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Examples of optimal solutions of infinite horizon variational problems arising in continuum mechanics

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Abstract. In this article we construct several important examples of optimal solutions of infinite horizon second order variational problems. The problems are related to a model in thermodynamics introduced in Coleman, Marcus and Mizel, *Arch. Rational Mech. Anal.* **117** (1992), 321–347.

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1. Introduction

In the present article we study optimal solutions of infinite horizon variational problems associated with the functional

$$I^f(D; w) = \int_D f\big(w(t), w'(t), w''(t)\big)\, dt \qquad \text{for all } w \in W^{2,1}(D),$$

where D is a bounded interval on the real line and $f \in C(\mathbb{R}^3)$ belongs to a space of functions $\mathscr{M}$ to be described in Section 2.

We consider the problems

$$\inf\{I^f(D; w) \mid w \in W^{2,1}(D), (w, w')(T_1) = x, (w, w')(T_2) = y\} \qquad (P_D^{x,y})$$

for $D = (T_1, T_2)$ and $x, y \in \mathbb{R}^2$ and also consider the following problem on the half line:

$$\inf\{J^f(w) \mid w \in W^{2,1}_{\text{loc}}(0, \infty), f(w, w', w'') \in L^1(0, T) \text{ for all } T > 0\}, \quad (P_\infty)$$

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where

$$J^f(w) = \liminf_{T\to\infty} T^{-1} I^f\big((0,T); w\big).$$

Variational problems of this type were introduced in [6] and investigated in [6], [9], [11], [12], [13], [14]. Similar constrained problems (involving a mass constraint) studied in [4], [7], [8], [10] were conceived as models for determining the thermodynamical equilibrium states of unidimensional bodies involving 'second order' materials for which the free energy density is given by f. A discussion of the physics underlying these models can be found in [2], [3] and in [4]. Properties of minimizers of the mass constrained problem on bounded intervals, and their relation to minimizers of the limiting problem on the full line were studied in [7], [8], [10].

In the present article we study the unconstrained problem (P_∞) and related problems on bounded intervals. It should be mentioned that several notions of minimizers to (P_∞) were introduced and studied in [6], [9], [11]. In particular, we consider the classes of periodic minimizers, c-optimal minimizers and perfect minimizers denoted, respectively, by $\mathscr{S}^f$, $\mathscr{T}^f$ and $\mathscr{P}^f$ with the following relation:

$$\mathscr{S}^f \subset \mathscr{P}^f \subset \mathscr{T}^f.$$

A function $w \in W^{2,\gamma}_{\mathrm{loc}}(0,\infty) \cap W^{1,\infty}(0,\infty)$ is *c-optimal* [9], [11], if, for every bounded interval $D = [T_1, T_2] \subset [0,\infty)$, the restriction $w|_D$ is a minimizer of $(P_D^{x,y})$ with $x = (w,w')(T_1)$, $y = (w,w')(T_2)$.

If a c-optimal function (minimizer) is periodic, we say that it is a periodic minimizer.

It is not difficult to show that for every $x \in \mathbb{R}^2$ there exists a c-optimal function w such that $\big(w(0), w'(0)\big) = x$. This c-optimal function can be constructed as a limit of solutions of the problems $(P_{(0,T)}^{x,0})$ where $T \to \infty$.

The existence of a periodic minimizer is a difficult problem which was solved in [6], [12]. It turns out (see [9], [14]) that for a typical integrand f there exists a unique (up to translations) periodic minimizer. More precisely, in [14] we considered certain complete metric spaces of integrands and showed that for most of their elements (in the sense of Baire category) the corresponding variational problems possess unique (up to translations) periodic minimizers. Note that if a periodic minimizer is unique, then solutions of the corresponding variational problems possess remarkable properties. In particular, all c-optimal functions converge (in some sense) to this periodic minimizer. Since a typical integrand possesses a unique minimizer it is interesting to construct integrands with a given number of periodic minimizers and even with an infinite number of periodic minimizers. This is the first goal of our article.

In [9], [11] we also studied a class of so-called perfect minimizers. Let f be an integrand and $\mu(f)$ be the infimum in (P_∞) which is finite [6]. In [6], [9], [10], [11], [12], [13], [14] we considered a continuous function π^f associated with the integrand f. For any $v \in W^{2,1}(D)$, where $D = (T_1, T_2)$, put

$$\Gamma^f(D, v) := I^f(D; v) - |D|\mu(f) + \pi^f\big((v, v')(T_2)\big) - \pi^f\big((v, v')(T_1)\big).$$

The functional $\Gamma^f(\cdot\,,\cdot)$ is very useful in the study of the problem (P_∞). It is not difficult to see that minimization problems with the functional I^f are equivalent to minimization problems with the functional Γ^f. On the other hand it is more convenient to work with the functional Γ^f because it is always nonnegative.

A c-optimal function w is called a perfect minimizer [11] if $\Gamma^f\big((0, T), w\big) = 0$ for any $T > 0$.

It turns out that for any $x \in \mathbb{R}^2$ there exists a perfect minimizer w such that $\big(w(0), w'(0)\big) = x$. Usage of perfect minimizers plays an important role in the theory developed in [9], [10], [11], [12], [13], [14]. The second goal of the article is to construct an integrand such that there exists a c-optimal minimizer which is not perfect.

2. Preliminaries

First we describe the space of integrands $\mathcal{M}$ that we are going to consider.

Let $a = (a_1, a_2, a_3, a_4) \in \mathbb{R}^4$, $a_i > 0$, $i = 1, 2, 3, 4$ and let α, β, γ be real numbers such that $1 \le \beta < \alpha$, $\beta \le \gamma$ and $\gamma > 1$. Denote by $\mathcal{M} = \mathcal{M}(\alpha, \beta, \gamma, a)$ the family of continuous functions $\{f\}$ such that

$$\begin{aligned}
&\text{(i)}\ \ f \in C^2(\mathbb{R}^3), \qquad \partial f/\partial x_2 \in C^2(\mathbb{R}^3), \qquad \partial f/\partial x_3 \in C^3(\mathbb{R}^3),\\
&\text{(ii)}\ \ \partial^2 f/\partial x_3^2 > 0,\\
&\text{(iii)}\ \ f(x) \ge a_1|x_1|^\alpha - a_2|x_2|^\beta + a_3|x_3|^\gamma - a_4,\\
&\text{(iv)}\ \ (|f| + |\nabla f|)(x) \le M_f(|x_1| + |x_2|)(1 + |x_3|^\gamma) \qquad \text{for all } x \in \mathbb{R}^3,
\end{aligned} \tag{2.1}$$

where $M_f : [0, \infty) \mapsto [0, \infty)$ is a continuous function depending on f.

In the sequel we assume that $f \in \mathcal{M} = \mathcal{M}(\alpha, \beta, \gamma, a)$ where $(\alpha, \beta, \gamma, a)$ is an arbitrary but fixed set of parameters satisfying the above conditions. Conditions (2.1) (iii), (iv) imply that

$$w \in W^{2,1}_{\mathrm{loc}}(\mathbb{R}_+) \text{ and } f(w, w', w'') \in L^1(0, T) \qquad \text{for all } T > 0 \;\Leftrightarrow\; w \in W^{2,\gamma}_{\mathrm{loc}}(\mathbb{R}_+),$$

where $\mathbb{R}_+ = [0, \infty)$ and

$$W^{2,\gamma}_{\text{loc}}(\mathbb{R}_+) = \{w \in W^{2,\gamma}_{\text{loc}}(0,\infty) \mid w \in W^{2,\gamma}(0,T) \text{ for all } T > 0\}.$$

For every $f \in \mathscr{M}$, the infimum in (P_∞) is finite (see [6] or Lemma 2.2 of [9]). Put

$$\mu(f) := \inf(P_\infty).$$

Leizarowitz and Mizel [6] showed that if f satisfies the condition

$$\mu(f) < \inf_{(w,s)\in\mathbb{R}^2} f(w,0,s),$$

then (P_∞) possesses a *periodic minimizer*. Later, Zaslavski [12] proved that this condition is not needed: the result holds for all $f \in \mathscr{M}$.

Let $f \in \mathscr{M}$ and denote by $\mathscr{S}^f$ the set of all periodic minimizers (P_∞).

For $w \in W^{2,\gamma}(D)$, D a bounded interval, put

$$E^f(D,w) := I^f(D,w) - \mu(f)|D|, \tag{2.2}$$

where $|D|$ is the Lebesgue measure of D.

By definition, $w \in W^{2,\gamma}_{\text{loc}}(\mathbb{R}_+)$ is a minimizer of (P_∞) iff

$$\liminf_{T\to\infty} \frac{1}{T} E^f\big((0,T),w\big) = 0.$$

If, in addition, $\big\{E^f\big((0,T),w\big) \mid T > 0\big\}$ is bounded we say that w is an (f)-*good minimizer*. This concept was first introduced by Leizarowitz [5] in a discrete context. More generally, if $v \in W^{2,\gamma}_{\text{loc}}(U)$ for some unbounded interval U, and if there exists a constant $M = M(U,v)$ such that $|E^f(D,v)| \le M$ for every bounded interval $D \subset U$, we say that v is an (f)-good function on U. The family of (f)-good functions on U is denoted by $\mathscr{G}^f(U)$; the family of (f)-good minimizers (i.e., $\mathscr{G}^f(\mathbb{R}_+)$) is denoted briefly by $\mathscr{G}^f$.

The following result was obtained in [12]; a discrete version was previously established in [5].

Lemma 2.1. *For every $w \in W^{2,\gamma}_{\text{loc}}(\mathbb{R}_+)$, either $\big\{\big|E^f\big((0,T),w\big)\big| \mid T > 0\big\}$ is bounded, i.e., $w \in \mathscr{G}^f$, or $\lim_{T\to\infty} E^f\big((0,T),w\big) = \infty$. If $w \in \mathscr{G}^f$ then $w \in W^{1,\infty}(\mathbb{R}_+)$.*

We have defined the class of (f)-good functions $\mathscr{G}^f$ which is rather "large" and the class of periodic minimizers $\mathscr{S}^f$ which is a "small" subset of $\mathscr{G}^f$. As it was shown in [9], [14] many integrands belonging to $\mathscr{M}$ possess a unique (up to translation) periodic minimizer. Now we define an important notion of c-optimal functions used in [9], [11].

If $w \in W^{2,\gamma}_{loc}(U) \cap W^{1,\infty}(U)$, where U is an unbounded interval, we say that v is *c-optimal* on U [9], [11], if, for every bounded interval $D = (T_1, T_2) \subset U$, the restriction $w|_D$ is a minimizer of $(P_D^{x,y})$ with $x = (w, w')(T_1)$, $y = (w, w')(T_2)$. The family of c-optimal functions on U is denoted by $\mathscr{T}^f(U)$; the family of c-optimal functions on $\mathbb{R}_+$ is denoted briefly by $\mathscr{T}^f$.

Note that the definition of a c-optimal function does not assume that it is a minimizer of (P_∞). However, by Proposition 2.3 of [9]:

Lemma 2.2. *If w is c-optimal on $\mathbb{R}_+$ then it is an (f)-good minimizer.*

The class of c-optimal minimizers $\mathscr{T}^f$ is, in a sense, a 'small' subset of $\mathscr{G}^f$. Obviously, a c-optimal minimizer cannot be modified on compact sets without losing the property of c-optimality. On the other hand the property of (f)-goodness is stable with respect to such modifications. Indeed, if $w_0 \in \mathscr{G}^f$ and if w_1 is a function in $W^{2,\gamma}_{loc}[0,\infty)$ such that $\{x \in \mathbb{R}_+ \mid w_0(x) \neq w_1(x)\}$ is bounded, then $w_1 \in \mathscr{G}^f$.

Nevertheless the class of c-optimal minimizers on $\mathbb{R}_+$ is a 'large' class in the following sense:

Proposition 2.3 ([11], Proposition 1.1). *For every point $x = (x_1, x_2) \in \mathbb{R}^2$ there exists a c-optimal minimizer w on $\mathbb{R}_+$ such that $\big(w(0), w'(0)\big) = x$.*

It is interesting to note that, in general, a c-optimal function on $\mathbb{R}_+$ cannot be extended to a c-optimal function on R. In fact, $\mathscr{T}^f(R)$ is a bounded set in $W^{1,\infty}(R)$ (see Lemma 3.7 of [11]) while, by our previous assertion, $\mathscr{T}^f$ is unbounded in $W^{1,\infty}(\mathbb{R}_+)$. In a generic sense the contrast is even more striking: $\mathscr{T}^f(R)$ is precisely the set of translates of a single periodic minimizer. Indeed, there exists a dense subset of $\mathscr{M}$ such that, for each f in this subset, problem (P_∞) possesses a unique (up to translation) periodic minimizer and every c-optimal function on R is a translate of the (unique) periodic minimizer [9].

Another class of minimizers that plays an important role in the theory developed in [9], [11] is the class of *perfect* minimizers, which is a subclass of $\mathscr{T}^f$. First we define the concept of a perfect function on an arbitrary interval. The definition requires some additional notation. For every $w \in \mathscr{G}^f$, put

$$E^f_\infty(w) := \liminf_{T\to\infty} E^f\big((0,T), w\big).$$

In a sense, $E^f_\infty(w)$ measures the distance between $I^f\big((0,T), w\big)$ and the target value $T\mu(f)$ as $T \to \infty$. For every $x \in \mathbb{R}^2$, put

$$\pi^f(x) := \inf\big\{E^f_\infty(w) \mid w \in \mathscr{G}^f, \big(w(0), w'(0)\big) = x\big\}. \tag{2.3}$$

It is known that $\pi^f \in C(\mathbb{R}^2)$ and $\pi^f(x) \to \infty$ as $|x| \to \infty$ [6], [11]. If $v \in W^{2,\gamma}(D)$, $D = (T_1, T_2)$, put

$$\Gamma^f(D,v) := I^f(D;v) - |D|\mu(f) + \pi^f\big((v,v')(T_2)\big) - \pi^f\big((v,v')(T_1)\big). \quad (2.4)$$

Given $x, y \in \mathbb{R}^2$ and $T > 0$, let $U_T^f(x,y)$ denote the infimum in problem $(P_{(0,T)}^{x,y})$. Then

$$\Gamma^f\big((0,T),v\big) \geq U_T^f(x,y) - T\mu(f) + \pi(y) - \pi(x) =: \Theta_T^f(x,y)$$

for every $v \in W^{2,\gamma}(0,T)$ such that $\big(v(0), v'(0)\big) = x$ and $\big(v(T), v'(T)\big) = y$. The following result, obtained by Leizarowitz and Mizel [6], adapts to the present problem a general principle concerning cost functions in infinite horizon problems, due to Leizarowitz ([5], Proposition 5.1).

Lemma 2.4. *Θ_T^f is non-negative and, for every $T > 0$ and every $x \in \mathbb{R}^2$, there exists $y \in \mathbb{R}^2$ such that $\Theta_T^f(x,y) = 0$.*

If D is a bounded interval and $w \in W^{2,\gamma}(D)$, then w is (f)*-perfect on* D if $\Gamma^f(D,w) = 0$. If U is an unbounded interval, we say that w is (f)*-perfect* on U if w is (f)-perfect on D for every bounded interval $D \subset U$. The family of (f)-perfect functions on U is denoted by $\mathscr{P}^f(U)$; the family of (f)-perfect functions on $\mathbb{R}_+$ is denoted briefly by $\mathscr{P}^f$.

If w is (f)-perfect on $D = (T_1, T_2)$ then: (a) w is a minimizer of problem $(P_D^{x,y})$ where $x = (w,w')(T_1)$, $y = (w,w')(T_2)$, and (b) w is (f)-perfect on every subinterval of D. These assertions follow immediately from the non-negativity of Θ_T^f and the additivity of Γ^f. Note also that the result of [6] quoted above implies the following.

Proposition 2.5 ([11], Proposition 1.2). *For every $x \in \mathbb{R}^2$ there exists an (f)-perfect function v on $\mathbb{R}_+$ such that $\big(v(0), v'(0)\big) = x$.*

The definition of a perfect function does not require boundedness. However the following result holds.

Proposition 2.6. (i) *If w is (f)-perfect on $\mathbb{R}_+$, then $w \in W^{1,\infty}(\mathbb{R}_+)$.*
(ii) *Every (f)-perfect function on $\mathbb{R}_+$ is a c-optimal minimizer of (P_∞).*

Obviously, every periodic minimizer of (P_∞) is (f)-perfect. Moreover,

$$\mathscr{S}^f \subset \mathscr{P}^f \subset \mathscr{T}^f \subset \mathscr{G}^f.$$

Note that $\mathscr{S}^f$ is a proper subset of $\mathscr{P}^f$. Indeed, by Proposition 2.3 of [9], $\mathscr{S}^f$ is bounded in $W^{1,\infty}(\mathbb{R}_+)$. On the other hand, by Proposition 2.5, $\mathscr{P}^f$ is unbounded in the norm of $W^{1,\infty}(\mathbb{R}_+)$. Obviously, $\mathscr{T}^f$ is a proper subset of $\mathscr{G}^f$. An interesting question is whether there exist c-optimal minimizers which are not perfect. (It was asked by the referee of the article [11].) The answer depends on the integrand f. If f possesses the periodic uniqueness property, i.e., (P_∞) has a unique (up to translation) periodic minimizer, then $\mathscr{P}^f = \mathscr{T}^f$ [11]. However, *there exists a family of integrands f for which $\mathscr{P}^f$ is a proper subset of $\mathscr{T}^f$.* A construction of such a family of integrands and other results pertaining to the non-uniqueness case are presented in this article. In order to obtain our results we use the following theorem of [11] which establish the *non-intersecting property* of c-optimal minimizers.

Theorem 2.7. (a) *Let v be a c-optimal minimizer of (P_∞). If there exists $T > 0$ such that $(v, v')(0) = (v, v')(T)$ then v is periodic with period T.*

(b) *Let v_1, v_2 be c-optimal minimizers of (P_∞) such that $(v_1, v_1')(0) = (v_2, v_2')(0)$. If there exist $t_1, t_2 \in [0, \infty)$ such that $(t_1, t_2) \neq (0, 0)$ and $(v_1, v_1')(t_1) = (v_2, v_2')(t_2)$, then $v_1 \equiv v_2$.*

Since in the present article we consider an arbitrary but fixed function $f \in \mathscr{M}$, the superscript f will be omitted in notation such as I^f, Γ^f, etc.

For every $T > 0$ and $x, y \in \mathbb{R}^2$ put

$$U_T(x, y) = \inf\{I(0, T, w) \mid w \in W^{2,\gamma}(0, T), (w, w')(0) = x, (w, w')(T) = y\}. \quad (2.5)$$

Denote by $|\cdot|$ the Euclidean norm. If $v \in W^{2,1}(D)$ put,

$$X_v(t) = \big(v(t), v'(t)\big), \qquad t \in D.$$

The following result, derived in [6], is based on a general principle concerning cost functions in infinite horizon problems, established by Proposition 5.1 of [5].

Proposition 2.8. *Let π be defined as in (2.3) and U_T as in (2.5). Then $\pi \in C(\mathbb{R}^2)$ and $(T, x, y) \to U_T^f(x, y)$ is continuous in $(0, \infty) \times \mathbb{R}^2 \times \mathbb{R}^2$. Furthermore, for every T, x, y as above,*

$$\Theta_T(x, y) = U_T(x, y) - T\mu(f) - \big(\pi(x) - \pi(y)\big) \geq 0, \quad (2.6)$$

and, for every $T > 0$ and every $x \in \mathbb{R}^2$, there exists $y \in \mathbb{R}^2$ such that $\Theta_T(x, y) = 0$.

The following simple but useful result was established in [9].

Proposition 2.9. *Let $D = (T_1, T_2)$ be a bounded interval and suppose that w_1, w_2 are perfect functions in D. If there exists $\tau \in D$ such that $(w_1, w_1')(\tau) = (w_2, w_2')(\tau)$ then $w_1 = w_2$ everywhere in D.*

3. Examples of periodic minimizers

We have already mentioned that every $f \in \mathcal{M}$ possesses a periodic minimizer [6], [12] and that many integrands f in $\mathcal{M}$ possess a unique (up to translation) periodic minimizer [9], [14]. One can ask if for any given natural number n there exists an integrand which possesses exactly (up to translation) n periodic minimizers and, moreover, if there exists an integrand with continuum different periodic minimizers. In this section we construct an example of an integrand which answers these questions in affirmative.

We proceed the construction of an example with three simple lemmas.

Note that for each pair of real numbers $d_2 > d_1 > 0$ there is C^∞-function $\phi : \mathbb{R}^1 \to [0, 1]$ such that

$$\begin{aligned} \phi(x) &= 1 \quad \text{if } |x| \le d_1, \\ 1 > \phi(x) &> 0 \quad \text{if } d_1 < |x| < d_2, \\ \phi(x) &= 0 \quad \text{if } |x| \ge d_2, \end{aligned}$$

and if

$$\psi(x) = 1 - \phi(x), \qquad x \in \mathbb{R}^1,$$

then $\psi \in C^\infty$ and

$$\begin{aligned} \psi(x) &= 0, \qquad x \in [-d_1, d_1], \\ 0 < \psi(x) &< 1 \quad \text{if } |x| \in (d_1, d_2), \\ \psi(x) &= 1 \quad \text{if } |x| \ge d_2. \end{aligned}$$

Now it is not difficult to prove the following auxiliary result.

Lemma 3.1. *Let $0 \le d_1 < d_2$. Then there is a C^∞-function $\phi : \mathbb{R}^1 \to [0, 1]$ such that*

$$\{x \in \mathbb{R}^1 \mid \phi(x) = 0\} = [d_1, d_2],$$

$\phi(x) \le 1$ for all $x \in \mathbb{R}^1$ and $\phi(x) = 1$ if $|x|$ is large enough.

Lemma 3.2. *Let $0 \le d_1 < d_2$. Then there is a C^∞-function $\psi : \mathbb{R}^1 \to [0, \infty)$ such that $\{x \in \mathbb{R}^1 \mid \psi(x) = 0\} = [d_1, d_2]$ and $\psi(x)/x^2 \to \infty$ as $|x| \to \infty$.*

Proof. Let ϕ be as guaranteed in Lemma 3.1. In order to prove the lemma it is sufficient to define

$$\psi(x) = \phi(x)(x^4 + 1), \qquad x \in \mathbb{R}^1. \qquad \square$$

Lemma 3.3. *Let $k \geq 1$ be an integer and let $l_i = [c_i, d_i]$, $i = 1, \dots, k$, where $0 \leq c_i \leq d_i$ for all $i = 1, \dots, k$ such that $l_i \cap l_j = \emptyset$ for all integers $i, j \in [1, \dots, k]$ satisfying $i \neq j$. Then there exists a C^∞-function $\phi : \mathbb{R}^1 \to [0, \infty)$ such that $\phi(x)/x^2 \to \infty$ as $|x| \to \infty$ and $\{x \in \mathbb{R}^1 \mid \phi(x) = 0\} = \bigcup_{i=1}^k l_i$.*

Proof. Let $i \in \{1, \dots, k\}$. We define $\psi_i : \mathbb{R}^1 \to [0, \infty)$ as follows. If $d_i > c_i$, then by Lemma 3.2 there is $\psi_i : \mathbb{R}^1 \to [0, \infty) \in C^\infty$ such that

$$\{x \in \mathbb{R}^1 \mid \psi_i(x) = 0\} = l_i, \qquad \psi_i(x)/x^2 \to \infty \qquad \text{as } |x| \to \infty.$$

If $c_i = d_i$ then set $\psi_i(x) = (x - c_i)^4$ for all $x \in \mathbb{R}^1$. Put

$$\phi(x) = \prod_{i=1}^k \phi_i(x), \qquad x \in \mathbb{R}^1.$$

It is not difficult to see that the lemma holds with the function ϕ. □

Construction of an integrand. Let k be a natural number and, for all $i = 1, \dots, k$, let $l_i = [c_i, d_i]$, $0 \leq c_i \leq d_i$, be such that $l_i \cap l_j = \emptyset$ for all integers $i, j \in \{1, \dots, k\}$ with $i \neq j$.

We construct an integrand f with the following properties:

(i) for each $s \in \bigcup_{i=1}^k l_i$ the function $v_s(t) = s^{1/2} \sin t$, $t \in \mathbb{R}^1$ is a periodic minimizer;

(ii) for each periodic minimizer v there exists $s \in \bigcup_{i=1}^k l_i$ such that v is a translation of v_s.

By Lemma 3.3 there is a function $\phi : \mathbb{R}^1 \to [0, \infty) \in C^\infty$ such that

$$\phi(x)/x^2 \to \infty \qquad \text{as } |x| \to \infty$$

and

$$\{x \in \mathbb{R}^1 \mid \phi(x) = 0\} = \bigcup_{i=1}^k l_i. \tag{3.1}$$

Define

$$f(w, p, r) = \phi(w^2 + p^2) + (w + r)^2, \qquad (w, p, r) \in \mathbb{R}^3. \tag{3.2}$$

Clearly there is $c_0 > 0$ such that $\phi(x) \geq 16x^2 - c_0$ for all $x \in \mathbb{R}^1$. Then

$$\begin{aligned} f(w,p,r) &\geq -c_0 + 16(w^2+p^2)^2 + w^2 + r^2 + 2wr \\ &\geq -c_0 + 16(w^2+p^2)^2 + w^2 + r^2 - (r^2/4 + 4w^2) \\ &\geq -c_0 + 3/4r^2 - 3w^2 + 16w^4 + 16p^4 \\ &\geq 3r^2/4 - c_0 + 10w^4 - 3 \end{aligned}$$

for any $(w,p,r) \in \mathbb{R}^3$. This implies that $f \in \mathscr{M}(\alpha,\beta,\gamma,a)$ where $\gamma = 2$, $a_3 \in (0,3/4)$, $1 < \alpha \leq 4$,

$$\min\{\alpha,\gamma\} > \beta \geq 1,$$

$a_2 > 0$, $a_1 \in (0,10)$, $a_4 > 3 + c_0$. Clearly $\mu(f) \geq 0$.

We show that the property (i) holds. Let $i \in \{1,\dots,k\}$ and $s \in [c_i,d_i]$. Consider the function

$$v_s(t) := s^{1/2}\sin t, \qquad t \in \mathbb{R}^1. \tag{3.3}$$

Clearly, $f\big(v_s(t), v_s'(t), v_s''(t)\big) = 0$ for all $t \in \mathbb{R}^1$ and thus v_s is a periodic (f)-minimizer. This implies that

$$\mu(f) = 0. \tag{3.4}$$

Let us show that the property (ii) hold. Assume that $v \in W^{2,1}_{\mathrm{loc}}(\mathbb{R}^1)$ is a periodic (f)-minimizer. Then, by (3.2) and (3.4),

$$f\big(v(t), v'(t), v''(t)\big) = 0$$

for almost all $t \in R$. Fix a real number t_0 such that

$$f\big(v(t_0), v'(t_0), v''(t_0)\big) = 0.$$

Combined with (3.2) this implies that

$$\phi\big(v(t_0)^2 + \big(v'(t_0)\big)^2\big) = 0.$$

It is clear that in view of (3.1) there exist $i \in \{1,\dots,k\}$ and $s \in [c_i,d_i]$ such that

$$v(t_0)^2 + \big(v'(t_0)\big)^2 = s.$$

Therefore there exists $\tau \in \mathbb{R}^1$ (depending only on s and t_0) such that $X_v(t_0) = X_{v_s}(t_0+\tau)$ and, by Proposition 2.9, $v(t) = v_s(t+\tau)$ for every t. Hence the properties (i) and (ii) hold.

It is easy to see that

$$\pi(x) \geq 0 \qquad \text{for all } x \in \mathbb{R}^2 \tag{3.5}$$

and that for each $i \in \{1, \dots k\}$, each $s \in [c_i, d_i]$ and each $z \in \{X_{v_s}(t) \mid t \in \mathbb{R}^1\}$ we have

$$\pi(z) = 0. \tag{3.6}$$

Proposition 3.4. *Let $v \in W^{2,1}_{\mathrm{loc}}(\mathbb{R}^1)$ be an (f)-perfect function such that*

$$\liminf_{t \to -\infty} |X_v(t)| < \infty.$$

Then there is $s \in \bigcup_{i=1}^{k} l_i$ such that v is a translation of v_s.

Proof. By Proposition 2.6 (i),

$$\sup\{|X_v(t)| \mid t \in \mathbb{R}^1\} < \infty.$$

Together with (2.4), the assumption that v is (f)-perfect, the continuity of π and (3.4), this implies that

$$\sup\{I(-T, T, v) \mid T \in (0, \infty)\} < \infty.$$

When combined with (3.2) the inequality above implies that for each $\varepsilon > 0$ there exist $T(\varepsilon) > 0$ such that for each $s_2 > s_1 \geq T(\varepsilon)$ and each $\sigma_1 < \sigma_2 < -T(\varepsilon)$,

$$I\big((s_1, s_2); v\big) \leq \varepsilon, \qquad I\big((\sigma_1, \sigma_2); v\big) \leq \varepsilon.$$

Let $z_0, z_1 \in \mathbb{R}^2$ be such that

$$z_0 = \lim_{i \to \infty} X_v(t_i), \qquad z_1 = \lim_{i \to \infty} X_v(s_i), \tag{3.7}$$

where $t_i \to \infty$ as $i \to \infty$, $s_i \to -\infty$ as $i \to \infty$.

Then using the standard arguments of our theory and the lower semicontinuity of integral functionals [1] (see, e.g., Proposition 2.3 of [11]) we obtain that there exists $u_0, u_1 \in W^{2,1}_{\mathrm{loc}}(\mathbb{R}^1)$ such that

$$X_{u_0}(0) = z_0, \qquad X_{u_1}(0) = z_1,$$
$$f\big(u_i(t), u_i'(t), u_i''(t)\big) = 0 \qquad \text{for a.e. } t \in \mathbb{R}^1,\ i = 1, 2.$$

Together with (3.1)–(3.3) and (3.6) this implies that

$$\phi\big(u_i(t)^2 + \big(u_i'(t)\big)^2\big) = 0, \qquad t \in \mathbb{R}^1,\ i = 1, 2,$$

$$z_0, z_1 \in \bigcup_{i=1}^{k} \bigcup_{s \in [c_i, d_i]} \{X_{v_s}(t) \mid t \in \mathbb{R}^1\},$$

and

$$\pi(z_0), \pi(z_1) = 0.$$

Together with (2.4), the assumptions of the proposition, (3.7) and (3.2) this implies that

$$\lim_{i\to\infty} I\big((s_i, t_i); v\big) = \lim_{i\to\infty} \pi\big(X_v(s_i)\big) - \pi\big(X_v(t_i)\big) = \pi(z_0) - \pi(z_1) = 0$$

and

$$f\big(v(t), v'(t), v''(t)\big) = 0 \qquad \text{for a.e. } t \in \mathbb{R}^1.$$

In view of (3.2),

$$\phi\big(v(t)^2 + \big(v'(t)\big)^2\big) = 0$$

for all $t \in \mathbb{R}^1$. Combined with (3.1) and (3.3) this implies that

$$\{X_v(t) \mid t \in \mathbb{R}^1\} \subset \bigcup_{i=1}^{k} \bigcup_{s \in [c_i, d_i]} \{X_{v_s}(t) \mid t \in \mathbb{R}^1\}.$$

Together with Proposition 2.9 this implies the validity of the proposition. □

4. An example of a c-optimal function which is not perfect

For any $v \in W^{2,1}_{\mathrm{loc}}(\mathbb{R}^1)$ put

$$\Omega(v) = \{X_v(t) \mid t \in \mathbb{R}^1\} = \big\{\big(v(t), v'(t)\big) \mid t \in R\big\}.$$

In this section we construct an example of an integrand $f \in \mathcal{M}(\alpha, \beta, \gamma, a)$, where $a = (a_1, a_2, a_3, a_4) \in \mathbb{R}^4$, $a_i > 0$, $i = 1, 2, 3, 4$ and α, β, γ are real numbers such that $1 \le \beta < \alpha, \beta \le \gamma$ and $\gamma > 1$. This integrand f will have the following properties:

(i) there exist two periodic minimizers v_1 and v_2 such that $\Omega(v_1) \cap \Omega(v_2) = \emptyset$;

(ii) every (f)-perfect function w on $\mathbb{R}^1$ which satisfies $\liminf_{t\to-\infty} |(w, w')(t)| < \infty$ is a translation of one of the periodic minimizers v_1 and v_2;

(iii) there exists a c-optimal minimizer w on $\mathbb{R}^1$ such that

$$\{(w, w')(t) \mid t \in \mathbb{R}^1\} \setminus \big(\Omega(v_1) \cup \Omega(v_2)\big) \neq \emptyset.$$

In view of (ii) and (iii) the integrand f possesses the c-optimal minimizer w which is not perfect.

Fix two real numbers c_1 and c_2 such that $0 \leq c_1 < c_2$ and define

$$\phi(x) = (x - c_1)^4 (x - c_2)^4, \qquad x \in \mathbb{R}^1,$$
$$f(w, p, r) = \phi(w^2 + p^2) + (w + r)^2, \qquad (w, p, r) \in \mathbb{R}^3.$$

Note that the integrand f belongs to the family of integrands introduced in Section 3 with $k = 2$, $d_i = c_i$, $i = 1, 2$.

Consider the functions

$$v_i(t) = c_i^{1/2} \sin t, \qquad t \in \mathbb{R}^1,$$

where $i = 1, 2$. As it was shown in Section 3, $\mu(f) = 0$, v_i is a periodic (f)-minimizer for $i = 1, 2$, $\pi(x) \geq 0$ for all $x \in \mathbb{R}^2$ and $\pi(z) = 0$ for each $z \in \Omega(v_1) \cup \Omega(v_2)$.

It is clear that Proposition 3.4 holds with the integrand f.

Proposition 4.1. *There exists a c-optimal minimizer v on $\mathbb{R}^1$ such that*

$$\{(v, v')(t) \mid t \in \mathbb{R}^1\} \setminus \big(\Omega(v_1) \cup \Omega(v_2)\big) \neq \emptyset.$$

Proof. Consider a sequence of positive numbers $T_i \to \infty$ as $i \to \infty$. For each integer $i \geq 1$ there exists $w_i \in W^{2,1}(-T_i, T_i)$ such that

$$(w_i, w_i')(-T_i) = (c_1^{1/2}, 0), \qquad (w_i, w_i')(T_i) = (c_2^{1/2}, 0) \tag{4.1}$$

and

$$I\big((-T_i, T_i); w_i\big) = U_{2T_i}\big((c_1^{1/2}, 0), (c_2^{1/2}, 0)\big). \tag{4.2}$$

By the mean value theorem for each integer $i \geq 1$ there exists $s_i \in (-T_i, T_i)$ such that

$$|(w_i, w_i')(s_i)| = (c_1^{1/2} + c_2^{1/2})/2. \tag{4.3}$$

By (4.1), (4.3) and Proposition 2.2 of [9] there exists $M_0 > 0$ such

$$\sup\{|(w_i, w_i')(t)| \mid t \in [-T_i, T_i]\} < M_0, \qquad i \geq 1. \tag{4.4}$$

We will show that $s_i + T_i \to \infty$. Let us assume the converse. Then, extracting a subsequence if necessary, we can assume without loss of generality that there exists $\lim_{i\to\infty} s_i + T_i < \infty$. For each integer $i \geq 1$ define $\tilde{w}_i \in W^{2,1}(0, T_i)$ by

$$\tilde{w}_i(t) = w_i(t - T_i), \qquad t \in [0, 2T_i]. \tag{4.5}$$

Clearly

$$\begin{aligned} (\tilde{w}_i, \tilde{w}_i')(s_i + T_i) &= (w_i, w_i')(s_i), \\ |(\tilde{w}_i, \tilde{w}_i')(s_i + T_i)| &= (c_2^{1/2} + c_1^{1/2})/2, \qquad i \geq 1. \end{aligned} \tag{4.6}$$

Using the standard arguments of our theory based on (4.1), (4.2), (4.4), continuity of the function $U_t(\cdot\,,\cdot)$ and Lemma 2.2 of [9] we can show that there exist a subsequence $\{\tilde{w}_{i_k}\}_{k=1}^{\infty}$ and $w \in W^{2,1}_{\mathrm{loc}}[0, \infty)$ such that for each integer $n \geq 1$,

$$\tilde{w}_{i_k} \to w \text{ as } k \to \infty \text{ weakly in } W^{2,\gamma}(0, n), \tag{4.7}$$

$$(\tilde{w}_{i_k}, \tilde{w}_{i_k}') \to (w, w') \text{ uniformly on } [0, n]. \tag{4.8}$$

Relations (4.8), (4.4), (4.5), (4.1) and (4.6) imply that

$$|(w, w')(t)| \leq M_0, \qquad t \in [0, \infty), \qquad \big(w(0), w'(0)\big) = (c_1^{1/2}, 0), \tag{4.9}$$

$$\Big|(w, w')\big(\lim_{i\to\infty}(s_i + T_i)\big)\Big| = (c_1^{1/2} + c_2^{1/2})/2. \tag{4.10}$$

By (4.1), (4.2), (4.7), (4.8), continuity of the function $U_t(\cdot\,,\cdot)$ and lower semicontinuity of the integral functional [1], w is a c-optimal minimizer. By (4.9),

$$(w, w')(0) = (v_1, v_1')(\pi/2) = (v_1, v_1')(\pi/2 + 2\pi).$$

By of Theorem 2.7 (b), $w(t) = v_1(t + \pi/2)$ for all $t \in \mathbb{R}^1$. This contradicts to (4.10). Therefore $s_i + T_i \to \infty$ as $i \to \infty$.

Using the fact that $f(w, p, r) = f(w, -p, r)$ for all $(w, p, r) \in \mathbb{R}^3$ and using the same arguments for the functions $\bar{w}_i(t) = w_i(-t)$, $t \in [-T_i, T_i]$, $i \geq 1$, we obtain that $T_i - s_i \to \infty$ as $i \to \infty$. Thus we have shown that

$$\lim_{i\to\infty}(s_i + T_i) = \infty \qquad \text{and} \qquad \lim_{i\to\infty}(T_i - s_i) = \infty. \tag{4.11}$$

For each integer $i \geq 1$ define $u_i \in W^{2,1}(-T_i - s_i, T_i - s_i)$ by

$$u_i(t) = w_i(t + s_i), \qquad t \in [-T_i - s_i, T_i - s_i]. \tag{4.12}$$

Using again the standard arguments of our theory based on (4.1), (4.2), (4.5), continuity of the function $U_t(\cdot\,,\cdot)$ and Lemma 2.2 of [9] we can show that there exists a subsequence $\{u_{i_k}\}_{k=1}^{\infty}$ and $u \in W^{2,1}_{\mathrm{loc}}(\mathbb{R}^1)$ such that for each integer $n \geq 1$,

$$u_{i_k} \to u \text{ as } k \to \infty \text{ weakly in } W^{2,\gamma}(-n,n), \tag{4.13}$$

$$(u_{i_k}, u'_{i_k}) \to (u,u') \text{ uniformly on } [-n,n]. \tag{4.14}$$

Relations (4.14), (4.4), (4.3) and (4.12) imply that

$$|(u,u')(t)| \le M_0, \qquad t \in \mathbb{R}^1, \qquad |(u,u')(0)| = (c_1^{1/2} + c_2^{1/2})/2. \tag{4.15}$$

By (4.1), (4.2), (4.13), (4.14), continuity of the function $U_t(\cdot,\cdot)$ and and lower semicontinuity of the integral functional, u is a c-optimal minimizer. By (4.15), $(u,u')(0) \notin \Omega(v_1) \cup \Omega(v_2)$. Proposition 4.1 is proved. □

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