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On the existence of positive solutions for a nonhomogeneous elliptic system

Ederson Moreira dos Santos*

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Abstract. The main goal of this work is to prove the existence and in some cases the uniqueness of positive solutions for some nonhomogeneous systems of Lane–Emden type. The nonhomogeneous part must satisfy a suitable positiveness condition, but it may change sign.

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1. Introduction and statement of the results

In the past years, a special attention has been devoted to the extension from equations to systems of known results concerning existence, uniqueness and multiplicity of solutions, as well as other qualitative properties of the solutions. Closely related to the problems treated in this paper, we cite the results in [13], [6], [20] concerning nonhomogeneous equations, and the results in [5], [7], [12], [14], [15], [17], [18], [19] either about homogeneous or about nonhomogeneous systems.

Let Ω be a smooth bounded domain in $\mathbb{R}^N$ with $N \geq 1$. We consider a nonhomogeneous elliptic system of the form

$$\begin{cases} -\Delta u = au + bv + v^p + \varepsilon f(x) & \text{in } \Omega, \\ -\Delta v = cu + dv + u^q + \varepsilon g(x) & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u, v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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which turns to be of Hamiltonian type in the case that $a = d$. Here a, b, c, d are constants, p and q are positive powers, $\varepsilon > 0$ is a parameter, $f, g \in C^1(\overline{\Omega})$ and we are concerned with the existence of at least one classical solution for the system (1.1).

Throughout in this paper we denote the 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

by A. Here λ_1 and φ_1 stand for the first eigenvalue of $(-\Delta, H_0^1(\Omega))$ and the first positive eigenfunction of $(-\Delta, H_0^1(\Omega))$ with $\int \varphi_1 \, dx = 1$, respectively. When necessary, we will write $\lambda_1 = \lambda_{1,\Omega}$ and $\varphi_1 = \varphi_{1,\Omega}$ to ensure that we are referring to λ_1 and φ_1 as above and to emphasize their dependence on Ω.

In the next three paragraphs we assume that $A \equiv 0$ in the system (1.1).

Collecting the results from this paper and from [14], [15] and comparing them with the results in [6] concerning the equation

$$\begin{cases} -\Delta u = \lambda u + u^p + \varepsilon f(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

with $\lambda = 0$, one can classify the system (1.1) with $A \equiv 0$ as: sublinear if $pq < 1$ and superlinear if $pq > 1$.

The case $pq < 1$ is fully treated in this paper.

Some contributions to the case $pq > 1$ can be found here and also in [14], [15], but some questions still remain as open problems. For example, in the case where one of the powers p or q belongs to $(0, 1)$, the nonexistence of solution for $\varepsilon > 0$ large enough is a conjecture.

Once we are interested on positive solutions, it is natural to impose some positiveness condition on the nonhomogeneous part. For that, given a function $h \in C^1(\overline{\Omega})$, we denote by u_h the solution of

$$\begin{cases} -\Delta u_h = h(x) & \text{in } \Omega, \\ u_h = 0 & \text{on } \partial\Omega. \end{cases}$$

In this work we assume:

(P) $f, g \in C^1(\overline{\Omega})$ are not simultaneously identically zero and $u_f, u_g \geq 0$ in Ω.

The matrix A, which appears in (1.1), plays the role that λ plays in (1.2). Here we assume:

(H1) A is cooperative, that is, $b, c \geq 0$.

(H2) The eigenvalues of A are smaller than λ_1.

The hypothesis $\lambda \in (-\infty, \lambda_1)$ appears in [13] in the study of (1.2). It is precisely the necessary and sufficient condition for the existence of a strong maximum principle for the linear equation

$$\begin{cases} -\Delta u = \lambda u + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

in the case with $f \geq 0$ in Ω.

In the case that A is cooperative, that is, when (H1) is satisfied, it is proved in [11] that the hypothesis (H2) is precisely the necessary and sufficient condition for the existence of a strong maximum principle for the linear cooperative system

$$\begin{cases} -\Delta u = au + bv + f(x) & \text{in } \Omega, \\ -\Delta v = cu + dv + g(x) & \text{in } \Omega, \\ u, v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

in the case with $f, g \geq 0$ in Ω.

In this way, the hypothesis (H2) on the cooperative matrix A for the study of the system (1.1) is just the natural extension of the hypothesis $\lambda \in (-\infty, \lambda_1)$ for the study of the equation (1.2).

In Section 2, we will see that if one replaces the condition $f \geq 0$ in Ω by the condition $u_f \geq 0$ in Ω, then a maximum principle for (1.3) is not valid for every $\lambda \in (-\infty, \lambda_1)$. In addition to that, we will also see that a maximum principle for (1.4) does not hold true under (H1), (H2) and (P).

At this point we can state our first result.

Theorem 1.1. *Assume* (P), (H1)–(H2), $p, q > 1$ *and*

(H3) $a \geq 0$ *if* f *changes sign;* $d \geq 0$ *if* g *changes sign*

are satisfied. Then there exists $\varepsilon^* \in (0, +\infty)$ *such that: the system* (1.1) *has a minimal positive solution* $(u_\varepsilon, v_\varepsilon)$ *for* $0 < \varepsilon < \varepsilon^*$ *and it has no solution for* $\varepsilon > \varepsilon^*$. *Furthermore,* $(u_\varepsilon, v_\varepsilon) \in C^{2,\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\overline{\Omega})$ *for all* $\alpha \in (0, 1)$.

Theorem 1.1 extends a result in [5], [17] and the hypothesis $p, q > 1$, with no surprise, makes possible the application of the method of subsolution and supersolution. Besides other arguments, our proof for Theorem 1.1 is based on the continuation of λ_1 with relation to Ω as presented in [4], [8]. We stress that such proof allows us to obtain all the results in [13] concerning (1.2), replacing the conditions $f \geq 0$ in Ω and $f \equiv 0$ on $\partial\Omega$, which are imposed in [13], by weaker conditions, namely: $u_f \geq 0$ in Ω for $\lambda \in [0, \lambda_1)$; $f \geq 0$ in Ω for $\lambda \in (-\infty, 0)$. Such proof also extends the application of the method of subsolution and supersolution, since we present a new class of supersolutions for certain problems. For example, for some problems in [1], [9], [10] and for the problem (1.2). At such examples, it is important to note that $\frac{1}{|e_1|_\infty} < \lambda_1$, with e_1 as defined at the beginning of Section 4 in this paper.

The next result, which deals with the system (1.1) with $A \equiv 0$, presents something more interesting. With the hypothesis $pq > 1$, we show that it is possible to treat it by means of the method of subsolution and supersolution. In this way, cases where one of the equations is sublinear and the other is superlinear are included.

Theorem 1.2. *Assume that* $p, q > 0$, $pq > 1$, (P) *holds and that* $A \equiv 0$.

(i) *If* $p, q \geq 1$, *then there exists* $\varepsilon^* \in (0, +\infty)$ *such that for each* $0 < \varepsilon < \varepsilon^*$ *the system* (1.1) *has a minimal positive solution* $(u_\varepsilon, v_\varepsilon)$ *and it has no solution for* $\varepsilon > \varepsilon^*$. *Furthermore,* $(u_\varepsilon, v_\varepsilon) \in C^{2,\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\overline{\Omega})$ *for all* $\alpha \in (0, 1)$.

(ii) *If* $p < 1$ *or* $q < 1$, *then there exists* $\varepsilon^* > 0$ *such that for each* $0 < \varepsilon < \varepsilon^*$ *the system* (1.1) *has a minimal positive solution* $(u_\varepsilon, v_\varepsilon)$. *Furthermore,* $(u_\varepsilon, v_\varepsilon) \in C^{2,\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\overline{\Omega})$ *with* $\alpha = \min\{p, q\}$.

For the system (1.1) with $A \equiv 0$, $pq > 1$ and in the case that one of the powers p, q belongs to $(0, 1)$, the question about the nonexistence of solution for $\varepsilon > 0$ large enough still remains as an open problem, but we conjecture that $\varepsilon^* < +\infty$ at item (ii) in Theorem 1.2.

We have one more result concerning (1.1), now in the case that $A \equiv 0$ and $pq < 1$. In such case, the system (1.1) has a sublinear behavior and the next theorem extends some of the results in [18] and [19].

Theorem 1.3. *In addition to* (P) *suppose that* $p, q > 0$, $pq < 1$ *and that* $A \equiv 0$. *Then for each* $\varepsilon > 0$, *the system* (1.1) *has a unique solution* (u, v). *Furthermore,* $(u, v) \in C^{2,\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\overline{\Omega})$ *with* $\alpha = \min\{p, q\}$.

For the context of this paper it is important to know that functions f, g satisfying (P) are not necessarily nonnegative. For instance, see [2], [3], [14] and also Section 2 in this paper.

In Section 2, we show some results concerning the existence of a maximum principle either for the equation (1.3) under $u_f \geq 0$ in Ω or for the system (1.4) under (P). In Sections 3–5, we present the proofs for Theorems 1.1–1.3, respectively.

2. On the maximum principle

In this section we discuss about necessary and sufficient conditions for the existence of a maximum principle, either for the equation (1.3) with $u_f \geq 0$ in Ω or for the cooperative system (1.4) under the positiveness hypothesis (P).

2.1. The scalar case. In the case with $f \geq 0$ in Ω, it is well known that a strong maximum principle exists for the equation (1.3) if and only if $\lambda \in (-\infty, \lambda_1)$.

Here we investigate for which values of λ the condition $u_f \geq 0$ in Ω implies that the solution u of the equation (1.3) is such that $u \geq 0$ in Ω. Actually we prove:

Proposition 2.1. *If $\lambda \in [0, \lambda_1)$, then for every $f \in C^1(\overline{\Omega})$ such that $u_f \geq 0$ in Ω, the solution u of the equation* (1.3) *satisfies $u \geq 0$ in Ω. The converse is also true.*

Proof. Suppose that $\lambda \in [0, \lambda_1)$. Given $f \in C^1(\overline{\Omega})$ such that $u_f \geq 0$ in Ω, let u be the classical solution of (1.3). So

$$\begin{cases} -\Delta(u - u_f) = \lambda(u - u_f) + \lambda u_f & \text{in } \Omega, \\ u - u_f = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $\lambda u_f \geq 0$ in Ω, by the classical maximum principle, one has that $u - u_f \geq 0$ in Ω, which implies that $u(x) \geq u_f(x) \geq 0$ for all $x \in \Omega$.

On the other hand, suppose that λ has the following property: given any $f \in C^1(\overline{\Omega})$ with $u_f \geq 0$ in Ω, the equation (1.3) has a solution u and it satisfies $u \geq 0$ in Ω. It is shown in the lines below show that λ must necessarily be in $[0, \lambda_1)$.

If λ has such property, then for $f = \varphi_1$ one has that

$$\lambda_1 \int u\varphi_1 \, dx = \int (-\Delta u)\varphi_1 \, dx = \int (\lambda u\varphi_1 + \varphi_1^2) \, dx > \lambda \int u\varphi_1 \, dx.$$

And the last inequality implies that $\lambda < \lambda_1$. Now, by contradiction, suppose that $\lambda < 0$. Fix $u_0 \in C^\infty(\overline{\Omega}) \backslash \{0\}$ such that $u_0 \geq 0$ in Ω, $u_0 \equiv 0$ on $\partial\Omega$ and $u_0(x_0) = 0$ for some $x_0 \in \Omega$. Take $f = -\Delta u_0$ and let u be the solution of the equation (1.3) associated to such f. So

$$\begin{cases} -\Delta(u - u_0) = \lambda(u - u_0) + \lambda u_0 & \text{in } \Omega, \\ u - u_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $\lambda u_0 \leq 0$ in Ω, by the classical strong maximum principle, one gets that $u - u_0 < 0$ in Ω. In particular, $u(x_0) < u_0(x_0) = 0$, which is a contradiction. □

2.2. The system case. Let $M = (m_{ij})$ be an $n \times n$ cooperative matrix and $F = (f_i)_{i=1}^n$, such that each f_i stands for a regular function defined on Ω. The linear system

$$\begin{cases} -\Delta U = MU + F(x) & \text{in } \Omega, \\ U = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

with $U = (u_i)_{i=1}^n$, was studied in [11] by de Figueiredo–Mitidieri. Theorem 2.1 in [11] gives the necessary and sufficient conditions for the existence of a maximum principle for the system (2.1), in the case with $f_i \geq 0$ in Ω for all $i = 1, \dots, n$. With $n = 2$, such conditions are equivalent to (H1) and (H2) in this paper.

Here we first present a sufficient condition for the existence of a maximum principle for the system (1.4) with the positiveness condition (P) satisfied.

Proposition 2.2. *Assume* (H1)–(H2) *and that* $a, d \geq 0$. *Then for every pair* (f, g) *satisfying* (P), *the solution pair* (u, v) *of the system* (1.4) *satisfies* $u, v \geq 0$ *in* Ω.

Proof. Let (u, v) be the solution of (1.4). Then

$$\begin{cases} -\Delta(u - u_f) = a(u - u_f) + b(v - u_g) + au_f + bu_g & \text{in } \Omega, \\ -\Delta(v - u_g) = c(u - u_f) + d(v - u_g) + cu_f + du_g & \text{in } \Omega, \\ u - u_f, v - u_g = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Since $a, b, c, d \geq 0$, one has that $au_f + bu_g, cu_f + du_g \geq 0$ in Ω. Then Theorem 2.1 in [11] guarantees that $u - u_f \geq 0$, $v - u_g \geq 0$ in Ω and so that $u \geq u_f \geq 0$, $v \geq u_g \geq 0$ in Ω. □

Theorem 2.1 in [11] guarantees that (H2) is a necessary condition for the existence of a maximum principle for the cooperative system (1.4), that is when (H1) is satisfied, in the case that the nonhomogeneous part satisfies (P). We strongly believe that the condition $a, d \geq 0$ is also necessary. This feeling arises from the examples below. But before that, note if $b = 0$ or $c = 0$, then the system (1.4) decouples and Proposition 2.1 guarantees the necessity of $a, d \geq 0$. So, from now to the end of this section we can assume that $b, c > 0$.

Example 2.3. Suppose that (H1), (H2), $a, d < 0$ and $ad - bc > 0$ are satisfied. Then for a suitable pair (f, g) satisfying (P), the solution (u, v) of the system (1.4) is such that u and v assume negative values.

Proof. The conditions $a, d < 0$ and $ad - bc > 0$ guarantee the existence of positive numbers σ such that $a + b\sigma < 0$ and $c + d\sigma < 0$. Fix such a number $\sigma > 0$. Let u_0 be as in the proof of Proposition 2.1, take $f = -\Delta u_0$ and $g = \sigma f$. Then $u_f = u_0$ and $u_g = \sigma u_0$. Let (u, v) be the solution of (1.4) associated to such pair (f, g). Since (H1)–(H2) are satisfied and $a + b\sigma < 0$, $c + d\sigma < 0$, one can apply to the system (2.2) the Theorem 2.1 in [11]. It guarantees that $u - u_0 < 0$, $v - \sigma u_0 < 0$ in Ω. In particular $u(x_0), v(x_0) < 0$. □

Example 2.4. Suppose (H1)–(H2), $a, d < 0$ and $ad - bc = 0$. Then for a suitable pair (f, g) satisfying (P), the solution (u, v) of the system (1.4) is such that v assumes negative values.

Proof. From the above hypotheses, there exists $\lambda < 0$ such that $c = \lambda a$ and $d = \lambda b$. Fix $\sigma > -\frac{a}{b}$, let u_0 be as in the proof of Proposition 2.1, $f = -\Delta u_0$, $g = \sigma f$ and let (u, v) be the solution of the system (1.4) associated to such pair (f, g). From the definition of λ, one has that

$$\begin{cases} -\Delta(v - \lambda u) = (\sigma - \lambda)f & \text{in } \Omega, \\ v - \lambda u = 0 & \text{on } \partial\Omega \end{cases}$$

and so that

$$v - \lambda u = (\sigma - \lambda)u_0. \tag{2.3}$$

On the other hand,

$$\begin{aligned} -\Delta\left(v + \frac{a}{b}u\right) &= (\lambda b + a)\left(v + \frac{a}{b}u\right) + \left(\sigma + \frac{a}{b}\right)f \qquad \text{in } \Omega \\ v + \frac{a}{b}u &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

Since $\lambda b + a < 0$ and $\sigma + \frac{a}{b} > 0$, arguing as in the proof of Proposition 2.1, one obtains that $v(x_0) < -\frac{a}{b}u(x_0)$. Applying the identity (2.3) to the last inequality one concludes that $v(x_0) < 0$. □

We have no general counterexample for the remaining case that remains to be analyzed, namely the case under the following hypotheses:

$$\text{(H1), (H2)}, \qquad \text{and} \qquad b, c > 0,\ ad - bc < 0 \qquad \text{with } a < 0 \text{ or } d < 0. \tag{2.4}$$

Just to make it clear, $a < 0$ or $d < 0$ in (2.4) means that at least one of the two numbers a, d is negative.

In one-dimensional case, we have some counterexamples in such case. But even in dimension one, our counterexamples do not cover all the range that (2.4) involves.

Before we present more counterexamples, let us discuss about some general aspects concerning the system (1.4). Suppose that (H2) is satisfied. In particular, for each pair $(f, g) \in C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ the system (1.4) admits a unique classical solution (u, v). Furthermore, if u (resp. v) assumes negatives values in Ω, then u_1 (resp. v_1) or u_2 (resp. v_2) assumes negative values in Ω, where (u_1, v_1) and (u_2, v_2) are the solutions of

$$\begin{cases} -\Delta u_1 = au_1 + bv_1 + f(x) & \text{in } \Omega, \\ -\Delta v_1 = cu_1 + dv_1 & \text{in } \Omega, \\ u_1, v_1 = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u_2 = au_2 + bv_2 & \text{in } \Omega, \\ -\Delta v_2 = cu_2 + dv_2 + g(x) & \text{in } \Omega, \\ u_2, v_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore, in order to find counterexamples we can restrict ourselves to simpler systems.

On the other hand, suppose that one is interested on the sign of (u, v), the solution of the system

$$\begin{cases} -\Delta u = au + bv + f(x) & \text{in } \Omega, \\ -\Delta v = cu + dv + \sigma f(x) & \text{in } \Omega, \\ u, v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.5}$$

under the following conditions:

- (H2), $ad - bc < 0$, $b, c > 0$;
- $u_f \geq 0$ in Ω, $f \not\equiv 0$ and $\sigma \geq 0$.

Associating the system (2.5) to one equation, one has that for every $\theta \in \mathbb{R}\backslash\{0\}$

$$\begin{cases} -\Delta(u + \theta v) = (a + \theta c)u + \left(\dfrac{b}{\theta} + d\right)\theta v + (1 + \theta\sigma)f(x) & \text{in } \Omega, \\ u + \theta v = 0 & \text{on } \partial\Omega. \end{cases}$$

On the other hand, the condition $b, c > 0$ guarantees the existence of θ^+, θ^-, the two solutions of

$$a + \theta c = \frac{b}{\theta} + d,$$

which are explicitly given by

$$\theta^+ = \frac{-(a - d) + \sqrt{(a - d)^2 + 4bc}}{2c}, \qquad \theta^- = \frac{-(a - d) - \sqrt{(a - d)^2 + 4bc}}{2c}.$$

Observe that $\theta^+ > 0$, $\theta^- < 0$ and

$$a + \theta^+ c = \lambda^+(A), \qquad a + \theta^- c = \lambda^-(A),$$

where $\lambda^+(A)$ and $\lambda^-(A)$ stand for the two eigenvalues of A. The condition $ad - bc < 0$ implies that $\lambda^+(A) > 0$ and $\lambda^-(A) < 0$. Arguing as in the proof of Proposition 2.1, by means of the classical maximum principle one obtains that:

- if $0 \leq \sigma < \frac{1}{-\theta^-}$, then $\begin{cases} v > \sigma u_f & \text{in } \Omega, \\ u_f - \theta^+(v - \sigma u_f) < u < u_f - \theta^-(v - \sigma u_f) & \text{in } \Omega; \end{cases}$

- if $\sigma = \frac{1}{-\theta^-}$, then $u > u_f$, $v > \sigma u_f$ in Ω;
- if $\sigma > \frac{1}{-\theta^-}$, then $\begin{cases} u > u_f & \text{in } \Omega, \\ \sigma u_f - \frac{1}{\theta^+}(u - u_f) < v < \sigma u_f + \frac{1}{-\theta^-}(u - u_f) & \text{in } \Omega. \end{cases}$

Based on such inequalities we present some more counterexamples, but now in dimension one.

Example 2.5. Assume all the conditions given by (2.4) and $\Omega = (0, 1)$.

- Let (u, v) be the solution of

$$\begin{cases} -u'' = au + bv + f(x) & \text{in } (0,1), \\ -v'' = cu + dv & \text{in } (0,1), \\ u, v = 0 & \text{on } \{0,1\}. \end{cases}$$

If $a\pi^2 < ad - bc$, then for a suitable $f \in C^1([0,1])$ with $u_f \geq 0$ in $(0,1)$, the component u assumes negative values on $(0,1)$.

- Let (u, v) be the solution of

$$\begin{cases} -u'' = au + bv & \text{in } (0,1), \\ -v'' = cu + dv + g(x) & \text{in } (0,1), \\ u, v = 0 & \text{on } \{0,1\}. \end{cases}$$

If $d\pi^2 < ad - bc$, then for a suitable $g \in C^1([0,1])$ with $u_g \geq 0$ in $(0,1)$, the component v assumes negative values on $(0,1)$.

Proof. It is well known that the eigenvalues of

$$\begin{cases} -u'' = \lambda u & \text{in } (0,1), \\ u = 0 & \text{on } \{0,1\}, \end{cases}$$

are $\lambda_k = (k\pi)^2$, all they are simple and they have $\varphi_k(t) = \sin(k\pi t)$ as associated eigenfunctions.

In order to find an appropriated function f, consider the identity

$$\sin(2^n \pi t) = 2^n \sin(\pi t) \prod_{k=0}^{n-1} \cos(2^k \pi t), \tag{2.6}$$

which holds true for all $n \geq 1$.

For each $n \geq 1$, let $f_n(t) := \sin(\pi t) + 2^n \sin(2^n \pi t)$. From the identity (2.6), one obtains that f_n changes sign in $(0,1)$, but $u_{f_n} > 0$ in $(0,1)$.

Now, suppose $a < 0$. For each $n \geq 1$, let (u_n, v_n) be the solution of

$$\begin{cases} -u_n'' = au_n + bv_n + f_n(x) & \text{in } (0,1), \\ -v_n'' = cu_n + dv_n & \text{in } (0,1), \\ u_n, v_n = 0 & \text{on } \{0,1\}. \end{cases}$$

By a straightforward calculation, one obtains that

$$u_n(t) = \sin(\pi t)\Big(\frac{\pi^2 - d}{(\pi^2 - a)(\pi^2 - d) - bc} + \frac{2^{4n}\pi^2 - 2^{2n}d}{(2^{2n}\pi^2 - a)(2^{2n}\pi^2 - d) - bc}\prod_{k=0}^{n-1}\cos(2^k \pi t)\Big).$$

From the explicit formula for u_n, it follows that u_n assume negative values in $(0,1)$ if and only if

$$\frac{\pi^2 - d}{(\pi^2 - a)(\pi^2 - d) - bc} < \frac{2^{4n}\pi^2 - 2^{2n}d}{(2^{2n}\pi^2 - a)(2^{2n}\pi^2 - d) - bc}, \tag{2.7}$$

which is equivalent to

$$\pi^2\big(a\pi^2 - (ad - bc)\big)2^{4n} + \big(-a\pi^4 + d(ad - bc)\big)2^{2n} + (\pi^2 - d)(ad - bc) < 0. \tag{2.8}$$

Let $p(t) := \pi^2\big(a\pi^2 - (ad - bc)\big)t^2 + \big(-a\pi^4 + d(ad - bc)\big)t + (\pi^2 - d)(ad - bc)$. So, u_n assumes negative values in $(0,1)$ if and only if $p(2^{2n}) < 0$.

If $a\pi^2 \geq ad - bc$, then one can check that $p(2^{2n}) > 0$ for all $n \in \mathbb{N}$.

On the other hand, the condition $a\pi^2 < ad - bc$ says that the leader coefficient of the second order polynomial $p(t)$ is negative and so $p(2^{2n}) < 0$ for n sufficiently large.

For the case with $d\pi^2 < ad - bc$, a counterexample is found in the same way, by analyzing the sign of the component v_n, where (u_n, v_n) stands for the solution of

$$\begin{cases} -u_n'' = au_n + bv_n & \text{in } (0,1), \\ -v_n'' = cu_n + dv_n + f_n(x) & \text{in } (0,1), \\ u_n, v_n = 0 & \text{on } \{0,1\}. \end{cases}$$

The matrices

$$\begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}, \quad \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -i & 1 \\ 1 & i \end{pmatrix} \quad \text{for } i = 1, \ldots, 9$$

satisfy all the conditions given by (2.4) with $\Omega = (0,1)$. Furthermore, for such matrices, either $a\pi^2 < ad - bc$ or $d\pi^2 < ad - bc$.

On the other hand, still with $\Omega = (0,1)$, the matrices

$$\begin{pmatrix} -\varepsilon & b \\ c & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & b \\ c & -\varepsilon \end{pmatrix} \quad \text{with } b, c > 0,\ bc = \pi^4,\ \varepsilon \in (0, \pi^2),$$

also satisfy all the conditions given by (2.4). Unfortunately, we have not found any counterexample for such matrices.

Remark 2.6. Consider the system (1.4) with (H1) and (H2) satisfied. Grouping the results of this subsection, a maximum principle for (1.4) in the case with (P) satisfied:

- does exist, if $a, d \geq 0$;
- does not exist, if:
  - $a < 0$ or $d < 0$ and with $b = 0$ or $c = 0$;
  - $a, d < 0$ and $ad - bc \geq 0$;
  - $ad - bc < 0$, in dimension one, provided: $a\pi^2 < ad - bc$ or $d\pi^2 < ad - bc$.

To end this section, we state the following proposition, which can be proved by an argument similar to that employed in the proof of Proposition 2.2.

Proposition 2.7. *Assume* (H1)–(H3) *and* (P). *Let* (u, v) *be the system* (1.4). *Then* $u, v \geq 0$ *in* Ω.

3. The proof of Theorem 1.1

To ease the reading of the proof of Theorem 3.2 below, we introduce:

Lemma 3.1. *Assume* (H1)–(H3) *and* (P). *Let* (u, v) *be a classical solution pair of the system*

$$\begin{cases} -\Delta u = au + bv + v^p + \varepsilon f(x) & \text{in } \Omega, \\ -\Delta v = cu + dv + u^q + \varepsilon g(x) & \text{in } \Omega, \\ u, v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

with $u, v \geq 0$ *in* Ω. *Then* $u, v > 0$ *in* Ω, *that is,* (u, v) *is a solution pair for* (1.1).

Proof. It is a straightforward application of Theorem 2.1 in [11]. For that, it is enough to analyze the systems that are satisfied by the pairs:

– (u, v), if $f, g \geq 0$ in Ω;
– $(u - \varepsilon u_f, v)$, if f changes sign in Ω and $g \geq 0$ in Ω;
– $(u, v - \varepsilon u_g)$, if $f \geq 0$ in Ω and g changes sign in Ω;
– $(u - \varepsilon u_f, v - \varepsilon u_g)$, if f and g change sign in Ω. □

The following theorem will be employed in the proof of all theorems stated at the first section of this paper.

Theorem 3.2. *Assume* (P), (H1)–(H3) *and that* $p, q > 0$. *If the system* (1.1) *has a subsolution pair* (u_s, v_s) *and a supersolution pair* (u_S, v_S) *such that* $0 \leq u_s \leq u_S$, $0 \leq v_s \leq v_S$ *in* Ω *and* $u_s, v_s, u_S, v_S \in C^2(\overline{\Omega})$, *then* (1.1) *has a classical solution pair* (u, v) *such that* $u_s \leq u \leq u_S$, $v_s \leq v \leq v_S$ *in* Ω. *Furthermore,* $u, v \in C^{2,\alpha}(\overline{\Omega})$ *for:* $\alpha = \min\{p, q\}$, *if* $p < 1$ *or* $q < 1$; *all* $\alpha \in (0, 1)$, *if* $p, q \geq 1$.

Proof. Let $(u_0, v_0) := (u_s, v_s)$. For each $n \geq 1$, let (u_n, v_n) be the solution of

$$\begin{cases} -\Delta u_n - a u_n = b v_{n-1} + v_{n-1}^p + \varepsilon f(x) & \text{in } \Omega, \\ -\Delta v_n - d v_n = c u_{n-1} + u_{n-1}^q + \varepsilon g(x) & \text{in } \Omega, \\ u_n, v_n = 0 & \text{on } \partial\Omega. \end{cases}$$

From (H1) and (H2) one has that $a, d < \lambda_1$. Such inequalities combined with (H1) allow one to prove that

$$u_s \leq u_n \leq u_S,\ v_s \leq v_n \leq v_S,\ u_{n-1} \leq u_n,\ v_{n-1} \leq v_n \text{ in } \Omega, \qquad \text{for all } n \geq 1.$$

By Lemma 9.17 in [16], u_n, v_n are bounded in $H^2(\Omega)$ and therefore in $H_0^1(\Omega)$. Hence there exist $u, v \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^\infty(\Omega)$ and a subsequence of $((u_n, v_n))$, here also denoted by $((u_n, v_n))$, such that

$$\begin{aligned} &u_n \rightharpoonup u, \quad v_n \rightharpoonup v \quad \text{in } H^2(\Omega) \text{ and } H_0^1(\Omega), \\ &u_n \to u, \quad v_n \to v \quad \text{in } L^2(\Omega), \\ &u_n \to u, \quad v_n \to v \quad \text{almost everywhere in } \Omega. \end{aligned}$$

Hence, by the Lebesgue's dominated convergence theorem,

$$u_n^q \to u^q, \qquad v_n^p \to v^p \quad \text{in } L^r(\Omega), \qquad \text{for all } r \geq 1.$$

Taking the limit as $n \to \infty$ in the identities

$$\int(\nabla u_n \nabla\varphi - au_n\varphi)\,dx = \int(bv_{n-1} + v_{n-1}^p)\varphi\,dx + \varepsilon\int f(x)\varphi\,dx,$$
$$\int(\nabla v_n \nabla\psi - dv_n\psi)\,dx = \int(cu_{n-1} + u_n^q)\psi\,dx + \varepsilon\int g(x)\psi\,dx,$$

which hold true for all φ, ψ $H_0^1(\Omega)$-functions, one gets

$$\int\nabla u\nabla\varphi\,dx = \int(au + bv + v^p)\varphi\,dx + \varepsilon\int f(x)\varphi\,dx,$$
$$\int\nabla v\nabla\psi\,dx = \int(cu + dv + u^q)\psi\,dx + \varepsilon\int g(x)\psi\,dx.$$

That is, $u, v \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^\infty(\Omega)$ is a solution for the system (3.1) in the sense of $H_0^1(\Omega)$. Hence $u, v \in W^{2,r}(\Omega)$ for all $1 \le r < +\infty$, by Lemma 9.17 in [16]. Hence, from the Sobolev embeddings, $u, v \in C^{1,\gamma}(\overline{\Omega})$ for all $0 < \gamma < 1$. By Lemma 6.36 (see also p. 53 and Theorem 9.15 in [16]) in [16] and Schauder's estimates, one gets that $u, v \in C^{2,\alpha}(\Omega)$ for: $\alpha = \min\{p, q\}$, if $p < 1$ or $q < 1$; all $\alpha \in (0, 1)$, if $p, q \ge 1$. Finally, by Lemma 3.1, one concludes that (u, v) is a solution of (1.1). □

Besides other facts, the proof of Theorem 1.1 is based on the continuity of λ_1 with respect to Ω. To be more explicit, we employ the fact that: given $\lambda < \lambda_{1,\Omega}$ there exists a smooth bounded domain $\Omega' \subset \mathbb{R}^N$ such that $\Omega \subset\subset \Omega'$ and $\lambda < \lambda_{1,\Omega'} < \lambda_{1,\Omega}$.

For each $t \in \mathbb{R}$, consider the 2×2 matrix

$$B(t) := \begin{pmatrix} -c & t-d \\ t-a & -b \end{pmatrix}.$$

The following lemma is the key for the proof of Theorem 1.1.

Lemma 3.3. *If* (H1) *and* (H2) *are satisfied, then there exists a smooth bounded domain Ω' such that $\Omega \subset\subset \Omega'$ and $B(\lambda_{1,\Omega'})$ has a positive eigenvalue with associated eigenvectors of entries of the same sign.*

Proof. From (H1) and (H2) one obtains that

$$a, d < \lambda_{1,\Omega} \quad \text{and} \quad (\lambda_{1,\Omega} - a)(\lambda_{1,\Omega} - d) - bc > 0.$$

Hence, the biggest eigenvalue of $B(\lambda_{1,\Omega})$, denoted here by $\lambda_{\lambda_{1,\Omega}}$ and for short by λ, is explicitly given by

$$\lambda = \lambda_{\lambda_{1,\Omega}} = \frac{-(b+c) + \sqrt{(b+c)^2 + 4\big((\lambda_{1,\Omega} - a)(\lambda_{1,\Omega} - d) - bc\big)}}{2},$$

$\lambda > 0$ and its associated eigenvectors are of the form

$$(\sigma, \theta) = \left(\sigma, \frac{\lambda_{1,\Omega} - a}{\lambda_{\lambda_{1,\Omega}} + b}\sigma\right), \qquad \sigma \in \mathbb{R}.$$

By the continuity of λ_1 with respect to Ω, there exists a smooth bounded domain $\Omega' \subset \mathbb{R}^N$ such that $\Omega \subset\subset \Omega'$ and all the above inequalities are verified if one replaces $\lambda_{1,\Omega}$ by $\lambda_{1,\Omega'}$. □

Proof of Theorem 1.1. By means of Theorem 3.2, it is enough to find a subsolution pair (u_s, v_s) and a supersolution pair (u_S, v_S) such that $0 \le u_s \le u_S$, $0 \le v_s \le v_S$ in Ω and $u_s, v_s, u_S, v_S \in C^2(\overline{\Omega})$.

If $f, g \ge 0$ in Ω, then $(u_s, v_s) := (0,0)$ is a subsolution pair. If f changes sign but $g \ge 0$ in Ω, then by (H1) and (H3), $(u_s, v_s) := (\varepsilon u_f, 0)$ is a subsolution pair. If $f \ge 0$ but g changes sign in Ω, then by (H1) and (H3), $(u_s, v_s) := (0, \varepsilon u_g)$ is a subsolution pair. Finally, if f and g change sign in Ω, then (H1) and (H3) guarantee that $(u_s, v_s) := (\varepsilon u_f, \varepsilon u_g)$ is a subsolution pair.

Fix Ω' as in Lemma 3.3 and set $(u_S, v_S) = (\sigma\varphi_{1,\Omega'}, \theta\varphi_{1,\Omega'})$, where (σ, θ) stands for an eigenvector of $B(\lambda_{1,\Omega'})$ with positive entries and associated to the positive eigenvalue $\lambda = \lambda_{\lambda_{1,\Omega'}}$.

It is obvious that $|(\sigma, \theta)|$ can be taken as small as one wants. In order to $(\sigma\varphi_{1,\Omega'}, \theta\varphi_{1,\Omega'})$ be a supersolution pair for (1.1), one imposes that

$$\begin{cases} \lambda_{1,\Omega'}(\sigma\varphi_{1,\Omega'}) = -\Delta(\sigma\varphi_{1,\Omega'}) \ge a(\sigma\varphi_{1,\Omega'}) + b(\theta\varphi_{1,\Omega'}) + (\theta\varphi_{1,\Omega'})^p + \varepsilon f(x) & \text{in } \Omega, \\ \lambda_{1,\Omega'}(\theta\varphi_{1,\Omega'}) = -\Delta(\theta\varphi_{1,\Omega'}) \ge c(\sigma\varphi_{1,\Omega'}) + d(\theta\varphi_{1,\Omega'}) + (\sigma\varphi_{1,\Omega'})^q + \varepsilon g(x) & \text{in } \Omega, \end{cases}$$

that is,

$$\begin{cases} \lambda(\theta\varphi_{1,\Omega'}) - (\theta\varphi_{1,\Omega'})^p = \big((\lambda_{1,\Omega'} - a)\sigma - b\theta\big)\varphi_{1,\Omega'} - (\theta\varphi_{1,\Omega'})^p \ge \varepsilon f(x) & \text{in } \Omega, \\ \lambda(\sigma\varphi_{1,\Omega'}) - (\sigma\varphi_{1,\Omega'})^q = \big(-c\sigma + (\lambda_{1,\Omega'} - d)\theta\big)\varphi_{1,\Omega'} - (\sigma\varphi_{1,\Omega'})^q \ge \varepsilon g(x) & \text{in } \Omega, \end{cases}$$

since (σ, θ) is an eigenvector of $B(\lambda_{1,\Omega'})$ associated to λ.

The above inequalities are satisfied if

$$\begin{cases} \lambda(\theta\varphi_{1,\Omega'}) - (\theta\varphi_{1,\Omega'})^p \ge \varepsilon|f|_\infty & \text{in } \Omega, \\ \lambda(\sigma\varphi_{1,\Omega'}) - (\sigma\varphi_{1,\Omega'})^q \ge \varepsilon|g|_\infty & \text{in } \Omega. \end{cases}$$

But the last inequalities are trivially verified for $\varepsilon > 0$ small enough and $|(\sigma, \theta)|$ suitable small, since $p, q > 1$ and $\varphi_{1,\Omega'} \ge \alpha > 0$ in $\overline{\Omega}$ for some constant α.

On the other hand, one needs $u_S \geq u_s$ and $v_S \geq v_s$ on $\overline{\Omega}$. Suppose that the case "f and g change sign" is being treated. Such inequalities are easier obtained in the other cases. Hence, it needs to be shown that $\sigma\varphi_{1,\Omega'} \geq \varepsilon u_f$ and $\theta\varphi_{1,\Omega'} \geq \varepsilon u_g$ in $\overline{\Omega}$. But such inequalities are also verified for $\varepsilon > 0$ sufficiently small because $\varphi_{1,\Omega'} \geq \alpha > 0$ in $\overline{\Omega}$.

In conclusion, Theorem 3.2 guarantees the existence of a classical solution (u, v) for (1.1), provided $\varepsilon > 0$ is small enough.

Denote by $(u_\varepsilon, v_\varepsilon)$ the above obtained solution pair. It is the minimal solution pair for (1.1). In fact, firstly note that any solution (u, v) of (1.1) satisfies $u > u_s$ and $v > v_s$ in Ω, with (u_s, v_s) as above, depending whether f, g change sign or not. From $u > u_s$ and $v > v_s$ in Ω, one shows that $u > u_n$ and $v > v_n$ in Ω, where (u_n, v_n) stands for any element of the sequence constructed by a monotonic iteration applied to the subsolution pair (u_s, v_s). See the proof of Theorem 3.2 for the precise monotonic iteration. Consequently, taking the limit, $u \geq u_\varepsilon$ and $v \geq v_\varepsilon$ in Ω, since one has pointwise convergence.

The condition (P) guarantees that

$$\int \big(f(x) + g(x)\big)\varphi_1\,dx = \int (\nabla u_f + \nabla u_g)\nabla\varphi_1\,dx = \lambda_1 \int (u_f + u_g)\varphi_1\,dx > 0.$$

On the other hand, (H1) and (H2) guarantee that the function $j(t, s) := (\lambda_1 - a)t - t^q + (\lambda_1 - d)s - s^p$, defined on $[0, +\infty) \times [0, +\infty)$, is bounded from above and its maximum value is $\alpha_{p,q} = (\lambda_1 - a)^{q/(q-1)} \frac{q-1}{q^{q/(q-1)}} + (\lambda_1 - d)^{p/(p-1)} \frac{p-1}{p^{p/(p-1)}}$.

If (1.1) has a solution, then using φ_1 as a multiplier and integrating (1.1) by parts, one gets by (H1) that

$$\int \big((\lambda_1 - a)u - u^q + (\lambda_1 - d)v - v^p\big)\varphi_1\,dx \geq \varepsilon \int \big(f(x) + g(x)\big)\varphi_1\,dx,$$

and so that

$$\varepsilon \leq \frac{\alpha_{p,q}}{\int \big(f(x) + g(x)\big)\varphi_1\,dx}.$$

Set

$$\varepsilon^* = \sup\{\varepsilon > 0 : (1.1) \text{ has a solution}\}. \tag{3.2}$$

The above estimate gives an upper bound for ε^*. Furthermore, one shows that the system (1.1) has a minimal positive solution for $0 < \varepsilon < \varepsilon^*$ and it has no solution for $\varepsilon > \varepsilon^*$. For that, thanks by Theorem 3.2, it is possible to argue similarly as in the proof of Lemma 3.1 in [6], with a slight change concerning the argument related to the subsolution, which depends on whether f, g change sign or not. $\square$

4. The proof of Theorem 1.2

In order to apply the method of subsolution and supersolution to the problems treated in this paper, the part that requires some work concerns the existence of a supersolution. We stress that we carry out this task even in the case (ii) of Theorem 1.2, where one of the equations of the system (1.1) is sublinear and the other one is superlinear.

To prove the existence of a supersolution, our technique is based on some ideas adopted in [9], [10]. To present it, the following notation is used:

$$\begin{cases} -\Delta e_1 = 1 & \text{in } \Omega, \\ e_1 = 0 & \text{on } \partial\Omega, \end{cases} \quad \begin{cases} -\Delta e_2 = e_1^p & \text{in } \Omega, \\ e_2 = 0 & \text{on } \partial\Omega, \end{cases} \quad \begin{cases} -\Delta e_3 = e_1^q & \text{in } \Omega, \\ e_3 = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} -\Delta e_4 = u_g^p & \text{in } \Omega, \\ e_4 = 0 & \text{on } \partial\Omega, \end{cases} \quad \begin{cases} -\Delta e_5 = u_f^q & \text{in } \Omega, \\ e_5 = 0 & \text{on } \partial\Omega. \end{cases}$$

With the aim of bringing our procedure to light, first we consider a particular case of the system (1.1), namely:

$$\begin{cases} -\Delta u = v^p & \text{in } \Omega, \\ -\Delta v = u^q + \varepsilon g(x) & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u, v = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

The reason for that is that the system (4.1) can be rewritten as

$$\begin{cases} -\Delta\big((-\Delta u)^{1/p}\big) = u^q + \varepsilon g(x) & \text{in } \Omega, \\ u, -\Delta u > 0 & \text{in } \Omega, \\ u, -\Delta u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Once more, (u_s, v_s) and (u_S, v_S) stand for candidates to be a subsolution and a supersolution respectively, either for (4.1) or for (1.1) according to the hypotheses of Theorem 1.2.

It is clear that $(u_s, v_s) := (0, \varepsilon u_g)$ is a subsolution pair for (4.1). As an additional comment, 0 may not be a subsolution for (4.2), but one can check that $\varepsilon^p e_4$ is indeed a subsolution for (4.2).

To find a supersolution pair for (4.1) we deal with (4.2) and we adapt the ideas of the authors of [9], [10], where they work with the problem

$$\begin{cases} -\Delta u = h(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

In order to find a supersolution for (4.3), under certain hypotheses on h, they impose that

$$\inf\left\{\frac{h(u)}{u} : u > 0\right\} < \frac{1}{|e_1|_\infty},$$

and they find a supersolution in the form of $w = \frac{M}{|e_1|_\infty} e_1$, provided M satisfies $\frac{h(M)}{M} < \frac{1}{|e_1|_\infty}$.

It leads us to look for a supersolution of (4.2) in the form of $u_S = m^p e_2$, which induces a supersolution pair for (4.1) in the form of $(u_S, v_S) = \left(u_S, (-\Delta u_S)^{1/p}\right) = (m^p e_2, m e_1)$. For that, the inequality $m \geq (m^p e_2)^q + \varepsilon g = m^{pq} e_2^q + \varepsilon g$ in Ω needs to be satisfied, which is certainly the case if

$$m(1 - m^{pq-1}|e_2|_\infty^q) \geq \varepsilon |g|_\infty. \tag{4.4}$$

On the other hand, the inequalities $u_S \geq u_s$, $v_S \geq v_s$, that is, $m^p e_2 \geq 0$, $m e_1 \geq \varepsilon u_g$ in Ω must also be satisfied. The second one is obtained imposing that $-\Delta(m e_1 - \varepsilon u_g) = m - \varepsilon g \geq 0$ in Ω. This holds provided

$$m \geq \varepsilon |g|_\infty. \tag{4.5}$$

It is easy to see that if (4.4) holds true, then (4.5) does. Furthermore, the inequality (4.4) is satisfied for certain values of m, provided

$$\varepsilon \leq \varepsilon_0, \qquad \text{with } \varepsilon_0 = \max\left\{\frac{m(1 - m^{pq-1}|e_2|_\infty^q)}{|g|_\infty} : m > 0\right\},$$

since $pq > 1$. Note that the superlinearity hypothesis $pq > 1$ is enough to guarantee the validity of (4.4). There is no need to suppose $p, q > 1$.

From the above calculation, $(u_s, v_s) = (0, \varepsilon u_g)$ is a subsolution pair and $(u_S, v_S) = (m^p e_2, m e_1)$ is a supersolution pair for (4.1), for certain values of m, provided $\varepsilon \leq \varepsilon_0$.

Following the ideas presented above, Theorem 3.2 can be applied to construct a solution of (4.1) provided $u_g \geq 0$ in Ω, $pq > 1$ and $0 < \varepsilon \leq \varepsilon_0$.

Remark 4.1. Suppose $0 < r < \frac{1}{p} < q$, $g(x) \geq \alpha > 0$ for all $x \in \Omega$, where α stands for a constant. Then the above method also works if one replaces in (4.2) $\varepsilon g(x)$ by $\varepsilon g(x) u^r$. In such case (4.2) becomes a problem of concave-convex type.

In fact, let $E = W^{2,(p+1)/p}(\Omega) \cap W_0^{1,(p+1)/p}(\Omega)$ and denote by $\varphi_{1,p}$ a function in E satisfying the following conditions: $\varphi_{1,p} > 0$, $-\Delta\varphi_{1,p} > 0$ in Ω, $\int |\varphi_{1,p}|^{(p+1)/p}\, dx = 1$ and

$$\lambda_{1,p} := \inf_{u \in E \setminus \{0\}} \frac{\int |\Delta u|^{(p+1)/p}\, dx}{\int |u|^{(p+1)/p}\, dx}$$

is attained by $\varphi_{1,p}$.

So, $\varphi_{1,p}$ is a solution of the eigenvalue problem

$$\begin{cases} -\Delta\big((-\Delta\varphi_{1,p})^{1/p}\big) = \lambda_{1,p}\varphi_{1,p}^{1/p} & \text{in } \Omega, \\ \varphi_{1,p}, -\Delta\varphi_{1,p} > 0 & \text{in } \Omega, \\ \varphi_{1,p}, -\Delta\varphi_{1,p} = 0 & \text{on } \partial\Omega. \end{cases}$$

Then the subsolution has the form of $u_s := \theta\varphi_{1,p}$. To show that u_s is in fact a subsolution, the hypotheses $g(x) \geq \alpha > 0$ for all $x \in \Omega$ and $0 < r < \frac{1}{p} < q$ play an important role. The supersolution has the form of $u_S := m^p e_2$.

Now we return to problem (1.1) according to the hypotheses of Theorem 1.2.

Proof of Theorem 1.2. *First case:* $q \geq 1$. The candidates to be subsolution and supersolution are

$$u_s = \varepsilon u_f, \qquad v_s = \varepsilon u_g, \qquad u_S = m^p e_2 + \varepsilon u_f \qquad \text{and} \qquad v_S = m e_1.$$

It is clear that (u_s, v_s) is a subsolution. The candidate to be supersolution is in fact a supersolution if ε is sufficiently small. For that,

$$\begin{cases} -\Delta u_S = m^p e_1^p + \varepsilon f(x) = (v_S)^p + \varepsilon f(x) & \text{in } \Omega, \\ -\Delta v_S = m \geq (u_S)^q + \varepsilon g(x) = (m^p e_2 + \varepsilon u_f)^q + \varepsilon g(x) & \text{in } \Omega, \end{cases}$$

where the last inequality has been imposed to be satisfied.

The following remains to be checked: $u_s \leq u_S$, $v_s \leq v_S$ and (u_S, v_S) is a supersolution pair. The last requirement is satisfied if

$$m \geq (m^p |e_2|_\infty + \varepsilon |u_f|_\infty)^q + \varepsilon |g|_\infty. \tag{4.6}$$

Since $q \geq 1$, $(a+b)^{1/q} \leq a^{1/q} + b^{1/q}$ for all $a, b \geq 0$. In this way, (4.6) is certainly satisfied if $m^{1/q} \geq m^p |e_2|_\infty + \varepsilon |u_f|_\infty + \varepsilon^{1/q} |g|_\infty^{1/q}$, which is satisfied if and only if

$$m^{1/q}(1 - m^{(pq-1)/q} |e_2|_\infty) \geq \varepsilon |u_f|_\infty + \varepsilon^{1/q} |g|_\infty^{1/q}. \tag{4.7}$$

Therefore, (4.6) is satisfied whenever (4.7) is. On the other hand, one needs $u_S \geq u_s$ and $v_S \geq v_s$, that is, $m^p e_2 + \varepsilon u_f \geq \varepsilon u_f$ and $me_1 \geq \varepsilon u_g$. For that, one imposes $-\Delta(me_1 - \varepsilon u_g) = m - \varepsilon g \geq 0$, which certainly holds true for

$$m \geq \varepsilon |g|_\infty. \tag{4.8}$$

But (4.8) is satisfied whenever (4.7) is. In this way, for the purpose of proving the above requirements on (u_S, v_S) it suffices to guarantee the validity of (4.7). But (4.7) is trivially verified for certain values of m, provided ε is sufficiently small. To be more precise, if $0 < \varepsilon \leq \varepsilon_0$ where ε_0 is such that

$$\varepsilon_0 |u_f|_\infty + (\varepsilon_0)^{1/q} |g|_\infty^{1/q} = \max\{m^{1/q}(1 - m^{(pq-1)/q}|e_2|_\infty) : m > 0\}.$$

Second case: $p \geq 1$. One can proceed exactly as in the case $q \geq 1$, but now taking

$$u_s = \varepsilon u_f, \quad v_s = \varepsilon u_g, \quad u_S = me_1 \quad \text{and} \quad v_S = m^q e_5 + \varepsilon u_g.$$

In both cases, $p \geq 1$ or $q \geq 1$, the Theorem 3.2 can be applied to guarantee that, under the hypotheses of Theorem 1.2, the system (1.1) has a solution for $\varepsilon > 0$ sufficiently small.

To finish, one defines ε^* by (3.2). □

5. The proof of Theorem 1.3

The condition $pq < 1$ implies that the positive parameter ε is not important in this case. The proof of Theorem 1.3 presented below shows this fact clearly. For this reason, without loss of generality, we take $\varepsilon = 1$.

To prove the uniqueness we adapt an argument that we found in [19]. To prove the existence of a solution for the system (1.1), under the conditions of Theorem 1.3, we employ an iterative method based on the following: starting from an obvious subsolution, we employ an iterative method to construct an increasing sequence of subsolutions. We prove that such a sequence is bounded from above by a constant and so, by some Sobolev embeddings, we prove that such sequence converges to a solution of the problem. The argument just described is based on some calculation that we found in [18]. A proof for Theorem 1.3 in the homogeneous case, that is, with $f \equiv g \equiv 0$, can be found in [18].

5.1. Uniqueness. In the case with $p, q < 1$ we can derive uniqueness by a very simple argument, similar to the one employed by the authors of [6] to treat a sublinear equation. In the general case, $pq < 1$, we adapt an argument found in [19]. Once more, we stress that to simplify our notation we consider $\varepsilon = 1$.

Here we prove the following theorem:

Theorem 5.1. *Under the hypotheses of Theorem* 1.3, *if a solution for* (1.1) *exists, then it must be unique.*

The proof of this theorem is split into six steps and is presented in the following six lemmas.

Lemma 5.2. *Assume all the hypotheses of Theorem* 1.3. *If* (u,v) *is a solution of* (1.1), *then* $u-u_f, v-u_g>0$ *in* Ω *and* $\frac{\partial u}{\partial\eta}, \frac{\partial}{\partial\eta}(u-u_f), \frac{\partial v}{\partial\eta}, \frac{\partial}{\partial\eta}(v-u_g)<0$ *on* $\partial\Omega$.

Proof. If (u,v) is a solution of (1.1), then

$$\begin{cases} -\Delta(u-u_f)=v^p+f(x)-f(x)=v^p>0 & \text{in } \Omega,\\ -\Delta(v-u_g)=u^q+g(x)-g(x)=u^q>0 & \text{in } \Omega,\\ u-u_f, v-u_g=0 & \text{on } \partial\Omega. \end{cases}$$

Then by the classical strong maximum principle $u-u_f, v-u_g>0$ in Ω and by the Hopf's Lemma $\frac{\partial}{\partial\eta}(u-u_f)<0$, $\frac{\partial}{\partial\eta}(v-u_g)<0$ on $\partial\Omega$.

On the other hand, $u_f, u_g\geq 0$ in Ω and $u_f, u_g\equiv 0$ on $\partial\Omega$ imply that $\frac{\partial u_f}{\partial\eta}, \frac{\partial u_g}{\partial\eta}\leq 0$ on $\partial\Omega$. Hence $\frac{\partial u}{\partial\eta}, \frac{\partial v}{\partial\eta}<0$ on $\partial\Omega$. □

Now, suppose that (u,v), $(\tilde{u},\tilde{v})$ are two solutions for the system (1.1) (recall that $A\equiv 0$ and $\varepsilon=1$). Let

$$S'=\{s>0 : u>s^{(q+1)/q}\tilde{u}, v>s^{(p+1)/p}\tilde{v} \text{ in } \Omega\} \quad \text{and} \quad s_*:=\sup S'.$$

To prove the uniqueness result, it is sufficient to show that S' is not empty and that $s_*\geq 1$.

For that, we consider the auxiliary set

$$S=\{s>0 : u-u_f>s^{(q+1)/q}(\tilde{u}-u_f), v-u_g>s^{(p+1)/p}(\tilde{v}-u_g) \text{ in } \Omega\}.$$

The next step is:

Lemma 5.3. *Let* $w,\tilde{w}\in C^2(\Omega)\cap C^1(\overline{\Omega})$ *be such that* $w,\tilde{w}>0$ *in* Ω, $w,\tilde{w}\equiv 0$ *on* $\partial\Omega$ *and* $\frac{\partial w}{\partial\eta}, \frac{\partial\tilde{w}}{\partial\eta}<0$ *on* $\partial\Omega$. *Then, for* $\tau>0$ *sufficiently small,* $w>\tau\tilde{w}$ *in* Ω.

Proof. Let

$$\alpha = \min_{\partial\Omega} \frac{\partial \tilde{w}}{\partial \eta}, \qquad \beta = \max_{\partial\Omega} \frac{\partial w}{\partial \eta}.$$

Then $\alpha, \beta < 0$. Fix $\tau > 0$ such that $\tau\alpha - \beta > 0$. Then

$$\frac{\partial}{\partial \eta}(\tau\tilde{w} - w) > 0 \qquad \text{on } \partial\Omega.$$

Since $\tau\tilde{w} - w = 0$ on $\partial\Omega$, it follows that $\tau\tilde{w} - w < 0$ in a certain neighborhood of $\partial\Omega$. Let $K \subset\subset \Omega$ be the compact set, the complement in Ω of such neighborhood. It is obvious that, for $\tau > 0$ even small if necessary, $\tau\tilde{w} - w < 0$ on K. In this way, for $\tau > 0$ sufficiently small, $\tau\tilde{w} - w < 0$ in Ω. □

As a consequence:

Lemma 5.4. *S' is not empty.*

Proof. It is a straightforward consequence of Lemmas 5.2 and 5.3. □

Lemma 5.5. *If $s_* < 1$, then $S' = S \cap (0, 1)$.*

Proof. If $s_* < 1$, then $S' \subset (0, 1)$. Furthermore, if $s \in S'$, then

$$\begin{cases} -\Delta\big(u - u_f - s^{(q+1)/q}(\tilde{u} - u_f)\big) = v^p - s^{(q+1)/q}\tilde{v}^p > (1 - s^{(1-pq)/q})v^p > 0 & \text{in } \Omega, \\ -\Delta\big(v - u_g - s^{(p+1)/p}(\tilde{v} - u_g)\big) = u^q - s^{(p+1)/p}\tilde{u}^q > (1 - s^{(1-pq)/p})u^q > 0 & \text{in } \Omega, \\ u - u_f - s^{(q+1)/q}(\tilde{u} - u_f), v - u_g - s^{(p+1)/p}(\tilde{v} - u_g) = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

Therefore, by the classical strong maximum principle one obtains that $s \in S$. On the other hand, it follows directly from the definitions of S and S' that $\big(S \cap (0, 1)\big) \subset S'$. □

Lemma 5.6. *If $s_* < 1$, then S' is an open interval.*

Proof. From the definition of S', it is clear that if $s \in S'$, then $(0, s) \subset S$. Such property implies that S' is an interval open from the left. Hence one just needs to show that for each $s \in S'$ there exists $\tau > 0$ such that $s + \tau \in S'$. Suppose $s_* < 1$. If $s \in S'$, then from (5.1) one gets by the Hopf's Lemma that

$$\frac{\partial}{\partial \eta}\big(u - u_f - s^{(q+1)/q}(\tilde{u} - u_f)\big) < 0$$

and

$$\frac{\partial}{\partial \eta}\big(v - u_g - s^{(p+1)/p}(\tilde{v} - u_g)\big) < 0 \qquad \text{on } \partial\Omega.$$

Then Lemma 5.2 and Lemma 5.3 guarantee the existence of $\tau > 0$ such that $s + \tau < 1$ and $s + \tau \in S$. Therefore, by Lemma 5.5, $s + \tau \in S'$. □

Lemma 5.7. *If $s_* < 1$, then $s_* \in S'$.*

Proof. If $s_* < 1$, then

$$
\begin{aligned}
-\Delta\big(u - u_f - s_*^{(q+1)/q}(\tilde{u} - u_f)\big) &= v^p - s_*^{(q+1)/q}\tilde{v}^p \geq v^p - s_*^{(q+1)/q}\frac{1}{s_*^{p+1}}v^p \\
&= (1 - s_*^{(1-pq)/q})v^p > 0, \\
-\Delta\big(v - u_g - s_*^{(p+1)/p}(\tilde{v} - u_g)\big) &= u^q - s_*^{(p+1)/p}\tilde{u}^q \geq u^q - s_*^{(p+1)/p}\frac{1}{s_*^{q+1}}u^q \\
&= (1 - s_*^{(1-pq)/p})u^q > 0.
\end{aligned}
$$

Taking in addition to this the fact that such functions vanish on $\partial\Omega$, the classical strong maximum principle guarantees that $s_* \in S$ and therefore in S', by Lemma 5.5. □

It is clear that Lemma 5.6 and Lemma 5.7 contradict each other. Hence we conclude that $s_* \geq 1$ and the proof of Theorem 5.1.

5.2. Existence. First of all, $(u_0, v_0) := (u_f, u_g)$ is a subsolution for (1.1) (recall that $A \equiv 0$ and $\varepsilon = 1$).

Now, for each $n \geq 0$, set inductively (u_{n+1}, v_{n+1}) as the solution of

$$
\begin{cases}
-\Delta u_{n+1} = v_n^p + f(x) & \text{in } \Omega, \\
-\Delta v_{n+1} = u_n^q + g(x) & \text{in } \Omega, \\
u_{n+1}, v_{n+1} = 0 & \text{on } \partial\Omega.
\end{cases}
$$

As claimed in the proof of Theorem 3.2, one has that $u_n \leq u_{n+1}$, $v_n \leq v_{n+1}$ in Ω for all $n \geq 0$.

Once the sequence (u_n, v_n) is constructed, the next step in proving the existence of solution is given by the following lemma.

Lemma 5.8. *There exists a constant $C > 0$ such that $0 \leq u_n, v_n < C$ in Ω for all $n \geq 0$.*

Proof. Suppose by contradiction that the above claim is not true. From the symmetry of the system (1.1) with $A \equiv 0$, one can suppose that $\alpha_n = |u_n|_\infty \to +\infty$ as $n \to +\infty$. It is important to remember that (α_n) is non-decreasing.

For each $n \in \mathbb{N}$, let $\beta_n = \alpha_n^{(q+1)/(p+1)}$, write $u_n = \alpha_n U_n$ and $v_n = \beta_n V_n$. Then

$$\begin{aligned} -\Delta U_{n+1} &= \frac{1}{\alpha_{n+1}}(-\Delta u_{n+1}) = \frac{1}{\alpha_{n+1}}\big(v_n^p + f(x)\big) = \frac{1}{\alpha_{n+1}}\big((\beta_n V_n)^p + f(x)\big), \\ -\Delta V_{n+1} &= \frac{1}{\beta_{n+1}}(-\Delta v_{n+1}) = \frac{1}{\beta_{n+1}}\big(u_n^q + g(x)\big) = \frac{1}{\beta_{n+1}}\big((\alpha_n U_n)^q + g(x)\big). \end{aligned} \tag{5.2}$$

Let $\theta := \frac{pq-1}{p+1} < 0$. Since α_n and β_n do not decrease with n

$$\begin{aligned} \frac{1}{\alpha_{n+1}}\big((\beta_n V_n)^p + f(x)\big) &\le \frac{|f|_\infty}{\alpha_{n+1}} + \frac{\alpha_n^{p(q+1)/(p+1)} V_n^p}{\alpha_{n+1}} \\ &\le \frac{|f|_\infty}{\alpha_n} + \alpha_n^{(pq+p-p-1)/p+1} V_n^p = \frac{|f|_\infty}{\alpha_n} + \alpha_n^{\theta} V_n^p \end{aligned} \tag{5.3}$$

and

$$\frac{1}{\beta_{n+1}}\big((\alpha_n U_n)^q + g(x)\big) \le \frac{|g|_\infty}{\beta_n} + \alpha_n^{q-(q+1)/(p+1)} U_n^q = \frac{|g|_\infty}{\beta_n} + \alpha_n^{\theta} U_n^q \xrightarrow{n\to+\infty} 0 \tag{5.4}$$

because $|U_n|_\infty = 1$ and $\alpha_n, \beta_n \to +\infty$ as $n \to +\infty$. From (5.4) and the second equation of (5.2) it follows that $|V_n|_\infty \to 0$ as $n \to +\infty$. Hence, from (5.3) and the first equation of (5.2), it follows that $|U_n|_\infty \to 0$, which is a contradiction. □

The above boundedness is employed to construct the solution, arguing exactly as in the proof of Theorem 3.2.

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Ederson Moreira dos Santos, Instituto de Matemática, Estatística e Computação Científica, Universidade Estadual de Campinas, Caixa Postal 6065, Campinas—SP, Brazil
E-mail: ederson81@gmail.com