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A note on common range of a class of co-analytic Toeplitz operators

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Abstract. We characterize the intersection of the ranges of a class of co-analytic Toeplitz operators by considering this set as the dual space of the Privalov space N^p, $1 < p < \infty$, in a certain topology. For a fixed p we define the class H_p consisting of those de Branges spaces $\mathscr{H}(b)$ such that the function b is not an extreme point of the unit ball of H^∞, and the associated measure μ_b for b satisfies an additional condition. It is proved that the function f analytic on $\mathbf{D}$ is a multiplier of every de Branges space from H_p if and only if f is in the intersection of the ranges of all Toeplitz operators belonging to the class H_p. We show that this is true if and only if the Taylor coefficients $\hat{f}(n)$ of f decay like $O\big(\exp(-cn^{1/(p+1)})\big)$ for a positive constant c.

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1. Introduction

Let $\mathbf{D}$ denote the open unit disk in the complex plane and let $\mathbf{T}$ denote the boundary of $\mathbf{D}$. Let $L^q(\mathbf{T}) = L^q(d\theta/2\pi)$, $0 < q \le \infty$, be the familiar Lebesgue spaces on the unit circle $\mathbf{T}$. For any fixed $p > 1$ the *Privalov class* N^p consists of all analytic functions f on $\mathbf{D}$ for which

$$\sup_{0<r<1} \int_0^{2\pi} \big(\log^+|f(re^{i\theta})|\big)^p \frac{d\theta}{2\pi} < +\infty.$$

The classes N^p were introduced by I. I. Privalov [21] where N^p is denoted as A_q.

The above inequality with 1 instead of p defines the condition for an analytic function f on $\mathbf{D}$ to be in the *Nevanlinna class* N. The *Smirnov class* N^+ is the set of all functions $f \in N$ such that

$$\lim_{r\to 1}\int_0^{2\pi}\log^+|f(re^{i\theta})|\frac{d\theta}{2\pi}=\int_0^{2\pi}\log^+|f^*(e^{i\theta})|\frac{d\theta}{2\pi}<+\infty,$$

where f^* is the *boundary function* of f on $\mathbf{T}$, i.e., $f^*(e^{i\theta})=\lim_{r\uparrow 1-}f(re^{i\theta})$ is the radial limit of f which exists for almost every $e^{i\theta}\in\mathbf{T}$. We denote by H^q, $0<q<\infty$, the classical *Hardy space* on $\mathbf{D}$, consisting of those analytic functions f on $\mathbf{D}$ for which

$$(\|f\|_q)^{\max\{1,q\}}:=\sup_{0<r<1}\int_0^{2\pi}|f(re^{i\theta})|^q,\qquad \frac{d\theta}{2\pi}<+\infty.$$

Recall that H^∞ is the space of bounded analytic functions on $\mathbf{D}$ with the supremum norm $\|\cdot\|_\infty$ defined as

$$\|f\|_\infty=\sup_{z\in\mathbf{D}}|f(z)|,\qquad f\in H^\infty.$$

It is known (see [18]) that

$$N^q\subset N^p\ (q>p),\qquad \bigcup_{q>0}H^q\subset\bigcap_{p>1}N^p,\qquad\text{and}\qquad \bigcup_{p>1}N^p\subset N^+,$$

where the above inclusion relations are proper.

Let P be the orthogonal projection from $L^2(\mathbf{T})$ onto H^2. Then for any function m in $L^\infty(\mathbf{T})$ the *Toeplitz operator* T_m on H^2 with symbol m is given by $T_m(g)=P(mg)$. Since the multiplication operator with symbol m is bounded on H^2, the Toeplitz operator T_m is a bounded linear operator on H^2 for each $m\in L^\infty(\mathbf{T})$. If m is the complex conjugate of a function in H^∞, T_m is called a *co-analytic* Toeplitz operator.

McCarthy proved [14], Theorem 2.2, that the function f with Taylor series $f(z)=\sum_{n=0}^\infty \hat{f}(n)z^n$ on $\mathbf{D}$ is in the range of every non-zero co-analytic Toeplitz operator if and only if there exists a constant $c>0$ such that $\hat{f}(n)=O(\exp(-c\sqrt{n}))$. The proof of this result is obtained by using Yanagihara's characterization of the dual of the space N^+ [27], Theorem 3.

In Section 2 we introduce just those functional and topological properties of Privalov spaces N^p relevant for present purposes. A *weight* on the unit circle $\mathbf{T}$ is a positive function in $L^1(d\theta/2\pi)$. Let $\mathscr{W}$ be the set of all weights w on $\mathbf{T}$ such that $\int_0^{2\pi}\log w(e^{i\theta})\,d\theta>-\infty$ (cf. [15]). For $p>1$, denote by $\mathscr{W}^p$ the set of all weights w on $\mathbf{T}$ such that $\int_0^{2\pi}|\log w(e^{i\theta})|^p\,d\theta<\infty$. We establish the fact that the set $(N^p)^{-1}$ of all invertible elements in N^p coincides with the set of all outer functions h such that $\log|h^*|\in L^p(\mathbf{T})$. It follows that a weight w is in $\mathscr{W}^p$ if and only if there is an outer function h in $H^2\cap(N^p)^{-1}$ with $|h^*(e^{i\theta})|^2=w(e^{i\theta})$ almost everywhere on $\mathbf{T}$.

Motivated by this fact and a result of McCarthy mentioned above, in Section 3 we consider a class of co-analytic Toeplitz operators $T_{\bar{h}}$ with a function $h \in H^\infty$ such that the function $|h^*|^2$ is in $\mathscr{W}^p$. It is shown (Theorem 3) that the intersection of the ranges of all Toeplitz operators belonging to that class consists of all analytic functions f on $\mathbf{D}$ whose Taylor coefficients $\hat{f}(n)$ decay like $O\big(\exp(-cn^{1/(p+1)})\big)$ for a positive constant c.

Let b be a fixed function in the unit ball of H^∞, and let $D_b = (1 - T_b T_{\bar{b}})^{1/2}$, where $T_b : H^2 \to H^2$ is the Toeplitz operator with symbol b. Then the *de Branges space* $\mathscr{H}(b)$ is defined to be the range of the operator D_b, with the range norm, i.e., the norm that makes $(1 - T_b T_{\bar{b}})^{1/2}$ a coisometry of H^2 onto $\mathscr{H}(b)$. An important result which characterizes the multipliers of $\mathscr{H}(b)$ is obtained in [2] by B. M. Davis and J. E. McCarthy. Namely, by [2, Theorem 4.2] the H^∞ function f is a multiplier of every $\mathscr{H}(b)$ space, when b is not an extreme point of the unit ball of H^∞, if and only if there is a constant $c > 0$ such that $\hat{f}(n) = O\big(\exp(-c\sqrt{n})\big)$.

In Section 4 we consider the class H_p, $1 < p < \infty$, consisting of those de Branges spaces $\mathscr{H}(b)$ for which a function b is not an extreme point of the unit ball of H^∞, and such that μ_b is the associated measure for b with the decomposition $\mu_b = |h^*(e^{i\theta})|^2\, d\theta/2\pi + \mu_s$ for which $|h^*|^2$ is in $\mathscr{W}^p$ (μ_s is a singular part of μ_b with respect to the measure $d\theta/2\pi$). We prove (the equivalences (iii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) of Theorem 4) that the function f analytic on $\mathbf{D}$ is a multiplier of every de Branges space from the class H_p if and only if this function is in the intersection of the ranges of all Toeplitz operators belonging to the class described above, i.e., if and only if $\hat{f}(n) = O\big(\exp(-cn^{1/(p+1)})\big)$ for a positive constant c.

Note that this result, Theorem 3 and Theorem C (Theorem 3 in [17]) are contained in our main result given by Theorem 4. This is in fact an extension of Theorem C which characterizes the universal multipliers of duals $H^2(w)$ with $w \in \mathscr{W}^p$.

2. Preliminaries

It is well known (see e.g., [6], p. 26) that a function $f \in N$ belongs to the Smirnov class N^+ if and only if $f = IF$, where I is an inner function on $\mathbf{D}$ and F is an outer function given by

$$F(z) = \exp\left(\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log|F^*(e^{i\theta})| \frac{d\theta}{2\pi}\right), \qquad z \in \mathbf{D},$$

where $\log|F^*| \in L^1(\mathbf{T})$. Furthermore (see e.g., [6]), a function $f = IF \in N^+$ with described factorization belongs to the Hardy space H^q, $0 < q < \infty$, if and only if $|F^*| \in L^q(\mathbf{T})$.

Privalov [21], p. 98, showed that a function $f \in N$ belongs to the class N^p, $1 < p < \infty$, if and only if $f = IF$, where I is an inner function and F is an outer function such that $\log^+|F^*| \in L^p(\mathbf{T})$. It follows immediately from this factorization that the set $(N^p)^{-1}$ of all invertible elements in N^p consists of those outer functions F for which $\log|F^*| \in L^p(\mathbf{T})$. This result is used in [7] to prove that a function $f \in N$ is in N^p if and only if it can be expressed as the ratio g/h, where g and h are in H^2 and h is an outer function such that $\log|h^*| \in L^p(\mathbf{T})$. Therefore, such a function h is in $(N^p)^{-1}$. Using this fact and Beurling's theorem for the space H^2, it is easy to show (see [7]) that

$$N^p = \bigcup_{h \in (N^p)^{-1} \cap H^2} H^2(|h^*|^2), \tag{2.1}$$

where $H^2(|h^*|^2)$ denotes the closure of the space of analytic polynomials in the space $L^2(|h^*|^2\, d\theta/2\pi)$.

In the general case, for any positive measure μ on the circle $\mathbf{T}$, $H^2(\mu)$ will denote the closure of the analytic polynomials in $L^2(\mu)$. Accordingly, $H^2(|h^*|^2)$ coincides with the space $H^2(|h^*|^2\, d\theta/2\pi)$.

The analogous representation of (2.1) for the class N^+ is given by McCarthy [15], p. 230; also see [8], V.4.4. Observe that $(N^+)^{-1}$ is the set of all outer functions.

Remark 1. Following R. Mortini [19], a ring R satisfying $H^\infty \subset R \subset N$ is said to be of *Nevanlinna–Smirnov type* if every function $f \in R$ can be written in the form g/h, where g and h belong to H^∞ and h is an invertible element in R. This is true for N itself and by an old theorem of the Nevanlinna brothers for the Smirnov class N^+ (see [20] or [6], Chapter 2); hence the name. By a result of Eoff [7] noticed previously (see also [16]), every space N^p, $1 < p < \infty$, is a ring of Nevanlinna–Smirnov type.

Stoll [25], Theorem 4.2, showed that the Privalov space N^p (with the notation $(\log^+ H)^\alpha$ in [25]) with the topology given by the metric ρ_p defined by

$$\rho_p(f,g) = \left(\int_0^{2\pi} \left(\log\left(1 + |f^*(e^{i\theta}) - g^*(e^{i\theta})|\right)\right)^p \frac{d\theta}{2\pi}\right)^{1/p}, \quad f,g \in N^p, \tag{2.2}$$

becomes an F-algebra, that is, an F-space in which multiplication is continuous.

Recall that (2.2) with 1 instead of p defines a translation invariant metric $\rho_1 = \rho$ on N^+ that makes N^+ into an F-space (see [27], Theorem 3).

On the other hand, the representation (2.1) allows one to define two other topologies on N^p: the usual locally convex inductive limit topology, which we shall call the *Helson topology* and denote $\mathscr{H}_p$, in which a neighborhood base for 0 is given by those balanced convex sets whose intersection with each space $H^2(|h^*|^2)$ is a neighborhood of zero in $H^2(|h^*|^2)$; and a not locally convex topology, which we shall denote I_p, in which a neighborhood base for zero is given by all sets whose intersection with each space $H^2(|h^*|^2)$ is a neighborhood of zero. For basic facts about inductive limits see Köthe [12], [13] and Wilansky [26]. It was proved in [7] that the topology I_p coincides with the metric topology ρ_p given by (2.2), which is not locally convex. Hence, I_p is strictly stronger than $\mathscr{H}_p$. Furthermore, the topology $\mathscr{H}_p$ is metrizable (see [17, Section 2]).

The analogous results for the space N^+ with the Helson topology $\mathscr{H}$ introduced in [11] are proved in [14], Section 2, and [15], pp. 230–31. Hence, we use the same name for the topology $\mathscr{H}_p$.

Recall that $\mathscr{W}$ is the set of all weights w on the circle $\mathbf{T}$ satisfying the condition $\int_0^{2\pi} \log w(e^{i\theta})\, d\theta > -\infty$. For given $p > 1$, $\mathscr{W}^p$ is defined as the set of all weights w on $\mathbf{T}$ such that $\int_0^{2\pi} |\log w(e^{i\theta})|^p\, d\theta < \infty$. For a weight $w \in \mathscr{W}^p$ consider the outer function h defined as

$$h(z) = \exp\left(\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log\sqrt{w(e^{i\theta})}\, \frac{d\theta}{2\pi}\right), \qquad z \in \mathbf{D}.$$

Then $w(e^{i\theta}) = |h^*(e^{i\theta})|^2$ almost everywhere on $\mathbf{T}$, and hence the boundary function h^* is in $L^2(d\theta/2\pi)$. So the function h is in H^2. As noticed previously, h also belongs to $(N^p)^{-1}$. Therefore, a weight w is in $\mathscr{W}^p$ if and only if there is an outer function h in $H^2 \cap (N^p)^{-1}$ with $|h^*(e^{i\theta})|^2 = w(e^{i\theta})$ almost everywhere on $\mathbf{T}$. This means that

$$\mathscr{W}^p = \{|h^*(e^{i\theta})|^2 : h \in H^2 \text{ is outer and } \log|h^*| \in L^p(\mathbf{T})\}.$$

Thus in view of (2.1), the topological dual of $(N^p, \mathscr{H}_p)$ consists of those functionals belonging to the topological dual $H^2(|h^*|^2)^*$ of $H^2(|h^*|^2)$ for every weight $|h^*(e^{i\theta})|^2$ in $\mathscr{W}^p$.

In terms of the polynomials, the topological dual of (N^p, ρ_p) is given as follows.

Theorem A ([17], Theorem D). *Let Γ be a linear functional defined on the set of polynomials by $\Gamma(\sum_{n=0}^N a_n z^n) = \sum_{n=0}^N a_n \gamma_n$. Then Γ extends to be continuous on N^p with respect to the metric ρ_p defined by* (2.2) *if and only if $\gamma_n = O(\exp(-cn^{1/(p+1)}))$ for some $c > 0$.*

Theorem B ([17], Theorem E). *The spaces $(N^p, \mathscr{H}_p)$ and (N^p, ρ_p) have the same topological duals. Hence, the dual of $(N^p, \mathscr{H}_p)$ is characterized by Theorem A.*

According to Theorems A and B, in the sequel we shall frequently identify the topological dual of $(N^p, \mathscr{H}_p)$ with the class of all analytic functions f whose Taylor coefficients $\hat{f}(n)$ satisfy $\hat{f}(n) = O(\exp(-cn^{1/(p+1)}))$ for some constant $c > 0$.

For given weight $w = |h^*|^2 \in \mathscr{W}$, we say that the function g is a *multiplier* of $H^2(|h^*|^2)^*$ if, for every f in $H^2(|h^*|^2)^*$, gf is also in $H^2(|h^*|^2)^*$. By [15, Theorem

3.2], the function g is a multiplier of $H^2(|h^*|^2)^*$ for every $w = |h^*|^2$ in $\mathscr{W}$ if and only if g is in $(N^+, \mathscr{H})^*$. The following result describes the universal multipliers of the set $\{H^2(|h^*|^2)^* : |h^*|^2 \in \mathscr{W}^p\}$.

Theorem C ([17], Theorem 3). *For any fixed $p > 1$, the following conditions about a function f analytic on* $\mathbf{D}$ *are equivalent:*

(i) *f is a multiplier of $H^2(|h^*|^2)^*$ for every $|h^*|^2$ in $\mathscr{W}^p$.*

(ii) *f is in the dual of N^p with respect to the topology $\mathscr{H}_p$.*

(iii) *Taylor coefficients $\hat{f}(n)$ of f satisfy $\hat{f}(n) = O\big(\exp(-cn^{1/(p+1)})\big)$ for a positive constant c.*

Remark 2. The N^+ analogues of Theorems A and B are given in [27], Theorem 3; see also [15], Theorems 1.7 and 2.1. Namely, Theorems A and B remain true if we replace N^p, ρ_p, $\mathscr{H}_p$, and p with N^+, ρ $(= \rho_1)$, $\mathscr{H}$ (the Helson topology defined on N^+ in [11]), and 1, respectively. Moreover, the N^+ analogue of Theorem C is given in [15], Theorem 3.2.

3. Common range of a class of co-analytic Toeplitz operators

Let $\langle\,,\rangle$ denote the formal inner product of two Fourier series, i.e., if $f(z) \sim \sum_{n=-\infty}^{\infty} a_n z^n$, $g(z) \sim \sum_{n=-\infty}^{\infty} b_n z^n$, then $\langle f, g\rangle = \sum_{n=-\infty}^{\infty} a_n \bar{b}_n$. Let $\langle f, g\rangle = \int_0^{2\pi} f^*(e^{i\theta})\overline{g^*(e^{i\theta})}\frac{d\theta}{2\pi}$ (note that, to avoid convergence problems, we think of this integral as $\lim_{r\uparrow 1-}\int_0^{2\pi} f(re^{i\theta})\overline{g(re^{i\theta})}\frac{d\theta}{2\pi}$). In particular, for any polynomial $p = \sum_{k=0}^{n} a_k z^k$ the action $\langle p, f\rangle$ of $f = \sum_{k=0}^{\infty} \hat{f}(k) z^k$ on p is given by

$$\langle p, f\rangle = \int_0^{2\pi} p(e^{i\theta})\overline{f^*(e^{i\theta})}\frac{d\theta}{2\pi} = \sum_{k=0}^{n} a_k \overline{\hat{f}(k)}.$$

Let S be the unilateral shift, which we think of as multiplication by the independent variable. Observe that S is a Toeplitz operator whose symbol is the coordinate function. Then by (1.1) of [2], the action of $T_{\bar{f}}$ on a polynomial p of degree d is given by

$$(T_{\bar{f}}p)(z) = \langle p, f\rangle + \langle p, Sf\rangle z + \cdots + \langle p, S^d f\rangle z^d. \tag{3.1}$$

A result of Helson in [11] given by the equivalence (i) $\Leftrightarrow$ (iv) of Theorem 0.1 in [14] asserts that a function f in H^2 defines a continuous linear functional on $(N^+, \mathscr{H})$ if and only if f is in the range of each Toeplitz operator $T_{\bar{h}}$ with h in H^∞. McCarthy [14], Section 1, gives another proof of this result, by using the fact that the dual of $(N^+, \mathscr{H})$ is the same as the intersection of the duals of the $H^2(w)$ spaces, where $w = |h^*|^2$ ranges over $\mathscr{W}$. Namely, it is proved in [14], Theorem 2.1, that the dual of N^+ is the same in both $\mathscr{H}$ and ρ topologies.

Suppose that μ is a *Szegő measure*. Then it can be written as $|h^*(e^{i\theta})|^2 \frac{d\theta}{2\pi}$ for some outer function h in H^2 (see e.g., [9], p. 144). Moreover [2], Section 2, μ being Szegő means that the functional that assigns to a polynomial its zeroth coefficient is continuous on $H^2(\mu)$. So if $T_{\bar{f}}$ is bounded on $H^2(\mu)$, the functional $p \mapsto (T_{\bar{f}}p)(0)$ is bounded, which by (3.1) means that there is a constant C such that

$$|\langle p, f \rangle| \le C \sqrt{\int_0^{2\pi} |p(e^{i\theta})|^2 \, d\mu}, \tag{3.2}$$

for any polynomial $p = \sum_{k=0}^n a_k z^k$. The set of all analytic functions f that satisfy (3.2) can be identified with the dual of $H^2(\mu)$, and we shall denote this set by $H^2(\mu)^*$.

For a measure μ on the circle $\mathbf{T}$ let

$$T(\mu) = \{f \in H^\infty : T_{\bar{f}} \text{ is bounded on } H^2(\mu)\},$$

and let $\mu = \mu_a + \mu_s$ be the decomposition of μ into absolutely continuous and singular parts (with respect to the Lebesgue measure $d\theta/2\pi$).

Proposition 1 ([2], Proposition 4.1). *Let μ be a measure on the circle* $\mathbf{T}$. *Then f is a multiplier of $H^2(\mu)^*$ if and only if f is in $T(\mu)$.*

Proposition 2. *A function f analytic on* $\mathbf{D}$ *is in the dual of N^p with respect to the topology $\mathscr{H}_p$ if and only if for any weight $w = |h^*|^2 \in \mathscr{W}^p$ there is a constant C (depending on w) such that for all polynomials p*

$$\left| \int_0^{2\pi} p(e^{i\theta}) \overline{f^*(e^{i\theta})} \frac{d\theta}{2\pi} \right|^2 \le C \int_0^{2\pi} |p(e^{i\theta})|^2 |h^*(e^{i\theta})|^2 \frac{d\theta}{2\pi}. \tag{3.3}$$

Proof. Let f be in the dual of $(N^p, \mathscr{H}_p)$. Then, by (ii) $\Rightarrow$ (i) of Theorem C, f is a multiplier of $H^2(w)^*$ for every weight $w = |h^*|^2$ in $\mathscr{W}^p$. By Proposition 1, f is in $T(|h^*|^2)$, i.e., the Toeplitz operator $T_{\bar{h}}$ is continuous on $H^2(|h^*|^2)$ for each $|h^*|^2 \in \mathscr{W}^p$. It follows that (3.2) holds for $\mu = |h^*(e^{i\theta})|^2 \, d\theta/2\pi$, which is in fact (3.3).

Conversely, suppose that for any weight $w = |h^*|^2 \in \mathscr{W}^p$ there is a constant C such that (3.3) holds for all polynomials p. This means that a function f is in the dual of $H^2(|h^*|^2)$ for any $|h^*|^2 \in \mathscr{W}^p$. Since the dual of $(N^p, \mathscr{H}_p)$ consists of those functionals belonging to the dual $H^2(w)^*$ for every w in $\mathscr{W}^p$, it follows that f is also in the dual of $(N^p, \mathscr{H}_p)$. This completes the proof. □

We are now ready to state the following result.

Theorem 3. *The function f with Taylor series $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ on* $\mathbf{D}$ *is in the range of every non-zero co-analytic Toeplitz operator $T_{\bar{h}}$ with $|h^*|^2 \in \mathscr{W}^p$ if and only if there exists a constant $c > 0$ such that $\hat{f}(n) = O\big(\exp(-cn^{1/(p+1)})\big)$.*

Proof. We follow the proof of the equivavalence of (i) and (iv) from Theorem 0.1 in [14]. First observe that if a weight $w \in \mathscr{W}^p$ is decreased, the space $H^2(w)$ increases, and its dual decreases. So the dual of $(N^p, \mathscr{H}_p)$ is the same as the intersection of the duals of the $H^2(w)$ spaces, where w ranges, not over all of $\mathscr{W}^p$ but only over those weights $w \in \mathscr{W}^p$ that satisfy $\|w\|_\infty \le 1$. Therefore, by Proposition 2, a function f belongs to the dual of $(N^p, \mathscr{H}_p)$ if and only if for each weight $w = |h^*|^2 \in \mathscr{W}^p$ with $\|w\|_\infty \le 1$, there exists a constant $C > 0$ such that for all polynomials p

$$\left| \int_0^{2\pi} p(e^{i\theta}) \overline{f^*(e^{i\theta})} \frac{d\theta}{2\pi} \right|^2 \le C \int_0^{2\pi} |p(e^{i\theta})|^2 |h^*(e^{i\theta})|^2 \frac{d\theta}{2\pi}.$$

This is true if and only if there exists a g in $H^2(|h^*|^2)$ of norm less or equal to $\sqrt{C}$ for which

$$\int_0^{2\pi} p(e^{i\theta}) \overline{f^*(e^{i\theta})} \frac{d\theta}{2\pi} = \int_0^{2\pi} p(e^{i\theta}) \overline{g^*(e^{i\theta})} |h^*(e^{i\theta})|^2 \frac{d\theta}{2\pi}. \tag{3.4}$$

As noticed in [14], p. 795, the set $H^2(|h^*|^2)$ is the same as $H^2/h = \{f/h : f \in H^2\}$. Hence g can be written as k/h for some k in H^2, which by putting in (3.4) yields

$$\int_0^{2\pi} p(e^{i\theta}) \overline{f^*(e^{i\theta})} \frac{d\theta}{2\pi} = \int_0^{2\pi} p(e^{i\theta}) \overline{k^*(e^{i\theta})} h^*(e^{i\theta}) \frac{d\theta}{2\pi} \tag{3.5}$$

for all polynomials p. Taking $p(z) = z^n$, $n = 0, 1, 2, \dots$ in (3.5), gives

$$\overline{\hat{f}(n)} = \int_0^{2\pi} e^{in\theta} \overline{k^*(e^{i\theta})} h^*(e^{i\theta}) \frac{d\theta}{2\pi}, \qquad n = 0, 1, 2, \dots,$$

whence

$$\hat{f}(n) = \int_0^{2\pi} k^*(e^{i\theta}) \overline{h^*(e^{i\theta}) e^{in\theta}} \frac{d\theta}{2\pi} = \langle k, S^n h \rangle.$$

Thus by (3.1) we obtain that

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n = \sum_{n=0}^{\infty} \langle k, S^n h \rangle z^n = T_{\bar{h}} k,$$

i.e., $f = T_{\bar{h}} k$. It follows that a function f is in the dual of $(N^p, \mathscr{H}_p)$ if and only if it is in the range of all non-zero co-analytic Toeplitz operator $T_{\bar{h}}$ with $|h^*|^2 \in \mathscr{W}^p$. By Theorems A and B, this is true if and only if $\hat{f}(n) = O\big(\exp(-cn^{1/(p+1)})\big)$ for some positive constant c. This completes the proof. □

4. Multipliers of de Branges spaces

For a function b in the unit ball of H^∞, de Branges space $\mathscr{H}(b)$ is by definition the range of the operator $(1 - T_b T_{\bar{b}})^{1/2}$, with the range norm, i.e., the norm that makes $(1 - T_b T_{\bar{b}})^{1/2}$ a coisometry of H^2 onto $\mathscr{H}(b)$. $\mathscr{H}(b)$ is a (not necessarily closed) subspace of H^2, on which the evaluation functionals at points in the disk are continuous. The spaces $\mathscr{H}(b)$ were originally introduced by de Branges and Rovnyak in [3] and [4], and have attracted much attention in the meantime. As noticed in [2], the spaces $\mathscr{H}(b)$ have been utilised in various contexts, ranging from model theory [1] and kernels of Toeplitz operators [10] to exposed points in H^1 [24] and complex function theory [23]. The structure of the spaces, however, is still not well understood; a natural question to ask is, given a specific b, what are the multipliers of $\mathscr{H}(b)$, i.e., what functions (necessarily in H^∞) multiply $\mathscr{H}(b)$ into itself? This question splits into two cases, depending on wheter b is an extreme point of the unit ball of H^∞ (and no non-constant polynomial is a multiplier), or it is not (and every polynomial is a multiplier) [22].

Let b be a fixed non-constant function in the unit ball of H^∞. The function $z \mapsto \frac{1+b(z)}{1-b(z)}$ then has a positive real part, so it can be reprezented as a Herglotz integral

$$\frac{1 + b(z)}{1 - b(z)} = \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu_b(e^{i\theta}) + ic, \tag{4.1}$$

where c is a real constant and μ_b is a unique positive Borel measure.

For each function f in $H^2(\mu_b)$, define a function $V_b f$ on $\mathbf{D}$ by

$$(V_b f)(z) = \left(\int_0^{2\pi} \frac{f(e^{i\theta})}{1 - ze^{-i\theta}} d\mu_b(e^{i\theta}) \right) \big(1 - b(z)\big).$$

Sarason proved in [24] that V_b is an isometry from $H^2(\mu_b)$ onto $\mathscr{H}(b)$. Considering the measure μ_b, notice that the absolutely continuous part of μ_b is

$$\frac{1 - |b^*(e^{i\theta})|^2}{|1 - b^*(e^{i\theta})|^2} \frac{d\theta}{2\pi}.$$

As noticed in [2], Section 1, the absolutely continuous part of a measure μ_b induced by (4.1) is a Szegő measure if and only if $\log(1 - |b^*(e^{i\theta})|^2)$ is integrable; this integrability in turn is equivalent to b not being an extreme point of the unit ball of H^∞; see [5].

For any given $p > 1$ let H_p denote the class of those de Branges spaces $\mathscr{H}(b)$ for which a function b is not an extreme point of the unit ball of H^∞ and the associated measure μ_b for b given by (4.1) has a decomposition $\mu_b = |h^*(e^{i\theta})|^2\, d\theta/2\pi + \mu_s$ such that $|h^*|^2$ is in $\mathscr{W}^p$ (μ_s is a singular part of μ_b with respect to the measure $d\theta/2\pi$).

We are now ready to extend Theorem C (Theorem 3 in [17]) in our main result as follows.

Theorem 4. *For any fixed $p > 1$, the following conditions about a function f analytic on $\mathbf{D}$ are equivalent:*

(i) *f is a multiplier of $H^2(|h^*|^2)^*$ for every $|h^*|^2$ in $\mathscr{W}^p$.*

(ii) *f is in the dual of N^p with respect to the topology $\mathscr{H}_p$.*

(iii) *Taylor coefficients $\hat{f}(n)$ of f satisfy $\hat{f}(n) = O\big(\exp(-cn^{1/(p+1)})\big)$ for a positive constant c.*

(iv) *f is in the range of every non-zero co-analytic Toeplitz operator $T_{\bar{h}}$ with $|h^*|^2 \in \mathscr{W}^p$.*

(v) *f is a multiplier of every de Branges space $\mathscr{H}(b)$ from the class H_p.*

(vi) *f is in $T\big(|h^*(e^{i\theta})|^2\, d\theta/2\pi\big)$ for every $|h^*|^2 \in \mathscr{W}^p$.*

Proof. The equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) are contained in Theorem C.

(iii) $\Leftrightarrow$ (v) may be proved analogously as Theorem 4.2. in [2].

(iii) $\Leftrightarrow$ (iv) is in fact the assertion of Theorem 3.

(i) $\Leftrightarrow$ (vi) follows immediately from Proposition 1.

This completes the proof. □

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