Portugal. Math. (N.S.)
Vol. 66, Fasc. 2, 2009, 175–190

Portugaliae Mathematica

Holomorphic extension theorem for tempered ultrahyperfunctions

Daniel H. T. Franco*

(Communicated by Miguel Ramos)

Abstract. In this article we are concerned with the space of tempered ultrahyperfunctions corresponding to a proper open convex cone. A holomorphic extension theorem (the version of the celebrated Edge-of-the-Wedge Theorem) will be given for this setting. As application, a version is also given of the principle of determination of an analytic function by its values on a non-empty open real set. The article finishes with the generalization of holomorphic extension theorem à la Martineau.

Mathematics Subject Classification (2000). 46F12, 46F15, 46F20.

Keywords. Tempered ultrahyperfunctions, Edge-of-the-Wedge Theorem, Fourier–Laplace transform.

1. Introduction

Sebastião e Silva [17], [18] and Hasumi [7] introduced the space of tempered ultradistributions, which has been studied by many authors, among others we refer the reader to [23], [11], [12], [2], [3], [4], [19], [20], [1], [15], [6]. Here, as Morimoto [11] and [12], we shall refer to the tempered ultradistributions as *tempered ultrahyperfunctions* in order to distinguish them from various other classes of ultradistributions which have been described as tempered (see, for example, Pathak [13] and Pilipovic [14]). Tempered ultrahyperfunctions are the strong dual of the space of test functions of rapidly decreasing entire functions in any horizontal strip. While Sebastião e Silva [17] used extension procedures for the Fourier transform combined with holomorphic representations and considered the 1-dimensional case, Hasumi [7] used duality arguments in order to extend the notion of tempered ultrahyperfunctions for the case of n dimensions (see also [18], Section 11). In a brief tour, Marimoto [11] gave some more precise informations concerning the

*This work was supported by the Fundação de Amparo à Pesquisa do Estado de Minas Gerais (FAPEMIG) agency, grant CEX00012/07.

work of Hasumi. More recently, the relation between the tempered ultrahyperfunctions and Schwartz distributions and some major results, as the kernel theorem and the Fourier–Laplace transform have been established by Brüning–Nagamachi in [1]. Earlier, some precisions on the Fourier–Laplace transform theorem for tempered ultrahyperfunctions were given by Carmichael [2] (see also [6]), by considering the theorem in its simplest form, i.e., the equivalence between support properties of a distribution in a closed convex cone and the holomorphy of its Fourier–Laplace transform in a suitable tube with conical basis. In this more general setting, which includes the results of Sebastião e Silva and Hasumi as special cases, Carmichael obtained new representations of tempered ultrahyperfunctions which were not considered in [17], [18], [7].

The purpose of this article is to prove the Edge-of-the-Wedge Theorem for the setting of tempered ultrahyperfunctions corresponding to a proper open convex cone. This classical theorem in complex analysis, discovered by theoretical physicists in 1950s [22], deals with the question about the principle of holomorphic continuation of functions of several complex variables, which arose in physics in the study of the Wightman functions and Green functions, or in connection with the dispersion relations in quantum field theory. It should be mentioned that other versions of the theorem for tempered ultrahyperfunctions can be found in [12], [20]. Our approach to this problem is different from that taken in [12], [20]. Our construction parallels that of Carmichael [2], [4], [3] and, in particular, the proof of the Edge-of-the-Wedge Theorem is inspired by Carmichael's work [3]. As an immediate application of the Edge-of-the-Wedge Theorem, we give also a proof of the principle of determination of an analytic function by its values on a non-empty open real set. We finish with a generalized version of holomorphic extension theorem à la Martineau.

We note that the results obtained here are of interest in the construction and study of *quasilocal* quantum field theories (where the fields are localizable only in regions greater than a certain scale of nonlocality), since Brüning–Nagamachi [1] have recently shown the importance of tempered ultrahyperfunctions for quantum field theories with a fundamental length. This is the case of a quantum field theory in non-commutative spacetimes [5].

2. Notation and definitions

The following multi-index notation is used without further explanation. Let $\mathbb{R}^n$ (resp. $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$) be the real (resp. complex) n-space whose generic points are denoted by $x = (x_1, \ldots, x_n)$ (resp. $z = (z_1, \ldots, z_n)$) such that $x + y = (x_1 + y_1, \ldots, x_n + y_n)$, $\lambda x = (\lambda x_1, \ldots, \lambda x_n)$, $x \geq 0$ means that $x_1 \geq 0, \ldots, x_n \geq 0$, $\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n$ and $|x| = |x_1| + \cdots + |x_n|$. Moreover, we define $\alpha =$

$(\alpha_1, \dots, \alpha_n) \in \mathbb{N}_o^n$, where $\mathbb{N}_o$ is the set of non-negative integers, such that the length of α is the corresponding ℓ^1-norm $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha + \beta$ denotes $(\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$, $\alpha \geq \beta$ means $(\alpha_1 \geq \beta_1, \dots, \alpha_n \geq \beta_n)$, $\alpha! = \alpha_1! \dots \alpha_n!$, $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$, and

$$D^\alpha \varphi(x) = \frac{\partial^{|\alpha|} \varphi(x_1, \dots, x_n)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_1} \dots \partial x_n^{\alpha_n}}.$$

Let Ω be a set in $\mathbb{R}^n$. Then we denote by Ω° the interior of Ω and by $\overline{\Omega}$ the closure of Ω. For $r > 0$, we denote by $B(x_o; r) = \{x \in \mathbb{R}^n \mid |x - x_o| < r\}$ a open ball and by $B[x_o; r] = \{x \in \mathbb{R}^n \mid |x - x_o| \leq r\}$ a closed ball, with center at point x_o and of radius r, respectively.

We consider two n-dimensional spaces—x-space and ξ-space—with the Fourier transform defined

$$\hat{f}(\xi) = \mathscr{F}[f(x)](\xi) = \int_{\mathbb{R}^n} f(x) e^{i\langle \xi, x \rangle} d^n x,$$

while the Fourier inversion formula is

$$f(x) = \mathscr{F}^{-1}[\hat{f}(\xi)](x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{-i\langle \xi, x \rangle} d^n \xi.$$

The variable ξ will always be taken real while x will also be complexified—when it is complex, it will be noted $z = x + iy$. The above formulas, in which we employ the symbolic "function notation", are to be understood in the sense of distribution theory.

3. Tempered ultrahyperfunctions

Since the theory of ultrahyperfunctions is not too well known, we shall introduce briefly in this section some definitions and basic properties of the tempered ultrahyperfunction space of Sebastião e Silva [17], [18] and Hasumi [7] (we indicate the References for more details) used throughout the article. To begin with, we shall consider the function

$$h_K(\xi) = \sup_{x \in K} \langle \xi, x \rangle, \qquad \xi \in \mathbb{R}^n,$$

where K is a compact set in $\mathbb{R}^n$. One calls $h_K(\xi)$ the *supporting function* of K. We note that $h_K(\xi) < \infty$ for every $\xi \in \mathbb{R}^n$ since K is bounded. For sets $K = [-k, k]^n$, $0 < k < \infty$, the supporting function $h_K(\xi)$ can be easily determined:

$$h_K(\xi) = \sup_{x \in K} \langle \xi, x \rangle = k|\xi|, \qquad \xi \in \mathbb{R}^n,\ |\xi| = \sum_{i=1}^{n} |\xi_i|.$$

Let K be a convex compact subset of $\mathbb{R}^n$, then $H_{\mathrm{b}}(\mathbb{R}^n; K)$ (b stands for bounded) defines the space of all functions in $C^\infty(\mathbb{R}^n)$ such that $e^{h_K(\xi)} D^\alpha \varphi(\xi)$ is bounded in $\mathbb{R}^n$ for any multi-index α. One defines in $H_{\mathrm{b}}(\mathbb{R}^n; K)$ seminorms

$$\|\varphi\|_{K,N} = \sup_{\substack{\xi \in \mathbb{R}^n \\ \alpha \leq N}} \{e^{h_K(\xi)} |D^\alpha \varphi(\xi)|\} < \infty, \qquad N = 0, 1, 2, \ldots.$$

If $K_1 \subset K_2$ are two compact convex sets, then $h_{K_1}(\xi) \leq h_{K_2}(\xi)$, and thus the canonical injection $H_{\mathrm{b}}(\mathbb{R}^n; K_2) \hookrightarrow H_{\mathrm{b}}(\mathbb{R}^n; K_1)$ is continuous. Let O be a convex open set of $\mathbb{R}^n$. To define the topology of $H(\mathbb{R}^n; O)$ it suffices to let K range over an increasing sequence of convex compact subsets $K_1, K_2, \ldots$ contained in O such that $K_i \subset K_{i+1}^\circ$ and $O = \bigcup_{i=1}^\infty K_i$ for each $i = 1, 2, \ldots$. Then the space $H(\mathbb{R}^n; O)$ is the projective limit of the spaces $H_{\mathrm{b}}(\mathbb{R}^n; K)$ according to restriction mappings above, i.e.,

$$H(\mathbb{R}^n; O) = \lim_{K \subset O} \operatorname{proj}\ H_{\mathrm{b}}(\mathbb{R}^n; K),$$

where K runs through the convex compact sets contained in O.

Theorem 3.1 ([7], [11], [1]). *The space $\mathscr{D}(\mathbb{R}^n)$ of all C^∞-functions on $\mathbb{R}^n$ with compact support is dense in $H(\mathbb{R}^n; K)$ and $H(\mathbb{R}^n; O)$. The space $H(\mathbb{R}^n; \mathbb{R}^n)$ is dense in $H(\mathbb{R}^n; O)$ and in $H(\mathbb{R}^n; K)$, and $H(\mathbb{R}^m; \mathbb{R}^m) \otimes H(\mathbb{R}^n; \mathbb{R}^n)$ is dense in $H(\mathbb{R}^{m+n}; \mathbb{R}^{m+n})$.*

From Theorem 3.1 we have the following injections [11]:

$$H'(\mathbb{R}^n; K) \hookrightarrow H'(\mathbb{R}^n; \mathbb{R}^n) \hookrightarrow \mathscr{D}'(\mathbb{R}^n),$$

and

$$H'(\mathbb{R}^n; O) \hookrightarrow H'(\mathbb{R}^n; \mathbb{R}^n) \hookrightarrow \mathscr{D}'(\mathbb{R}^n).$$

Definition 3.2. The dual space $H'(\mathbb{R}^n; O)$ of $H(\mathbb{R}^n; O)$ is the space of distributions of exponential growth, V, represented as a finite order derivative of continuous functions of exponential growth

$$V = D_\xi^\gamma [e^{h_K(\xi)} g(\xi)],$$

where $g(\xi)$ is a bounded continuous function.

In the space $\mathbb{C}^n$ of n complex variables $z_i = x_i + iy_i$, $1 \le i \le n$, we denote by $T(\Omega) = \mathbb{R}^n + i\Omega \subset \mathbb{C}^n$ the tubular set of all points z such that $y_i = \operatorname{Im} z_i$ belongs to the domain Ω, i.e., Ω is a connected open set in $\mathbb{R}^n$ called the basis of the tube $T(\Omega)$. Let K be a convex compact subset of $\mathbb{R}^n$. Then $\mathfrak{H}_b(T(K))$ defines the space of all continuous functions φ on $T(K)$ which are holomorphic in the interior $T(K^\circ)$ of $T(K)$ such that the estimate

$$|\varphi(z)| \le M_{K,N}(\varphi)(1+|z|)^{-N} \tag{1}$$

is valid. The best possible constants in (1) are given by a family of seminorms in $\mathfrak{H}_b(T(K))$

$$\|\varphi\|_{K,N} = \inf\{M_{K,N}(\varphi) \mid \sup_{z \in T(K)} \{(1+|z|)^N |\varphi(z)|\} < \infty, N = 0,1,2,\dots\}. \tag{2}$$

If $K_1 \subset K_2$ are two convex compact sets, we have that the canonical injection

$$\mathfrak{H}_b(T(K_2)) \hookrightarrow \mathfrak{H}_b(T(K_1)), \tag{3}$$

is continuous.

Given that the spaces $\mathfrak{H}_b(T(K_i))$ are Fréchet spaces, with topology defined by the seminorms (2), the space $\mathfrak{H}(T(O))$ is characterized as a projective limit of Fréchet spaces:

$$\mathfrak{H}(T(O)) = \lim_{K \subset O} \operatorname{proj} \mathfrak{H}_b(T(K)),$$

where K runs through the convex compact sets contained in O and the projective limit is taken following the restriction mappings above.

Let K be a convex compact set in $\mathbb{R}^n$. Then the space $\mathfrak{H}(T(K))$ is characterized as a inductive limit

$$\mathfrak{H}(T(K)) = \lim_{K_1 \supset K} \operatorname{ind} \mathfrak{H}_b(T(K_1)),$$

where K_1 runs through the convex compact sets such that K is contained in the interior of K_1 and the inductive limit is taken following the restriction mappings (3).

For any element $U \in \mathfrak{H}'$, its Fourier transform is defined to be a distribution V of exponential growth such that the Parseval-type relation

$$\langle V, \varphi \rangle = \langle U, \psi \rangle, \qquad \varphi \in H,\ \psi = \mathscr{F}[\varphi] \in \mathfrak{H},$$

holds. In the same way, the inverse Fourier transform of a distribution V of exponential growth is defined by the relation

$$\langle U, \psi \rangle = \langle V, \varphi \rangle, \qquad \psi \in \mathfrak{H}, \; \varphi = \mathscr{F}^{-1}[\psi] \in H.$$

It follows from the Fourier transform and Theorem 3.1 the

Theorem 3.3 ([11], [1]). *$\mathfrak{H}(T(\mathbb{R}^n))$ is dense in $\mathfrak{H}(T(O))$ and in $\mathfrak{H}(T(K))$, and $\mathfrak{H}(T(\mathbb{R}^{m+n}))$ is dense in $\mathfrak{H}(T(O))$.*

Proposition 3.4 ([11]). *If $f \in H(\mathbb{R}^n; O)$, the Fourier transform of f belongs to the space $\mathfrak{H}(T(O))$, for any open convex non-empty set $O \subset \mathbb{R}^n$. By the dual Fourier transform $H'(\mathbb{R}^n; O)$ is topologically isomorphic with the space $\mathfrak{H}'(T(-O))$.*

Definition 3.5. A tempered ultrahyperfunction is a continuous linear functional defined on the space of test functions $\mathfrak{H}(T(\mathbb{R}^n))$ of rapidly decreasing entire functions in any horizontal strip.

The space of all tempered ultrahyperfunctions is denoted by $\mathscr{U}(\mathbb{R}^n)$. As a matter of fact, these objects are equivalence classes of holomorphic functions defined by a certain space of functions which are analytic in the 2^n octants in $\mathbb{C}^n$ and represent a natural generalization of the notion of hyperfunctions on $\mathbb{R}^n$ but are *non-localizable*. The space $\mathscr{U}(\mathbb{R}^n)$ is characterized in the following way [7]: Let $\mathscr{H}_\omega$ be the space of all functions $f(z)$ such that (i) $f(z)$ is analytic for $\{z \in \mathbb{C}^n \mid |\mathrm{Im}\, z_1| > p, |\mathrm{Im}\, z_2| > p, \ldots, |\mathrm{Im}\, z_n| > p\}$, (ii) $f(z)/z^p$ is bounded continuous in $\{z \in \mathbb{C}^n \mid |\mathrm{Im}\, z_1| \geqq p, |\mathrm{Im}\, z_2| \geqq p, \ldots, |\mathrm{Im}\, z_n| \geqq p\}$, where $p = 0, 1, 2, \ldots$ depends on $f(z)$ and (iii) $f(z)$ is bounded by a power of z, $|f(z)| \leq \mathrm{C}(1 + |z|)^N$, where C and N depend on $f(z)$. Define the *kernel* of the mapping $f : \mathfrak{H}(T(\mathbb{R}^n)) \to \mathbb{C}$ by Π, the set of all z-dependent pseudo-polynomials, $z \in \mathbb{C}^n$ (a pseudo-polynomial is a function of z of the form $\sum_s z_j^s G(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n)$, such that $G(z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) \in \mathscr{H}_\omega$). Then $f(z) \in \mathscr{H}_\omega$ belongs to the kernel $\boldsymbol{\Pi}$ if and only if $\langle f(z), \psi(x) \rangle = 0$, with $\psi(x) \in \mathfrak{H}(T(\mathbb{R}^n))$ and $x = \mathrm{Re}\, z$. The space of tempered ultrahyperfunctions is the quotient space $\mathscr{U} = \mathscr{H}_\omega / \Pi$. Thus, we have the

Theorem 3.6 (Hasumi [7], Proposition 5). *The space of tempered ultrahyperfunctions $\mathscr{U}$ is algebraically isomorphic to the space of generalized functions $\mathfrak{H}'$.*

4. The space of holomorphic functions $\mathscr{H}_c^o$

We start by introducing some terminology and simple facts concerning cones. An open set $C \subset \mathbb{R}^n$ is called a cone if $\mathbb{R}_+ \cdot C \subset C$. A cone C is an open connected cone if C is an open connected set. Moreover, C is called convex if $C + C \subset C$ and *proper* if it contains no any straight line. A cone C' is called compact in C—we write $C' \Subset C$—if the projection $\mathrm{pr}\, \bar{C}' \stackrel{\mathrm{def}}{=} \bar{C}' \cap S^{n-1} \subset \mathrm{pr}\, C \stackrel{\mathrm{def}}{=} C \cap S^{n-1}$, where S^{n-1} is the unit sphere in $\mathbb{R}^n$. Being given a cone C in y-space, we associate with C a closed convex cone C^* in ξ-space which is the set $C^* = \{\xi \in \mathbb{R}^n \mid \langle \xi, y \rangle \geq 0 \text{ for all } y \in C\}$. The cone C^* is called the *dual cone* of C. In the sequel, it will be sufficient to assume for our purposes that the open connected cone C in $\mathbb{R}^n$ is an open convex cone with vertex at the origin and proper. By $T(C)$ we will denote the set $\mathbb{R}^n + iC \subset \mathbb{C}^n$. If C is open and connected, $T(C)$ is called the tubular radial domain in $\mathbb{C}^n$, while if C is only open $T(C)$ is referred to as a tubular cone. In the former case we say that $f(z)$ has a boundary value $U = BV(f(z))$ in $\mathfrak{H}'$ as $y \to 0$, $y \in C$ or $y \in C' \Subset C$, respectively, if for all $\psi \in \mathfrak{H}$ the limit

$$\langle U, \psi \rangle = \lim_{\substack{y \to 0 \\ y \in C \text{ or } C'}} \int_{\mathbb{R}^n} f(x+iy)\psi(x) d^n x$$

exists. An important example of tubular radial domain used in quantum field theory is the tubular radial domain with the forward light-cone, V_+, as its basis

$$V_+ = \left\{ z \in \mathbb{C}^n \mid \operatorname{Im} z_1 > \left(\sum_{i=2}^{n} \operatorname{Im}^2 z_i \right)^{1/2}, \operatorname{Im} z_1 > 0 \right\}.$$

We will deal with tubes defined as the set of all points $z \in \mathbb{C}^n$ such that

$$T(C) = \{x + iy \in \mathbb{C}^n \mid x \in \mathbb{R}^n, y \in C, |y| < \delta\},$$

where $\delta > 0$ is an arbitrary number.

Let C be a proper open convex cone, and let $C' \Subset C$. Let $B[0; r]$ denote a *closed* ball of the origin in $\mathbb{R}^n$ of radius r, where r is an arbitrary positive real number. Denote $T(C'; r) = \mathbb{R}^n + i(C' \backslash (C' \cap B[0; r]))$. We are going to introduce a space of holomorphic functions which satisfy a certain estimate according to Carmichael [2]. We want to consider the space consisting of holomorphic functions $f(z)$ such that

$$|f(z)| \leq M(C')(1 + |z|)^N e^{h_{C^*}(y)}, \qquad z \in T(C'; r), \tag{4}$$

where $h_{C^*}(y) = \sup_{\xi \in C^*} \langle \xi, y \rangle$ is the supporting function of C^*, $M(C')$ is a constant that depends on an arbitrary compact cone C' and N is a non-negative real number. The set of all functions $f(z)$ which are holomorphic in $T(C'; r)$ and satisfy the estimate (4) will be denoted by $\mathscr{H}_c^o$.

Remark 4.1. The space of functions $\mathscr{H}_c^o$ constitutes a generalization of the space $\mathfrak{A}_\omega^i$ of Sebastião e Silva [17] and the space a_ω of Hasumi [7] to arbitrary tubular radial domains in $\mathbb{C}^n$.

Lemma 4.2 ([2], [6]). *Let C be an open convex cone, and let $C' \Subset C$. Let $h(\xi) = e^{k|\xi|}g(\xi)$, $\xi \in \mathbb{R}^n$, be a function with support in C^*, where $g(\xi)$ is a bounded continuous function on $\mathbb{R}^n$. Let y be an arbitrary but fixed point of $\big(C' \backslash (C' \cap B[0;r])\big)$. Then $e^{-\langle \xi, y\rangle}h(\xi) \in L^2$, as a function of $\xi \in \mathbb{R}^n$.*

Definition 4.3. We denote by $H'_{C^*}(\mathbb{R}^n; O)$ the subspace of $H'(\mathbb{R}^n; O)$ of distributions of exponential growth with support in the cone C^*:

$$H'_{C^*}(\mathbb{R}^n; O) = \{V \in H'(\mathbb{R}^n; O) \,|\, \operatorname{supp}(V) \subseteq C^*\}.$$

Lemma 4.4 ([2], [6]). *Let C be an open convex cone, and let $C' \Subset C$. Let $V = D_\xi^\gamma[e^{h_K(\xi)}g(\xi)]$, where $g(\xi)$ is a bounded continuous function on $\mathbb{R}^n$ and $h_K(\xi) = k|\xi|$ for a convex compact set $K = [-k, k]^n$. Let $V \in H'_{C^*}(\mathbb{R}^n; O)$. Then $f(z) = (2\pi)^{-n}\langle V, e^{-i\langle \xi, z\rangle}\rangle$ is an element of $\mathscr{H}_c^o$.*

5. The space of holomorphic functions $\mathscr{H}_c^{*o}$

We now shall introduce another space of holomorphic functions whose elements are analytic in a domain $T(C')$ which is larger than $T(C';r)$ and has boundary values in $\mathbb{R}^n$. The boundary values so obtained are of importance in the representation of vacuum expectation values in the case of a quantum field theory in noncommutatives spacetimes [5].

Let C be a proper open convex cone, and let $C' \Subset C$. Let $B(0;r)$ denote an *open* ball of the origin in $\mathbb{R}^n$ of radius r, where r is an arbitrary positive real number. Denote $T(C';r) = \mathbb{R}^n + i\big(C' \backslash \big(C' \cap B(0;r)\big)\big)$. Throughout this section, we consider functions $f(z)$ which are holomorphic in $T(C') = \mathbb{R}^n + iC'$ and which satisfy the estimate (4), with $B[0;r]$ replaced by $B(0;r)$. We denote this space by $\mathscr{H}_c^{*o}$. We note that $\mathscr{H}_c^{*o} \subset \mathscr{H}_c^o$ for any open convex cone C. Put $\mathscr{U}_c = \mathscr{H}_c^{*o}/\Pi$, that is, $\mathscr{U}_c$ is the quotient space of $\mathscr{H}_c^{*o}$ by set of pseudo-polynomials Π.

Definition 5.1. The set $\mathscr{U}_c$ is the subspace of the tempered ultrahyperfunctions generated by $\mathscr{H}_c^{*o}$ corresponding to a proper open convex cone $C \subset \mathbb{R}^n$.

The following theorems will be important to us in the proof of Edge-of-the-Wedge Theorem.

Theorem 5.2. *Let C be an open convex cone, and let $C' \Subset C$. Let $V = D_\xi^\gamma h(\xi)$, where $h(\xi) = e^{h_K(\xi)}g(\xi)$ with $g(\xi)$ being a bounded continuous function on $\mathbb{R}^n$ and $h_K(\xi) = k|\xi|$ for a convex compact set $K = [-k, k]^n$. Let $V \in H'_{C^*}(\mathbb{R}^n; O)$. Then:*

(i) *$f(z) = (2\pi)^{-n}\langle V, e^{-i\langle \xi, z\rangle}\rangle$ is an element of $\mathscr{H}_c^{*o}$,*

(ii) *$\{f(z) \,|\, y = \operatorname{Im} z \in C' \Subset C, |y| \le Q\}$ is a strongly bounded set in $\mathfrak{H}'\big(T(O)\big)$, where Q is an arbitrarily but fixed positive real number,*

(iii) $f(z) \to \mathscr{F}^{-1}[V] \in \mathfrak{H}'(T(O))$ *in the strong (and weak) topology of* $\mathfrak{H}'(T(O))$ *as* $y = \operatorname{Im} z \to 0$, $y \in C' \Subset C$.

Theorem 5.3. *Let* $f(z) \in \mathscr{H}_c^{*o}$ *where* C *is an open convex cone. Then the distribution* $V \in H'_{C^*}(\mathbb{R}^n; O)$ *has a uniquely determined inverse Fourier–Laplace transform* $f(z) = (2\pi)^{-n}\langle V, e^{-i\langle \xi, z\rangle}\rangle$, *which is holomorphic in* $T(C')$ *and satisfies the estimate (4), with* $B[0;r]$ *replaced by* $B(0;r)$.

The Theorem 5.2 shows that functions in $\mathscr{H}_c^{*o}$ have distributional boundary values in $\mathfrak{H}'(T(O))$. Further, it shows that functions in $\mathscr{H}_c^{*o}$ satisfy a strong boundedness property in $\mathfrak{H}'(T(O))$. On the other hand, the Theorem 5.3 shows that the functions $f(z) \in \mathscr{H}_c^{*o}$ can be recovered as the (inverse) Fourier–Laplace transform of the constructed distribution $V \in H'_{C^*}(\mathbb{R}^n; O)$. This result is a version of the Paley–Wiener–Schwartz theorem in the tempered ultrahyperfunction set-up.

Remark 5.4. It is important to note that in Theorems 5.2 and 5.3 we are considering the inverse Fourier–Laplace transform $f(z) = (2\pi)^{-n}\langle V, e^{-i\langle \xi, z\rangle}\rangle$, in contrast to the Fourier–Laplace transform used in [2], [4].

Sketch of Proof of Theorem 5.2. In order to prove (i), we can proceed as in the proof of [6], Lemma 2, and obtain the equality

$$f(z) = (2\pi)^{-n}\langle V, e^{-i\langle \xi, z\rangle}\rangle, \qquad z \in T(C'; r), \tag{5}$$

with $B[0;r]$ replaced by $B(0;r)$ in the estimate (4). The equality (5) holds pointwise for arbitrary compact subcones C' of C and for arbitrary $r > 0$. Since C is open, for any $y \in C$ there is a compact subcone C' of C and a $r > 0$ such that $y \in (C' \backslash (C' \cap B(0;r)))$. Hence any $z \in T(C')$ is in $T(C';r)$ for some $C' \subset C$ and some $r > 0$. Thus we can conclude that (i) is obtained from (5). The proofs of (ii) and (iii) are similar to the proofs of the eqs. (35) and (36) in [2], Theorem 3. □

Proof of Theorem 5.3. Consider

$$h_y(\xi) = \int_{\mathbb{R}^n} \frac{f(z)}{P(iz)} e^{i\langle \xi, z\rangle} d^n x, \qquad z \in T(C'; r), \tag{6}$$

with $h_y(\xi) = e^{k|\xi|} g_y(\xi)$, where $g(\xi)$ is a bounded continuous function on $\mathbb{R}^n$ and $P(iz) = (-i)^{|\gamma|} z^\gamma$. By hypothesis $f(z) \in \mathscr{H}_c^{*o}$ and satisfies (4), with $B[0;r]$ replaced by $B(0;r)$. For this reason, for an n-tuple $\gamma = (\gamma_1, \ldots, \gamma_n)$ of non-negative integers conveniently chosen, we obtain that

$$\left|\frac{f(z)}{P(iz)}\right| \le M(C')(1+|z|)^{-n-\varepsilon}e^{h_{c^*}(y)}, \tag{7}$$

where n is the dimension and ε is any fixed positive real number. This implies that the function $h_y(\xi)$ exists and is a continuous function of ξ. Further, by using arguments paralleling the analysis in [21], p. 225, and the Cauchy–Poincaré Theorem [21], p. 198, we can show that the function $h_y(\xi)$ is independent of $y = \operatorname{Im} z$. Therefore, we denote the function $h_y(\xi)$ by $h(\xi)$.

From (7) we have that $f(z)/P(iz) \in L^2$ as a function of $x = \operatorname{Re} z \in \mathbb{R}^n$, $y \in C' \backslash (C' \cap B(0;r))$. Hence, from (6) and the Plancherel theorem we have that $e^{-\langle \xi, y\rangle}h(\xi) \in L^2$ as a function of $\xi \in \mathbb{R}^n$, and

$$\frac{f(z)}{P(iz)} = \mathscr{F}^{-1}[e^{-\langle \xi, y\rangle}h(\xi)](x), \qquad z \in T(C';r), \tag{8}$$

where the inverse Fourier transform is in the L^2 sense. Here, Parseval's equation holds:

$$(2\pi)^{-n}\int_{\mathbb{R}^n} |e^{-\langle \xi, y\rangle}h(\xi)|^2 d^n\xi = \int_{\mathbb{R}^n}\left|\frac{f(z)}{P(iz)}\right|^2 d^n x.$$

In this case for the eq. (8) to be true, ξ must belong to the open half-space $\{\xi \in C^* \mid \langle \xi, y\rangle < 0\}$ for $y \in C' \backslash (C' \cap B(0;r))$, since by hypothesis $f(z) \in \mathscr{H}_c^{*o}$. Then there is $\delta(C')$ such that for $y \in C' \backslash (C' \cap B(0;r))$ we have $\langle \xi, y\rangle \le -\delta(C')|\xi|\,|y|$. This justifies the negative sign in (8) (see Remark 5.4).

Now, if $h(\xi) \in H'_{C^*}(\mathbb{R}^n;O)$, then $V = D_\xi^\gamma h(\xi) \in H'_{C^*}(\mathbb{R}^n;O)$. Since C^* is a regular set [16], pp. 98–99, it follows that $\operatorname{supp}(h) = \operatorname{supp}(V)$. By Theorem 5.2, $\langle V, e^{-i\langle \xi, z\rangle}\rangle$ exists as a holomorphic function of $z \in T(C')$ and satisfies the estimate (4), with $B[0;r]$ replaced by $B(0;r)$. A simple calculation yields that

$$(2\pi)^{-n}\langle V, e^{-i\langle \xi, z\rangle}\rangle = P(iz)\mathscr{F}^{-1}[e^{-\langle \xi, y\rangle}h(\xi)](x), \qquad z \in T(C';r). \tag{9}$$

In view of Lemma 4.2, the inverse Fourier transform can be interpreted in L^2 sense. Combining (8) and (9), we have $f(z) = (2\pi)^{-n}\langle V, e^{-i\langle \xi, z\rangle}\rangle$ for $z \in T(C';r)$. Since $r > 0$ is arbitrary, this equality holds for each $z \in T(C')$. The uniqueness follows from the isomorphism of the dual Fourier transform, according to Proposition 3.4. This completes the proof of the theorem. □

6. Edge-of-the-Wedge Theorem

In what follows, we formulate a version of the Edge-of-the-Wedge Theorem for the space of the tempered ultrahyperfunctions in its simplest form: the common analytic continuation of two functions $f_1(z)$ and $f_2(z)$ holomorphic respectively in the two tubes $\mathbb{R}^n + iC_j$, $j = 1, 2$, where each C_j is an open convex cone.

Theorem 6.1 (Edge-of-the-Wedge Theorem). *Let C be an open cone of the form $C = C_1 \cup C_2$, where each C_j, $j = 1, 2$, is a proper open convex cone. Denote by* $\mathrm{ch}(C)$ *the convex hull of the cone C. Assume that the distributional boundary values of two holomorphic functions $f_j(z) \in \mathscr{H}^{*o}_{c_j}$ $(j = 1, 2)$ agree, that is, $U = BV(f_1(z)) = BV(f_2(z))$, where $U \in \mathfrak{H}'(T(O))$ in accordance with the Theorem 5.2. Then there exists $F(z) \in \mathscr{H}^{o}_{\mathrm{ch}(C)}$ such that $F(z) = f_j(z)$ on the domain of definition of each $f_j(z)$, $j = 1, 2$.*

Proof. By hypothesis $BV(f_1(z)) = BV(f_2(z))$ in $\mathfrak{H}'(T(O))$, and we call this common value U. By Theorem 5.2, we have that $BV(f_j(z)) = \mathscr{F}^{-1}[V_j]$, $j = 1, 2$. On the other hand, this implies that $V_j = \mathscr{F}[BV(f_j(z))]$. But, according to Theorem 5.3 there exists a unique $V_j \in H'_{C_j^*}(\mathbb{R}^n; O)$, $j = 1, 2$, such that $f_j(z) = (2\pi)^{-n}\langle V_j, e^{-i\langle \xi, z\rangle}\rangle$. Using these facts we have that $V_1 = V_2$ in $H'_{C^*}(\mathbb{R}^n; O)$. We call this common value V and thus have $U = \mathscr{F}^{-1}[V]$. By Theorem 2 in [3], $\mathrm{supp}(V) \subseteq \{\xi \in \mathbb{R}^n \mid \langle \xi, y\rangle \geq 0 \text{ for all } y \in \mathrm{ch}(C)\}$, then by Definition 4.3 $V \in H'_{(\mathrm{ch}(C))^*}(\mathbb{R}^n; O)$.

We now put

$$F(z) = (2\pi)^{-n}\langle V, e^{-i\langle \xi, z\rangle}\rangle, \qquad z \in T(\mathrm{ch}(C)) = \mathbb{R}^n + i\,\mathrm{ch}(C). \tag{10}$$

with $V \in H'_{(\mathrm{ch}(C))^*}(\mathbb{R}^n; O)$. Since $\mathrm{ch}(C)$ is an open convex cone, we have by exactly the proof of [6], Lemma 2, that $F(z) \in \mathscr{H}^{o}_{\mathrm{ch}(C)}$. Further, using the fact that $V_1 = V_2 = V$, from Theorem 5.2 we have that

$$f_j(z) = (2\pi)^{-n}\langle V_j, e^{-i\langle \xi, z\rangle}\rangle = (2\pi)^{-n}\langle V, e^{-i\langle \xi, z\rangle}\rangle, \qquad z \in T(C_j'). \tag{11}$$

Thus combining (10) and (11) we have that $F(z)$ coincides with $f_j(z)$, $j = 1, 2$, on the domain of definition of each $f_j(z)$. □

Corollary 6.2. *Suppose that the hypotheses of Theorem* 6.1 *hold with C_1 and C_2 opposite to each other. Then $F(z)$ is a polynomial in $z \in \mathbb{C}^n$.*

Proof. Similar to the proof of [3], Corollary 1. □

The following theorem is an immediate consequence of the Edge-of-the-Wedge Theorem and reflects a of the most important principle governing the behaviour of analytic functions, that is, the determination of a function by its values on a non-empty open real set.

Theorem 6.3. *Let C be some open convex cone. Let $f(z) \in \mathscr{H}_c^{*o}$. If the distributional boundary value $BV(f(z))$ of $f(z)$ in the sense of tempered ultrahyperfunctions vanishes, then the function $f(z)$ itself vanishes.*

Proof. Define $g(x+iy) = \overline{f(x-iy)}$. The function $g(z)$ is holomorphic in $\overline{T(C')} = \mathbb{R}^n - iC'$, satisfies (4), with $B[0;r]$ replaced by $B(0;r)$, in $\overline{T(C';r)} = \mathbb{R}^n - i\big(C' \backslash \big(C' \cap B(0;r)\big)\big)$, and approaches 0 as $y \to 0$. Thus we can apply the Edge-of-the-Wedge Theorem to f and g. Since $\mathrm{ch}\big(C \cup (-C)\big) = \mathbb{R}^n$, by Corollary 6.2, $F(z)$, the common analytic continuation of f and g, is a polynomial in $z \in \mathbb{C}^n$. But by hypothesis $BV\big(F(z)\big)$ vanishes as a distribution and therefore as a function together with $f(z)$ identically. □

7. The Martineau Edge-of-the-Wedge Theorem for tempered ultrahyperfunctions

The great advance in the theory of the Edge-of-the-Wedge Theorem came with the realization due to Martineau [8], [9], [10], who was able to prove its version for the case involving more than two functions holomorphic respectively in the tubes $\mathbb{R}^n + iC_j$, $j = 1, \dots, m$. In what follows, we formulate a version of the Martineau's Edge-of-the-Wedge Theorem for the space of the tempered ultrahyperfunctions.

Theorem 7.1 (Generalized Edge of the Wedge Theorem). *Let $C_1, \dots, C_m$ be proper open convex cones in $\mathbb{R}^n$. Given any set of m open convex cones C_j' such that $C_j' \Subset C_j$, $j = 1, \dots, m$, then the following two properties of a set of m functions $f_j(z) \in \mathscr{H}_{C_j}^{*o}$ $(j = 1, \dots, m)$ are equivalent:*

P$_1$: *The distributional boundary value $U = \sum_{j=1}^m BV\big(f_j(z)\big) \in \mathfrak{H}'\big(T(O)\big)$ vanishes identically.*

P$_2$: *Denote by $\mathrm{ch}(C_j \cup C_k)$ the convex hull of $C_j \cup C_k$. For each pair of indices (j,k), $1 \le j,k \le m$, there is a holomorphic function $g_{jk}(z) \in \mathscr{H}^o_{\mathrm{ch}(C_j \cup C_k)}$ such that $g_{jk}(z) + g_{kj}(z) = 0$ for all $j,k = 1, \dots, m$—thus $g_{jj}(z) = 0$ for all $j,k = 1, \dots, m$—and such that $f_j(z) = \sum_{k=1}^m g_{jk}(z)$ on $T(C_j') = \mathbb{R}^n + iC_j'$ for each $j = 1, \dots, m$.*

For our proof of Theorem 7.1 we prepare a lemma on the analytic decomposability of $\mathfrak{H}'\big(T(O)\big)$. Let C be an open cone of the form $C = \bigcup_{j=1}^m C_j$, $m < \infty$, where each C_j is an proper open convex cone. If we write $C' \Subset C$, we mean $C' = \bigcup_{j=1}^m C_j'$ with $C_j' \Subset C_j$. Furthermore, we define by $C_j^* = \{\xi \in \mathbb{R}^n \mid \langle \xi, x \rangle \ge 0 \text{ for all } x \in C_j\}$ the dual cones of C_j such that the dual cones C_j^*, $j = 1, \dots, m$, have the properties

$$\mathbb{R}^n \Big\backslash \bigcup_{j=1}^{m} C_j^*, \tag{12}$$

and

$$C_j^* \cap C_k^*, \qquad j \neq k,\ j,k = 1,\ldots,m, \tag{13}$$

are sets of Lebesgue measure zero. Assume that $V \in H'_{C^*}(\mathbb{R}^n; O)$ can be written as $V = \sum_{j=1}^m V_j$, where we define

$$V_j = D_\xi^\gamma[e^{h_K(\xi)}\lambda_j(\xi)g(\xi)],$$

with $\lambda_j(\xi)$ denoting the characteristic function of C_j^*, $j = 1,\ldots,m$, $g(\xi)$ being a bounded continuous function on $\mathbb{R}^n$, and $h_K(\xi) = k|\xi|$ for a convex compact set $K = [-k,k]^n$.

Lemma 7.2. *Let C be an open cone of the form $C = \bigcup_{j=1}^m C_j$, $m < \infty$, where the C_j are proper open convex cones such that* (12) *and* (13) *are satisfied. Let $U \in \mathfrak{H}'(T(O))$. Then $U = \sum_{j=1}^m BV(f_j(z))$, where each $BV(f_j(z))$ is the strong boundary value in $\mathfrak{H}'(T(O))$ of a function $f_j(z) \in \mathscr{H}_{c_j}^{*o}$ and such that each $BV(f_j(z)) = \mathscr{F}^{-1}[V_j]$, with $V_j \in H'_{C_j^*}(\mathbb{R}^n; O)$, $j = 1,\ldots,m$.*

Proof. This result follows using the same method adopted in the proof of [2], Theorem 4, by replacing the reference to Theorems 2 and 3 with a reference to the Theorem 5.2 of this article. □

Proof of Theorem 7.1. $P_2 \Rightarrow P_1$. Assume that $g_{jk}(z) \in \mathscr{H}^o_{\mathrm{ch}(C_j \cup C_k)}$. By hypothesis, we have

$$f_j(z) = \sum_{k=1}^m g_{jk}(z), \qquad z \in T(C_j').$$

Then

$$BV(f_j(z)) = BV\Big(\sum_{k=1}^m g_{jk}(z)\Big) \qquad \text{as } C_j' \ni y \to 0.$$

Hence,

$$\sum_{j=1}^m BV(f_j(z)) = \sum_{j=1}^m \Big(BV\Big(\sum_{k=1}^m g_{jk}(z)\Big)\Big) = BV\Big(\sum_{j=1}^m\sum_{k=1}^m g_{jk}(z)\Big) \equiv 0,$$

taking into account the anti-symmetry of the functions $g_{jk}(z)$.

Proof that $P_1 \Rightarrow P_2$. If $m = 1$, $P_1 \Rightarrow f_1 \equiv 0$ by Theorem 6.3. Henceforth we assume that $m \geq 2$. Let $U_j = BV(f_j(z)) \in \mathfrak{H}'(T(O))$ as $C'_j \ni y \to 0$. Then there exists $U_{jk} \in \mathfrak{H}'(T(O))$ such that $U_j = \sum_{k=1}^m U_{jk}$ with the restriction that $U_{jk} + U_{kj} = 0$. Thus $\sum_{j=1}^m U_j \equiv 0$. By Theorem 5.2, we have that $U_j = \mathscr{F}^{-1}[V_j]$, $j = 1, \ldots, m$. On the other hand, this implies that $V_j = \mathscr{F}[U_j] = \mathscr{F}[\sum_{k=1}^m U_{jk}] = \sum_{k=1}^m \mathscr{F}[U_{jk}] = \sum_{k=1}^m V_{jk}$. Since $\sum_{j=1}^m U_j \equiv 0$, it follows that $\mathscr{F}[\sum_{j=1}^m U_j] = \sum_{j=1}^m \mathscr{F}[U_j] = \sum_{j=1}^m V_j = \sum_{j=1}^m \sum_{k=1}^m V_{jk} \equiv 0$. This yields that $V_{jk} + V_{kj} = 0$. According to Theorem 5.3 there exists a unique $V_j \in H'_{C_j^*}(\mathbb{R}^n; O)$, $j = 1, \ldots, m$, such that

$$
\begin{aligned}
f_j(z) &= (2\pi)^{-n} \langle V_j, e^{-i\langle \xi, z\rangle} \rangle \\
&= (2\pi)^{-n} \Big\langle \sum_{k=1}^m V_{jk}, e^{-i\langle \xi, z\rangle} \Big\rangle = \sum_{k=1}^m \big((2\pi)^{-n} \langle V_{jk}, e^{-i\langle \xi, z\rangle} \rangle \big).
\end{aligned}
$$

We now put

$$
g_{jk}(z) = (2\pi)^{-n} \langle V_{jk}, e^{-i\langle \xi, z\rangle} \rangle, \qquad z \in T\big(\mathrm{ch}(C_j \cup C_k)\big), \tag{14}
$$

with $V_{jk} \in H'_{(\mathrm{ch}(C_j \cup C_k))^*}(\mathbb{R}^n; O)$ and $\mathrm{supp}(V_{jk}) \subseteq \{\xi \in \mathbb{R}^n \mid \langle \xi, y\rangle \geq 0 \text{ for all } y \in \mathrm{ch}(C_j \cup C_k)\}$. Since $\mathrm{ch}(C_j \cup C_k)$ is an open convex cone, we again have by exactly the proof as of [6], Lemma 2, that $g_{jk}(z) \in \mathscr{H}^o_{\mathrm{ch}(C_j \cup C_k)}$. Further, from Theorem 5.2 it follows that

$$
f_j(z) = (2\pi)^{-n} \langle V_j, e^{-i\langle \xi, z\rangle} \rangle = \sum_{k=1}^m \big((2\pi)^{-n} \langle V_{jk}, e^{-i\langle \xi, z\rangle} \rangle \big), \qquad z \in T(C'_j). \tag{15}
$$

Thus combining (14) and (15) we have that $f_j(z) = \sum_{k=1}^m g_{jk}(z)$ on $T(C'_j)$ with the restriction that $g_{jk} + g_{kj} = 0$. This completes the proof. □

Acknowledgments. The author would like to express his gratitude to Afrânio R. Pereira, Winder A. M. Melo, and to the Departament of Physics of the Universidade Federal de Viçosa (UFV) for the opportunity of serving as Visiting Researcher. The author thanks to the referee for her/his remarks which improved the presentation of the results.

References

[1] E. Brüning and S. Nagamachi, Relativistic quantum field theory with a fundamental length. *J. Math. Phys.* **45** (2004), 2199–2231. Zbl 1071.81078 MR 2059688

[2] R. D. Carmichael, Distributions of exponential growth and their Fourier transforms. *Duke Math. J.* **40** (1973), 765–783. Zbl 0276.46019 MR 0341074

[3] R. D. Carmichael, Distributional boundary values and the tempered ultra-distributions. *Rend. Sem. Mat. Univ. Padova* **56** (1976), 101–123. Zbl 0384.46028 MR 0492350

[4] R. D. Carmichael, The tempered ultra-distributions of J. Sebastião e Silva. *Portugal. Math.* **36** (1977), 119–137. Zbl 0445.46029 MR 577417

[5] D. H. T. Franco, J. A. Lourenço, and L. H. Renoldi, The ultrahyperfunctional approach to non-commutative quantum field theory. *J. Phys. A: Math. Theor.* **41** (2008), 095402. Zbl 1134.81046 MR 2453745

[6] D. H. T. Franco and L. H. Renoldi, A note on Fourier-Laplace transform and analytic wave front set in theory of tempered ultrahyperfunctions. *J. Math. Anal. Appl.* **325** (2007), 819–829. Zbl 1105.42005 MR 2270053

[7] M. Hasumi, Note on the n-dimensional tempered ultra-distributions. *Tôhoku Math. J.* (2) **13** (1961), 94–104. Zbl 0103.09201 MR 0131759

[8] A. Martineau, Distributions et valeurs au bord des fonctions holomorphes. In *Theory of distributions* (Proc. Internat. Summer Inst., Lisbon, 1964), Inst. Gulbenkian Ci., Lisbon 1964, 193–326. MR 0219754

[9] A. Martineau, Théorèmes sur le prolongement analitique du type "edge of the wedge theorem", Sémin. Bourbaki 10 (1966–1968), Exp. No. 340. Zbl 0209.14802 MR 201676

[10] A. Martineau, Le "edge of the wedge theorem" en théorie des hyperfonctions de Sato. In *Proc. Internat. Conf. on Functional Analysis and Related Topics* (Tokyo, 1969), Univ. of Tokyo Press, Tokyo 1970, 95–106. Zbl 0193.41503 MR 0267128

[11] M. Morimoto, Theory of tempered ultrahyperfunctions. I, II. *Proc. Japan Acad.* **51** (1975), 87–91; ibid. **51** (1975), 213–218. Zbl 0348.46029 0348.46030 MR 0380396

[12] M. Morimoto, Convolutors for ultrahyperfunctions. In *Internat. Symp. Math. Probl. Theoret. Phys.* (Kyoto Univ., Kyoto, 1975), Lecture Notes in Phys. 39., Springer, Berlin 1975, 49–54. Zbl 0323.46036 MR 0631540

[13] R. S. Pathak, Tempered ultradistributions as boundary values of analytic functions. *Trans. Amer. Math. Soc.* **286** (1984), 537–556. Zbl 0529.46031 MR 760974

[14] S. Pilipović, Tempered ultradistributions. *Boll. Un. Mat. Ital. B* (7) **2** (1988), 235–251. Zbl 0657.46030 MR 946067

[15] A. U. Schmidt, Asymptotic hyperfunctions, tempered hyperfunctions, and asymptotic expansions. *Internat. J. Math. Math. Sci.* (2005), 755–788. Zbl 1098.46030 MR 2173691

[16] L. Schwartz, *Méthodes mathématiques pour les sciences physiques*. Enseignement des Sciences, Hermann, Paris 1961. Zbl 0101.41301 MR 0143360

[17] J. Sebastião e Silva, Les fonctions analytiques comme ultra-distributions dans le calcul opérationnel. *Math. Ann.* **136** (1958), 58–96. Zbl 0195.41302 MR 0105615

[18] J. Sebastião e Silva, Les séries de multipôles des physiciens et la théorie des ultradistributions. *Math. Ann.* **174** (1967), 109–142. Zbl 0152.13102 MR 0217597

[19] J. Sousa Pinto, Silva tempered ultradistributions. *Portugal. Math.* **47** (1990), 267–292. Zbl 0739.46028 MR 1090268

[20] M. Suwa, Distributions of exponential growth with support in a proper convex cone. *Publ. Res. Inst. Math. Sci.* **40** (2004), 565–603. Zbl 1069.46023 MR 2049647

[21] V. S. Vladimirov, *Methods of the theory of functions of many complex variables*. The M.I.T. Press, Cambridge, Mass., 1966. Zbl 0125.31904 MR 0201669

[22] V. S. Vladimirov, V. V. Zharinov, and A. G. Sergeev, Bogolyubov's "edge of the wedge" theorem, its development and applications. *Uspekhi Mat. Nauk* **49** (1994), 47–60; English transl. *Russian Math. Surveys* **49**, No. 5 (1994), 51–65. Zbl 0861.32005 MR 1311229

[23] Z. Zieleźny, On the space of convolution operators in $\mathscr{K}_1'$. *Studia Math.* **31** (1968), 111–124. Zbl 0182.45603 MR 0248520

Received February 29, 2008; revised June 18, 2008

D. H. T. Franco, Universidade Federal de Viçosa, Departamento de Física, Avenida Peter Henry Rolfs s/n, Campus Universitário, CEP 36570-000, Viçosa—MG, Brasil
E-mail: dhtfranco@gmail.com