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Abel's method on summation by parts and well-poised bilateral ${}_5\psi_5$-series identities

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Abstract. The modified Abel lemma on summation by parts is employed to establish two well-poised bilateral ${}_5\psi_5$-series identities. Several quadratic, cubic and quartic transformation formulae are derived for the truncated partial sums of Bailey's very well-poised ${}_6\psi_6$-series.

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For two complex x and q, the shifted-factorial of x with base q is defined by

$$(x;q)_0 = 1 \quad \text{and} \quad (x;q)_n = (1-x)(1-xq)\dots(1-xq^{n-1}) \quad \text{for } n \in \mathbb{N}.$$

When $|q| < 1$, we have two well-defined infinite products

$$(x;q)_\infty = \prod_{k=0}^{\infty}(1-q^k x) \quad \text{and} \quad (x;q)_n = (x;q)_\infty/(xq^n;q)_\infty.$$

In particular, the shifted factorial with negative integer order can be written explicitly from the last fraction as

$$(x;q)_{-n} = \frac{(-1)^n q^{\binom{1+n}{2}} x^{-n}}{(q/x;q)_n} \quad \text{for } n \in \mathbb{N}.$$

The product and fraction of shifted factorials are abbreviated respectively to

$$[\alpha,\beta,\dots,\gamma;q]_n = (\alpha;q)_n(\beta;q)_n\dots(\gamma;q)_n,$$

$$\left[\begin{matrix}\alpha,\beta,\dots,\gamma\\ A,B,\dots,C\end{matrix}\middle| q\right]_n = \frac{(\alpha;q)_n(\beta;q)_n\dots(\gamma;q)_n}{(A;q)_n(B;q)_n\dots(C;q)_n}.$$

Following Gasper–Rahman [13], the unilateral and bilateral basic hypergeometric series (shortly as q-series) are defined respectively by

$$ {}_{1+r}\phi_s\left[\begin{matrix} a_0, & a_1, \dots, a_r \\ & b_1, \dots, b_s \end{matrix}\middle| q; z\right] = \sum_{n=0}^{\infty} \{(-1)^n q^{\binom{n}{2}}\}^{s-r} \left[\begin{matrix} a_0, & a_1, \dots, a_r \\ q, & b_1, \dots, b_s \end{matrix}\middle| q\right]_n z^n, $$

$$ {}_r\psi_s\left[\begin{matrix} a_1, & a_2, \dots, a_r \\ b_1, & b_2, \dots, b_s \end{matrix}\middle| q; z\right] = \sum_{n=-\infty}^{+\infty} \{(-1)^n q^{\binom{n}{2}}\}^{s-r} \left[\begin{matrix} a_1, & a_2, \dots, a_r \\ b_1, & b_2, \dots, b_s \end{matrix}\middle| q\right]_n z^n; $$

where the base q will be restricted to $|q| < 1$ for non-terminating q-series. When $r = s$, the most important case, the definitions of the ϕ and ψ-series just displayed coincide with those due to Bailey [2], Chapter 6, and Slater [17], who collected most of the classical summation and transformation formulae on q-series.

One of the most important and useful identities in the theory of basic hypergeometric series is Bailey's summation formula [3] (cf. Gasper–Rahman [13], II-33, also) for a nonterminating very-well-poised bilateral ${}_6\psi_6$-series. For complex parameters a, b, c, d, e satisfying the condition $|qa^2/bcde| < 1$, Bailey's identity may be reproduced as

$$ {}_6\psi_6\left[\begin{matrix} qa^{1/2}, & -qa^{1/2}, & b, & c, & d, & e \\ a^{1/2}, & -a^{1/2}, & qa/b, & qa/c, & qa/d, & qa/e \end{matrix}\middle| q; \frac{qa^2}{bcde}\right] \tag{1a} $$

$$ = \left[\begin{matrix} q, qa, q/a, qa/bc, qa/bd, qa/be, qa/cd, qa/ce, qa/de \\ qa/b, qa/c, qa/d, qa/e, q/b, q/c, q/d, q/e, qa^2/bcde \end{matrix}\middle| q\right]_\infty. \tag{1b} $$

It has recently been provided a completely new and simple proof by Chu [11] through Abel's lemma on summation by parts. By employing this approach further, this paper will prove two nonterminating well-poised bilateral ${}_5\psi_5$-series identities. Several quadratic, cubic and quartic transformation formulae are derived consequently for the truncated partial sums of Bailey's very well-poised ${}_6\psi_6$-series.

Abel's lemma on summation by parts has been shown very useful and important in classical analysis. For an arbitrary complex sequence $\{\tau_k\}$, define the backward and forward difference operators ∇ and $\triangle$, respectively, by

$$ \nabla\tau_k = \tau_k - \tau_{k-1} \quad \text{and} \quad \triangle\tau_k = \tau_k - \tau_{k+1} \tag{2} $$

where $\triangle$ is adopted for convenience in the present paper, which differs from the usual operator Δ only in the minus sign.

Then Abel's lemma on summation by parts may be reformulated as

$$ \sum_{k=-\infty}^{+\infty} B_k \nabla A_k = [AB]_{+\infty} - [AB]_{-\infty} + \sum_{k=-\infty}^{+\infty} A_k \triangle B_k, \tag{3} $$

provided that the two limits $[AB]_{\pm\infty} := \lim_{n\to\pm\infty} A_n B_{n+1}$ exist and one of both series just displayed is convergent.

Proof. Let m and n be two integers. According to the definition of the backward difference, we have

$$\sum_{k=m}^{n} B_k \nabla A_k = \sum_{k=m}^{n} B_k\{A_k - A_{k-1}\} = \sum_{k=m}^{n} A_k B_k - \sum_{k=m}^{n} A_{k-1} B_k.$$

Replacing k by $k+1$ for the last sum, we get the following expression:

$$\begin{aligned}\sum_{k=m}^{n} B_k \nabla A_k &= A_n B_{n+1} - A_{m-1} B_m + \sum_{k=m}^{n} A_k\{B_k - B_{k+1}\} \\ &= A_n B_{n+1} - A_{m-1} B_m + \sum_{k=m}^{n} A_k \triangle B_k.\end{aligned}$$

Letting $m \to -\infty$ and $n \to +\infty$, we get the identity stated in the lemma. □

1. Nonterminating bilateral ${}_5\psi_5$-series identities

Now we state two general bilateral series identities and their implications.

1.1. Zero well-poised bilateral series. For the bilateral series, there holds the following general statement:

$${}_{1+2\kappa}\psi_{2\kappa+1}\left[\begin{matrix} c_1, & c_2, \dots, & c_{1+2\kappa} \\ 1/c_1, & 1/c_2, \dots, & 1/c_{1+2\kappa} \end{matrix} \middle| q; \frac{1}{c_1 c_2 \dots c_{1+2\kappa}}\right] = 0. \tag{4}$$

A very special case has been obtained by Joshi and Verma [20], eq. 3.18.

In fact, denote by Θ the bilateral ψ-series on the left hand side. Its reversal with the summation index shifted by $k \to k-1$ can be stated as

$$\begin{aligned}\Theta &= {}_{1+2\kappa}\psi_{2\kappa+1}\left[\begin{matrix} qc_1, & qc_2, \dots, & qc_{1+2\kappa} \\ q/c_1, & q/c_2, \dots, & q/c_{1+2\kappa} \end{matrix} \middle| q; \prod_{\imath=1}^{1+2\kappa} \frac{1}{c_\imath}\right] \\ &= {}_{1+2\kappa}\psi_{2\kappa+1}\left[\begin{matrix} c_1, & c_2, \dots, & c_{1+2\kappa} \\ 1/c_1, & 1/c_2, \dots, & 1/c_{1+2\kappa} \end{matrix} \middle| q; \prod_{\imath=1}^{1+2\kappa} \frac{1}{c_\imath}\right] \times \prod_{\imath=1}^{1+2\kappa} \frac{1-1/c_\imath}{1-c_\imath} \left\{\prod_{\imath=1}^{1+2\kappa} \frac{1}{c_\imath}\right\}^{-1}.\end{aligned}$$

Simplifying the last factor-product, we find that

$$\Theta = (-1)^{1+2\kappa}\Theta = 0$$

which is exactly the identity displayed in (4).

1.2. The first well-poised bilateral series identity. Denote by ψ^+ the partial sum of the terms with nonnegative indices from the corresponding bilateral ψ-series. The first well-poised bilateral series identity is given by the following surprising theorem which expresses a well-poised bilateral ${}_5\psi_5$-series in terms of Bailey's well-poised ${}_6\psi_6$-partial sum.

Theorem 1. *For four indeterminate $\{a, b, c, d\}$ satisfying the condition $|q/bcd| < 1$, there holds the following transformation formula:*

$$ {}_5\psi_5\left[\begin{matrix} a, & b, & c, & d, & q/abcd \\ q/a, & q/b, & q/c, & q/d, & abcd \end{matrix}\middle| q; q\right] \tag{5a}$$

$$ = {}_5\psi_5\left[\begin{matrix} a, & b, & c, & d, & q/abcd \\ q/a, & q/b, & q/c, & q/d, & abcd \end{matrix}\middle| q; q^2\right] \tag{5b}$$

$$ = \left[\begin{matrix} q, q/ab, q/ac, q/ad, q/bc, q/bd, q/cd, q/abcd \\ q/a, q/b, q/c, q/d, q/abc, q/abd, q/acd, q/bcd \end{matrix}\middle| q\right]_\infty + \lambda(a; b, c, d) \tag{5c}$$

$$ \times\, {}_6\psi_6^+\left[\begin{matrix} q^2/a\sqrt{bcd}, & -q^2/a\sqrt{bcd}, & q/a, & q/ab, & q/ac, & q/ad \\ q/a\sqrt{bcd}, & -q/a\sqrt{bcd}, & q^2/abcd, & q^2/abc, & q^2/abd, & q^2/acd \end{matrix}\middle| q; \frac{q}{bcd}\right] \tag{5d}$$

where the λ-function is defined by infinite product

$$\lambda(a; b, c, d) = \frac{(q/abcd)\{1 - q^2/a^2bcd\}}{[q/abc, q/abd, q/acd; q]_1} \left[\begin{matrix} a, & b, & c, & d, & q^2/abcd \\ q/a, & q/b, & q/c, & q/d, & abcd \end{matrix}\middle| q\right]_\infty. \tag{6}$$

The equality between two ${}_5\psi_5$-series displayed in (5a) and (5b) is justified by the series reversal. Theorem 1 is the common generalization of the following important examples.

First, letting $a \to \infty$ through the replacement $a \to a/q^n$ and then $n \to \infty$ in Theorem 1, the limiting case gives two well-poised bilateral ${}_3\psi_3$-series identities.

Corollary 2 (Bailey [5], eq. 2.2, and [19], eq. 5.5). *For $\varepsilon = 1, 2$, there hold the following nonterminating series identities:*

$$ {}_3\psi_3\left[\begin{matrix} b, & c, & d \\ q/b, & q/c, & q/d \end{matrix}\middle| q; \frac{q^\varepsilon}{bcd}\right] = \left[\begin{matrix} q, q/bc, q/bd, q/cd \\ q/b, q/c, q/d, q/bcd \end{matrix}\middle| q\right]_\infty.$$

Then taking $a = q^{-n}$ in Theorem 1, we recover the following terminating series identities.

Corollary 3 (Well-poised ${}_5\phi_4$-series identities: Bailey [5], eq. 3.1, and Jackson [15], eq. 1). *For $\delta = 0, 1$, there hold the following identities:*

$$
{}_5\phi_4\left[\begin{matrix} q^{-2n}, & b, & c, & d, & q^{1-3n}/bcd \\ & q^{1-2n}/b, & q^{1-2n}/c, & q^{1-2n}/d, & q^n bcd \end{matrix}\middle| q; q^{1+\delta}\right]
$$
$$
= q^{n(\delta-1)}\left[\begin{matrix} b, c, d, bcd \\ q, bc, bd, cd \end{matrix}\middle| q\right]_n \times \left[\begin{matrix} q, bc, bd, cd \\ b, c, d, bcd \end{matrix}\middle| q\right]_{2n}.
$$

Different proofs may be found in Bressoud [6], eq. 1, Carlitz [7], eqs. 3.4 and 3.6, Guo [14], eq. 4.1, and Verma-Joshi [20], eq. 3.8. Carlitz [8], eqs. 15 and 16, worked out also the results corresponding to the limiting case $q \to 1$.

In view of the fact that

$$
\frac{(qw;q)_k}{(w;q)_k} = \frac{1 - wq^k}{1 - w} = \frac{1}{1 - w} - \frac{q^k w}{1 - w}
$$

the linear combination of (5a) and (5b) leads us to bilateral identity with an extra w-parameter:

Proposition 4. *For four indeterminate $\{a, b, c, d\}$ satisfying the condition $|q/bcd| < 1$, there holds the following transformation formula:*

$$
{}_6\psi_6\left[\begin{matrix} qw, & a, & b, & c, & d, & q/abcd \\ w, & q/a, & q/b, & q/c, & q/d, & abcd \end{matrix}\middle| q; q\right]
$$
$$
= \left[\begin{matrix} q, q/ab, q/ac, q/ad, q/bc, q/bd, q/cd, q/abcd \\ q/a, q/b, q/c, q/d, q/abc, q/abd, q/acd, q/bcd \end{matrix}\middle| q\right]_\infty + \lambda(a; b, c, d)
$$
$$
\times\, {}_6\psi_6^+\left[\begin{matrix} q^2/a\sqrt{bcd}, & -q^2/a\sqrt{bcd}, & q/a, & q/ab, & q/ac, & q/ad \\ q/a\sqrt{bcd}, & -q/a\sqrt{bcd}, & q^2/abcd, & q^2/abc, & q^2/abd, & q^2/acd \end{matrix}\middle| q; \frac{q}{bcd}\right].
$$

Special terminating cases have been investigated by Bailey [4], eq. 3, Chu [9], §2, and Jain-Verma [19], eqs. 5.1 and 5.2.

1.3. The second well-poised bilateral series identity. It expresses another well-poised bilateral ${}_5\psi_5$-series in terms of Bailey's well-poised ${}_6\psi_6$-partial sum.

Theorem 5. *For four indeterminate $\{a, b, c, d\}$ with $|q^2/bcd| < 1$, there holds the following transformation:*

$$
{}_5\psi_5\left[\begin{matrix} a, & b, & c, & d, & q^3/abcd \\ q^2/a, & q^2/b, & q^2/c, & q^2/d, & abcd/q \end{matrix}\middle| q;q\right] \tag{7a}
$$

$$
= -q \times {}_5\psi_5\left[\begin{matrix} a, & b, & c, & d, & q^3/abcd \\ q^2/a, & q^2/b, & q^2/c, & q^2/d, & abcd/q \end{matrix}\middle| q;q^3\right] \tag{7b}
$$

$$
= \left[\begin{matrix} q, q^2/ab, q^2/ac, q^2/ad, q^2/bc, q^2/bd, q^2/cd, q^2/abcd \\ q^2/a, q^2/b, q^2/c, q^2/d, q^2/abc, q^2/abd, q^2/acd, q^2/bcd \end{matrix}\middle| q\right]_\infty + \mu(a;b,c,d) \tag{7c}
$$

$$
\times {}_6\psi_6^+\left[\begin{matrix} q^3/a\sqrt{bcd}, & -q^3/a\sqrt{bcd}, & q/a, & q^2/ab, & q^2/ac, & q^2/ad \\ q^2/a\sqrt{bcd}, & -q^2/a\sqrt{bcd}, & q^4/abcd, & q^3/abc, & q^3/abd, & q^3/acd \end{matrix}\middle| q;\frac{q^2}{bcd}\right] \tag{7d}
$$

where the μ-function is defined by infinite product

$$
\mu(a;b,c,d) = \frac{(q^2/abcd)\{1-q^4/a^2bcd\}}{[q^2/abc, q^2/abd, q^2/acd; q]_1}\left[\begin{matrix} a, & b, & c, & d, & q^4/abcd \\ q^2/a, & q^2/b, & q^2/c, & q^2/d, & abcd/q \end{matrix}\middle| q\right]_\infty. \tag{8}
$$

Again, the first equality between two ${}_5\psi_5$-series displayed in (7a) and (7b) is confirmed by the series reversal. From Theorem 5, we can deduce the following important examples.

First, letting $a \to \infty$ through the replacement $a \to a/q^n$ and then $n \to \infty$ in Theorem 5, the limiting case gives two well-poised bilateral ${}_3\psi_3$-series identities.

Corollary 6 (Bailey [5], eq. 2.3). *For $\varepsilon = 1, 2$, there hold the following nonterminating series identities:*

$$
{}_3\psi_3\left[\begin{matrix} b, & c, & d \\ q^2/b, & q^2/c, & q^2/d \end{matrix}\middle| q;\frac{q^{2\varepsilon}}{bcd}\right] = (-q)^{1-\varepsilon}\left[\begin{matrix} q, q^2/bc, q^2/bd, q^2/cd \\ q^2/b, q^2/c, q^2/d, q^2/bcd \end{matrix}\middle| q\right]_\infty.
$$

If we take $a = q^{-n}$ in Theorem 5, then the following terminating series identities are recovered.

Corollary 7 (Well-poised ${}_5\phi_4$-series identities: Bailey [5], eq. 3.2, and Carlitz [7], eqs. 3.4 and 3.7). *For $\delta = 0, 1$, there hold the following identities:*

$$
{}_5\phi_4\left[\begin{matrix} q^{-1-2n}, & b, & c, & d, & q^{-1-3n}/bcd \\ & q^{-2n}/b, & q^{-2n}/c, & q^{-2n}/d, & q^{1+n}bcd \end{matrix}\middle| q;q^{1+2\delta}\right]
$$

$$
= (-q^{1+2n})^{\delta-1}(1-q)\left[\begin{matrix} qb, qc, qd, qbcd \\ q, qbc, qbd, qcd \end{matrix}\middle| q\right]_n \times \left[\begin{matrix} q^2, qbc, qbd, qcd \\ qb, qc, qd, qbcd \end{matrix}\middle| q\right]_{2n}.
$$

Refer to Guo [14], eq. 4.5, and Verma-Joshi [20], eq. 3.12, for different proofs.

Recall from (4) that

$$ {}_5\psi_5\left[\begin{matrix} a, & b, & c, & d, & q^3/abcd \\ q^2/a, & q^2/b, & q^2/c, & q^2/d, & abcd/q \end{matrix}\middle| q; q^2\right] = 0. \tag{9} $$

By means of three terms relation

$$ \begin{aligned} \frac{(qu;q)_k}{(u;q)_k}\frac{(qv;q)_k}{(v;q)_k} &= \frac{1-uq^k}{1-u}\frac{1-vq^k}{1-v} \\ &= \frac{1}{(1-u)(1-v)} - \frac{q^k(u+v)}{(1-u)(1-v)} + \frac{q^{2k}uv}{(1-u)(1-v)} \end{aligned} $$

we derive from the combination of (7a) and (7b) the following general identity with two extra free-parameters:

Proposition 8. *For four indeterminate $\{a,b,c,d\}$ satisfying the condition $|q^2/bcd| < 1$, there holds the following transformation formula:*

$$ \begin{aligned} &{}_7\psi_7\left[\begin{matrix} qu, & qv, & a, & b, & c, & d, & q^3/abcd \\ u, & v, & q^2/a, & q^2/b, & q^2/c, & q^2/d, & abcd/q \end{matrix}\middle| q; q\right]\frac{(1-u)(1-v)}{1-uv/q} \\ &= \left[\begin{matrix} q, q^2/ab, q^2/ac, q^2/ad, q^2/bc, q^2/bd, q^2/cd, q^2/abcd \\ q^2/a, q^2/b, q^2/c, q^2/d, q^2/abc, q^2/abd, q^2/acd, q^2/bcd \end{matrix}\middle| q\right]_\infty + \mu(a;b,c,d) \\ &\times {}_6\psi_6^+\left[\begin{matrix} q^3/a\sqrt{bcd}, & -q^3/a\sqrt{bcd}, & q/a, & q^2/ab, & q^2/ac, & q^2/ad \\ q^2/a\sqrt{bcd}, & -q^2/a\sqrt{bcd}, & q^4/abcd, & q^3/abc, & q^3/abd, & q^3/acd \end{matrix}\middle| q; \frac{q^2}{bcd}\right]. \end{aligned} $$

1.4. Partial sums of well-poised bilateral series. Denote by ψ^- the partial sum of the terms with negative indices from the corresponding bilateral ψ-series. By reversing the summation index $k \to -1-k$, it is not hard to check the following relation:

$$ \begin{aligned} &{}_7\psi_7^-\left[\begin{matrix} qu, & qv, & a, & b, & c, & d, & q^3/abcd \\ u, & v, & q^2/a, & q^2/b, & q^2/c, & q^2/d, & abcd/q \end{matrix}\middle| q; q\right] \\ &= \frac{-uv(1-q/u)(1-q/v)}{q(1-u)(1-v)} \\ &\times {}_7\psi_7^+\left[\begin{matrix} q^2/u, & q^2/v, & a, & b, & c, & d, & q^3/abcd \\ q/u, & q/v, & q^2/a, & q^2/b, & q^2/c, & q^2/d, & abcd/q \end{matrix}\middle| q; q\right]. \end{aligned} $$

In view of ${}_7\psi_7 = {}_7\psi_7^+ + {}_7\psi_7^-$, we may express the case $u, v = \pm q^{1/2}$ of Proposition 8 in terms of unilateral series:

Corollary 9. *For four indeterminate $\{a,b,c,d\}$ satisfying the condition $|q^2/bcd| < 1$, there holds the following transformation formula:*

$$
{}_7\psi_7^+\left[\begin{matrix} q^{3/2}, & -q^{3/2}, & a, & b, & c, & d, & q^3/abcd \\ q^{1/2}, & -q^{1/2}, & q^2/a, & q^2/b, & q^2/c, & q^2/d, & abcd/q \end{matrix}\middle| q;q\right]
$$

$$
= \left[\begin{matrix} q^2, q^2/ab, q^2/ac, q^2/ad, q^2/bc, q^2/bd, q^2/cd, q^2/abcd \\ q^2/a, q^2/b, q^2/c, q^2/d, q^2/abc, q^2/abd, q^2/acd, q^2/bcd \end{matrix}\middle| q\right]_\infty + \frac{\mu(a;b,c,d)}{1-q}
$$

$$
\times\, {}_6\psi_6^+\left[\begin{matrix} q^3/a\sqrt{bcd}, & -q^3/a\sqrt{bcd}, & q/a, & q^2/ab, & q^2/ac, & q^2/ad \\ q^2/a\sqrt{bcd}, & -q^2/a\sqrt{bcd}, & q^4/abcd, & q^3/abc, & q^3/abd, & q^3/acd \end{matrix}\middle| q;\frac{q^2}{bcd}\right].
$$

When $a = q$, the last ${}_6\psi_6^+$-series reduces to one. We have therefore established the following strange summation formula.

$$
{}_6\psi_6^+\left[\begin{matrix} q^{3/2}, & -q^{3/2}, & b, & c, & d, & q^2/bcd \\ q^{1/2}, & -q^{1/2}, & q^2/b, & q^2/c, & q^2/d, & bcd \end{matrix}\middle| q;q\right] \tag{10a}
$$

$$
= \frac{(1-q/b)(1-q/c)(1-q/d)(1-q/bcd)}{(1-q)(1-q/bc)(1-q/bd)(1-q/cd)}\left\{1-\left[\begin{matrix} b,c,d,q^2/bcd \\ q/b,q/c,q/d,bcd/q \end{matrix}\middle| q\right]_\infty\right\} \tag{10b}
$$

which can also be verified by means of telescoping method.

In fact, noting the finite difference

$$
\triangle\left[\begin{matrix} b, & c, & d, & q^2/bcd \\ q/b, & q/c, & q/d, & bcd/q \end{matrix}\middle| q\right]_k = (1-q^{1+2k})\left[\begin{matrix} b,c,d,q^2/bcd \\ q^2/b,q^2/c,q^2/d,bcd \end{matrix}\middle| q\right]_k q^k
$$

$$
\times \frac{(1-q/bc)(1-q/bd)(1-q/cd)}{(1-q/b)(1-q/c)(1-q/d)(1-q/bcd)}
$$

we can confirm the ${}_6\psi_6^+$-series identity as follows:

$$
{}_6\psi_6^+\left[\begin{matrix} q^{3/2}, & -q^{3/2}, & b, & c, & d, & q^2/bcd \\ q^{1/2}, & -q^{1/2}, & q^2/b, & q^2/c, & q^2/d, & bcd \end{matrix}\middle| q;q\right]
$$

$$
= \sum_{k=0}^{\infty}\frac{1-q^{1+2k}}{1-q}\left[\begin{matrix} b,c,d,q^2/bcd \\ q^2/b,q^2/c,q^2/d,bcd \end{matrix}\middle| q\right]_k q^k
$$

$$
= \frac{(1-q/b)(1-q/c)(1-q/d)(1-q/bcd)}{(1-q)(1-q/bc)(1-q/bd)(1-q/cd)}
$$

$$
\times \sum_{k=0}^{\infty}\triangle\left[\begin{matrix} b, & c, & d, & q^2/bcd \\ q/b, & q/c, & q/d, & bcd/q \end{matrix}\middle| q\right]_k
$$

$$
= \frac{(1-q/b)(1-q/c)(1-q/d)(1-q/bcd)}{(1-q)(1-q/bc)(1-q/bd)(1-q/cd)}
$$

$$
\times\left\{1-\left[\begin{matrix} b, & c, & d, & q^2/bcd \\ q/b, & q/c, & q/d, & bcd/q \end{matrix}\middle| q\right]_\infty\right\}.
$$

1.5. Case $q \to 1$: Classical Hypergeometric Identities. Recall the q-Gamma function [1], §10.3,

$$\Gamma_q(x) = (1-q)^{1-x}\frac{(q;q)_\infty}{(q^x;q)_\infty} \quad \text{and} \quad \lim_{q\to 1^-}\Gamma_q(x) = \Gamma(x) \tag{11}$$

as well as the notation for classical bilateral hypergeometric series [17], Chapter 6. Performing replacements $a \to q^a$, $b \to q^b$, $c \to q^c$, $d \to q^d$ and $w \to q^w$ in Proposition 4 and then letting $q \to 1$, we derive the following classical hypergeometric series transformation formula.

Theorem 10. *For four indeterminate $\{a,b,c,d\}$ satisfying the condition $\Re(b+c+d) < 1$, there holds the following transformation formula:*

$$\begin{aligned}
&{}_6H_6\left[\begin{matrix}1+w, & a, & b, & c, & d, & 1-a-b-c-d\\ w, & 1-a, & 1-b, & 1-c, & 1-d, & a+b+c+d\end{matrix}\middle|1\right]\\
&\quad=\Gamma\left[\begin{matrix}1-a,1-b,1-c,1-d,1-a-b-c,1-a-b-d,1-a-c-d,1-b-c-d\\ 1-a-b,1-a-c,1-a-d,1-b-c,1-b-d,1-c-d,1-a-b-c-d\end{matrix}\right]\\
&\qquad+\frac{2-2a-b-c-d}{(1-a-b-c)(1-a-b-d)(1-a-c-d)}\Gamma\left[\begin{matrix}1-a,1-b,1-c,1-d,a+b+c+d\\ a,b,c,d,2-a-b-c-d\end{matrix}\right]\\
&\qquad\times\sum_{k=0}^{\infty}\frac{2-2a-b-c-d+2k}{2-2a-b-c-d}\left[\begin{matrix}1-a,1-a-b,1-a-c,1-a-d\\ 2-a-b-c-d,2-a-b-c,2-a-b-d,2-a-c-d\end{matrix}\right]_k.
\end{aligned}$$

Similarly, we can get from Proposition 8 another classical well-poised bilateral series identity.

Theorem 11. *For four indeterminate $\{a,b,c,d\}$ satisfying the condition $\Re(b+c+d) < 2$, there holds the following transformation formula:*

$$\begin{aligned}
&{}_7H_7\left[\begin{matrix}1+u, & 1+v, & a, & b, & c, & d, & 3-a-b-c-d\\ u, & v, & 2-a, & 2-b, & 2-c, & 2-d, & a+b+c+d-1\end{matrix}\middle|1\right]\\
&=\frac{u+v-1}{uv}\Gamma\left[\begin{matrix}2-a,2-b,2-c,2-d,2-a-b-c,2-a-b-d,2-a-c-d,2-b-c-d\\ 2-a-b,2-a-c,2-a-d,2-b-c,2-b-d,2-c-d,2-a-b-c-d\end{matrix}\right]\\
&\quad+\frac{4-2a-b-c-d}{(2-a-b-c)(2-a-b-d)(2-a-c-d)}\Gamma\left[\begin{matrix}2-a,2-b,2-c,2-d,a+b+c+d-1\\ a,b,c,d,4-a-b-c-d\end{matrix}\right]\\
&\quad\times\frac{u+v-1}{uv}\sum_{k=0}^{\infty}\frac{4-2a-b-c-d+2k}{4-2a-b-c-d}\left[\begin{matrix}1-a,2-a-b,2-a-c,2-a-d\\ 4-a-b-c-d,3-a-b-c,3-a-b-d,3-a-c-d\end{matrix}\right]_k.
\end{aligned}$$

It seems that both identities just displayed have not appeared previously in the literature on classical hypergeometric series.

2. Proofs via the modified Abel Lemma on summation by parts

First, let us define two functions by

$$\mathrm{U}(a,b,c,d) := {}_5\psi_5\left[\begin{matrix} a, & b, & c, & d, & q/abcd \\ q/a, & q/b, & q/c, & q/d, & abcd \end{matrix}\middle| q;q\right], \tag{12a}$$

$$\mathrm{V}(a,b,c,d) := {}_5\psi_5\left[\begin{matrix} a, & b, & c, & d, & q^3/abcd \\ q^2/a, & q^2/b, & q^2/c, & q^2/d, & abcd/q \end{matrix}\middle| q;q\right]. \tag{12b}$$

2.1. For two sequences $\{A_k, B_k\}$ defined by

$$A_k = \left[\begin{matrix} abd, & qc, & q^2/abcd \\ q^2/abd, & q/c, & abcd \end{matrix}\middle| q\right]_k \quad \text{and} \quad B_k = \left[\begin{matrix} a, & b, & d, & q^2/abd \\ q/a, & q/b, & q/d, & abd/q \end{matrix}\middle| q\right]_k$$

it is not hard to compute the limiting relations

$$[AB]_{+\infty} = -[AB]_{-\infty} = \frac{1}{1-abd/q}\left[\begin{matrix} a, & b, & qc, & d, & q^2/abcd \\ q/a, & q/b, & q/c, & q/d, & abcd \end{matrix}\middle| q\right]_\infty$$

and the following differences

$$\nabla A_k = (1+q^k)\left[\begin{matrix} abd/q, & c, & q/abcd \\ q^2/abd, & q/c, & abcd \end{matrix}\middle| q\right]_k q^k,$$

$$\begin{aligned} \triangle B_k = (1-q^{1+2k})&\left[\begin{matrix} a, & b, & d, & q^2/abd \\ q^2/a, & q^2/b, & q^2/d, & abd \end{matrix}\middle| q\right]_k q^k \\ &\times \frac{(1-q/ab)(1-q/ad)(1-q/bd)}{(1-q/a)(1-q/b)(1-q/d)(1-q/abd)}. \end{aligned}$$

By means of (5a) and (5b), we can manipulate, through the modified Abel lemma on summation by parts, the following series

$$\begin{aligned} 2\mathrm{U}(a,b,c,d) &= \sum_k (1+q^k)\left[\begin{matrix} a, & b, & c, & d, & q/abcd \\ q/a, & q/b, & q/c, & q/d, & abcd \end{matrix}\middle| q\right]_k q^k \\ &= \sum_k B_k \nabla A_k = 2[AB]_{+\infty} + \sum_k A_k \triangle B_k \\ &= \frac{2}{1-abd/q}\left[\begin{matrix} a, & b, & qc, & d, & q^2/abcd \\ q/a, & q/b, & q/c, & q/d, & abcd \end{matrix}\middle| q\right]_\infty \\ &\quad + \frac{(1-q/ab)(1-q/ad)(1-q/bd)}{(1-q/a)(1-q/b)(1-q/d)(1-q/abd)} \\ &\quad \times \sum_k (1-q^{1+2k})\left[\begin{matrix} a,b,qc,d,q^2/abcd \\ q^2/a,q^2/b,q/c,q^2/d,abcd \end{matrix}\middle| q\right]_k q^k. \end{aligned}$$

Applying (7a) and (7b) to the last line, we therefore establish the following relation:

$$\mathrm{U}(a,b,c,d) = \mathrm{V}(a,b,qc,d)\frac{(1-q/ab)(1-q/ad)(1-q/bd)}{(1-q/a)(1-q/b)(1-q/d)(1-q/abd)} \tag{13a}$$

$$+\frac{1}{1-abd/q}\left[\begin{matrix} a, & b, & qc, & d, & q^2/abcd \\ q/a, & q/b, & q/c, & q/d, & abcd \end{matrix}\middle| q\right]_\infty. \tag{13b}$$

2.2. Similarly for two sequences $\{C_k, D_k\}$ defined by

$$C_k = \left[\begin{matrix} qa, & qc, & q/ac \\ q/a, & q/c, & qac \end{matrix}\middle| q\right]_k \quad \text{and} \quad D_k = \left[\begin{matrix} qac, & b, & d, & q/abcd \\ 1/ac, & q/b, & q/d, & abcd \end{matrix}\middle| q\right]_k$$

we can calculate without difficulty the limiting relations

$$[CD]_{+\infty} = -[CD]_{-\infty} = \frac{1}{1-1/ac}\left[\begin{matrix} qa, & b, & qc, & d, & q/abcd \\ q/a, & q/b, & q/c, & q/d, & abcd \end{matrix}\middle| q\right]_\infty$$

and the following differences

$$\nabla C_k = (1+q^k)\left[\begin{matrix} a, & c, & 1/ac \\ q/a, & q/c, & qac \end{matrix}\middle| q\right]_k q^k,$$

$$\triangle D_k = (1-q^{1+2k})\left[\begin{matrix} qac, & b, & d, & q/abcd \\ q/ac, & q^2/b, & q^2/d, & qabcd \end{matrix}\middle| q\right]_k q^k$$

$$\times\frac{(1-1/abc)(1-1/acd)(1-q/bd)}{(1-1/ac)(1-q/b)(1-q/d)(1-1/abcd)}.$$

Taking into account of (5a) and (5b), we can apply the Abel lemma on summation by parts to reformulate the following series:

$$2\mathrm{U}(a,b,c,d) = \sum_k (1+q^k)\left[\begin{matrix} a, & b, & c, & d, & q/abcd \\ q/a, & q/b, & q/c, & q/d, & abcd \end{matrix}\middle| q\right]_k q^k$$

$$= \sum_k D_k \nabla C_k = 2[CD]_{+\infty} + \sum_k C_k \triangle D_k$$

$$= \frac{2}{1-1/ac}\left[\begin{matrix} qa, & b, & qc, & d, & q/abcd \\ q/a, & q/b, & q/c, & q/d, & abcd \end{matrix}\middle| q\right]_\infty$$

$$+\frac{(1-1/abc)(1-1/acd)(1-q/bd)}{(1-1/ac)(1-q/b)(1-q/d)(1-1/abcd)}$$

$$\times\sum_k (1-q^{1+2k})\left[\begin{matrix} qa, b, qc, d, q/abcd \\ q/a, q^2/b, q/c, q^2/d, qabcd \end{matrix}\middle| q\right]_k q^k.$$

Recalling (7a) and (7b), we derive the following crossing relation:

$$\mathrm{U}(a,b,c,d)=\mathrm{V}(qa,b,qc,d)\frac{(1-1/abc)(1-1/acd)(1-q/bd)}{(1-1/ac)(1-q/b)(1-q/d)(1-1/abcd)} \tag{14a}$$

$$+\frac{1}{1-1/ac}\left[\begin{matrix} qa, & b, & qc, & d, & q/abcd \\ q/a, & q/b, & q/c, & q/d, & abcd \end{matrix}\middle| q\right]_\infty. \tag{14b}$$

2.3. Now combining (13a)–(13b) with (14a)–(14b) under the replacement $a \to a/q$ and then canceling $\mathrm{V}(a,b,c,d)$, we derive the following independent relation for $\mathrm{U}(a,b,c,d)$:

$$\mathrm{U}(a,b,c,d)=\lambda(a;b,c,d)+\mathrm{U}(a/q,b,c,d) \tag{15a}$$

$$\times\frac{(1-q/ab)(1-q/ac)(1-q/ad)(1-q/abcd)}{(1-q/a)(1-q/abc)(1-q/abd)(1-q/acd)}. \tag{15b}$$

Observing that the relation displayed in (15a)–(15b) results from shifting parameter a by q in the U-function. Iterating it m-times, we derive the relation with an extra natural number parameter m as follows:

$$\mathrm{U}(a,b,c,d)=\mathrm{U}(a/q^m,b,c,d)\left[\begin{matrix} q/ab, q/ac, q/ad, q/abcd \\ q/a, q/abc, q/abd, q/acd \end{matrix}\middle| q\right]_m$$

$$+\sum_{k=0}^{m-1}\lambda(a/q^k;b,c,d)\left[\begin{matrix} q/ab, q/ac, q/ad, q/abcd \\ q/a, q/abc, q/abd, q/acd \end{matrix}\middle| q\right]_k$$

which can be further simplified as

$$\mathrm{U}(a,b,c,d)=\mathrm{U}(a/q^m,b,c,d)\left[\begin{matrix} q/ab, q/ac, q/ad, q/abcd \\ q/a, q/abc, q/abd, q/acd \end{matrix}\middle| q\right]_m+\lambda(a;b,c,d) \tag{16a}$$

$$\times\sum_{k=0}^{m-1}\frac{1-q^{2+2k}/a^2bcd}{1-q^2/a^2bcd}$$

$$\times\left[\begin{matrix} q/a, q/ab, q/ac, q/ad \\ q^2/abcd, q^2/abc, q^2/abd, q^2/acd \end{matrix}\middle| q\right]_k\left(\frac{q}{bcd}\right)^k. \tag{16b}$$

According to the Weierstrass M-test on uniformly convergent series (cf. Stromberg [18], p. 141), we may compute the following limit

$$\lim_{m\to\infty}\mathrm{U}(a/q^m,b,c,d)={}_3\psi_3\left[\begin{matrix} b, & c, & d \\ q/b, & q/c, & q/d \end{matrix}\middle| q;\frac{q}{bcd}\right]$$

$$=\left[\begin{matrix} q, q/bc, q/bd, q/cd \\ q/b, q/c, q/d, q/bcd \end{matrix}\middle| q\right]_\infty$$

thanks to Corollary 2 for the last equality.

Letting $m \to \infty$ in (16a)–(16b), we find the following transformation formula:

$$\mathrm{U}(a,b,c,d) = \left[\begin{matrix} q, q/ab, q/ac, q/ad, q/bc, q/bd, q/cd, q/abcd \\ q/a, q/b, q/c, q/d, q/abc, q/abd, q/acd, q/bcd \end{matrix} \middle| q\right]_\infty + \lambda(a;b,c,d)$$
$$\times \sum_{k=0}^{\infty} \frac{1-q^{2+2k}/a^2bcd}{1-q^2/a^2bcd}$$
$$\times \left[\begin{matrix} q/a, q/ab, q/ac, q/ad \\ q^2/abcd, q^2/abc, q^2/abd, q^2/acd \end{matrix} \middle| q\right]_k \left(\frac{q}{bcd}\right)^k.$$

This is the transformation formula stated in Theorem 1. □

2.4. Similarly, we can work out the corresponding results for $\mathrm{V}(a,b,c,d)$. Under the replacements $a \to a/q$ and $c \to c/q$, the difference between (13a)–(13b) and (14a)–(14b) leads us to another independent relation:

$$\mathrm{V}(a,b,c,d) = \mu(a;b,c,d) + \mathrm{V}(a/q,b,c,d) \tag{17a}$$
$$\times \frac{(1-q^2/ab)(1-q^2/ac)(1-q^2/ad)(1-q^2/abcd)}{(1-q^2/a)(1-q^2/abc)(1-q^2/abd)(1-q^2/acd)}. \tag{17b}$$

Iterating (17a)–(17b) m-times gives rise to the following relation with an extra natural number parameter m:

$$\mathrm{V}(a,b,c,d) = \mathrm{V}(a/q^m,b,c,d)\left[\begin{matrix} q^2/ab, q^2/ac, q^2/ad, q^2/abcd \\ q^2/a, q^2/abc, q^2/abd, q^2/acd \end{matrix} \middle| q\right]_m$$
$$+ \sum_{k=0}^{m-1} \mu(a/q^k;b,c,d)\left[\begin{matrix} q^2/ab, q^2/ac, q^2/ad, q^2/abcd \\ q^2/a, q^2/abc, q^2/abd, q^2/acd \end{matrix} \middle| q\right]_k$$

which can be further simplified as

$$\mathrm{V}(a,b,c,d)$$
$$= \mathrm{V}(a/q^m,b,c,d)\left[\begin{matrix} q^2/ab, q^2/ac, q^2/ad, q^2/abcd \\ q^2/a, q^2/abc, q^2/abd, q^2/acd \end{matrix} \middle| q\right]_m + \mu(a;b,c,d) \tag{18a}$$
$$\times \sum_{k=0}^{m-1} \frac{1-q^{4+2k}/a^2bcd}{1-q^4/a^2bcd} \left[\begin{matrix} q/a, q^2/ab, q^2/ac, q^2/ad \\ q^4/abcd, q^3/abc, q^3/abd, q^3/acd \end{matrix} \middle| q\right]_k \left(\frac{q^2}{bcd}\right)^k. \tag{18b}$$

According to the Weierstrass M-test on uniformly convergent series, we may compute the following limit

$$\lim_{m\to\infty} \mathrm{V}(a/q^m, b, c, d) = {}_3\psi_3\left[\begin{matrix} b, & c, & d \\ q^2/b, & q^2/c, & q^2/d \end{matrix}\middle| q; \frac{q^2}{bcd}\right]$$
$$= \left[\begin{matrix} q, q^2/bc, q^2/bd, q^2/cd \\ q^2/b, q^2/c, q^2/d, q^2/bcd \end{matrix}\middle| q\right]_\infty$$

thanks to Corollary 6 for the last equality.

Letting $m \to \infty$ in (18a)–(18b), we find the following transformation formula:

$$\mathrm{V}(a, b, c, d)$$
$$= \left[\begin{matrix} q, q^2/ab, q^2/ac, q^2/ad, q^2/bc, q^2/bd, q^2/cd, q^2/abcd \\ q^2/a, q^2/b, q^2/c, q^2/d, q^2/abc, q^2/abd, q^2/acd, q^2/bcd \end{matrix}\middle| q\right]_\infty + \mu(a; b, c, d)$$
$$\times \sum_{k=0}^{\infty} \frac{1 - q^{4+2k}/a^2bcd}{1 - q^4/a^2bcd} \left[\begin{matrix} q/a, q^2/ab, q^2/ac, q^2/ad \\ q^4/abcd, q^3/abc, q^3/abd, q^3/acd \end{matrix}\middle| q\right]_k \left(\frac{q^2}{bcd}\right)^k.$$

This proves the transformation formula stated in Theorem 5. □

2.5. In addition, Theorem 1 can further be confirmed by Bailey's very well-poised bilateral ${}_6\psi_6$-series identity. The same can be done for Theorem 5.

In fact, reversing the summation index $k \to -1-k$ for Bailey's well-poised ${}_6\psi_6$-partial sum, we have no difficulty to check the following relation:

$${}_6\psi_6^-\left[\begin{matrix} q\sqrt{A}, & -q\sqrt{A}, & B, & C, & D, & E \\ \sqrt{A}, & -\sqrt{A}, & qA/B, & qA/C, & qA/D, & qA/E \end{matrix}\middle| q; \frac{qA^2}{BCDE}\right]$$
$$= \frac{-q}{A} \frac{(1-q^2/A)(1-A/B)(1-A/C)(1-A/D)(1-A/E)}{(1-A)(1-q/B)(1-q/C)(1-q/D)(1-q/E)}$$
$$\times {}_6\psi_6^+\left[\begin{matrix} q^2/\sqrt{A}, & -q^2/\sqrt{A}, & qB/A, & qC/A, & qD/A, & qE/A \\ q/\sqrt{A}, & -q/\sqrt{A}, & q^2/B, & q^2/C, & q^2/D, & q^2/E \end{matrix}\middle| q; \frac{qA^2}{BCDE}\right].$$

Observe that the ${}_5\psi_5$-series in Theorem 1 is invariant under the replacement $a \to q/abcd$. Equating the corresponding right members, we get, after lengthy simplification, the following equation:

$${}_6\psi_6\left[\begin{matrix} q^2/a\sqrt{bcd}, & -q^2/a\sqrt{bcd}, & q/a, & q/ab, & q/ac, & q/ad \\ q/a\sqrt{bcd}, & -q/a\sqrt{bcd}, & q^2/abcd, & q^2/abc, & q^2/abd, & q^2/acd \end{matrix}\middle| q; \frac{q}{bcd}\right] \tag{19a}$$
$$= \frac{(1-q/abc)(1-q/abd)(1-q/acd)}{(q/abcd)(1-q^2/a^2bcd)} \left[\begin{matrix} q, q/a, q/bc, q/bd, q/cd, abcd \\ a, b, c, d, q/bcd, q^2/abcd \end{matrix}\middle| q\right]_\infty \tag{19b}$$
$$\times \left\{ \left[\begin{matrix} a, abc, abd, acd \\ ab, ac, ad, abcd \end{matrix}\middle| q\right]_\infty - \left[\begin{matrix} q/ab, q/ac, q/ad, q/abcd \\ q/a, q/abc, q/abd, q/acd \end{matrix}\middle| q\right]_\infty \right\}. \tag{19c}$$

Evaluating the last ${}_6\psi_6$-series displayed in (19a) Bailey's summation formula (1a)–(1b), we recover from (19a)–(19c) the following difference equation:

$$\begin{aligned}&[ab, q/ab, ac, q/ac, ad, q/ad, abcd, q/abcd; q]_\infty\\ &\qquad - [a, q/a, abc, q/abc, abd, q/abd, acd, q/acd; q]_\infty\\ &\quad = a[b, q/b, c, q/c, d, q/d, a^2bcd, q/a^2bcd; q]_\infty\end{aligned}$$

whose equivalent form has explicitly appeared in Chu [10], Theorem 1.1:

$$\langle ab, ac, ad, abcd; q\rangle_\infty - \langle a, abc, abd, acd; q\rangle_\infty = a\langle b, c, d, a^2bcd; q\rangle_\infty.$$

3. Further transformations for U(*a*, *b*, *c*, *d*)

This section will further repeat the iterating process for other parameters and derive three interesting transformations of $\mathrm{U}(a, b, c, d)$ into quadratic, cubic and quartic series.

3.1. Applying (15a)–(15b) again to $\mathrm{U}(b, a/q, c, d) = \mathrm{U}(a/q, b, c, d)$ displayed in (15a) and then simplifying the result, we get the relation:

$$\begin{aligned}&\mathrm{U}(a, b, c, d)\\ &\quad = \mathrm{U}(a/q, b/q, c, d)\left[\begin{matrix}q/ac, q/ad, q/bc, q/bd\\ q/a, q/b, q/acd, q/bcd\end{matrix}\middle| q\right]_1 \left[\begin{matrix}q/ab, q/abcd\\ q/abc, q/abd\end{matrix}\middle| q\right]_2 &\text{(20a)}\\ &\quad + \lambda(a; b, c, d) + \lambda(b; a/q, c, d)\left[\begin{matrix}q/ab, q/ac, q/ad, q/abcd\\ q/a, q/abc, q/abd, q/acd\end{matrix}\middle| q\right]_1. &\text{(20b)}\end{aligned}$$

Iterating further this relation m-times, we derive the recurrence relation.

Theorem 12 (Recurrence relation). *Define R_m and $R^\star_m$-functions by*

$$R^\star_m(a, b, c, d) = \frac{\lambda(a; b, c, d)}{1 - q^2/a^2bcd} \times R_m(a, b, c, d), \tag{21a}$$

$$R_m(a, b, c, d) = \sum_{k=0}^{m-1}\{1 - q^{2+3k}/a^2bcd\}\frac{(q/ab; q)_{2k}}{[q^2/acd, q/bcd; q]_k} \tag{21b}$$

$$\times \frac{[q/a, q/b, q/ac, q/ad, q/bc, q/bd; q]_k}{[q^2/abc, q^2/abd, q^2/abcd; q]_{2k}}\left\{\frac{q^{2+k}}{abc^2d^2}\right\}^k. \tag{21c}$$

Then there holds the following relation:

$$\begin{aligned}
&\mathrm{U}(a,b,c,d)\\
&\quad= \mathrm{U}(a/q^m, b/q^m, c, d)\left[\begin{matrix} q/ac, q/ad, q/bc, q/bd \\ q/a, q/b, q/acd, q/bcd \end{matrix}\middle| q\right]_m \left[\begin{matrix} q/ab, q/abcd \\ q/abc, q/abd \end{matrix}\middle| q\right]_{2m}\\
&\qquad + R_m^\star(a,b,c,d) + R_m^\star(b, a/q, c, d)\left[\begin{matrix} q/ab, q/ac, q/ad, q/abcd \\ q/a, q/abc, q/abd, q/acd \end{matrix}\middle| q\right]_1.
\end{aligned}$$

In view of Corollary 2, we have the following limit:

$$\lim_{m\to\infty} \mathrm{U}(a/q^m, b/q^m, c, d) = {}_2\psi_3\left[\begin{matrix} -, & c, & d \\ 0, & q/c, & q/d \end{matrix}\middle| q; \frac{q}{cd}\right] = \left[\begin{matrix} q, q/cd \\ q/c, q/d \end{matrix}\middle| q\right]_\infty$$

which leads us to the following quadratic transformation formula.

Proposition 13. *Let $R^\star = \lim_{m\to\infty} R_m^\star$, where the $R_m^\star$-function is defined in Theorem* 12*. Then there holds the nonterminating series transformation:*

$$\begin{aligned}
\mathrm{U}(a,b,c,d) &= \left[\begin{matrix} q, q/ab, q/ac, q/ad, q/bc, q/bd, q/cd, q/abcd \\ q/a, q/b, q/c, q/d, q/abc, q/abd, q/acd, q/bcd \end{matrix}\middle| q\right]_\infty\\
&\qquad + R^\star(a,b,c,d) + R^\star(b, a/q, c, d)\left[\begin{matrix} q/ab, q/ac, q/ad, q/abcd \\ q/a, q/abc, q/abd, q/acd \end{matrix}\middle| q\right]_1.
\end{aligned}$$

3.2. Applying (15a)–(15b) again to $\mathrm{U}(c, b/q, a/q, d) = \mathrm{U}(a/q, b/q, c, d)$ displayed in (20a) and then simplifying the equation corresponding to (20a)–(20b), we get the relation:

$$\mathrm{U}(a,b,c,d)$$

$$= \mathrm{U}(a/q, b/q, c/q, d)\frac{(q/abcd;q)_3}{(q/abc;q)_3} \tag{22a}$$

$$\times\left[\begin{matrix} q/ad, q/bd, q/cd \\ q/a, q/b, q/c \end{matrix}\middle| q\right]_1 \left[\begin{matrix} q/ab, q/ac, q/bc \\ q/abd, q/acd, q/bcd \end{matrix}\middle| q\right]_2 \tag{22b}$$

$$+ \lambda(a; b, c, d) + \lambda(b; a/q, c, d)\left[\begin{matrix} q/ab, q/ac, q/ad, q/abcd \\ q/a, q/abc, q/abd, q/acd \end{matrix}\middle| q\right]_1 \tag{22c}$$

$$+ \lambda(c; a/q, b/q, d)\left[\begin{matrix} q/ac, q/ad, q/bc, q/bd \\ q/a, q/b, q/acd, q/bcd \end{matrix}\middle| q\right]_1 \left[\begin{matrix} q/ab, q/abcd \\ q/abc, q/abd \end{matrix}\middle| q\right]_2. \tag{22d}$$

Iterating further this relation m-times, we derive the recurrence relation.

Theorem 14 (Recurrence relation). *Define S_m and $S^\star_m$-functions by*

$$S^\star_m(a,b,c,d)=\frac{\lambda(a;b,c,d)}{1-q^2/a^2bcd}\times S_m(a,b,c,d), \tag{23a}$$

$$S_m(a,b,c,d)=\sum_{k=0}^{m-1}\{1-q^{2+4k}/a^2bcd\}\left[\begin{matrix}q/ab,q/ac,q/bc\\ q^2/abd,q^2/acd,q/bcd\end{matrix}\middle|q\right]_{2k} \tag{23b}$$

$$\times\frac{[q/a,q/b,q/c,q/ad,q/bd,q/cd;q]_k}{[q^2/abc,q^2/abcd;q]_{3k}}\left\{\frac{q^{3+3k}}{a^2b^2c^2d^3}\right\}^k. \tag{23c}$$

Then there holds the following relation:

$$\begin{aligned}\mathrm{U}(a,b,c,d)&=\mathrm{U}(a/q^m,b/q^m,c/q^m,d)\left[\begin{matrix}q/ad,q/bd,q/cd\\ q/a,q/b,q/c\end{matrix}\middle|q\right]_m\\&\times\frac{(q/abcd;q)_{3m}}{(q/abc;q)_{3m}}\left[\begin{matrix}q/ab,q/ac,q/bc\\ q/abd,q/acd,q/bcd\end{matrix}\middle|q\right]_{2m}\\&+S^\star_m(a,b,c,d)+S^\star_m(b,a/q,c,d)\left[\begin{matrix}q/ab,q/ac,q/ad,q/abcd\\ q/a,q/abc,q/abd,q/acd\end{matrix}\middle|q\right]_1\\&+S^\star_m(c,a/q,b/q,d)\left[\begin{matrix}q/ac,q/ad,q/bc,q/bd\\ q/a,q/b,q/acd,q/bcd\end{matrix}\middle|q\right]_1\left[\begin{matrix}q/ab,q/abcd\\ q/abc,q/abd\end{matrix}\middle|q\right]_2.\end{aligned}$$

In view of Corollary 2, we have the following limit:

$$\lim_{m\to\infty}\mathrm{U}(a/q^m,b/q^m,c/q^m,d)={}_1\psi_3\left[\begin{matrix}-, & -, & d\\ 0, & 0, & q/d\end{matrix}\middle|q;\frac{q}{d}\right]=\frac{(q;q)_\infty}{(q/d;q)_\infty}$$

which leads us to the following cubic transformation formula.

Proposition 15. *Let $S^\star=\lim_{m\to\infty}S^\star_m$, where the $S^\star_m$-function is defined in Theorem* 14. *Then there holds the nonterminating series transformation:*

$$\begin{aligned}\mathrm{U}(a,b,c,d)&=\left[\begin{matrix}q,q/ab,q/ac,q/ad,q/bc,q/bd,q/cd,q/abcd\\ q/a,q/b,q/c,q/d,q/abc,q/abd,q/acd,q/bcd\end{matrix}\middle|q\right]_\infty\\&+S^\star(a,b,c,d)+S^\star(b,a/q,c,d)\left[\begin{matrix}q/ab,q/ac,q/ad,q/abcd\\ q/a,q/abc,q/abd,q/acd\end{matrix}\middle|q\right]_1\\&+S^\star(c,a/q,b/q,d)\left[\begin{matrix}q/ac,q/ad,q/bc,q/bd\\ q/a,q/b,q/acd,q/bcd\end{matrix}\middle|q\right]_1\left[\begin{matrix}q/ab,q/abcd\\ q/abc,q/abd\end{matrix}\middle|q\right]_2.\end{aligned}$$

3.3. Applying (15a)–(15b) again to $\mathrm{U}(d,a/q,b/q,c/q)=\mathrm{U}(a/q,b/q,c/q,d)$ displayed in (22a) and then simplifying the equation corresponding to (22a–22d), we get the relation:

$$\begin{aligned}
&\mathrm{U}(a,b,c,d)\\
&\quad=\mathrm{U}(a/q,b/q,c/q,d/q) && \text{(24a)}\\
&\qquad\times\frac{[q/ab,q/ac,q/ad,q/bc,q/bd,q/cd;q]_2(q/abcd;q)_4}{[q/a,q/b,q/c,q/d;q]_1[q/abc,q/abd,q/acd,q/bcd;q]_3} && \text{(24b)}\\
&\qquad+\lambda(a;b,c,d)+\lambda(b;a/q,c,d)\left[\begin{matrix}q/ab,q/ac,q/ad,q/abcd\\ q/a,q/abc,q/abd,q/acd\end{matrix}\middle|q\right]_1 && \text{(24c)}\\
&\qquad+\lambda(c;a/q,b/q,d)\left[\begin{matrix}q/ac,q/ad,q/bc,q/bd\\ q/a,q/b,q/acd,q/bcd\end{matrix}\middle|q\right]_1\left[\begin{matrix}q/ab,q/abcd\\ q/abc,q/abd\end{matrix}\middle|q\right]_2 && \text{(24d)}\\
&\qquad+\lambda\left(d;\frac{a}{q},\frac{b}{q},\frac{c}{q}\right)\left[\begin{matrix}q/ad,q/bd,q/cd\\ q/a,q/b,q/c\end{matrix}\middle|q\right]_1\\
&\qquad\times\left[\begin{matrix}q/ab,q/ac,q/bc\\ q/abd,q/acd,q/bcd\end{matrix}\middle|q\right]_2\frac{(q/abcd;q)_3}{(q/abc;q)_3}. && \text{(24e)}
\end{aligned}$$

Iterating further this relation m-times, we derive the recurrence relation.

Theorem 16 (Recurrence relation). *Define T_m and $T_m^\star$-functions by*

$$T_m^\star(a,b,c,d)=\frac{\lambda(a;b,c,d)}{1-q^2/a^2bcd}\times T_m(a,b,c,d), \tag{25a}$$

$$\begin{aligned}
T_m(a,b,c,d)&=\sum_{k=0}^{m-1}\{1-q^{2+5k}/a^2bcd\}\frac{[q/a,q/b,q/c,q/d;q]_k}{(q^2/abcd;q)_{4k}} && \text{(25b)}\\
&\quad\times\frac{[q/ab,q/ac,q/ad,q/bc,q/bd,q/cd;q]_{2k}}{[q^2/abc,q^2/abd,q^2/acd,q/bcd;q]_{3k}}\left\{\frac{q^{4+6k}}{a^3b^3c^3d^3}\right\}^k. && \text{(25c)}
\end{aligned}$$

Then there holds the following relation:

$$\begin{aligned}
&\mathrm{U}(a,b,c,d)\\
&\quad=\mathrm{U}(a/q^m,b/q^m,c/q^m,d/q^m)\\
&\qquad\times\frac{[q/ab,q/ac,q/ad,q/bc,q/bd,q/cd;q]_{2m}(q/abcd;q)_{4m}}{[q/a,q/b,q/c,q/d;q]_m[q/abc,q/abd,q/acd,q/bcd;q]_{3m}}\\
&\qquad+T_m^\star(a,b,c,d)+T_m^\star(b,a/q,c,d)\left[\begin{matrix}q/ab,q/ac,q/ad,q/abcd\\ q/a,q/abc,q/abd,q/acd\end{matrix}\middle|q\right]_1\\
&\qquad+T_m^\star(c,a/q,b/q,d)\left[\begin{matrix}q/ac,q/ad,q/bc,q/bd\\ q/a,q/b,q/acd,q/bcd\end{matrix}\middle|q\right]_1\left[\begin{matrix}q/ab,q/abcd\\ q/abc,q/abd\end{matrix}\middle|q\right]_2\\
&\qquad+T_m^\star\left(d,\frac{a}{q},\frac{b}{q},\frac{c}{q}\right)\left[\begin{matrix}q/ad,q/bd,q/cd\\ q/a,q/b,q/c\end{matrix}\middle|q\right]_1\left[\begin{matrix}q/ab,q/ac,q/bc\\ q/abd,q/acd,q/bcd\end{matrix}\middle|q\right]_2\frac{(q/abcd;q)_3}{(q/abc;q)_3}.
\end{aligned}$$

Recall the Jacobi triple product identity (cf. [13], II-28)

$$[q, x, q/x; q]_\infty = \sum_{k=-\infty}^{+\infty} (-1)^k q^{\binom{k}{2}} x^k \quad \text{for } |q| < 1. \tag{26}$$

We may compute the following limit

$$\lim_{m\to\infty} \mathrm{U}(a/q^m, b/q^m, c/q^m, d/q^m) = \sum_k (-1)^k q^{3\binom{k}{2}+k} = [q^3, q, q^2; q^3]_\infty = (q;q)_\infty.$$

This leads us to the following quartic transformation formula.

Proposition 17. *Let* $T^\star = \lim_{m\to\infty} T^\star_m$*, where the* $T^\star_m$*-function is defined in Theorem* 16*. Then there holds the nonterminating series transformation:*

$$\begin{aligned}\mathrm{U}(a,b,c,d) = &\left[\begin{matrix} q, q/ab, q/ac, q/ad, q/bc, q/bd, q/cd, q/abcd \\ q/a, q/b, q/c, q/d, q/abc, q/abd, q/acd, q/bcd \end{matrix}\middle| q\right]_\infty \\ &+ T^\star(a,b,c,d) + T^\star(b, a/q, c, d)\left[\begin{matrix} q/ab, q/ac, q/ad, q/abcd \\ q/a, q/abc, q/abd, q/acd \end{matrix}\middle| q\right]_1 \\ &+ T^\star(c, a/q, b/q, d)\left[\begin{matrix} q/ac, q/ad, q/bc, q/bd \\ q/a, q/b, q/acd, q/bcd \end{matrix}\middle| q\right]_1 \left[\begin{matrix} q/ab, q/abcd \\ q/abc, q/abd \end{matrix}\middle| q\right]_2 \\ &+ T^\star\left(d, \frac{a}{q}, \frac{b}{q}, \frac{c}{q}\right)\left[\begin{matrix} q/ad, q/bd, q/cd \\ q/a, q/b, q/c \end{matrix}\middle| q\right]_1 \\ &\times \left[\begin{matrix} q/ab, q/ac, q/bc \\ q/abd, q/acd, q/bcd \end{matrix}\middle| q\right]_2 \frac{(q/abcd;q)_3}{(q/abc;q)_3}.\end{aligned}$$

4. Further transformations for $\mathrm{V}(a, b, c, d)$

Similarly, we can derive other three transformation formulae for $\mathrm{V}(a, b, c, d)$.

4.1. Applying (17a)–(17b) again to $\mathrm{V}(b, a/q, c, d) = \mathrm{V}(a/q, b, c, d)$ displayed in (17a) and then simplifying the result, we get the relation:

$$\begin{aligned}\mathrm{V}(a,b,c,d) = \mathrm{V}(a/q, b/q, c, d)&\left[\begin{matrix} q^2/ac, q^2/ad, q^2/bc, q^2/bd \\ q^2/a, q^2/b, q^2/acd, q^2/bcd \end{matrix}\middle| q\right]_1 \\ \times &\left[\begin{matrix} q^2/ab, q^2/abcd \\ q^2/abc, q^2/abd \end{matrix}\middle| q\right]_2 + \mu(a; b, c, d) \qquad (27a) \\ + \mu(b; a/q, c, d)&\left[\begin{matrix} q^2/ab, q^2/ac, q^2/ad, q^2/abcd \\ q^2/a, q^2/abc, q^2/abd, q^2/acd \end{matrix}\middle| q\right]_1. \qquad (27b)\end{aligned}$$

Iterating further this relation m-times, we derive the recurrence relation.

Theorem 18 (Recurrence relation). *Define $\mathscr{R}_m$ and $\mathscr{R}^\star_m$-functions by*

$$\mathscr{R}^\star_m(a,b,c,d)=\frac{\mu(a;b,c,d)}{1-q^4/a^2bcd}\times\mathscr{R}_m(a,b,c,d), \tag{28a}$$

$$\mathscr{R}_m(a,b,c,d)=\sum_{k=0}^{m-1}\{1-q^{4+3k}/a^2bcd\}\frac{(q^2/ab;q)_{2k}}{[q^3/acd,q^2/bcd;q]_k} \tag{28b}$$

$$\times\frac{[q/a,q/b,q^2/ac,q^2/ad,q^2/bc,q^2/bd;q]_k}{[q^3/abc,q^3/abd,q^4/abcd;q]_{2k}}\left\{\frac{q^{4+k}}{abc^2d^2}\right\}^k. \tag{28c}$$

Then there holds the following relation:

$$\begin{aligned}\mathrm{V}(a,b,c,d)&\\ &=\mathrm{V}(a/q^m,b/q^m,c,d)\left[\begin{matrix}q^2/ac,q^2/ad,q^2/bc,q^2/bd\\ q^2/a,q^2/b,q^2/acd,q^2/bcd\end{matrix}\middle|q\right]_m\left[\begin{matrix}q^2/ab,q^2/abcd\\ q^2/abc,q^2/abd\end{matrix}\middle|q\right]_{2m}\\ &\quad+\mathscr{R}^\star_m(a,b,c,d)+\mathscr{R}^\star_m(b,a/q,c,d)\left[\begin{matrix}q^2/ab,q^2/ac,q^2/ad,q^2/abcd\\ q^2/a,q^2/abc,q^2/abd,q^2/acd\end{matrix}\middle|q\right]_1.\end{aligned}$$

In view of Corollary 6, we have the following limit:

$$\lim_{m\to\infty}\mathrm{V}(a/q^m,b/q^m,c,d)={}_2\psi_3\left[\begin{matrix}-, & c, & d\\ 0, & q^2/c, & q^2/d\end{matrix}\middle|q;\frac{q^2}{cd}\right]=\left[\begin{matrix}q,q^2/cd\\ q^2/c,q^2/d\end{matrix}\middle|q\right]_\infty$$

which leads us to the following quadratic transformation formula.

Proposition 19. *Let $\mathscr{R}^\star=\lim_{m\to\infty}\mathscr{R}^\star_m$, where the $\mathscr{R}^\star_m$-function is defined in Theorem 18. Then there holds the nonterminating series transformation:*

$$\begin{aligned}\mathrm{V}(a,b,c,d)&=\left[\begin{matrix}q,q^2/ab,q^2/ac,q^2/ad,q^2/bc,q^2/bd,q^2/cd,q^2/abcd\\ q^2/a,q^2/b,q^2/c,q^2/d,q^2/abc,q^2/abd,q^2/acd,q^2/bcd\end{matrix}\middle|q\right]_\infty\\ &\quad+\mathscr{R}^\star(a,b,c,d)\\ &\quad+\mathscr{R}^\star(b,a/q,c,d)\left[\begin{matrix}q^2/ab,q^2/ac,q^2/ad,q^2/abcd\\ q^2/a,q^2/abc,q^2/abd,q^2/acd\end{matrix}\middle|q\right]_1.\end{aligned}$$

4.2. Applying (17a)–(17b) again to $\mathrm{V}(c,b/q,a/q,d)=\mathrm{V}(a/q,b/q,c,d)$ displayed in (27a) and then simplifying the equation corresponding to (27a)–(27b), we get the relation:

$$\mathrm{V}(a,b,c,d)$$

$$= \mathrm{V}(a/q,b/q,c/q,d)\frac{(q^2/abcd;q)_3}{(q^2/abc;q)_3} \tag{29a}$$

$$\times \left[\begin{matrix} q^2/ad,q^2/bd,q^2/cd \\ q^2/a,q^2/b,q^2/c \end{matrix}\middle| q\right]_1 \left[\begin{matrix} q^2/ab,q^2/ac,q^2/bc \\ q^2/abd,q^2/acd,q^2/bcd \end{matrix}\middle| q\right]_2 \tag{29b}$$

$$+ \mu(a;b,c,d) + \mu(b;a/q,c,d)\left[\begin{matrix} q^2/ab,q^2/ac,q^2/ad,q^2/abcd \\ q^2/a,q^2/abc,q^2/abd,q^2/acd \end{matrix}\middle| q\right]_1 \tag{29c}$$

$$+ \mu(c;a/q,b/q,d)\left[\begin{matrix} q^2/ac,q^2/ad,q^2/bc,q^2/bd \\ q^2/a,q^2/b,q^2/acd,q^2/bcd \end{matrix}\middle| q\right]_1 \left[\begin{matrix} q^2/ab,q^2/abcd \\ q^2/abc,q^2/abd \end{matrix}\middle| q\right]_2. \tag{29d}$$

Iterating further this relation m-times, we derive the recurrence relation.

Theorem 20 (Recurrence relation). *Define $\mathscr{S}_m$ and $\mathscr{S}^\star_m$-functions by*

$$\mathscr{S}^\star_m(a,b,c,d) = \frac{\mu(a;b,c,d)}{1-q^4/a^2bcd} \times \mathscr{S}_m(a,b,c,d), \tag{30a}$$

$$\mathscr{S}_m(a,b,c,d) = \sum_{k=0}^{m-1}\{1-q^{4+4k}/a^2bcd\}\left[\begin{matrix} q^2/ab,q^2/ac,q^2/bc \\ q^3/abd,q^3/acd,q^2/bcd \end{matrix}\middle| q\right]_{2k} \tag{30b}$$

$$\times \frac{[q/a,q/b,q/c,q^2/ad,q^2/bd,q^2/cd;q]_k}{[q^3/abc,q^4/abcd;q]_{3k}}\left\{\frac{q^{6+3k}}{a^2b^2c^2d^3}\right\}^k. \tag{30c}$$

Then there holds the following relation:

$$\mathrm{V}(a,b,c,d)$$

$$= \mathrm{V}(a/q^m,b/q^m,c/q^m,d)\left[\begin{matrix} q^2/ad,q^2/bd,q^2/cd \\ q^2/a,q^2/b,q^2/c \end{matrix}\middle| q\right]_m$$

$$\times \frac{(q^2/abcd;q)_{3m}}{(q^2/abc;q)_{3m}}\left[\begin{matrix} q^2/ab,q^2/ac,q^2/bc \\ q^2/abd,q^2/acd,q^2/bcd \end{matrix}\middle| q\right]_{2m}$$

$$+ \mathscr{S}^\star_m(a,b,c,d) + \mathscr{S}^\star_m(b,a/q,c,d)\left[\begin{matrix} q^2/ab,q^2/ac,q^2/ad,q^2/abcd \\ q^2/a,q^2/abc,q^2/abd,q^2/acd \end{matrix}\middle| q\right]_1$$

$$+ \mathscr{S}^\star_m(c,a/q,b/q,d)\left[\begin{matrix} q^2/ac,q^2/ad,q^2/bc,q^2/bd \\ q^2/a,q^2/b,q^2/acd,q^2/bcd \end{matrix}\middle| q\right]_1 \left[\begin{matrix} q^2/ab,q^2/abcd \\ q^2/abc,q^2/abd \end{matrix}\middle| q\right]_2.$$

In view of Corollary 6, we have the following limit:

$$\lim_{m\to\infty} \mathrm{V}(a/q^m,b/q^m,c/q^m,d) = {}_1\psi_3\left[\begin{matrix} -, & -, & d \\ 0, & 0, & q^2/d \end{matrix}\middle| q;\frac{q^2}{d}\right] = \frac{(q;q)_\infty}{(q^2/d;q)_\infty}$$

which leads us to the following cubic transformation formula.

Proposition 21. *Let* $\mathscr{S}^\star = \lim_{m\to\infty} \mathscr{S}^\star_m$, *where the* $\mathscr{S}^\star_m$*-function is defined in Theorem* 20. *Then there holds the nonterminating series transformation:*

$$\begin{aligned}
&\mathrm{V}(a,b,c,d)\\
&\quad = \left[\begin{matrix} q, q^2/ab, q^2/ac, q^2/ad, q^2/bc, q^2/bd, q^2/cd, q^2/abcd \\ q^2/a, q^2/b, q^2/c, q^2/d, q^2/abc, q^2/abd, q^2/acd, q^2/bcd \end{matrix}\middle| q\right]_\infty \\
&\qquad + \mathscr{S}^\star(a,b,c,d) + \mathscr{S}^\star(b,a/q,c,d)\left[\begin{matrix} q^2/ab, q^2/ac, q^2/ad, q^2/abcd \\ q^2/a, q^2/abc, q^2/abd, q^2/acd \end{matrix}\middle| q\right]_1 \\
&\qquad + \mathscr{S}^\star(c,a/q,b/q,d)\left[\begin{matrix} q^2/ac, q^2/ad, q^2/bc, q^2/bd \\ q^2/a, q^2/b, q^2/acd, q^2/bcd \end{matrix}\middle| q\right]_1 \left[\begin{matrix} q^2/ab, q^2/abcd \\ q^2/abc, q^2/abd \end{matrix}\middle| q\right]_2.
\end{aligned}$$

4.3. Applying (17a) and (17b) again to $\mathrm{V}(d,a/q,b/q,c/q) = \mathrm{V}(a/q,b/q,c/q,d)$ displayed in (29a) and then simplifying the equation corresponding to (29a)–(29d), we get the relation:

$$\mathrm{V}(a,b,c,d)$$

$$= \mathrm{V}(a/q,b/q,c/q,d/q) \tag{31a}$$

$$\times \frac{[q^2/ab, q^2/ac, q^2/ad, q^2/bc, q^2/bd, q^2/cd; q]_2 (q^2/abcd; q)_4}{[q^2/a, q^2/b, q^2/c, q^2/d; q]_1 [q^2/abc, q^2/abd, q^2/acd, q^2/bcd; q]_3} \tag{31b}$$

$$+ \mu(a;b,c,d) + \mu(b;a/q,c,d)\left[\begin{matrix} q^2/ab, q^2/ac, q^2/ad, q^2/abcd \\ q^2/a, q^2/abc, q^2/abd, q^2/acd \end{matrix}\middle| q\right]_1 \tag{31c}$$

$$+ \mu(c;a/q,b/q,d)\left[\begin{matrix} q^2/ac, q^2/ad, q^2/bc, q^2/bd \\ q^2/a, q^2/b, q^2/acd, q^2/bcd \end{matrix}\middle| q\right]_1 \left[\begin{matrix} q^2/ab, q^2/abcd \\ q^2/abc, q^2/abd \end{matrix}\middle| q\right]_2 \tag{31d}$$

$$\begin{aligned}
&+ \mu\left(d;\frac{a}{q},\frac{b}{q},\frac{c}{q}\right)\left[\begin{matrix} q^2/ad, q^2/bd, q^2/cd \\ q^2/a, q^2/b, q^2/c \end{matrix}\middle| q\right]_1 \\
&\times \left[\begin{matrix} q^2/ab, q^2/ac, q^2/bc \\ q^2/abd, q^2/acd, q^2/bcd \end{matrix}\middle| q\right]_2 \frac{(q^2/abcd;q)_3}{(q^2/abc;q)_3}.
\end{aligned} \tag{31e}$$

Iterating further this relation m-times, we derive the recurrence relation.

Theorem 22 (Recurrence relation). *Define* $\mathscr{T}_m$ *and* $\mathscr{T}^\star_m$*-functions by*

$$\mathscr{T}^\star_m(a,b,c,d) = \frac{\mu(a;b,c,d)}{1-q^4/a^2bcd} \times \mathscr{T}_m(a,b,c,d), \tag{32a}$$

$$\mathscr{T}_m(a,b,c,d) = \sum_{k=0}^{m-1}\{1-q^{4+5k}/a^2bcd\}\frac{[q/a,q/b,q/c,q/d;q]_k}{(q^4/abcd;q)_{4k}} \tag{32b}$$

$$\times \frac{[q^2/ab, q^2/ac, q^2/ad, q^2/bc, q^2/bd, q^2/cd; q]_{2k}}{[q^3/abc, q^3/abd, q^3/acd, q^2/bcd; q]_{3k}}\left\{\frac{q^{8+6k}}{a^3b^3c^3d^3}\right\}^k. \tag{32c}$$

Then there holds the following relation:

$$
\begin{aligned}
&\mathrm{V}(a,b,c,d)\\
&\quad= \mathrm{V}(a/q^m, b/q^m, c/q^m, d/q^m)\\
&\qquad\times \frac{[q^2/ab, q^2/ac, q^2/ad, q^2/bc, q^2/bd, q^2/cd; q]_{2m}(q^2/abcd;q)_{4m}}{[q^2/a, q^2/b, q^2/c, q^2/d; q]_m [q^2/abc, q^2/abd, q^2/acd, q^2/bcd; q]_{3m}}\\
&\qquad+ \mathscr{T}_m^\star(a,b,c,d) + \mathscr{T}_m^\star(b, a/q, c, d) \left[\begin{matrix} q^2/ab, q^2/ac, q^2/ad, q^2/abcd \\ q^2/a, q^2/abc, q^2/abd, q^2/acd \end{matrix} \middle| q\right]_1\\
&\qquad+ \mathscr{T}_m^\star(c, a/q, b/q, d) \left[\begin{matrix} q^2/ac, q^2/ad, q^2/bc, q^2/bd \\ q^2/a, q^2/b, q^2/acd, q^2/bcd \end{matrix} \middle| q\right]_1 \left[\begin{matrix} q^2/ab, q^2/abcd \\ q^2/abc, q^2/abd \end{matrix} \middle| q\right]_2\\
&\qquad+ \mathscr{T}_m^\star\left(d, \frac{a}{q}, \frac{b}{q}, \frac{c}{q}\right) \left[\begin{matrix} q^2/ad, q^2/bd, q^2/cd \\ q^2/a, q^2/b, q^2/c \end{matrix} \middle| q\right]_1\\
&\qquad\times \left[\begin{matrix} q^2/ab, q^2/ac, q^2/bc \\ q^2/abd, q^2/acd, q^2/bcd \end{matrix} \middle| q\right]_2 \frac{(q^2/abcd;q)_3}{(q^2/abc;q)_3}.
\end{aligned}
$$

By means of Jacobi's triple product identity (26), we have the following limit

$$
\lim_{m\to\infty} \mathrm{V}(a/q^m, b/q^m, c/q^m, d/q^m) = \sum_k (-1)^k q^{3\binom{k}{2}+2k} = [q^3, q, q^2; q^3]_\infty = (q;q)_\infty.
$$

This leads us to the following quartic transformation formula.

Proposition 23. *Let* $\mathscr{T}^\star = \lim_{m\to\infty} \mathscr{T}_m^\star$*, where the* $\mathscr{T}_m^\star$*-function is defined in Theorem 22. Then there holds the nonterminating series transformation:*

$$
\begin{aligned}
&\mathrm{V}(a,b,c,d)\\
&\quad= \left[\begin{matrix} q, q^2/ab, q^2/ac, q^2/ad, q^2/bc, q^2/bd, q^2/cd, q^2/abcd \\ q^2/a, q^2/b, q^2/c, q^2/d, q^2/abc, q^2/abd, q^2/acd, q^2/bcd \end{matrix} \middle| q\right]_\infty + \mathscr{T}^\star(a,b,c,d)\\
&\qquad+ \mathscr{T}^\star(b, a/q, c, d) \left[\begin{matrix} q^2/ab, q^2/ac, q^2/ad, q^2/abcd \\ q^2/a, q^2/abc, q^2/abd, q^2/acd \end{matrix} \middle| q\right]_1 + \mathscr{T}^\star(c, a/q, b/q, d)\\
&\qquad\times \left[\begin{matrix} q^2/ac, q^2/ad, q^2/bc, q^2/bd \\ q^2/a, q^2/b, q^2/acd, q^2/bcd \end{matrix} \middle| q\right]_1 \left[\begin{matrix} q^2/ab, q^2/abcd \\ q^2/abc, q^2/abd \end{matrix} \middle| q\right]_2 + \mathscr{T}^\star\left(d, \frac{a}{q}, \frac{b}{q}, \frac{c}{q}\right)\\
&\qquad\times \left[\begin{matrix} q^2/ad, q^2/bd, q^2/cd \\ q^2/a, q^2/b, q^2/c \end{matrix} \middle| q\right]_1 \left[\begin{matrix} q^2/ab, q^2/ac, q^2/bc \\ q^2/abd, q^2/acd, q^2/bcd \end{matrix} \middle| q\right]_2 \frac{(q^2/abcd;q)_3}{(q^2/abc;q)_3}.
\end{aligned}
$$

5. Relations on nonterminating partial well-poised series

Based on the relations established for $\mathrm{U}(a,b,c,d)$ and $\mathrm{V}(a,b,c,d)$, we can further derive six curious relations which transform the partial well-poised series into the series with the same character, but convergent faster. They resemble somehow the quadratic, cubic, quartic transformation formulae discovered by Gasper and Rahman [12], [16].

5.1. Comparing the expression for $\mathrm{U}(a,b,c,d)$ displayed in Theorem 1 with those in Propositions 13, 15, 17 and then equating the right members, we find the following three curious transformation formulae.

Theorem 24 (Partial well-poised series transformation).

$$
\begin{aligned}
&{}_6\psi_6^{+}\left[\begin{matrix} q^2/a\sqrt{bcd}, & -q^2/a\sqrt{bcd}, & q/a, & q/ab, & q/ac, & q/ad \\ q/a\sqrt{bcd}, & -q/a\sqrt{bcd}, & q^2/abcd, & q^2/abc, & q^2/abd, & q^2/acd \end{matrix}\middle| q; \frac{q}{bcd}\right] \\
&\quad\times\left\{1-\frac{q^2}{a^2bcd}\right\} \\
&\quad = R(a,b,c,d) + \frac{qR(b,a/q,c,d)}{bcd}\left[\begin{matrix} q/a, q/ab, q/ac, q/ad \\ q^2/abc, q^2/abd, q/bcd, q^2/abcd \end{matrix}\middle| q\right]_1.
\end{aligned}
$$

According to the definitions of λ and R-functions in Theorem 1 and Proposition 13, we may write the last theorem explicitly as follows:

$$
\begin{aligned}
&{}_6\psi_6^{+}\left[\begin{matrix} q^2/a\sqrt{bcd}, & -q^2/a\sqrt{bcd}, & q/a, & q/ab, & q/ac, & q/ad \\ q/a\sqrt{bcd}, & -q/a\sqrt{bcd}, & q^2/abcd, & q^2/abc, & q^2/abd, & q^2/acd \end{matrix}\middle| q; \frac{q}{bcd}\right] \\
&\quad\times\left\{1-\frac{q^2}{a^2bcd}\right\} \\
&= \sum_{k\geq 0}\{1-q^{2+3k}/a^2bcd\}\frac{(q/ab;q)_{2k}[q/a,q/b,q/ac,q/ad,q/bc,q/bd;q]_k}{[q^2/acd,q/bcd;q]_k[q^2/abc,q^2/abd,q^2/abcd;q]_{2k}} \\
&\quad\times\left\{\frac{q^{2+k}}{abc^2d^2}\right\}^k + \frac{q}{bcd}\left[\begin{matrix} q/a, q/ab, q/ac, q/ad \\ q^2/abc, q^2/abd, q/bcd, q^2/abcd \end{matrix}\middle| q\right]_1 \\
&\quad\times\sum_{k\geq 0}\{1-q^{3+3k}/ab^2cd\}\frac{(q^2/ab;q)_{2k}[q^2/a,q/b,q^2/ac,q^2/ad,q/bc,q/bd;q]_k}{[q^2/acd,q^2/bcd;q]_k[q^3/abc,q^3/abd,q^3/abcd;q]_{2k}} \\
&\quad\times\left\{\frac{q^{3+k}}{abc^2d^2}\right\}^k.
\end{aligned}
$$

When $acd = q$, the corresponding partial ${}_6\psi_6^+$-series may be evaluated by the q-Dougall sum (cf. [13], II-20):

$$ {}_6\phi_5\left[\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d \\ & \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d \end{matrix}\middle| q; \frac{qa}{bcd}\right] \tag{33a} $$

$$ = \left[\begin{matrix} qa, qa/bc, qa/bd, qa/cd \\ qa/b, qa/c, qa/d, qa/bcd \end{matrix}\middle| q\right]_\infty \quad \text{where } |qa/bcd| < 1. \tag{33b} $$

We therefore establish the following interesting identity.

Corollary 25 (Quadratic summation formula).

$$ \left[\begin{matrix} q/ab, q/bd, ad/b \\ a/b, qd/b, q^2/abd \end{matrix}\middle| q\right]_\infty $$
$$ = \sum_{k\geq 0}\left\{1 - \frac{q^{1+3k}}{ab}\right\} \frac{(q/ab;q)_{2k}}{[q, a/b;q]_k} \frac{[q/a, q/b, q/ad, q/bd, d, ad/b;q]_k}{[q/b, qd/b, q^2/abd;q]_{2k}} \left(q^k \frac{a}{b}\right)^k $$
$$ - \left[\begin{matrix} q/a, d, q/ab, q/ad \\ q/b, b/a, qd/b, q^2/abd \end{matrix}\middle| q\right]_1 \sum_{k\geq 0}\{1 - q^{2+3k}/b^2\} \left(q^{1+k}\frac{a}{b}\right)^k $$
$$ \times \frac{(q^2/ab;q)_{2k}}{[q, qa/b;q]_k} \frac{[q^2/a, q/b, q^2/ad, q/bd, qd, ad/b;q]_k}{[q^2/b, q^2d/b, q^3/abd;q]_{2k}}. $$

For each nonterminating series on the right hand side, there is no closed formula available. However, their combination results in a closed factorial fraction expression.

Theorem 26 (Partial well-poised series transformation).

$$ {}_6\psi_6^+\left[\begin{matrix} q^2/a\sqrt{bcd}, & -q^2/a\sqrt{bcd}, & q/a, & q/ab, & q/ac, & q/ad \\ q/a\sqrt{bcd}, & -q/a\sqrt{bcd}, & q^2/abcd, & q^2/abc, & q^2/abd, & q^2/acd \end{matrix}\middle| q; \frac{q}{bcd}\right] $$
$$ \times\left\{1 - \frac{q^2}{a^2bcd}\right\} $$
$$ = S(a,b,c,d) + \frac{qS(b, a/q, c, d)}{bcd}\left[\begin{matrix} q/a, q/ab, q/ac, q/ad \\ q^2/abc, q^2/abd, q/bcd, q^2/abcd \end{matrix}\middle| q\right]_1 $$
$$ + \frac{q^3S(c, a/q, b/q, d)}{abc^2d^2} \frac{[q/a, q/b, q/ac, q/bc, q/ad, q/bd;q]_1 (q/ab;q)_2}{[q^2/abd, q^2/acd;q]_1 [q^2/abc, q/bcd, q^2/abcd;q]_2}. $$

Theorem 27 (Partial well-poised series transformation).

$$
{}_6\psi_6^+\left[\begin{matrix} q^2/a\sqrt{bcd}, & -q^2/a\sqrt{bcd}, & q/a, & q/ab, & q/ac, & q/ad \\ q/a\sqrt{bcd}, & -q/a\sqrt{bcd}, & q^2/abcd, & q^2/abc, & q^2/abd, & q^2/acd \end{matrix}\middle| q; \frac{q}{bcd}\right]
\times\left\{1-\frac{q^2}{a^2bcd}\right\}
$$

$$
= T(a,b,c,d) + \frac{qT(b,a/q,c,d)}{bcd}\left[\begin{matrix} q/a, q/ab, q/ac, q/ad \\ q^2/abc, q^2/abd, q/bcd, q^2/abcd \end{matrix}\middle| q\right]_1
$$

$$
+\frac{q^3T(c,a/q,b/q,d)}{abc^2d^2}\frac{[q/a,q/b,q/ac,q/bc,q/ad,q/bd;q]_1(q/ab;q)_2}{[q^2/abd,q^2/acd;q]_1[q^2/abc,q/bcd,q^2/abcd;q]_2}
$$

$$
+\frac{q^6T(d,a/q,b/q,c/q)}{a^2b^2c^2d^3}\frac{[q/a,q/b,q/c,q/ad,q/bd,q/cd;q]_1}{[q/bcd,q^2/abcd;q]_3}
\times\left[\begin{matrix} q/ab, q/ac, q/bc \\ q^2/abc, q^2/abd, q^2/acd \end{matrix}\middle| q\right]_2.
$$

5.2. Similarly comparing the expression for $\mathrm{V}(a,b,c,d)$ displayed in Theorem 5 with those in Propositions 19, 21, 23 and then equating the right members, we find other transformation formulae.

Theorem 28 (Partial well-poised series transformation).

$$
{}_6\psi_6^+\left[\begin{matrix} q^3/a\sqrt{bcd}, & -q^3/a\sqrt{bcd}, & q/a, & q^2/ab, & q^2/ac, & q^2/ad \\ q^2/a\sqrt{bcd}, & -q^2/a\sqrt{bcd}, & q^4/abcd, & q^3/abc, & q^3/abd, & q^3/acd \end{matrix}\middle| q; \frac{q^2}{bcd}\right]
\times\left\{1-\frac{q^4}{a^2bcd}\right\}
$$

$$
= \mathscr{R}(a,b,c,d) + \frac{q^2\mathscr{R}(b,a/q,c,d)}{bcd}\left[\begin{matrix} q/a, q^2/ab, q^2/ac, q^2/ad \\ q^3/abc, q^3/abd, q^2/bcd, q^4/abcd \end{matrix}\middle| q\right]_1.
$$

Theorem 29 (Partial well-poised series transformation).

$$
{}_6\psi_6^+\left[\begin{matrix} q^3/a\sqrt{bcd}, & -q^3/a\sqrt{bcd}, & q/a, & q^2/ab, & q^2/ac, & q^2/ad \\ q^2/a\sqrt{bcd}, & -q^2/a\sqrt{bcd}, & q^4/abcd, & q^3/abc, & q^3/abd, & q^3/acd \end{matrix}\middle| q; \frac{q^2}{bcd}\right]
\times\left\{1-\frac{q^4}{a^2bcd}\right\}
$$

$$
= \mathscr{S}(a,b,c,d) + \frac{q^2\mathscr{S}(b,a/q,c,d)}{bcd}\left[\begin{matrix} q/a, q^2/ab, q^2/ac, q^2/ad \\ q^3/abc, q^3/abd, q^2/bcd, q^4/abcd \end{matrix}\middle| q\right]_1
$$

$$
+\frac{q^5\mathscr{S}(c,a/q,b/q,d)}{abc^2d^2}\frac{(q^2/ab;q)_2}{[q^3/abd,q^3/acd;q]_1}
\times\frac{[q/a,q/b,q^2/ac,q^2/ad,q^2/bc,q^2/bd;q]_1}{[q^3/abc,q^2/bcd,q^4/abcd;q]_2}.
$$

Theorem 30 (Partial well-poised series transformation).

$$
\begin{aligned}
&{}_6\psi_6^{+}\left[\begin{matrix} q^3/a\sqrt{bcd}, & -q^3/a\sqrt{bcd}, & q/a, & q^2/ab, & q^2/ac, & q^2/ad \\ q^2/a\sqrt{bcd}, & -q^2/a\sqrt{bcd}, & q^4/abcd, & q^3/abc, & q^3/abd, & q^3/acd \end{matrix}\,\middle|\, q; \frac{q^2}{bcd}\right] \\
&\quad \times \left\{1 - \frac{q^4}{a^2bcd}\right\} \\
&= \mathscr{T}(a,b,c,d) + \frac{q^2\mathscr{T}(b,a/q,c,d)}{bcd}\left[\begin{matrix} q/a, q^2/ab, q^2/ac, q^2/ad \\ q^3/abc, q^3/abd, q^2/bcd, q^4/abcd \end{matrix}\,\middle|\, q\right]_1 \\
&\quad + \frac{q^5\mathscr{T}(c,a/q,b/q,d)}{abc^2d^2}\,\frac{(q^2/ab;q)_2}{[q^3/abd,q^3/acd;q]_1} \\
&\quad \times \frac{[q/a,q/b,q^2/ac,q^2/ad,q^2/bc,q^2/bd;q]_1}{[q^3/abc,q^2/bcd,q^4/abcd;q]_2} \\
&\quad + \frac{q^9\mathscr{T}(d,a/q,b/q,c/q)}{a^2b^2c^2d^3}\,\frac{[q/a,q/b,q/c,q^2/ad,q^2/bd,q^2/cd;q]_1}{[q^2/bcd,q^4/abcd;q]_3} \\
&\quad \times \left[\begin{matrix} q^2/ab, q^2/ac, q^2/bc \\ q^3/abc, q^3/abd, q^3/acd \end{matrix}\,\middle|\, q\right]_2 .
\end{aligned}
$$

As illustrated in Corollary 25, one can write down other reciprocal formulae from Theorems 26–30. They will not be reproduced here for the limit of space.

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