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Existence of solutions to a one-dimensional model of bounded piezoelectric material

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Abstract. We study the dynamic of piezoelectric bodies with switchable domains, described by a coupling of an hyperbolic equation and a parabolic one. We consider the one dimensional case and prove local existence of solutions.

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1. Introduction

We consider a piezoelectric material in one space dimension occupying the region $\omega = (0,1)$. Following Daví [2], the switching domain phenomena is described by the effective number of aligned dipoles $f(t,x)$ which satisfies the equation

$$\partial_t f - \partial_x^2 f = -\partial_x u. \tag{1}$$

$u(t,x)$ is the electric displacement field related to f by the equation

$$\partial_t^2 u - \partial_x\big((1+f)\partial_x u\big) = 0 \tag{2}$$

where $t \geq 0$ is the time variable and $x \in \omega$ is the space variable. The indices denote partial derivatives and in order to simplify the presentation, all physical constants had been taken equal to one. The coupling (1)–(2) is completed by the boundary conditions

$$\begin{aligned} &f(t,0) = f(t,1) = \alpha, \qquad \alpha > 0,\\ &\partial_x u(t,0) = \partial_x u(t,1) = 0 \end{aligned} \tag{3}$$

for $t \geq 0$, and the initial conditions

$$f(0,x) = f_0(x),$$
$$u(0,x) = u_0(x), \qquad \partial_t u(0,x) = u_1(x) \tag{4}$$

for $x \in \omega$. In the case of free space, this problem was studied in [4]. The main difficulty in dealing with this system is associated with the structure of the velocity of the wave equation which is time dependent and depends of the solution to the heat equation. As it will became clear in Section 2, this fact prevents a global existence result.

In order to state our main result, let us introduce some notations. For $T > 0$, we set $\omega_T = (0,T) \times \omega$. We denote by $|\cdot|$, $|\cdot|_T$ and $\|\cdot\|_T$ the norms in the Lebesgue spaces $L^2(\omega)$, $L^2(\omega_T)$ and $L^\infty\big(0,T;L^2(\omega)\big)$ respectively. $\mathscr{F}$ and $\mathscr{U}$ will denote the spaces

$$\mathscr{F} = C\big([0,T];H^2(\omega)\big) \cap H^1\big(0,T;H^1(\omega)\big),$$
$$\mathscr{U} = H^1\big(0,T;H^1(\omega)\big) \cap C^1\big([0,T];L^2(\omega)\big). \tag{5}$$

For a function g depending only in the space variable x, the prime in g' will denote the usual derivative and finally, the problem given by (1)–(2) and (3)–(4) will be shortly referred to as $(\mathscr{P})$. Our main result is the following:

Theorem 1.1. *Suppose that the following assumptions hold:*

$$u_0 \in H^1(\omega), \qquad u_1 \in L^2(\omega), \qquad f_0 \in H^2(\omega),$$
$$f_0(0) = f_0(1) = \alpha, \qquad f_0(x) \geq \alpha \qquad \text{in } \omega \tag{6}$$

with

$$3\sqrt{2}\alpha > \max(E_0, 1), \qquad E_0 \equiv |f_0'' - u_0'|^2 + |u_1|^2 + \int_\omega \big(1 + f_0(x)\big)|u_0'(x)|^2\,dx. \tag{7}$$

Then problem $(\mathscr{P})$ admits a weak solution (f,u) defined on a time interval $(0,T)$ depending on the data, such that $f \geq 0$ in ω_T and

$$f \in H^1\big(0,T;H^1(\omega)\big) \cap C\big([0,T];H^2(\omega)\big),$$
$$u \in H^1(\omega_T) \cap C\big([0,T];H^1(\omega)\big) \cap C^1\big([0,T];L^2(\omega)\big).$$

Let us explain briefly why these assumptions are required. The solution f of the heat equation needs to be nonnegative. As the sign of the source term $\partial_x u$ is variable, if the initial data f_0 is positive and big enough, the positivity of f is conserved for a while. For a non trivial example of data satisfying the above assumptions, one can take $f_0(x) = \alpha + \frac{1}{4} - \big(x - \frac{1}{2}\big)^2$, $u_0(x) = -2x$ and $u_1(x) = a$, $(a \in \mathbb{R})$.

The idea of the proof is to obtain a solution to our problem as limit of approximated solutions, as was done in [4] for the case $\omega = \mathbb{R}$. However, in the case ω is a bounded domain this is more difficult and the arguments used [4] do not quite work here. Indeed the existence proof given in [4] is essentially based on an estimate in L^∞, in time and space, for the time derivative $\partial_t f$, together with its negativity ($\partial_t f \leq 0$ almost everywhere) obtained by using the estimate in $L^\infty(0, T; L^2)$ of $\partial_t u$ and the regularity of the heat kernel in the free space. This leads to an energy estimate of the solution u of the wave equation alone, which permits to construct a sequence of approximated solutions (using linearization and decoupling procedure) satisfying uniform estimates, allowing to get at the limit a solution to the coupled system. Unfortunately, such estimate on $\partial_t f$ can not be obtained in the case of bounded domain (only an energy estimate on the pair (f, u) can be obtained making the dissociation of the two equations of the problem inoperative). This is the reason why we use here a fixed point theorem to solve the problem stated in a bounded domain.

This paper is organized as follows. In Section 2 we give some formal estimates satisfied by solutions of $(\mathscr{P})$. Their local nature also constitute an obstacle to establish global existence. In Section 3, we define the approximated problems $(\mathscr{P}_\nu^\varepsilon)$ using a regularization of the wave velocity and introducing an artificial viscosity via small parameters ν and ε. We then establish Theorem 3.1 which gives existence and uniqueness of a global solution $(f_\nu^\varepsilon, u_\nu^\varepsilon)$, and which is proved using the Leray–Schauder fixed point theorem. An intermediary study of the solutions of the heat and the wave equations, leads us to some uniform estimates on $(f_\nu^\varepsilon, u_\nu^\varepsilon)$. We apply them in Section 4, to pass to the limit when $\nu \to 0$ and $\varepsilon \to 0$, which leads to a solution of our problem.

2. Formal estimates

We consider a regular solution (f, u) of problem $(\mathscr{P})$ defined on $(0, T)$. The time derivative $\partial_t f$ satisfies

$$\left.\begin{aligned} &\partial_t(\partial_t f) - \partial_x^2(\partial_t f) = -\partial_{tx}^2 u \quad \text{in } \omega_T, \\ &\partial_t f(t, 0) = \partial_t f(t, 1) = 0 \quad \text{in } (0, T), \\ &\partial_t f(0, x) = f_0''(x) - u_0'(x) \quad \text{in } \omega. \end{aligned}\right\} \tag{8}$$

Multiplying this equation by $\partial_t f$ and integrating over ω, we get

$$\frac{1}{2}\frac{d}{dt}|\partial_t f|^2 + |\partial_{tx}^2 f|^2 = -\int_\omega \partial_{tx}^2 u \partial_t f \, dx = \int_\omega \partial_t u \partial_{tx}^2 f \, dx,$$

so

$$|\partial_t f(t)|^2 + \int_0^t |\partial_{tx}^2 f(s)|^2\, ds \le |f_0'' - u_0'|^2 + \int_0^t |\partial_t u(s)|^2\, ds. \tag{9}$$

Now, the solution u of the wave equation (2) satisfies

$$\begin{aligned} &|\partial_t u(t)|^2 + \int_\omega \big(1 + f(t,x)\big)|\partial_x u(t,x)|^2\, dx \\ &\quad = |u_1|^2 + \int_\omega \big(1 + f_0(x)\big)|u_0'(x)|^2\, dx + \int_0^t\int_\omega \partial_t f(s,x)|\partial_x u(s,x)|^2\, dx\, ds. \end{aligned} \tag{10}$$

Adding (9) and (10) leads to

$$\begin{aligned} &|\partial_t f(t)|^2 + \int_0^t |\partial_{tx}^2 f(s)|^2\, ds + |\partial_t u(t)|^2 + \int_\omega \big(1 + f(t,x)\big)|\partial_x u(t,x)|^2\, dx \\ &\quad \le E_0 + \int_0^t |\partial_t u(s)|^2\, ds + \int_0^t\int_\omega |\partial_t f(s,x)|\,|\partial_x u(s,x)|^2\, dx\, ds. \end{aligned} \tag{11}$$

Since $\partial_t f(t,0) = 0$, we can write $|\partial_t f(t,x)|^2 = \int_0^x 2\partial_t f(t,y)\partial_{tx}^2 f(t,y)\, dy$ for all $(t,x) \in \omega_T$ so

$$|\partial_t f(t)|_{L^\infty(\omega)}^2 \le 2|\partial_t f(t)|\,|\partial_{tx}^2 f(t)|. \tag{12}$$

Using Young's inequality, the last term of (11) can be bounded by

$$\int_0^t |\partial_{tx}^2 f(s)|^2\, ds + \frac{3}{4}\int_0^t |\partial_t f(s)|^{2/3}|\partial_x u(s)|^{8/3}\, ds,$$

therefore, (11) becomes

$$\begin{aligned} &|\partial_t f(t)|^2 + \int_\omega \big(1 + f(t,x)\big)|\partial_x u(t,x)|^2\, dx + |\partial_t u(t)|^2 \\ &\quad \le E_0 + \int_0^t |\partial_t u(s)|^2\, ds + \frac{3}{4}\int_0^t |\partial_t f(s)|^{2/3}|\partial_x u(s)|^{8/3}\, ds. \end{aligned} \tag{13}$$

Assume that $f(t,x) \ge 0$ in ω_T and let

$$\varphi(t) = \max\big(|\partial_t u(t)|^2, |\partial_x u(t)|^2, |\partial_t f(t)|^2, 1\big). \tag{14}$$

The last inequality leads to $\varphi(t) \le E_1 + \int_0^t \varphi(s)\, ds + \frac{3}{4}\int_0^t \varphi^{5/3}(s)\, ds$, and so

$$\varphi(t) \le E_1 + \frac{7}{4}\int_0^t \varphi^{5/3}(s)\, ds \tag{15}$$

where $E_1 = \max(E_0, 1)$. Then consider the o.d.e

$$\lambda'(t) = \tfrac{7}{4}\lambda^{5/3}(t), \qquad \lambda(0) = E_1 \tag{16}$$

which admits a local solution given by

$$\lambda(t) = \left(E_1{}^{-2/3} - \tfrac{7}{6}t\right)^{-3/2}, \qquad t \in \left[0, \tfrac{6}{7}E_1^{-2/3}\right[. \tag{17}$$

It follows by comparison between (15) and (16) that

$$\varphi(t) \leq \lambda(t) \qquad \text{for all } t \in \left[0, \min\left(T, \tfrac{6}{7}E_1^{-2/3}\right)\right[,$$

that is, for the values of t

$$|\partial_t u(t)|^2, |\partial_x u(t)|^2, |\partial_t f(t)|^2 \leq \lambda(t). \tag{18}$$

As we see it, the key fact is to obtain an L^∞ estimate of the time derivative $\partial_t f$ of the wave velocity but this can not be obtained directly and the best we get is (12). This leads to an exponent greater than unity in (15) and provides only a finite time existence.

Let us now make precise the approximating procedure used to prove Theorem 1.1

3. The approximated problem

Let $T > 0$ be fixed and let $\varepsilon > 0$, $\nu > 0$ be small parameters. We introduce the following regularization $(\mathscr{P}_\nu^\varepsilon)$ of the problem $(\mathscr{P})$:

$$\left.\begin{array}{l}
\partial_t f - \partial_x^2 f = -\partial_x u \quad \text{in } \omega_T,\\
f(t,0) = f(t,1) = \alpha \quad \text{in } (0,T),\\
f(0,x) = f_0(x) \quad \text{in } \omega,\\
\partial_t^2 u - \partial_x[(1 + \rho_\nu \star \tilde{f}^+)\partial_x u] - \varepsilon\partial_x\left[\left(1 + (\rho_\nu \star \tilde{f}^+)^2\right)\partial_{tx}^2 u\right] = 0 \quad \text{in } \omega_T,\\
\partial_x u(t,0) = \partial_x u(t,1) = 0 \quad \text{in } (0,T),\\
u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x) \quad \text{in } \omega,
\end{array}\right\} \tag{19}$$

where $\star$ is the convolution product, $f^+ = \max(f, 0)$ is the positive part of f and $\tilde{f}^+$ is its extension by 0 outside ω_T. Moreover, $\rho_\nu = \rho_\nu(t,x)$ is a regularizing sequence supported in the ball $B(0,\nu)$ of radius ν centered at the origin with

$$\rho_\nu \in \mathscr{D}(\mathbb{R}^2), \qquad \int_{\mathbb{R}^2} \rho_\nu(t,x)\,dx\,dt = 1.$$

We have the following result.

Theorem 3.1. *Under the assumptions of Theorem* 1.1, *problem* $(\mathscr{P}_\nu^\varepsilon)$ *admits a unique weak solution* $(f_\nu^\varepsilon, u_\nu^\varepsilon)$ *in the space* $(\mathscr{F}, \mathscr{U})$ *defined by* (5).

The proof is based on a fixed point argument. Let $f \in L^2(\omega_T)$ be fixed, consider the solution u of the regularized wave equation

$$\left.\begin{array}{l} \partial_t^2 u - \partial_x[(1 + \rho_\nu \star \tilde{f}^+)\partial_x u] - \varepsilon\partial_x\big[\big(1 + (\rho_\nu \star \tilde{f}^+)^2\big)\partial_{tx}^2 u\big] = 0 \quad \text{in } \omega_T, \\ \partial_x u(t,0) = \partial_x u(t,1) = 0 \quad \text{in } (0,T), \\ u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x) \quad \text{in } \omega, \end{array}\right\} \tag{20}$$

and define the operator $\mathscr{S}$ on $L^2(\omega_T)$ by

$$u = \mathscr{S}(f). \tag{21}$$

Then we define the operator $\mathscr{T}$ on $\mathscr{U}$ by setting

$$\mathscr{T}(u) = g \tag{22}$$

where g is the solution to the heat equation

$$\left.\begin{array}{l} \partial_t g - \partial_x^2 g = -\partial_x u \quad \text{in } \omega_T, \\ g(t,0) = g(t,1) = \alpha \quad \text{in } (0,T), \\ g(0,x) = f_0(x) \quad \text{in } \omega. \end{array}\right\} \tag{23}$$

We want to show using Leray–Schauder Theorem that the operator

$$\mathscr{K} = i \circ \mathscr{T} \circ \mathscr{S} \tag{24}$$

possesses a fixed point in $L^2(\omega_T)$, $i : H^1(\omega_T) \to L^2(\omega_T)$ being the canonical injection. First we have to verify that $\mathscr{K}$ is well defined, so we begin in the two following subsections by making a full study of the regularized wave equation (20) and the heat equation (23).

3.1. The regularized wave equation. Let f be a nonnegative function defined on $\mathbb{R}^2$, $\nu > 0$ and $\varepsilon > 0$ be fixed. We consider the hyperbolic problem

$$\left.\begin{array}{l} \partial_t^2 u - \partial_x[(1 + \rho_\nu \star f)\partial_x u] - \varepsilon\partial_x\big[\big(1 + (\rho_\nu \star f)^2\big)\partial_{tx}^2 u\big] = 0 \quad \text{in } \omega_T, \\ \partial_x u(t,0) = \partial_x u(t,1) = 0 \quad \text{in } (0,T), \\ u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x) \quad \text{in } \omega, \end{array}\right\} \tag{25}$$

and prove the following result.

Proposition 3.2. *Let u_0, u_1 as in Theorem* 1.1 *and let $f \in L^\infty(\mathbb{R}^2)$ be such that $f \geq 0$ a.e. in $\mathbb{R}^2$. Then for every $T > 0$, $v > 0$, $\varepsilon > 0$, problem* (25) *has a unique solution $u \in \mathscr{U}$, $\partial_t^2 u \in L^2(0, T; H^{-1}(\omega))$.*

Proof. To prove the existence of a solution, we apply the variational method using the Theorem of Lions (see [6], [7]).

We set $\partial_t u = v$, $(U, V) = e^{-kt}(u, v)$ where $k > 0$ is a constant, so (25) may be written as

$$\left.\begin{array}{l} \partial_t U + kU - V = 0, \\ \partial_t V + kV - \partial_x[(1 + \rho_v \star f)\partial_x U] - \varepsilon\partial_x\big[\big(1 + (\rho_v \star f)^2\big)\partial_x V\big] = 0, \\ \partial_x U(t,0) = \partial_x U(t,1) = \partial_x V(t,0) = \partial_x V(t,1) = 0, \\ U(0,x) = u_0(x),\ V(0,x) = u_1(x). \end{array}\right\} \tag{26}$$

We introduce the Hilbert space $\mathscr{H}$ and its subspace $\mathscr{V}$:

$$\mathscr{H} = L^2\big(0, T; H^1(\omega)\big) \times L^2\big(0, T; H^1(\omega)\big), \qquad \mathscr{V} = D([0, T[\times \bar{\omega}) \times D([0, T[\times \bar{\omega}),$$

equipped with the norms

$$|(w_1, w_2)|^2_{\mathscr{H}} = \int_0^T \big(\|w_1(t)\|^2_{H^1(\omega)} + \|w_2(t)\|^2_{H^1(\omega)}\big)\, dt,$$

$$\|(w_1, w_2)\|^2_{\mathscr{V}} = |(w_1, w_2)|^2_{\mathscr{H}} + \frac{1}{2}\big(\|w_1(0)\|^2_{H^1(\omega)} + |w_2(0)|^2\big),$$

so that the injection $\mathscr{V} \subset \mathscr{H}$ is continuous. Thus we define a bilinear form B on $\mathscr{H} \times \mathscr{V}$ and a linear form L on $\mathscr{V}$ by

$$\begin{aligned} B\big((U, V), (\varphi, \psi)\big) = \int_{\omega_T} & \big(-\partial_x U \partial^2_{tx}\varphi + \partial_x(kU - V)\partial_x\varphi - kU\partial_t\varphi \\ & + (k^2 U - kV)\varphi\big)\, dx\, dt + \int_{\omega_T} \big(-V\partial_t\psi + kV\psi \\ & + (1 + \rho_v \star f)\partial_x U \partial_x\psi + \varepsilon[1 + (\rho_v \star f)^2]\partial_x V \partial_x\psi\big)\, dx\, dt, \end{aligned}$$

$$L(\varphi, \psi) = \int_\omega \big(u_0'(x)\partial_x\varphi(0, x) + ku_0(x)\varphi(0, x) + u_1(x)\psi(0, x)\big)\, dx.$$

Clearly $B\big(\cdot, (\varphi, \psi)\big)$ is continuous on $\mathscr{H}$, for any fixed $(\varphi, \psi) \in \mathscr{V}$ and

$$B\big((\varphi,\psi),(\varphi,\psi)\big) = \frac{1}{2}\big(|\partial_x\varphi(0)|^2 + k|\varphi(0)|^2 + |\psi(0)|^2\big) + k^2|\varphi|_T^2 + k|\psi|_T^2 + k|\partial_x\varphi|_T^2 + \varepsilon\int_{\omega_T}\big(1+(\rho_\nu\star f)^2\big)|\partial_x\psi|^2 + \int_{\omega_T}\big(-k\varphi\psi + (\rho_\nu\star f)\partial_x\varphi\partial_x\psi\big)\,dx\,dt.$$

Using Young's inequality, we get

$$\Big|\int_{\omega_T}\big(-k\varphi\psi + (\rho_\nu\star f)\partial_x\varphi\partial_x\psi\big)\,dx\,dt\Big| \le \frac{k}{2}(|\varphi|_T^2 + |\psi|_T^2) + \frac{1}{2\varepsilon}|\partial_x\varphi|_T^2 + \frac{\varepsilon}{2}\int_{\omega_T}(\rho_\nu\star f)^2|\partial_x\psi|^2\,dx\,dt$$

so

$$B\big((\varphi,\psi),(\varphi,\psi)\big) \ge \frac{1}{2}\big(|\partial_x\varphi(0)|^2 + k|\varphi(0)|^2 + |\psi(0)|^2\big) + \Big(k^2 - \frac{k}{2}\Big)|\varphi|_T^2 + k|\psi|_T^2 + \Big(k - \frac{1}{2\varepsilon}\Big)|\partial_x\varphi|_T^2 + \frac{\varepsilon}{2}\int_{\omega_T}\big(1+(\rho_\nu\star f)^2\big)|\partial_x\psi|^2\,dx\,dt. \tag{27}$$

Choosing $k > \max\big(\frac{1}{2},\frac{1}{2\varepsilon}\big)$, we get the coerciveness inequality

$$B\big((\varphi,\psi),(\varphi,\psi)\big) \ge \beta\|(\varphi,\psi)\|_{\mathscr{V}}^2 \qquad \text{for all } (\varphi,\psi)\in\mathscr{V} \tag{28}$$

with $\beta = \min\big(1, k^2-\frac{k}{2}, k-\frac{1}{2\varepsilon}, \frac{\varepsilon}{2}\big) > 0$. Then, since L is continuous on $\mathscr{V}$, applying the Theorem of Lions, we conclude that there exists a solution (U,V) in $\mathscr{H}$ to the variational equation

$$B\big((U,V),(\varphi,\psi)\big) = L(\varphi,\psi) \qquad \text{for all } (\varphi,\psi)\in\mathscr{V}. \tag{29}$$

Therefore

$$-\partial_x^2(\partial_t U + kU - V) + k(\partial_t U + kU - V) = 0 \tag{30}$$

and

$$\partial_t V + kV - \partial_x[(1+\rho_\nu\star f)\partial_x U] - \varepsilon\partial_x\big[\big(1+(\rho_\nu\star f)^2\big)\partial_x V\big] = 0$$

in the sense of distributions. In particular $\partial_t V \in L^2\big(0,T;H^{-1}(\omega)\big)$ so the traces $V(0,.)$ and $V(T,.)$ are well defined in $L^2(\omega)$. Moreover $\partial_t(-\partial_x^2 U + kU) =$

$\partial_x^2(kU - V) - k(kU - V)$ so $\partial_t(-\partial_x^2 U + kU) \in L^2(0, T; H^{-1}(\omega))$ then $-\partial_x^2 U + kU \in H^1(0, T; H^{-1}(\omega)) \subset C([0, T]; H^{-1}(\omega))$ and the traces $(-\partial_x^2 U + kU)(0, .)$ and $(-\partial_x^2 U + kU)(T, .)$ are well defined in $H^{-1}(\omega)$. Now, we multiply the equation (30) by $\varphi \in D([0, T[\times \bar{\omega})$ and integrate over ω_T to get

$$\int_{\omega_T} \big(-\partial_x U \partial_{tx}^2 \varphi + \partial_x(kU - V)\partial_x \varphi - kU\partial_t \varphi + (k^2 U - kV)\varphi\big)\, dx\, dt$$
$$= L(\varphi, 0) = \langle (-\partial_x^2 U + kU)(0, .), \varphi(0, .)\rangle$$
$$+ \int_0^T \big(\partial_x(kU - V)(t, 1)\varphi(t, 1) - \partial_x(kU - V)(t, 0)\varphi(t, 0)\big)\, dt.$$

Thus, taking first $\varphi \in D(]0, T[\times \bar{\omega})$, we get

$$\int_0^T \big(\partial_x(kU - V)(t, 1)\varphi(t, 1) - \partial_x(kU - V)(t, 0)\varphi(t, 0)\big)\, dt = 0$$

and since φ is arbitrary, we conclude that

$$\partial_x(kU - V)(t, 0) = \partial_x(kU - V)(t, 1) = 0 \tag{31}$$

then for $\varphi \in D([0, T[\times \omega)$ we see that $\langle (-\partial_x^2 U + kU)(0, .), \varphi(0, .)\rangle = L(\varphi, 0)$ but $\langle (-\partial_x^2 U + kU)(0, .), \varphi(0, .)\rangle = \int_\omega \big(\partial_x U(0, x)\partial_x \varphi(0, x) + kU(0, x)\varphi(0, x)\big)\, dx$ which leads to $U(0, x) = u_0(x)$ and $\partial_x U(0, x) = u_0'(x)$. Similarly, we get $V(0, x) = u_1(x)$ and

$$\left.\begin{aligned} &\big(1 + \rho_v \star f(t, 0)\big)\partial_x U(t, 0) + \varepsilon\big(1 + (\rho_v \star f)^2(t, 0)\big)\partial_x V(t, 0) = 0,\\ &\big(1 + \rho_v \star f(t, 1)\big)\partial_x U(t, 1) + \varepsilon\big(1 + (\rho_v \star f)^2(t, 1)\big)\partial_x V(t, 1) = 0. \end{aligned}\right\} \tag{32}$$

Comparing (31) and (32), we obtain

$$\partial_x U(t, 0) = \partial_x U(t, 1) = \partial_x V(t, 0) = \partial_x V(t, 1) = 0.$$

Therefore, using again (30), we see that

$$-\partial_x^2(\partial_t U) + k(\partial_t U) = \partial_x^2(kU - V) - k(kU - V) \in L^2\big(0, T; H^{-1}(\omega)\big),$$
$$\partial_x(\partial_t U)(t, 0) = \partial_x(\partial_t U)(t, 1) = 0,$$

so $\partial_t U \in L^2(0, T; H^1(\omega))$ and so is $\partial_t U + kU - V$, and since it is the solution to

$$-\partial_x^2(\partial_t U + kU - V) + k(\partial_t U + kU - V) = 0,$$
$$\partial_x(\partial_t U + kU - V)(t, 0) = \partial_x(\partial_t U + kU - V)(t, 1) = 0$$

we conclude that $\partial_t U + kU - V = 0$. Hence (U, V) is a solution to (26) and leads to a solution u to our problem.

For the uniqueness, since (25) is linear, it is enough to check that if $u_0 = u_1 = 0$, then $u = 0$. We multiply this equation by $\partial_t u$ and integrate on ω to get

$$\frac{1}{2}\frac{d}{dt}(|\partial_t u|^2 + |\partial_x u|^2) + \varepsilon\int_\omega \big(1 + (\rho_v \star f)^2\big)|\partial^2_{tx} u|^2\,dx = -\int_\omega (\rho_v \star f)\partial_x u \partial^2_{tx} u\,dx$$

since $|\int_\omega (\rho_v \star f)\partial_x u \partial^2_{tx} u\,dx| \le \frac{\varepsilon}{2}\int_\omega \big(1 + (\rho_v \star f)^2\big)|\partial^2_{tx} u|^2\,dx + \frac{1}{2\varepsilon}|\partial_x u|^2$, we get

$$\frac{d}{dt}(|\partial_t u|^2 + |\partial_x u|^2) + \varepsilon\int_\omega \big(1 + (\rho_v \star f)^2\big)|\partial^2_{tx} u|^2\,dx \le \frac{1}{\varepsilon}|\partial_x u|^2$$

therefore, using Gronwall's inequality, we get $\partial_x u = 0$, $\partial_t u = 0$ and so $u = 0$. □

3.2. The heat equation. We consider the heat equation

$$\left.\begin{array}{l} \partial_t f - \partial_x^2 f = -\partial_x v \quad \text{in } \omega_T, \\ f(t,0) = f(t,1) = \alpha \quad \text{in } (0,T), \\ f(0,x) = f_0(x) \quad \text{in } \omega. \end{array}\right\} \tag{33}$$

where v is a fixed function. The following result holds

Proposition 3.3. *Let $v \in L^\infty\big(0,T;H^1(\omega)\big)$ such that $\partial_t v \in L^2\big(0,T;H^1(\omega)\big)$ and $v(0,.) = u_0$. Under assumptions of Theorem* 1.1, *there exists a unique solution $f \in \mathscr{F}$ to problem* (33) *satisfying the following estimates in* $(0,T)$*:*

$$|f(t) - \alpha|^2 + \int_0^t |\partial_x f(s)|^2\,ds \le |f_0 - \alpha|^2 + \int_0^t |v(s)|^2\,ds, \tag{34}$$

$$|\partial_x f(t)|^2 + \int_0^t |\partial_t f(s)|^2\,ds \le |f_0'|^2 + \int_0^t |\partial_x v(s)|^2\,ds, \tag{35}$$

$$|\partial_t f(t)|^2 + \int_0^t |\partial^2_{tx} f(s)|^2\,ds \le |f_0'' - u_0'|^2 + \int_0^t |\partial_t v(s)|^2\,ds, \tag{36}$$

as well as

$$f(t,x) \ge \alpha - \frac{\sqrt{2}}{6}\|\partial_x v\|_T \qquad \text{in } \omega_T, \tag{37}$$

$$|f(t,x)| \le |f_0|_{L^\infty(\omega)} + \frac{\sqrt{2}}{6}\|\partial_x v\|_T \qquad \text{in } \omega_T. \tag{38}$$

Proof. We multiply the equation (33) by $(f - \alpha)$ and integrate to get

$$\frac{1}{2}\frac{d}{dt}|f-\alpha|^2 + |\partial_x f|^2 = -\int_\omega \partial_x v(f-\alpha)\,dx = \int_\omega v\partial_x f\,dx \le \frac{1}{2}(|v|^2 + |\partial_x f|^2)$$

which leads to (34). Multiplying (33) by $\partial_t f$ permits to get (35) whereas (36) is obtained as (9). Note that from the equation, we can deduce a bound of $\|\partial_x^2 f\|_T$. Now, let us write $f = h + k$ where h and k are the solutions of the problems

$$\left.\begin{array}{l} \partial_t h - \partial_x^2 h = 0 \quad \text{in } \omega_T, \\ h(t,0) = h(t,1) = \alpha \quad \text{in } (0,T), \\ h(0,x) = f_0(x) \quad \text{in } \omega, \end{array}\right\} \tag{39}$$

and

$$\left.\begin{array}{l} \partial_t k - \partial_x^2 k = -\partial_x v \quad \text{in } \omega_T, \\ k(t,0) = k(t,1) = 0 \quad \text{in } (0,T), \\ k(0,x) = 0 \quad \text{in } \omega, \end{array}\right\} \tag{40}$$

and let $v_n = \sqrt{2}\sin(n\pi x)$, $n \in \mathbb{N}^\star$. $(v_n)_{n\ge 1}$ is an orthogonal basis of $H_0^1(\omega)$ which is orthonormal in $L^2(\omega)$. Then the solution to (40) is given by

$$\begin{aligned} k(t,x) &= -\sum_{n\ge 1} k_n(t,x) \\ k_n(t,x) &= v_n(x)\int_0^t \big(\partial_x v(s,.), v_n\big)_{L^2(\omega)} \exp\big(-n^2\pi^2(t-s)\big)\,ds \end{aligned} \tag{41}$$

where $(.,.)_{L^2(\omega)}$ denotes the scalar product in $L^2(\omega)$. Since $|v_n(x)| \le \sqrt{2}$ then using Cauchy-Schwartz inequality, we easily deduce that

$$|k_n(t,x)| \le \frac{\sqrt{2}}{n^2\pi^2}\|\partial_x v\|_T \tag{42}$$

which leads to

$$|k(t,x)| \le \frac{\sqrt{2}}{6}\|\partial_x v\|_T. \tag{43}$$

Finally, we easily verify, see [1], [3], [8] that the solution to (39) satisfies $\alpha \le h(t,x) \le \sup_\omega f_0$ a.e. in ω_T so (37) and (38) follow. □

3.3. Solving the approximated problem. We will check that the operator $\mathcal{K}$ defined by (24) is continuous. First, we have

Lemma 3.4. *$\mathscr{S}$ defined by* (21) *is continuous from $L^2(\omega_T)$ to $\mathscr{U}$.*

Proof. Let $f_1, f_2 \in L^2(\omega_T)$, $u_i = \mathscr{S}(f_i)$, $i = 1, 2$ and set $f = f_1 - f_2$, $u = u_1 - u_2$. We have

$$\left.\begin{aligned}
&\partial_t^2 u - \partial_x[(1+\rho_\nu \star \tilde{f}_1^+)\partial_x u] - \varepsilon\partial_x\big[\big(1+(\rho_\nu \star \tilde{f}_1^+)^2\big)\partial_{tx}^2 u\big]\\
&= \partial_x[\rho_\nu \star (\tilde{f}_1^+ - \tilde{f}_2^+)\partial_x u_2] + \varepsilon\partial_x\big[\big((\rho_\nu \star \tilde{f}_1^+)^2 - (\rho_\nu \star \tilde{f}_2^+)^2\big)\partial_{tx}^2 u_2\big] \quad \text{in } \omega_T,\\
&\partial_x u(t,0) = \partial_x u(t,1) = 0 \quad \text{in } (0,T),\\
&u(0,x) = 0,\ \partial_t u(0,x) = 0 \quad \text{in } \omega.
\end{aligned}\right\} \tag{44}$$

Multiplying this equation by $\partial_t u$ and integrating over ω, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(|\partial_t u(t)|^2 + \int_\omega (1+\rho_\nu \star \tilde{f}_1^+)|\partial_x u(t,x)|^2\,dx\Big) + \varepsilon\int_\omega |\partial_{tx}^2 u(t,x)|^2\,dx\\
&\quad = -\varepsilon\int_\omega (\rho_\nu \star \tilde{f}_1^+)^2|\partial_{tx}^2 u(t,x)|^2\,dx + \frac{1}{2}\int_\omega (\partial_t\rho_\nu \star \tilde{f}_1^+)|\partial_x u(t,x)|^2\,dx\\
&\qquad - \int_\omega \rho_\nu \star (\tilde{f}_1^+ - \tilde{f}_2^+)\partial_x u_2 \partial_{tx}^2 u\,dx\\
&\qquad - \varepsilon\int_\omega \big((\rho_\nu \star \tilde{f}_1^+)^2 - (\rho_\nu \star \tilde{f}_2^+)^2\big)\partial_{tx}^2 u_2 \partial_{tx}^2 u\,dx.
\end{aligned}$$

We have

$$\Big|\int_0^t\int_\omega (\partial_t\rho_\nu \star \tilde{f}_1^+)|\partial_x u(s,x)|^2\,dx\,ds\Big| \le |\partial_t\rho_\nu|_T|f_1^+|_T\int_0^t |\partial_x u(s)|^2\,ds.$$

Since $|f_1^+ - f_2^+|_T \le |f_1 - f_2|_T$, we get using Young's inequality

$$\begin{aligned}
\Big|\int_0^t\int_\omega \rho_\nu \star (\tilde{f}_1^+ - \tilde{f}_2^+)\partial_x u_2\partial_{tx}^2 u\,dx\,ds\Big| &\le |\rho_\nu|_T|f|_T|\partial_x u_2|_T\Big(\int_0^t |\partial_{tx}^2 u|^2\,ds\Big)^{1/2}\\
&\le \frac{\varepsilon}{4}\int_0^t |\partial_{tx}^2 u|^2\,ds + \frac{1}{\varepsilon}|\rho_\nu|_T^2|\partial_x u_2|_T^2|f|_T^2.
\end{aligned}$$

In the same manner, we obtain

$$\begin{aligned}
&\varepsilon\int_0^t\int_\omega \big((\rho_\nu \star \tilde{f}_1^+)^2 - (\rho_\nu \star \tilde{f}_2^+)^2\big)\partial_{tx}^2 u_2\partial_{tx}^2 u\,dx\,ds\\
&\quad \le \frac{\varepsilon}{2}\int_0^t |\partial_{tx}^2 u|^2\,ds + \frac{1}{2\varepsilon}|\rho_\nu|_T^4|f_1 + f_2|_T^2|\partial_{tx}^2 u_2|_T^2|f|_T^2.
\end{aligned}$$

Gathering all these inequalities, we get

$$|\partial_t u(t)|^2 + |\partial_x u(t)|^2 + \frac{\varepsilon}{2}\int_0^t |\partial_{tx}^2 u(s)|^2 \, ds \le A_1 |f|_T^2 + A_2 \int_0^t |\partial_x u|^2 \, ds \tag{45}$$

where A_1, A_2 are positive constants depending on ε, ν, f_1, f_2, u_2. Using Gronwall's inequality this yields

$$|\partial_x u(t)|^2 \le A_1 \big(1 + A_2 t \exp(A_2 t)\big) |f|_T^2.$$

So when $|f|_T^2 \to 0$, we get first that $\|\partial_x u\|_T \to 0$, then, coming back to (45), $\|\partial_t u\|_T \to 0$ and $|\partial_{tx}^2 u|_T \to 0$. □

Now, we turn our attention to the operator $\mathscr{T}$. We have

Lemma 3.5. *$\mathscr{T}$ defined by* (22) *is continuous from $\mathscr{U}$ to $H^1(\omega_T)$.*

Proof. Let $u_1, u_2 \in \mathscr{U}$, $g_i = \mathscr{T}(u_i)$, $i = 1, 2$ and set $u = u_1 - u_2$, $g = g_1 - g_2$. Then g is the solution to the problem

$$\left.\begin{array}{l} \partial_t g - \partial_x^2 g = -\partial_x u \quad \text{in } \omega_T, \\ g(t,0) = g(t,1) = 0 \quad \text{in } (0,T), \\ g(0,x) = 0 \quad \text{in } \omega, \end{array}\right\} \tag{46}$$

and satisfies the following estimate:

$$|g(t)|^2 + 2\int_0^t |\partial_x g(s)|^2 \, ds \le \int_0^t \big(|g(s)|^2 + |\partial_x u(s)|^2\big) \, ds.$$

Making use of Gronwall's inequality we easily deduce that when $u \to 0$ in $H^1(\omega_T)$, $g \to 0$ in $L^\infty\big(0,T; L^2(\omega)\big)$ and $\partial_x g \to 0$ in $L^2(\omega_T)$. Moreover, multiplying the equation (46) by $\partial_t g$, we see that the following estimate holds:

$$\int_0^t |\partial_t g(s)|^2 \, ds + |\partial_x g(t)|^2 \le \int_0^t |\partial_x u(s)|^2 \, ds.$$

So $\partial_x g \to 0$ in $L^\infty\big(0,T; L^2(\omega)\big)$ and $\partial_t g \to 0$ in $L^2(\omega_T)$ when $u \to 0$ in $H^1(\omega_T)$. □

We continue to verify the properties of the operator $\mathscr{K}$. For $\lambda \in [0,1]$, we define

$$\mathscr{F}_\lambda = \{f \in L^2(\omega_T); f = \lambda \mathscr{K}(f)\}.$$

Lemma 3.6. *There exists $C > 0$ independent of λ such that*

$$|f|_T \leq C \quad \text{for all } f \in \mathscr{F}_\lambda.$$

Proof. Since $\mathscr{F}_0 = \{0\}$, consider $\lambda \in]0,1]$ then $f \in \mathscr{F}_\lambda$ is such that $f \in H^1(\omega_T)$ and satisfies

$$\left.\begin{array}{l} \partial_t f - \partial_x^2 f = -\lambda \partial_x u \quad \text{in } \omega_T, \\ f(t,0) = f(t,1) = \lambda\alpha \quad \text{in } (0,T), \\ f(0,x) = \lambda f_0(x) \quad \text{in } \omega, \end{array}\right\} \tag{47}$$

where $u = \mathscr{S}(f)$ is the solution to (20). Multiplying this equation by $\partial_t u$ and integrating over ω, we get

$$\frac{1}{2}\frac{d}{dt}\left(|\partial_t u(t)|^2 + |\partial_x u(t)|^2\right) + \varepsilon \int_\omega \left(1 + (\rho_\nu \star \tilde{f}^+)^2\right)|\partial_{tx}^2 u|^2\, dx$$
$$= -\int_\omega (\rho_\nu \star \tilde{f}^+)\partial_x u \partial_{tx}^2 u\, dx \leq \frac{\varepsilon}{2}\int_\omega (\rho_\nu \star \tilde{f}^+)^2 |\partial_{tx}^2 u|^2\, dx + \frac{1}{2\varepsilon}\int_\omega |\partial_x u|^2\, dx.$$

Hence,

$$|\partial_t u(t)|^2 + |\partial_x u(t)|^2 + \varepsilon \int_0^t \int_\omega \left(1 + (\rho_\nu \star \tilde{f}^+)^2\right)|\partial_{tx}^2 u(s,x)|^2\, dx\, ds$$
$$\leq |u_1|^2 + |u_0'|^2 + \frac{1}{\varepsilon}\int_0^t |\partial_x u(s)|^2\, ds.$$

Therefore, using Gronwall's inequality we get for $t \in (0,T)$

$$|\partial_x u(t)|^2 \leq C \tag{48}$$

where $C > 0$ depends on u_0, u_1, ε, T but not on ν nor on λ, then

$$\|\partial_x u\|_T, \|\partial_t u\|_T, \|u\|_T, |\partial_{tx}^2 u|_T \leq C. \tag{49}$$

Moreover, following the proof of Proposition 3.3, we get that f satisfies

$$|f(t) - \lambda\alpha|^2 + \int_0^t |\partial_x f(s)|^2\, ds \leq |\lambda(f_0 - \alpha)|^2 + \int_0^t |\lambda u(s)|^2\, ds$$

and we conclude using (49) that

$$|f|_T \leq C_1$$

where $C_1 > 0$ depends upon u_0, u_1, f_0, α, ε and T but not on ν nor on λ. □

Proof of Theorem 3.1. Since $H^1(\omega_T)$ is compactly imbedded in $L^2(\omega_T)$, using Leray–Schauder fixed point theorem [5], we deduce from the previous lemmas, that operator $\mathscr{K}$ admits a fixed point $f_\nu^\varepsilon \in L^2(\omega_T)$, and denoting $u_\nu^\varepsilon = \mathscr{S}(f_\nu^\varepsilon)$, we conclude that $(f_\nu^\varepsilon, u_\nu^\varepsilon) \in (\mathscr{F}, \mathscr{U})$ is a solution to $(\mathscr{P}_\nu^\varepsilon)$.

In order to show the uniqueness, we consider two solutions (f_1, u_1), (f_2, u_2) of $(\mathscr{P}_\nu^\varepsilon)$ and set $g = f_1 - f_2$, $u = u_1 - u_2$. As u satisfies (44), following the proof of Lemma 3.4, we get

$$|\partial_t u(t)|^2 + |\partial_x u(t)|^2 + \varepsilon \int_0^t |\partial_{tx}^2 u(s)|^2 \, ds \leq A_1 \int_0^t |g(s)|^2 \, ds + A_2 \int_0^t |\partial_x u|^2 \, ds \quad (50)$$

where A_1, A_2 are positive constants depending on ε, ν, f_1, f_2, u_2. Now, since g is the solution to (46), it satisfies

$$|g(t)|^2 + 2 \int_0^t |\partial_x g(s)|^2 \, ds \leq \int_0^t |g(s)|^2 \, ds + \int_0^t |\partial_x u|^2 \, ds \quad (51)$$

and adding (50) and (51) leads to

$$|g(t)|^2 + |\partial_x u(t)|^2 \leq A \int_0^t \left(|g(s)|^2 + |\partial_x u(s)|^2 \right) ds$$

with $A > 0$. Then, by Gronwall's inequality, we obtain first $|g(t)| = |\partial_x u(t)| = 0$ and coming back to (50), we deduce that $|\partial_t u(t)| = 0$ and therefore $g = 0$, $u = 0$ a.e. □

4. Passing to the limit: End of proof of Theorem 1.1

Let $T > 0$ and $\varepsilon > 0$ be fixed. We want to pass to the limit as $\nu \to 0$ in the problem $(\mathscr{P}_\nu^\varepsilon)$. We have

Lemma 4.1. *The solutions $(f_\nu^\varepsilon, u_\nu^\varepsilon)$ of problem $(\mathscr{P}_\nu^\varepsilon)$ are bounded in $(\mathscr{F}, \mathscr{U})$ uniformly with respect to ν. More precisely the following estimates are satisfied*

$$\|u_\nu^\varepsilon\|_T + \|\partial_x u_\nu^\varepsilon\|_T + \|\partial_t u_\nu^\varepsilon\|_T + |\partial_{tx}^2 u_\nu^\varepsilon|_T \leq C, \quad (52)$$

$$\|f_\nu^\varepsilon\|_T + \|\partial_x f_\nu^\varepsilon\|_T + \|\partial_t f_\nu^\varepsilon\|_T + \|\partial_x^2 f_\nu^\varepsilon\|_T + |\partial_{tx}^2 f_\nu^\varepsilon|_T \leq C \quad (53)$$

with $C > 0$ depending on u_0, u_1, f_0, α, ε and T but not on ν.

Proof. The estimates (52) result from (49) because $u_\nu^\varepsilon = \mathscr{S}(f_\nu^\varepsilon)$ and $f_\nu^\varepsilon \in \mathscr{F}_1$. We deduce (53) using (34), (35) and (36). □

As a consequence of Lemma 4.1, we can extract subsequences, still denoted $(f_\nu^\varepsilon, u_\nu^\varepsilon)$, for which we have at least the following weak convergences as $\nu \to 0$

$$f_\nu^\varepsilon \rightharpoonup f^\varepsilon, \quad u_\nu^\varepsilon \rightharpoonup u^\varepsilon \quad \text{weakly in } H^1(\omega_T),$$
$$\partial_{tx}^2 u_\nu^\varepsilon \rightharpoonup \partial_{tx}^2 u^\varepsilon \quad \text{weakly in } L^2(\omega_T),$$

and the strong convergences

$$f_\nu^\varepsilon \to f^\varepsilon, \quad u_\nu^\varepsilon \to u^\varepsilon \quad \text{strongly in } L^2(\omega_T)$$

with $(f^\varepsilon, u^\varepsilon) \in (\mathscr{F}, \mathscr{U})$. Therefore

$$\rho_\nu \star \widetilde{(f_\nu^\varepsilon)}^+ \to \widetilde{(f^\varepsilon)}^+ \quad \text{strongly in } L^2(\mathbb{R}^2)$$

and passing to the limit in the problem $(\mathscr{P}_\nu^\varepsilon)$, we easily get that $(f^\varepsilon, u^\varepsilon)$ is a solution to the problem $(\mathscr{P}^\varepsilon)$ below

$$\left.\begin{array}{l}
\partial_t f^\varepsilon - \partial_x^2 f^\varepsilon = -\partial_x u^\varepsilon \quad \text{in } \omega_T,\\
f^\varepsilon(t,0) = f^\varepsilon(t,1) = \alpha \quad \text{in } (0,T),\\
f^\varepsilon(0,x) = f_0(x) \quad \text{in } \omega,\\
\partial_t^2 u^\varepsilon - \partial_x[(1+f^{\varepsilon,+})\partial_x u^\varepsilon] - \varepsilon\partial_x\big[\big(1+(f^{\varepsilon,+})^2\big)\partial_{tx}^2 u^\varepsilon\big] = 0 \quad \text{in } \omega_T,\\
\partial_x u^\varepsilon(t,0) = \partial_x u^\varepsilon(t,1) = 0 \quad \text{in } (0,T),\\
u^\varepsilon(0,x) = u_0(x),\ \partial_t u^\varepsilon(0,x) = u_1(x) \quad \text{in } \omega.
\end{array}\right\} \tag{54}$$

Moreover, we have

Lemma 4.2. *Let assumptions of Theorem* 1.1 *hold. Then, there exists $T^\star > 0$, depending only on the data u_0, u_1, f_0, such that for all $\varepsilon > 0$ and $T \in\,]0, T^\star[$, the problem $(\mathscr{P}^\varepsilon)$ admits a weak solution defined on $(0,T)$, $(f^\varepsilon, u^\varepsilon) \in (\mathscr{F}, \mathscr{U})$ satisfying*

$$\|u^\varepsilon\|_T + \|\partial_x u^\varepsilon\|_T + \|\partial_t u^\varepsilon\|_T \le C \tag{55}$$
$$\|f^\varepsilon\|_T + \|\partial_x f^\varepsilon\|_T + \|\partial_x^2 f^\varepsilon\|_T + |\partial_{tx}^2 f^\varepsilon|_T + \|\partial_t f^\varepsilon\|_T \le C \tag{56}$$

with $C > 0$ independent of ε.

Proof. Indeed, proceeding as in Section 2 we get

$$|\partial_t f^\varepsilon(t)|^2 + \int_0^t |\partial_{tx}^2 f^\varepsilon(s)|^2\, ds \le |f_0'' - u_0'|^2 + \int_0^t |\partial_t u^\varepsilon(s)|^2\, ds, \tag{57}$$

$$\begin{aligned}
|\partial_t u^\varepsilon(t)|^2 &+ \int_\omega (1+f^{\varepsilon,+})|\partial_x u^\varepsilon|^2\, dx + \varepsilon\int_0^t\int_\omega \big(1+(f^{\varepsilon,+})^2\big)|\partial_{tx}^2 u^\varepsilon|^2\, dx\, ds\\
&= |u_1|^2 + \int_\omega \big(1+f_0(x)\big)|u_0'(x)|^2\, dx + \int_0^t\int_\omega \partial_t f^{\varepsilon,+}|d_x u^\varepsilon|^2\, dx\, ds.
\end{aligned} \tag{58}$$

Therefore as well as for (59) we get

$$|\partial_t f^\varepsilon(t)|^2 + |\partial_t u^\varepsilon(t)|^2 + \int_\omega \big(1 + f^{\varepsilon,+}(t,x)\big)|\partial_x u^\varepsilon(t,x)|^2\,dx \\ \le E_0 + \int_0^t |\partial_t u^\varepsilon(s)|^2\,ds + \frac{3}{4}\int_0^t |\partial_t f^\varepsilon(s)|^{2/3}|\partial_x u^\varepsilon(s)|^{8/3}\,ds. \tag{59}$$

Now, we set

$$T^\star = \tfrac{6}{7}E_1^{-2/3} \qquad \text{where } E_1 = \max(E_0, 1), \tag{60}$$

and

$$\lambda(t) = \big(E_1^{-2/3} - \tfrac{7}{6}t\big)^{-3/2}. \tag{61}$$

Hence, (59) leads for $t \in (0, T)$ to

$$|\partial_t u^\varepsilon(t)|^2, |\partial_x u^\varepsilon(t)|^2, |\partial_t f^\varepsilon(t)|^2 \le \lambda(t) \tag{62}$$

for all $T \in]0, T^\star[$ then $\|u^\varepsilon\|_T \le C$ with $C > 0$ independent of ε. Coming back to (34), (35) and (36), we deduce that

$$\|f^\varepsilon\|_T, \|\partial_x f^\varepsilon\|_T, |\partial^2_{tx} f^\varepsilon|_T \le C \tag{63}$$

and since $\partial^2_x f^\varepsilon = \partial_t f^\varepsilon + \partial_x u^\varepsilon$, we have also $\|\partial^2_x f^\varepsilon\|_T \le C$. □

Now, let $T \in]0, T^\star[$. The estimates of Lemma 4.2 allow to pass to the limit as $\varepsilon \to 0$ in the problem $(\mathscr{P}^\varepsilon)$ and we get easily $(f, u) \in \mathscr{F} \times \big(H^1(\omega_T) \cap C\big([0,T]; H^1(\omega)\big) \cap C^1\big([0,T]; L^2(\omega)\big)$ solution to

$$\left.\begin{array}{l} \partial_t f - \partial^2_x f = -\partial_x u \quad \text{in } \omega_T, \\ f(t,0) = f(t,1) = \alpha \quad \text{in } (0,T), \\ f(0,x) = f_0(x) \quad \text{in } \omega, \\ \partial^2_t u - \partial_x[(1+f^+)\partial_x u] = 0 \quad \text{in } \omega_T, \\ \partial_x u(t,0) = \partial_x u(t,1) = 0 \quad \text{in } (0,T), \\ u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x) \quad \text{in } \omega. \end{array}\right\} \tag{64}$$

Then, it follows from (37) that $f(t,x) \ge \alpha - \frac{\sqrt{2}}{6}\|\partial_x u\|_T$, a.e. in ω_T. So, since u satisfies (62) and λ is increasing, we deduce that $f(t,x) \ge \alpha - \frac{\sqrt{2}}{6}\lambda(T)$, a.e. in ω_T. Therefore, if α is such that $E_1 < 3\sqrt{2}\,\alpha$ then, choosing $T < T^\star$ satisfying $\alpha - \frac{\sqrt{2}}{6}\lambda(T) = 0$, that is $T = \frac{7}{6}\big(E_1^{-2/3} - (3\sqrt{2}\alpha)^{-2/3}\big)$, we get $f(t,x) \ge 0$ a.e. in ω_T and we conclude that (f, u) is a solution to problem $(\mathscr{P})$ on the time interval $(0, T)$.

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