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Inequalities for Riemann's zeta function

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Abstract. Let ζ and Λ the the Riemann zeta function and the von Mangoldt function, respectively. Further, let $c > 0$. We prove that the double-inequality

$$\exp\left(-c\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{s+\alpha}}\right) < \frac{\zeta(s+c)}{\zeta(s)} < \exp\left(-c\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{s+\beta}}\right)$$

holds for all $s > 1$ with the best possible constants

$$\alpha = 0 \quad \text{and} \quad \beta = \frac{1}{\log 2}\log\left(\frac{c\log 2}{1-2^{-c}}\right).$$

This extends and refines a recent result of Cerone and Dragomir.

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1. Introduction

In 2005, Cerone and Dragomir [4] presented remarkable inequalities for the ratio $\zeta(s+1)/\zeta(s)$, where ζ denotes the classical Riemann zeta function,

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} \qquad (s > 1).$$

They proved that the double-inequality

$$\exp\left(-\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{s}}\right) \le \frac{\zeta(s+1)}{\zeta(s)} \le \exp\left(-\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{s+1}}\right) \tag{1.1}$$

holds for all $s > 1$. Here, Λ is the von Mangoldt function, defined by

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^m \text{ for some prime number } p \text{ and integer } m \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$

which plays an important role in the distribution of prime numbers. Properties of this function are given, for instance, in [2], Sec. 2.8. The logarithmic derivative of ζ can be expressed as a Dirichlet series with coefficients $-\Lambda(n)$. We have

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \qquad (s > 1); \tag{1.2}$$

see [2], p. 236. Many further facts on the ζ-function are collected in the monographs [5], [6], [7], [8]. Noteworthy historical remarks can be found in [3].

The aim of this note is to generalize (1.1). We are interested in sharp upper and lower bounds for the ratio $\zeta(s+c)/\zeta(s)$, where c is a (fixed) positive real number. More precisely, we ask for the largest number $\alpha = \alpha(c)$ and the smallest number $\beta = \beta(c)$ such that for all $s > 1$ we have

$$\exp\Big(-c\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{s+\alpha}}\Big) < \frac{\zeta(s+c)}{\zeta(s)} < \exp\Big(-c\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{s+\beta}}\Big).$$

In the next section, we give a complete answer to this question. It turns out that if $c = 1$, then the best possible constants are $\alpha = 0$ and $\beta = (\log\log 4)/\log 2 = 0.4712\ldots$. This yields an improvement of the right-hand side of (1.1).

2. Main result

The following theorem extends and refines (1.1).

Theorem. *Let $c > 0$ be a real number. For all real numbers $s > 1$ we have*

$$\exp\Big(-c\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{s+\alpha}}\Big) < \frac{\zeta(s+c)}{\zeta(s)} < \exp\Big(-c\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{s+\beta}}\Big) \tag{2.1}$$

with the best possible constants

$$\alpha = 0 \qquad \textit{and} \qquad \beta = \frac{1}{\log 2}\log\Big(\frac{c\log 2}{1-2^{-c}}\Big). \tag{2.2}$$

Proof. From (1.2) we get

$$[\log \zeta(s)]'' = \sum_{n=2}^{\infty} \frac{\Lambda(n) \log n}{n^s} \qquad (s > 1).$$

This representation implies that ζ is strictly log-convex on $(1, \infty)$. Applying the mean value theorem we obtain

$$\log \zeta(s+c) - \log \zeta(s) > c \frac{\zeta'(s)}{\zeta(s)},$$

which is equivalent to the left-hand side of (2.1) with $\alpha = 0$.

To prove the right-hand side of (2.1) with β as given in (2.2) we define

$$F(s) = -c \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{s+\beta}} - \log \zeta(s+c) + \log \zeta(s) \qquad (s > 1).$$

Differentiation yields

$$F'(s) = c \sum_{n=1}^{\infty} \frac{\Lambda(n) \log n}{n^{s+\beta}} - \frac{\zeta'(s+c)}{\zeta(s+c)} + \frac{\zeta'(s)}{\zeta(s)} = \sum_{n=2}^{\infty} \frac{\Lambda(n) f(n)}{n^{s+\beta}}, \tag{2.3}$$

where

$$f(x) = c \log x + x^{\beta - c} - x^{\beta}.$$

Let $x \geq 2$ and

$$g(x) = x f'(x) = c + (\beta - c) x^{\beta - c} - \beta x^{\beta}. \tag{2.4}$$

Then

$$x^{1+c-\beta} \beta^{-2} g'(x) = \left(\frac{c}{\beta} - 1 \right)^2 - x^c \leq \left(\frac{c}{\beta} - 1 \right)^2 - 2^c.$$

We have $0 < \beta < c$. Therefore, in order to prove that $g'(x)$ is negative, it suffices to show that

$$\frac{c}{\beta} - 1 < 2^{c/2}. \tag{2.5}$$

We set $c = 2(\log t)/\log 2$ with $t > 1$. Then (2.5) is equivalent to

$$0 < \log 2 + \frac{2t}{t+1} \log t + \log\log t - \log(t-1) - \log(t+1) = h(t), \text{ say.} \quad (2.6)$$

Differentiation gives

$$t(t-1)(t+1)^2(\log t)h'(t) = t^3 + t^2 - t - 1 - (2t^2 + 2t)\log t + (2t^2 - 2t)(\log t)^2 = j(t), \text{ say.}$$

Since $j(1) = j'(1) = 0$ and

$$tj''(t) = 6(t^2 - 1) + (8t - 4)\log t + 4t(\log t)^2 > 0,$$

we get $j(t) > 0$ for $t > 1$. This implies that h is strictly increasing on $(1, \infty)$. Since $h(1) = 0$, we conclude that (2.6) is valid. Thus, g is strictly decreasing on $[2, \infty)$, so that we obtain

$$g(x) \le g(2) \qquad \text{for } x \ge 2.$$

Next we prove that $g(2) < 0$. We set $c = (\log u)/\log 2$ with $u > 1$. Then we get

$$\frac{1}{c} g(2) = 1 + \left(\frac{\beta}{c} - 1\right) 2^{\beta - c} - \frac{\beta}{c} 2^{\beta} = 1 - \log\log u - \frac{u}{u-1} \log u + \log(u-1) = k(u), \text{ say.}$$

We have

$$(\log u)k'(u) = \left(\frac{\log u}{u-1}\right)^2 - \frac{1}{u} < 0.$$

This leads to $k(u) < k(1) = 0$ for $u > 1$. Thus, $g(x) < 0$ for $x \ge 2$, so that (2.4) yields

$$f(x) < f(2) = 0 \qquad \text{for } x > 2. \quad (2.7)$$

From (2.3) and (2.7) we get $F'(s) < 0$ for $s > 1$. Applying the limit relations

$$\lim_{s \to \infty} \zeta(s) = 1 \qquad \text{and} \qquad \lim_{s \to \infty} \zeta'(s) = 0$$

gives for $s > 1$ that

$$F(s) > \lim_{x\to\infty} F(x) = \lim_{x\to\infty}\left(c\frac{\zeta'(x+\beta)}{\zeta(x+\beta)} - \log\zeta(x+c) + \log\zeta(x)\right) = 0.$$

This proves the right-hand side of (2.1).

It remains to show that the constants given in (2.2) are best possible. First, we assume that there exists a number $\alpha > 0$ such that the first inequality in (2.1) holds for all $s > 1$. Then we have

$$\exp\frac{c\zeta'(s+\alpha)}{\zeta(s+\alpha)} < \frac{\zeta(s+c)}{\zeta(s)} \qquad (s > 1).$$

We let s tend to 1 and obtain

$$\exp\frac{c\zeta'(1+\alpha)}{\zeta(1+\alpha)} \le 0,$$

a contradiction! This implies that the largest constant α in the left-hand inequality of (2.1) is given by $\alpha = 0$.

We suppose that the right-hand side of (2.1) is valid for all $s > 1$. Then we get

$$2^{\beta}\left[\frac{y(s+c)}{2^c}\frac{\log\bigl(1+z(s+c)\bigr)}{z(s+c)} - y(s)\frac{\log\bigl(1+z(s)\bigr)}{z(s)}\right] < \frac{c2^{s+\beta}\zeta'(s+\beta)}{\zeta(s+\beta)}, \quad (2.8)$$

where

$$y(s) = 2^s[\zeta(s)-1] \quad \text{and} \quad z(s) = \frac{y(s)}{2^s} = \zeta(s) - 1.$$

In [1] it is proved that

$$\lim_{s\to\infty} y(s) = 1. \quad (2.9)$$

For $v > 0$ we have

$$0 < -2^{v+2}\zeta'(v+2) - \log 2 = \sum_{n=3}^{\infty}(\log n)\left(\frac{2}{n}\right)^{v+2}$$
$$< \left(\frac{2}{3}\right)^{v}\sum_{n=3}^{\infty}(\log n)\left(\frac{2}{n}\right)^{2} = \left(\frac{2}{3}\right)^{v}[-\log 2 - 4\zeta'(2)].$$

This leads to

$$\lim_{s\to\infty} 2^s\zeta'(s) = -\log 2. \tag{2.10}$$

Applying (2.9) and (2.10) we conclude from (2.8) that

$$2^\beta\left(\frac{1}{2^c}-1\right) \le -c\log 2$$

or, equivalently,

$$\beta \ge \frac{1}{\log 2}\log\left(\frac{c\log 2}{1-2^{-c}}\right).$$

Hence, the smallest constant β in the second inequality of (2.1) is given in (2.2). □

Remark. Euler's totient function $\varphi(n)$ is defined to be the number of positive integers not exceeding n, which are relatively prime to n. The main properties of this function are collected in [2], Sec. 2.3–2.5. In view of

$$\frac{\zeta(s)}{\zeta(s+1)} = \sum_{n=1}^{\infty}\frac{\varphi(n)}{n^{s+1}} \qquad (s>1)$$

(see [2], p. 229), the theorem (with $c=1$) provides the following double-inequality for Dirichlet series:

$$\exp\left(\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{s+a}}\right) < \sum_{n=1}^{\infty}\frac{\varphi(n)}{n^{s+1}} < \exp\left(\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{s}}\right) \qquad (s>1),$$

where $a = (\log\log 4)/\log 2$.

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