$$\Delta \mathscr{E}_{\text{KdV}}(t) \geq D_0 \|v\|_1^2 - D_1 \|v\|_1^3 - D_2 \|v\|_1^4,$$

where D_0, D_1, D_2 are positive constants. Hence, from (3.14) it follows that

$$\Delta \mathscr{E}_{\text{KdV}}(t) \geq g\big(\rho_c\big(u(t), \mathcal{O}_{\phi_c}\big)\big) \tag{3.22}$$

for all $t \in [0, T]$, where $g(s) = As^2 - Bs^3 - Cs^4$ with $A, B, C > 0$. The essential properties of g are $g(0) = 0$ and $g(s) > 0$ for s small. The stability result can be derived from (3.22) as follows. Let $\varepsilon > 0$ be sufficiently small. Then, by the continuity of $\mathscr{E}_{\text{KdV}}$ in $\{u \in H^1_{\text{per}}([0, L]) \mid \|u\| = \|\phi_c\|\}$, there is a $\delta = \delta(\varepsilon)$ such that if $\rho_c(u_0, \mathcal{O}_{\phi_c}) < \delta$, then

$$\Delta \mathscr{E}_{\text{KdV}}(0) < g(\varepsilon)$$

for $t \in [0, T]$. Since $\Delta \mathscr{E}_{\text{KdV}}$ is constant in time, $g\big(\rho_c\big(u(t), \mathcal{O}_{\phi_c}\big)\big) < g(\varepsilon)$. Therefore, since $\rho_c\big(u(t), \mathcal{O}_{\phi_c}\big)$ is a continuous function of time, $\rho_c\big(u(t), \mathcal{O}_{\phi_c}\big) < \varepsilon$ for all $t \in [0, T]$, i.e., ϕ_c is orbitally stable in $H^1_{\text{per}}([0, L])$ with regard to small perturbations that preserve the L^2_{per}-norm. To prove stability relative to general small perturbations we use that the mapping $c \in \left(\frac{4\pi^2}{L^2}\right) \to \phi_c$ is continuous and the preceding theory. To see this, fix c and let ϕ_c be the cnoidal wave whose stability is in question. Let $\varepsilon > 0$ be given and let u_0 be the initial data for (1.1) for which $\|u_0 - \phi_c\|_1 \leq \delta$, where δ will be determined conveniently. For δ small enough there exists d near c such that $F(u_0) = F(\phi_d)$ and δ' such that $\|\phi_c - \phi_d\|_1 < \delta'$, by the continuity of the mapping $c \in \left(\frac{4\pi^2}{L^2}\right) \to \phi_c$. Moreover, $\|u_0 - \phi_d\|_1 \leq \|u_0 - \phi_c\|_1 + \|\phi_c - \phi_d\|_1 \leq \delta + \delta'$. Making use of the stability result for perturbations preserving the L^2_{per}-norm if δ is small enough, we obtain that $\rho_c\big(u(t), \mathcal{O}_{\phi_d}\big) < \frac{1}{2}\varepsilon$ for all $t \in [0, T]$, since δ' is independent of t. Thus, $\rho_c\big(u(t), \mathcal{O}_{\phi_c}\big) \leq \rho_c\big(u(t), \mathcal{O}_{\phi_d}\big) + \rho_c(\phi_d, \mathcal{O}_{\phi_c}) < \frac{1}{2}\varepsilon + \bar{\delta}$ for all $t \in [0, T]$, with $\bar{\delta}$ independent of t. The desired result follows.

4. Stability of *L*-periodic travelling wave solutions for the Boussinesq system

Let $X := H^1_{\text{per}}([0, L]) \times L^2_{\text{per}}([0, L])$ and $\vec{\phi} = \big(\phi_c(x - ct), \psi_c(x - ct)\big) \in X$ be an L-periodic travelling wave solution for the system (1.3). Now we show that the orbit $\mathcal{O}_{\vec{\phi}}$ is stable in the X sense by the flow of system (1.3), that is, for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $\inf_{s\in\mathbb{R}} \|\vec{u}_0 - \tau_s(\vec{\phi})\|_X < \delta$, then the solution $\vec{u}(t)$ of (1.3) with $\vec{u}(0) = \vec{u}_0$ satisfies

$$\inf_{s\in\mathbb{R}} \|\vec{u}(t) - \tau_s(\vec{\phi})\|_X < \varepsilon$$

for all t for which $\vec{u} = (u, v)$ exists.

4.1. Local existence theory. In the present section a theorem asserting the local well-posedness of the initial value problem (1.2)–(1.13) is stated. The well-posedness theorem is a straightforward consequence of the abstract techniques of Kato [22], [23] for quasi-linear evolution equations, and consequently the proof is omitted.

To apply Kato's theory to the initial value problem (1.2)–(1.13), we consider the equivalent formulation (1.3)–(4.1) with

$$\begin{cases} u(x,0) = u_0(x), \\ v(x,0) = v_0(x) \end{cases} \tag{4.1}$$

for $x \in \mathbb{R}$.

For $T > 0$ and $s \in \mathbb{R}$ define the following spaces of solutions and initial conditions

$$\begin{cases} X_s(T) = C\big(0, T; H^{s+2}_{\text{per}}([0, L])\big) \cap C^1\big(0, T; H^{s}_{\text{per}}([0, L])\big), \\ Y = H^{s+2}_{\text{per}}([0, L]) \times H^{s+1}_{\text{per}}([0, L]). \end{cases} \tag{4.2}$$

Theorem 4.1. *Let $(u_0, v_0) \in Y$ for some $s > 1/2$. Then there exist $T > 0$, which depends only on $\|(u_0, v_0)\|_Y$, and unique functions $u \in X_s(T)$ and $v \in X_{s-1}(T)$, which solve the initial value problem* (1.3)–(4.1). *Moreover, the pair (u, v) depends continuously on (u_0, v_0) in the sense that the associated mapping $(u_0, v_0) \to (u, v)$ is continuous from Y into the space $X_s(T) \times X_{s-1}(T)$.*

This theorem follows directly from the general results of Kato (1974, 1983) on quasi-linear evolution equations. The functional analytic setting for Kato's theory consists of a pair of reflexive Banach spaces X and Y, with Y continuously and densely imbedded in X. A central role in the theory is played by a Banach space isomorphism S of Y onto X, and the norms on these two spaces are chosen in such a way that S is an isometry. The theory applies to the abstract, quasi-linear evolution equation

$$\vec{U}_t + A(t, \vec{U})\vec{U} = F(t, \vec{U}) \qquad \text{for } t > 0 \text{ with } \vec{U}(0) = \vec{\phi}, \tag{4.3}$$

where $\vec{\phi}$ is a given initial value. The theory asserts that there exists a positive time T such that (4.3) possesses a unique solution in $C(0,T;Y) \cap C^1(0,T;X)$ under certain assumptions.

To apply Kato's theory to the situation envisaged in Theorem 4.1, take $X = H^s_{\text{per}}([0,L]) \times H^{s-1}_{\text{per}}([0,L])$ with $s > 1/2$, and take Y as in (4.2). Also, let $S = (I_d - \partial_x^2, I_d - \partial_x^2)$ with I_d denoting the identity operator, let A be the matrix of differential operators

$$A = \begin{pmatrix} 0 & -\partial_x \\ -\partial_x + \partial_x^3 & 0 \end{pmatrix},$$

and take the nonlinear operator F to be

$$F = F(t,u,v) = \begin{pmatrix} 0 \\ -\left(\frac{u^2}{2}\right)_x \end{pmatrix}.$$

With this choice of A and F, and writing

$$\vec{U} = \begin{pmatrix} u \\ v \end{pmatrix},$$

(4.3) reduces to (1.3)–(4.1) if $\vec{\phi} = (u_0, v_0)$, and it is straightforward to verify that the hypotheses required in Kato's theory are satisfied.

A consequence of Theorem 4.1 is stated in the following corollary. Define for $T > 0$ and $s \in \mathbb{R}$

$$Y_s(T) = X_s(T) \cap C^2\big(0, T, H^{s-2}_{\text{per}}([0,L])\big).$$

Corollary 4.2. *Let $(u_0, v_0) \in Y$ for some $s > 1/2$. Then there exist $T > 0$, which depends only on $\|(u_0, v_0)\|_Y$, and a unique function $u \in Y_s(T)$ which is a solution of eq. (1.2) in the distributional sense on $\mathbb{R} \times [0,T]$, and for which $u(\cdot,0) = u_0$ and $u_t(\cdot,0) = v_0'$. The solution u depends continuously on (u_0, v_0) in the sense that the associated mapping $(u_0, v_0) \to u$ is continuous from Y into the space $Y_s(T)$.*

Remark 4.3. If $s > 5/2$, then the solution is classical, which means that all derivatives featured in the equation exist pointwise and are jointly continuous functions of x and t.

4.2. Spectral analysis. In this section we study the spectral properties associated to the linear operator

$$\mathscr{L}_c = (H'' + cI'')(\phi_{w(c)}, \psi_{w(c)}) \tag{4.4}$$

determined by the periodic solutions $(\phi_{w(c)}, \psi_{w(c)})$ found in Theorem 2.4. We compute the Hessian operator $\mathscr{L}_c$ by calculating the associated quadratic form, which is denoted by $\mathscr{Q}_c$. By definition, $\mathscr{Q}_c(g,h)$ is the coefficient of ε^2 in

$$H(\phi_{w(c)} + \varepsilon g, \psi_{w(c)} + \varepsilon h) + cI(\phi_{w(c)} + \varepsilon g, \psi_{w(c)} + \varepsilon h),$$

and so is given by

$$\begin{aligned}\mathscr{Q}_c(g,h) &= \int_0^L \left\{\frac{1}{2}(g^2 + g_x^2 + h^2) - \frac{1}{2}\phi_{w(c)}g^2 + cgh\right\} dx \\ &= \int_0^L \left\{\frac{1}{2}[(1-c^2)g^2 + g'^2 - \phi_{w(c)}g^2] + \frac{1}{2}(h+cg)^2\right\} dx \\ &:= \mathscr{Q}_c^1(g) + \frac{1}{2}\|h+cg\|_0^2. \end{aligned} \tag{4.5}$$

Note that $\mathscr{Q}_c$ is the sum of the quadratic form $\mathscr{Q}_c^1$ associated to the operator $-\frac{d^2}{dx^2} + 1 - c^2 - \phi_w$ and the non-negative term $\frac{1}{2}\|h+cg\|_0^2$. From the equations (1.8) for the cnoidal wave $(\phi_{w(c)}, \psi_{w(c)})$, it follows that $g = \phi'_{w(c)}$ and $h = \psi'_{w(c)}$ satisfy $\mathscr{L}_c(g,h) = 0$. To see that this is the only eigenfunction corresponding to the eigenvalue zero and the other expected properties of the operator $\mathscr{L}_c$, we will first consider the following periodic eigenvalue problem:

$$\begin{cases}\mathscr{L}'_{\mathrm{cn}}v := \left(-\frac{d^2}{dx^2} + 1 - c^2 - \phi_{w(c)}\right)v = \mu v, \\ v(0) = v(L),\ v'(0) = v'(L),\end{cases}$$

where ϕ_w is given by Theorem 2.1. The operator $\mathscr{L}'_{\mathrm{cn}}$ has the same spectral structure of $\mathscr{L}_{\mathrm{cn}}$. We can see this by replacing c with $1-c^2$ in the proof of Theorem 3.2.

To prove that the kernel of $\mathscr{L}_c$ is spanned by $\frac{d}{dx}(\phi_{w(c)}, \psi_{w(c)})$, consider the quadratic form $\mathscr{Q}_c(g,h)$ as the pairing of (g,h) against $(\tilde{g},\tilde{h})$ in the $H^1_{\mathrm{per}}([0,L]) \times L^2_{\mathrm{per}}([0,L]) - H^{-1}_{\mathrm{per}}([0,L]) \times L^2_{\mathrm{per}}([0,L])$ duality, where $(\tilde{g},\tilde{h})^t$ is the unbounded operator

$$\tilde{\mathscr{L}}_c := \begin{pmatrix} 1 - \partial_{xx} - \phi_{w(c)} & c \\ c & 1 \end{pmatrix}$$

applied to $(g,h)^t$. Then $\tilde{\mathscr{L}}_c(g,h)^t = 0$ implies that

$$\begin{cases} -g'' + (1-c^2)g - \phi_{w(c)}g = 0, \\ h = -cg. \end{cases}$$

From the properties of the operator $\mathscr{L}_{\rm cn} = -\partial_x^2 + w - \phi_w$ established in Theorem 3.2, it follows that $g = \lambda\phi'_{w(c)}$ and $h = -cg = -c\lambda\phi'_{w(c)} = \lambda\psi'_{w(c)}$, where $0 \neq \lambda \in \mathbb{R}$.

To show that there is a single, simple, negative eigenvalue, consider $\mathscr{Q}_c^1$ defined in (4.5) above. By Theorem 3.2, the operator $\mathscr{L}_{\rm cn}$ has exactly one negative eigenvalue which is simple, say μ_0, with associated eigenfunction χ. Thus, $\mathscr{Q}_c^1$ takes on a negative value and so does $\mathscr{Q}_c$. In fact, considering $\vec{\zeta} = (\chi, -c\chi)$, we have

$$\mathscr{L}_c(\vec{\chi}) = Q_c^1(\chi) + \frac{1}{2}\|c\chi - c\chi\|^2 = Q_c^1(\chi) = \frac{1}{2}\lambda_0 < 0.$$

Denoting by λ_0 the lowest eigenvalue of $\mathscr{L}_c$, we will show that the next eigenvalue λ_1 is 0, which is known to be simple, and so λ_2 is in fact strictly positive. These results are proved using the (min-max) Rayley–Ritz characterization of eigenvalues (see [15], [33]), namely

$$\lambda_1 = \max_{(\phi_1,\psi_1)\in X} \min_{\substack{(g,h)\in X\setminus\{0\} \\ \langle g,\phi_1\rangle_1 + \langle h,\psi_1\rangle = 0}} \frac{Q_c(g,h)}{\|g\|_1^2 + \|h\|^2}.$$

Choosing $\phi_1 = \zeta$, $\psi_1 = 0$ we obtain the lower estimate

$$\lambda_1 \geq \min_{\substack{(g,h)\in X\setminus\{0\} \\ \langle g,\chi\rangle_1 = 0}} \frac{Q_c(g,h)}{\|g\|_1^2 + \|h\|^2}. \tag{4.6}$$

The right-hand side of (4.6) is non-negative on the subspace

$$\{(g,h) \in X\setminus\{0\} \mid \langle g,\zeta\rangle_1 = 0\},$$

since $Q_c^1(g) \geq 0$ by Theorem 3.2. Thus, $\lambda_1 = 0$ and, from earlier considerations, λ_1 is simple and $\lambda_2 > 0$.

The above analysis can be summarized in the form of the following theorem:

Theorem 4.4. *Let $\mathscr{L}_c$ be the linear operator defined on $H^2_{\rm per}([0,L]) \times H^1_{\rm per}([0,L])$ by* (4.4). *Then the first two eigenvalues λ_0 and λ_1 of $\mathscr{L}_c$ are simple and satisfy $\lambda_0 < \lambda_1 = 0$. Moreover, $\vec{\phi}'_{w(c)}$ is the eigenfunction of λ_1.*

In analogy with the analysis of Section 3, we prove now the key ingredient of Theorem 4.6 below.

Lemma 4.5 (Convexity of $m(c)$). *Let $c \in (-1,1)$ and $L > 2\pi$. Then the function $m(c)$ is convex, provided that $c^2 > \frac{1}{3}$ and $1 - c^2 > \frac{4\pi^2}{L^2}$.*

Proof. Lemma 3.4 and relation $(H+cI)'(\phi_w,\psi_w)=0$ imply that

$$\begin{aligned}
m''(c) &= \frac{d}{dc} I(\phi_w,\psi_w) \\
&= \frac{d}{dc}\int_0^L \phi_w\psi_w \\
&= -\frac{d}{dc}\int_0^L c\phi_w^2 \\
&= -\int_0^L \phi_w^2\,dx - c\frac{d}{dc}\left[\int_0^L \phi_w^2\,dx\right] \\
&= -\int_0^L \phi_w^2\,dx - c\frac{d}{dw}\left[\int_0^L \phi_w^2\,dx\right]\frac{dw}{dc} \\
&= -\int_0^L \phi_w^2\,dx + 2c^2\frac{d}{dw}\left[\int_0^L \phi_w^2\,dx\right] \\
&= 2\left[-w\int_0^L \phi_w\,dx + 2c^2\frac{d}{dw}\left(w\int_0^L \phi_w\,dx\right)\right] \\
&= 2\left[-w\int_0^L \phi_w\,dx + 2c^2\left(\int_0^L \phi_w\,dx + w\frac{d}{dw}\int_0^L \phi_w\,dx\right)\right] \\
&= 2\left[(3c^2-1)\int_0^L \phi_w\,dx + 2c^2 w\frac{d}{dw}\left(\int_0^L \phi_w\,dx\right)\right] > 0
\end{aligned}$$

if $c^2>\frac{1}{3}$, since $\frac{d}{dw}\left(\int_0^L \phi_w\,dx\right)>0$. □

Theorem 4.6. *Let* $\vec{\phi}_w=(\phi_{w(c)},\psi_{w(c)})=(\phi_{w(c)},-c\phi_{w(c)})$ *be the cnoidal wave solution given by Theorem* 2.4, *and let* $w=w(c)=1-c^2\in\left(\frac{4\pi^2}{L^2},1\right)$. *Let*

$$\Gamma=\inf\{\langle\mathscr{L}_c(\vec{\varphi}),(\vec{\varphi})\rangle \mid (\vec{\varphi})\in H^1_{\text{per}}([0,L])\times L^2_{\text{per}}([0,L]),\|\vec{\varphi}\|=1,\langle\vec{\varphi},I'(\vec{\phi_w})\rangle=0\}$$

and

$$\begin{aligned}
\Upsilon=\inf\{\langle\mathscr{L}_c(\vec{\varphi}),(\vec{\varphi})\rangle \mid (\vec{\varphi})\in H^1_{\text{per}}([0,L])\times L^2_{\text{per}}([0,L]),\\
\|\vec{\varphi}\|=1,\langle\vec{\varphi},I'(\vec{\phi_w})\rangle=0,\langle\vec{\varphi},(\phi_w\phi_w',-c\phi_w\phi_w')\rangle=0\}.
\end{aligned}$$

Then for $c^2>\frac{1}{3}$ *the linear operator* $\mathscr{L}_c$ *satisfies* $\Gamma=0$ *and* $\Upsilon>0$.

Proof. We first observe that $\langle\mathscr{L}_c\vec{\varphi},\vec{\varphi}\rangle=\langle\mathscr{L}_{\text{cn}}\varphi,\varphi\rangle+\int_0^L(c\varphi+\chi)^2\,dx$ for $\vec{\varphi}=(\varphi,\chi)$. From this and Theorem 3.2 it follows that Γ is finite. Since $\langle\vec{\phi_w'},I'(\vec{\phi_w})\rangle=0$ and $\mathscr{L}_c(\vec{\phi_w'})=0$, it follows that $\Gamma\le 0$.

Now we show that the minimum is attained. To this end, let $\{(g_j, h_j)\}$ be a sequence of $H^1_{\text{per}}([0,L]) \times L^2_{\text{per}}([0,L])$-functions with $\|(g_j,h_j)\| = 1$ for all j, $\langle (g_j,h_j), I'(\phi_w, \psi_w)\rangle = 0$ for all j and

$$\lim_{j\to\infty} \langle \mathscr{L}_c(g_j,h_j),(g_j,h_j)\rangle = \Gamma. \tag{4.7}$$

It follows that $\|(g_j,h_j)\|$ is uniformly bounded in $H^1_{\text{per}}([0,L]) \times L^2_{\text{per}}([0,L])$ as j varies. So there is a subsequence of $\{(g_j,h_j)\}$, which we denote again by $\{(g_j,h_j)\}$, and a function $(g,h) \in H^1_{\text{per}}([0,L]) \times L^2_{\text{per}}([0,L])$ such that $(g_j,h_j) \rightharpoonup (g,h)$ weakly in $H^1_{\text{per}}([0,L]) \times L^2_{\text{per}}([0,L]) \subset L^2_{\text{per}}([0,L]) \times L^2_{\text{per}}([0,L])$. Since the embedding $H^1_{\text{per}}([0,L]) \subset L^2_{\text{per}}([0,L])$ is compact, we also obtain a subsequence of $\{(g_j)\}$, which we denote again by $\{(g_j)\}$, such that

$$g_j \to g \text{ in } L^2_{\text{per}}([0,L]).$$

Now from (4.7) it follows that

$$\langle \mathscr{L}_c(g_j,h_j),(g_j,h_j)\rangle = \int_0^L \{(g_j')^2 + g_j^2 + h_j^2 - \phi_w g_j^2 + 2cg_jh_j\}\,dx \to \Gamma$$

as $j \to \infty$. So, for all $\varepsilon > 0$, there exists $J \in \mathbb{N}$ such that

$$\left|\int_0^L (g_j')^2\,dx + \int_0^L g_j^2\,dx + \int_0^L h_j^2\,dx - \int_0^L \phi_w g_j^2\,dx + \int_0^L 2cg_jh_j\,dx - \Gamma\right| < \varepsilon$$

for all $j > J$, or in other words,

$$0 < 1 \le \int_0^L g_j^2\,dx + \int_0^L (g_j')^2 + \int_0^L h_j^2\,dx < \int_0^L \phi_w g_j^2\,dx - \int_0^L 2cg_jh_j\,dx + \Gamma + \varepsilon.$$

Since $g_j \to g$ in $L^2_{\text{per}}([0,L])$ and $h_j \rightharpoonup h$ in $L^2_{\text{per}}([0,L])$ as $j \to \infty$, we have $\int_0^L g_jh_j\,dx \to \int_0^L gh\,dx$. This together with the boundedness of ϕ_w implies that

$$1 < \int_0^L \phi_w g^2\,dx - \int_0^L 2cgh\,dx + \Gamma + \varepsilon. \tag{4.8}$$

We conclude that $(g,h) \neq 0$ by (4.8), since ε is arbitrary.

By Fatou's Lemma, $\|(g,h)\|_{L^2_{\text{per}}([0,L])\times L^2_{\text{per}}([0,L])} \le 1$. Now we shall divide the proof that $\Gamma = 0$ into two parts: (a) $\|(g,h)\| < 1$ and (b) $\|(g,h)\| = 1$.

(a) Suppose that $\|(g,h)\| < 1$ and $\Gamma < 0$. Then define $(g_*, h_*) = \frac{(g,h)}{\|(g,h)\|}$, which is admissible. By weak convergence of (g_j,h_j) to (g,h) in $H^1_{\text{per}}([0,L]) \times L^2_{\text{per}}([0,L])$ we get

$$\Gamma \leq \langle \mathscr{L}_c(g_*, h_*), (g_*, h_*) \rangle = \frac{1}{\|(g, h)\|^2} \Gamma < \Gamma,$$

a contradiction. Hence, in this case $\Gamma = 0$ and the minimum is attained at (g_*, h_*).

(b) If $\|(g, h)\|_{L^2_{\text{per}}([0, L]) \times L^2_{\text{per}}([0, L])} = 1$, then (g, h) is the minimum.

Now we prove that $\Gamma \geq 0$ if $c^2 > \frac{1}{3}$ and conclude that $\Gamma = 0$. Here we will apply [34], Lemma E1, in the case that $A = \mathscr{L}_c$ and $R = I'(\vec{\phi}_{w(c)})$. In fact, from the analysis made above we have that $\mathscr{L}_c$ has the necessary spectral properties required by Lemma E1. Then $\Gamma \geq 0$ if $\langle \mathscr{L}_c^{-1} I'(\vec{\phi}_{w(c)}), I'(\vec{\phi}_{w(c)}) \rangle \leq 0$. Now, from Theorem 2.4 we have that the mapping $\left(-\sqrt{1 - \frac{4\pi^2}{L^2}}, \sqrt{1 - \frac{4\pi^2}{L^2}}\right) \to (\phi_{w(c)}, \psi_{w(c)})$ is of class C^1, so differentiating the system (1.3) with regard to c, we obtain that $(g, h) = -\frac{d}{dc}(\phi_{w(c)}, \psi_{w(c)})$ satisfies

$$\mathscr{L}_c \begin{pmatrix} \frac{d}{dc}\phi_w \\ \frac{d}{dc}\psi_w \end{pmatrix} = \begin{pmatrix} -\frac{d^2}{dx^2} + 1 - \phi_w & c \\ c & 1 \end{pmatrix} \begin{pmatrix} \frac{d}{dc}\phi_w \\ \frac{d}{dc}\psi_w \end{pmatrix} = \begin{pmatrix} -\psi_w \\ -\phi_w \end{pmatrix} = -I'(\phi_w, \psi_w).$$

We then have

$$\begin{aligned}
\langle \mathscr{L}_c^{-1} I'(\phi_w, \psi_w), I'(\phi_w, \psi_w) \rangle &= \left\langle \left(-\frac{d}{dc}\phi_w, -\frac{d}{dc}\psi_w \right), (\psi_w, \phi_w) \right\rangle \\
&= -\int_0^L \psi_w \frac{d}{dc}\phi_w \, dx - \int_0^L \phi_w \frac{d}{dc}\psi_w \, dx \\
&= -\int_0^L \frac{d}{dc}(\phi_w \psi_w) \, dx \\
&= -\frac{d}{dc}\int_0^L \phi_w \psi_w \, dx \\
&= -\frac{d}{dc} I(\phi_w, \psi_w). \qquad (4.9)
\end{aligned}$$

Thus, from (4.9) we deduce that

$$\langle \mathscr{L}_c^{-1} I'(\phi_w, \psi_w), I'(\phi_w, \psi_w) \rangle \leq 0 \;\Leftrightarrow\; \frac{d}{dc} I(\phi_w, \psi_w) \geq 0.$$

Now Lemma 4.5 implies that $\frac{d}{dc} I(\phi_w, \psi_w) > 0$ if $c^2 > \frac{1}{3}$. Thus $\Gamma = 0$ if $c^2 > 1/3$.

Now we show that $\Upsilon > 0$. It is easy to see that $\Upsilon \geq 0$. We will prove $\Upsilon > 0$ by showing that the assumption $\Upsilon = 0$ leads to a contradiction. Suppose that $\Upsilon = 0$. Then, by similar argument to that in Theorem 3.5 (b), there exists a function $\vec{\varphi} = (\varphi, \upsilon)$ which satisfies $\|\vec{\varphi}\|_{L^2_{\text{per}}([0, L]) \times L^2_{\text{per}}([0, L])} = 1$, $\langle \vec{\varphi}, I'(\phi, \psi) \rangle = 0$,

$\left\langle \vec{\varphi}, \left(\left(\frac{\phi_w^2}{2}\right)', -c\left(\frac{\phi_w^2}{2}\right)' \right) \right\rangle = 0$ and $\langle \mathscr{L}_c\vec{\varphi}, \vec{\varphi}\rangle = 0$. Moreover, there are λ_*, θ_*, μ_* such that

$$\mathscr{L}_c\vec{\varphi} = \lambda_*\vec{\varphi} + \theta_* I'(\vec{\varphi}) + \mu_*\left(\left(\frac{\phi_w^2}{2}\right)', -c\left(\frac{\phi_w^2}{2}\right)' \right). \tag{4.10}$$

Taking the inner product of (4.10) with $\vec{\varphi}$, we get $\lambda_* = 0$. Taking again the inner product of (4.10) with $\frac{d}{dx}\overrightarrow{\phi_w} = (\phi_w', -c\phi_w')$, we obtain that $\mu_* = 0$. Then $\mathscr{L}_c\vec{\varphi} = \theta_* I'(\vec{\varphi})$. Now, since $\mathscr{L}_c\big((g,h)\big) = I'(\overrightarrow{\phi_w})$, with $(g,h) = -\frac{d}{dc}\overrightarrow{\phi_w}$ it follows that $\mathscr{L}_c\big(\vec{\varphi} - \theta_*(g,h)\big) = 0$ and so there is $\alpha \in \mathbb{R}$ such that $\vec{\varphi} - \theta_*(g,h) = \alpha\overrightarrow{\phi_w'}$. Since $\frac{d}{dc}I(\overrightarrow{\phi_w}) > 0$ for all c such that $c^2 > \frac{1}{3}$, we have $\langle (g,h), I'(\overrightarrow{\phi_w})\rangle \neq 0$ for all c such that $c^2 > \frac{1}{3}$, from which we conclude that $\theta_* = 0$. Thus, $\vec{\varphi} = \alpha\overrightarrow{\phi_w'}$ is orthogonal to $\left(\left(\frac{\phi_w^2}{2}\right)', -c\left(\frac{\phi_w^2}{2}\right)' \right)$, a contradiction. This completes the proof of the theorem. □

4.3. Proof of Theorem 1.4. We define

$$\vec{u}(x+y,t) - \overrightarrow{\phi_w}(x) = mI'\big(\overrightarrow{\phi_w}(x)\big) + \vec{z}(x,t), \tag{4.11}$$

where $\big\langle \vec{z}(\cdot,t), I'\big(\overrightarrow{\phi_w}(\cdot)\big)\big\rangle = 0$, $m \in \mathbb{R}$, and $y = y(t)$ is chosen to be the minimum for

$$\Theta_t(y) = \|\vec{u}'(\cdot + y,t) - \vec{\phi}_{w(c)}'\|^2 + c\|\vec{u}(\cdot + y,t) - \vec{\phi}_{w(c)}\|^2,$$

with $\vec{u}(x,t) = \big(u(x,t), v(x,t)\big)$ and $\vec{z}(x,t) = \big(a(x,t), b(x,t)\big)$. Here the deviation of the solution $\vec{u}(t)$ from the orbit $\mathcal{O}_{\vec{\phi}_{w(c)}}$ is measured by

$$\rho_{w(c)}\big(\vec{u}(t), \mathcal{O}_{\vec{\phi}_{w(c)}}\big) = \sqrt{\Theta_t\big(y(t)\big)}.$$

Therefore $\vec{z}(t)$ satisfies the compatibility condition

$$\int_0^L a(x,t)\phi_w\phi_w'\,dx - c\int_0^L b(x,t)\phi_w\phi_w'\,dx = 0, \tag{4.12}$$

or in other words, $\big\langle \big(a(x,t), b(x,t)\big), (\phi_w\phi_w', -c\phi_w\phi_w')\big\rangle = 0$. Next, using that H and I are invariant by translation, the representation (4.11), the classical embedding $H^1_{\text{per}}([0,L]) \hookrightarrow L^r_{\text{per}}([0,L])$ for every $r \geq 2$, and the fact that ϕ_w satisfies (1.10), we have the following variation for $\mathscr{E}[\vec{u}] = H(\vec{u}) + cI(\vec{u})$ and $\vec{v}(\cdot) := \vec{u}\big(\cdot + y(t), t\big) - \overrightarrow{\phi_w}(\cdot)$:

$$\begin{aligned}\Delta\mathscr{E} &= \mathscr{E}(\vec{u}_0) - \mathscr{E}(\overrightarrow{\phi_w})\\ &= \mathscr{E}\big(\vec{u}(\cdot,t)\big) - \mathscr{E}(\overrightarrow{\phi_w})\end{aligned}$$

$$
\begin{aligned}
&= \mathscr{E}\big(\vec{u}(\cdot + y(t), t)\big) - \mathscr{E}(\vec{\phi_w}) \\
&= \frac{1}{2}\langle \mathscr{E}''(\vec{\phi_w})\vec{v}, \vec{v}\rangle + o(\|\vec{v}\|_X^3) \\
&= \frac{1}{2}\langle \mathscr{L}_c\vec{v}, \vec{v}\rangle + o(\|\vec{v}\|_X^3) \\
&\geq \frac{m^2}{2}\langle \mathscr{L}_c I'(\vec{\phi}_w), I'(\vec{\phi}_w)\rangle + m\langle \mathscr{L}_c I'(\vec{\phi}_w), \vec{z}\rangle + \frac{1}{2}\langle \mathscr{L}_c(\vec{z}), \vec{z}\rangle \\
&\quad - C\|\vec{z}\|_X^3 + o(\|\vec{v}\|_X^3), \qquad C > 0.
\end{aligned} \tag{4.13}
$$

The inequality in (4.13) is arrived at as follows. First Taylor expand the third equality in the first line about $\vec{\phi}_w$. The first variation of $\mathscr{E}$ at $\vec{\phi}_w$ vanishes by (1.9). The second variation is the quadratic function in $\vec{v}$.

So, using the spectral structure of $\mathscr{L}_c$, Theorem 4.6 and constraint (4.12) we get from (4.13)

$$\Delta\mathscr{E}(t) \geq \tilde{D}_0\|\vec{z}\|_X^2 - \tilde{D}_1\|\vec{z}\|_X^3 + o(\|\vec{v}\|_X^2), \tag{4.14}$$

where $\tilde{D}_0$, $\tilde{D}_1$ are positive constants, since $m = o(\|\vec{v}\|_X)$. In fact, we have that

$$m\langle \mathscr{L}_c I'(\vec{\phi}_w), \vec{z}\rangle = m\langle \mathscr{L}_c I'(\vec{\phi}_w), \vec{v}\rangle - m^2\langle \mathscr{L}_c I'(\vec{\phi}_w), I'(\vec{\phi}_w)\rangle.$$

Now, since $|m\langle \mathscr{L}_c I'(\vec{\phi}_w), \vec{v}\rangle| \leq |m|\,\|\mathscr{L}_c I'(\vec{\phi}_w)\|\,\|\vec{v}\|_X$, it follows that

$$m\langle \mathscr{L}_c I'(\vec{\phi}_w), \vec{v}\rangle = O(|m|\,\|\vec{v}\|_X) = o(\|\vec{v}\|_X^2),$$

since $m = o(\|\vec{v}\|_X)$. Similarly, it is easy to see that $m^2\langle \mathscr{L}_c I'(\vec{\phi}_w), I'(\vec{\phi}_w)\rangle = O(m^2) = o(\|\vec{v}\|_X^2)$.

Finally, since

$$\|\vec{z}\|_X = \|\vec{v} - mI'(\vec{\phi}_w)\|_X \geq \|\vec{v}\|_X - |m|\,\|I'(\vec{\phi}_w)\|_X \geq \|\vec{v}\|_X - o(\|\vec{v}\|_X)$$

we have that $\|\vec{z}\|_X^2 \geq \|\vec{v}\|_X^2 + o(\|\vec{v}\|_X^2)$ and since $\|\vec{z}\|_X = \|\vec{v} - mI'(\vec{\phi}_w)\|_X \leq \|\vec{v}\|_X + |m|\,\|I'(\vec{\phi}_w)\|_X$, it follows that $\|\vec{z}\|_X^3 \leq \|\vec{v}\|_X^3 + o(\|\vec{v}\|_X^3)$. Hence, from (4.14) it follows that

$$\Delta\mathscr{E}(t) \geq h\big(\rho_{w(c)}\big(\vec{u}(t), \mathcal{O}_{\vec{\phi}_{w(c)}}\big)\big),$$

where $h(s) = F_0 s^2 - F_1 s^3$ with $F_0, F_1 > 0$. The theorem follows by similar arguments used in proof of Theorem 1.1. In fact, let $\varepsilon > 0$ be sufficiently small. Then, by the continuity of $\mathscr{E}$ in $\vec{u} \in H^1_{\mathrm{per}}([0, L]) \times L^2_{\mathrm{per}}([0, L])$, there is a $\delta = \delta(\varepsilon)$ such that if $\rho_{w(c)}(\vec{u}_0, \mathcal{O}_{\vec{\phi}_{w(c)}}) < \delta$, then for $t \in [0, T]$,

$$\Delta\mathscr{E}(0) < h(\varepsilon).$$

Since $\Delta\mathscr{E}$ is constant in time, $h\big(\rho_{w(c)}\big(\vec{u}(t), \mathscr{O}_{\vec{\phi}_{w(c)}}\big)\big) < h(\varepsilon)$. Therefore, since $\rho_{w(c)}\big(\vec{u}(t), \mathscr{O}_{\vec{\phi}_c}\big)$ is a continuous function of time, $\rho_{w(c)}\big(\vec{u}(t), \mathscr{O}_{\vec{\phi}_c}\big) < \varepsilon$ for all $t \in [0, T]$. The desired result follows.

4.4. A global existence theorem. In this section it is shown that if the initial data (u_0, v_0) lies close enough to the initial data $(\phi_{w(c)}, \psi_{w(c)})$ corresponding to a stable cnoidal wave, then the local solution of (1.3)–(4.1), guaranteed by Theorem 4.1, admits a unique extension to a global smooth solution. The precise statement is as follows.

Theorem 4.7. *Let* $L^2 > 4\pi^2$ *such that* $1 - c^2 > 4\pi^2/L^2$ *and* $c \in \left(-1, -\frac{\sqrt{3}}{3}\right) \cup \left(\frac{\sqrt{3}}{3}, 1\right)$. *Let* $(\phi_{w(c)}, \psi_{w(c)})$ *denote a cnoidal wave solution of* (1.3)–(4.1), *with* $w(c) = 1 - c^2$. *Then there exists* $\delta = \delta(c) > 0$ *such that for all* $(u_0, v_0) \in Y$ *and* $\vartheta \in \mathbb{R}$ *with*

$$\|u_0(\cdot) - \phi_{w(c)}(\cdot + \vartheta)\|_1 + \|v_0(\cdot) - \psi_{w(c)}(\cdot + \vartheta)\|_0 \le \delta,$$

the solution (u, v) *of* (1.3)–(4.1) *corresponding to the initial data* (u_0, v_0) *is global and lies in* $X_s(T) \times X_{s-1}(T)$ *for all positive* T. *Moreover, for all* $T > 0$, *the mapping sending* (u_0, v_0) *to the solution* (u, v) *of* (1.3)–(4.1) *is continuous from* Y *into* $X_s(T) \times X_{s-1}(T)$.

Proof. Let T^* be the maximal time of existence of the solution (u, v). The goal is to show that $T^* = +\infty$. It suffices to show that the pair (u, v) remains bounded in X for all $0 \le t \le T < T^*$ with bound independent of T. This is true for all initial values sufficiently close to a stable cnoidal wave by Theorem 1.4. Thus the proof is finished. □

5. Appendix

In this appendix we recall some properties of the Jacobi elliptic integrals that have been used in this work (see [13]).

First, we define *the normal elliptic integral of the first kind*,

$$\int_0^y \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}} = \int_0^\varphi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \equiv F(\varphi, k),$$

where $y = \sin\varphi$, and *the normal elliptic integral of the second kind*,

$$\int_0^y \sqrt{\frac{1 - k^2 t^2}{1 - t^2}}\, dt = \int_0^\varphi \sqrt{1 - k^2 \sin^2 \theta}\, d\theta \equiv E(\varphi, k).$$

In their algebraic forms, these two integrals possess the following properties: the first is finite for all real (or complex) values of y, including infinity; the second has a simple pole of order 1 for $y = \infty$. The number k is called the *modulus*. This number may take any real or imaginary value. Here we wish to take $0 < k < 1$. The number k' is called the *complementary modulus* and is related to k by $k' = \sqrt{1-k^2}$. The variable φ is the *argument* of the normal elliptic integrals.

When $y = 1$, the integrals above are said to be *complete*. In this case, one writes:

$$\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = F(\pi/2, k) \equiv K(k) \equiv K,$$

and

$$\int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}}\,dt = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\,d\theta = E(\pi/2, k) \equiv E(k) \equiv E.$$

Some special values of K and E are: $K(0) = E(0) = \pi/2$, $E(1) = 1$ and $K(1) = +\infty$. For $k \in (0,1)$, one has $K'(k) > 0$, $K''(k) > 0$, $E'(k) < 0$, $E''(k) < 0$ and $E(k) < K(k)$. Moreover, $E(k) + K(k)$ and $E(k)K(k)$ are strictly increasing functions on $(0,1)$.

Now we give some derivatives of the complete elliptical integrals K and E, that we used in this work:

$$\frac{dK}{dk} = \frac{E - k'^2K}{kk'^2};$$
$$\frac{dE}{dk} = \frac{E-K}{k};$$
$$\frac{d^2E}{dk^2} = -\frac{1}{k}\frac{dK}{dk} = -\frac{E-k'^2K}{k^2k'^2}.$$

We will now define the *Jacobian Elliptic Functions*. The elliptic integral

$$u(y_1; k) \equiv u = \int_0^{y_1} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{\varphi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = F(\varphi, k) \quad (5.1)$$

is a strictly increasing function of the real variable y_1, hence we can define its inverse function by $y_1 = \sin\varphi \equiv \operatorname{sn}(u; k)$ (or briefly $y_1 = \operatorname{sn} u$, when it is not necessary to emphasize the modulus). The function $\operatorname{sn} u$ is an odd elliptic function. Other two basic functions can be defined by

$$\operatorname{cn}(u;k) = \sqrt{1 - y_1^2} = \sqrt{1 - \operatorname{sn}^2(u;k)},$$

$$\operatorname{dn}(u;k) = \sqrt{1 - k^2 y_1^2} = \sqrt{1 - k^2 \operatorname{sn}^2(u;k)},$$

requiring that $\operatorname{sn}(0,k) = 0$, $\operatorname{cn}(0,k) = 1$ and $\operatorname{dn}(0,k) = 1$. The functions $\operatorname{cn} u$ and $\operatorname{dn} u$ are therefore even functions. The functions $\operatorname{sn} u$, $\operatorname{cn} u$, and $\operatorname{dn} u$ are called *Jacobian elliptic functions* and are one-valued functions of the argument u. These functions have a real period, namely $4K$, $4K$ and $2K$, respectively. The most important properties of the Jacobian elliptic functions which have been used in this work are summarized by the formulas given below.

1. Fundamental relations:

$$sn^2 u + \operatorname{cn}^2 u = 1,$$

$$k^2 sn^2 u + dn^2 u = 1,$$

$$k'^2 sn^2 u + \operatorname{cn}^2 u = dn^2 u,$$

$$-1 \leqq \operatorname{sn} u \leqq 1,\ -1 \leqq \operatorname{cn} u \leqq 1,\ k'^2 \leqq \operatorname{dn} u \leqq 1.$$

2. Special values:

$$\operatorname{sn}(-u) = -\operatorname{sn} u,\ \operatorname{cn}(-u) = \operatorname{cn} u,\ \operatorname{dn}(-u) = \operatorname{dn} u,\ \operatorname{sn} 0 = 0,$$

$$\operatorname{cn} 0 = 1,\ \operatorname{sn} K = 1,\ \operatorname{cn} K = 0,$$

$$\operatorname{sn}(u + 4K) = \operatorname{sn} u,\ \operatorname{cn}(u + 4K) = \operatorname{cn} u,\ \operatorname{dn}(u + 2K) = \operatorname{dn} u,$$

$$\operatorname{sn}(u + 2K) = -\operatorname{sn} u,\ \operatorname{cn}(u + 2K) = -\operatorname{cn} u.$$

Finally, we have

$$\operatorname{sn}(u,0) = \sin u, \qquad \operatorname{cn}(u,0) = \cos u,$$

$$\operatorname{sn}(u,1) = \tanh u, \qquad \operatorname{cn}(u,1) = \operatorname{sech} u.$$

3. Differentiation of the Jacobian elliptic functions:

$$\frac{\partial}{\partial u}\operatorname{sn}(u) = \operatorname{cn} u \operatorname{dn} u, \qquad \frac{\partial}{\partial u}\operatorname{cn}(u) = -\operatorname{sn} u \operatorname{dn} u,$$

$$\frac{\partial}{\partial u}\operatorname{dn}(u) = -k^2 \operatorname{sn} u \operatorname{cn} u.$$

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