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The geometry of 2-calibrated manifolds

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Abstract. We define 2-calibrated structures, which are analogs of symplectic structures in odd dimensions. We show the existence of differential topological constructions compatible with the structure by developing an appropriate approximately holomorphic geometry for 2-calibrated structures.

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1. Introduction and statement of results

In recent years there has been an enormous success in the study of symplectic manifolds using approximately holomorphic methods. These methods—introduced by S. Donaldson in 1996 [11]—amount to treating symplectic manifolds as generalizations of Kähler manifolds. To this end it is convenient to think of a symplectic manifold, once a compatible almost complex structure J has been fixed, as a Kähler manifold (P, J, Ω) for which the integrability condition for J has been dropped.

Let M be any hypersurface of the Kähler manifold (P, J, Ω). M inherits on the one hand a codimension 1 distribution $D := JTM \cap TM$ endowed with an integrable almost complex structure $J : D \to D$ (i.e., a CR structure of hypersurface type), and on the other hand a closed 2-form $\omega := \Omega_{|M}$ which is nowhere degenerate when restricted to D. A 2-calibrated structure on M, together with a compatible almost complex structure, is the structure obtained when the integrability assumption on $J : D \to D$ is dropped.

Let us assume that the CR distribution of the $(2n+1)$-dimensional CR manifold (of hypersurface type) (M, D, J) is co-oriented (i.e., the real line bundle TM/D is trivial and a positive side has been chosen). The Levi form is the symmetric tensor

$$\mathscr{L} : D \times D \to TM/D, \qquad (u, v) \to [U, JV]/\sim,$$

where U, V are local sections of D extending $u, v \in T_xM$, and we consider the class of the above Lie bracket at x in the quotient real line bundle TM/D, where we can make sense of positive and negative values. We can distinguish several interesting geometries according to the behavior of the Levi form:

(1) If $\mathscr{L}$ is strictly positive (resp. negative) we get a strictly pseudo-convex (resp. pseudo-concave) CR structure. If we drop J what remains is a co-oriented contact structure (they always carry almost complex structures along the contact distribution).

(2) If $\mathscr{L} \equiv 0$ then D integrates into a codimension 1 foliation whose leaves inherit a Kähler structure. If J is dropped what we obtain is a class of regular Poisson manifolds that include mapping tori associated to symplectomorphisms and more generally cosymplectic structures (defined by a closed 1-form α and a closed 2-form ω such that $\alpha \wedge \omega^n$ is a volume form). When $n = 1$ the latter are nothing but smooth taut foliations.

(3) If $n = 1$ and $\mathscr{L} \geq 0$, by dropping J we obtain a class of structures that include all taut confoliations (see Section 3.5 in [15]).

Definition 1.1. A 2-calibrated structure on M^{2n+1} is a pair (D, ω), where D is a codimension 1 distribution and ω a closed 2-form nowhere degenerate on D.

We call the triple (M, D, ω) a 2-calibrated manifold. We also say that ω is positive on D. If D is integrable we speak of 2-calibrated foliations.

(M, D, ω) is said to be integral if $[\omega] \in H^2(M; \mathbb{R})$ is in the image of the integer cohomology, in which case we choose a lift $h \in H^2(M; \mathbb{Z})$ of $[\omega]$ that we fix once and for all. The pre-quantum line bundle (L, ∇) is the unique—up to isomorphism—Hermitian line bundle with compatible connection with Chern class h and curvature $-2\pi i \omega$.

As we saw 2-calibrated structures do contain contact structures, cosymplectic structures and 3-dimensional taut confoliations.

A 2-calibrated manifold (M, D, ω) always admits compatible almost complex structures $J : D \to D$. The purpose of this article is to explore how to adapt approximately holomorphic geometry to the tuple (M, D, ω, J), and to see how we can apply this theory to know more about (M, D, ω).

In what follows all our manifolds will be closed and smooth, and all tensors and maps smooth unless otherwise stated.

The first application we will obtain is an analog of the existence of transverse cycles through any point of a 3-dimensional taut foliation.

The appropriate generalization of a transverse cycle is as follows.

Definition 1.2. W is a 2-calibrated submanifold of (M, D, ω) if $TW \cap D$ has codimension 1 inside TW and ω is positive when restricted to it. In other words, W must intersect D transversely and in a symplectic sub-distribution of (D, ω).

The existence of submanifolds—which extends the main result for contact manifolds in [24]—is the content of the following result:

Proposition 1.1. *Let (M^{2n+1}, D, ω) be an integral 2-calibrated manifold and $L^{\otimes k}$ the sequence of powers of its pre-quantum line bundle (Definition 1.1). For any fixed point $y \in M$, any $m = 1, \ldots, n$, and any rank m complex vector bundle $E \to M$, if $k \in \mathbb{N}$ is large enough it is possible to find 2-calibrated submanifolds W_k of M of codimension $2m$ through y with the following properties:*

- *The inclusion $l : W_k \hookrightarrow M$ induces maps $l_* : \pi_j(W_k) \to \pi_j(M)$ which are isomorphisms for $j = 0, \ldots, n-m-1$, and an epimorphism for $j = n-m$. The same result holds for the homology groups.*
- *The Poincaré dual of $[W_k]$ is $c_m(E \otimes L^{\otimes k})$.*

The submanifolds in Proposition 1.1 are obtained by pulling back the **0** section of a vector bundle. Something similar can be done with the determinantal loci of a homomorphism of complex vector bundles (see Theorem 1.6 in [32] and Corollary 5.2 in [4]).

Proposition 1.2. *Let (M, D, ω) be an integral 2-calibrated manifold and $L^{\otimes k}$ the sequence of powers of its pre-quantum line bundle. Let E, F be Hermitian vector bundles with connections and consider the sequence of bundles $I_k = E^* \otimes F \otimes L^{\otimes k} = \mathrm{Hom}(E, F \otimes L^{\otimes k})$. Then for all $k \in \mathbb{N}$ large enough there exist sections τ_k of I_k for which the determinantal loci $\Sigma^i(\tau_k) = \{x \in M \mid \mathrm{rank}(\tau_k(x)) = i\}$ are integral 2-calibrated submanifolds stratifying M.*

The Poincaré dual of the closure of $\Sigma^i(\tau_k)$ is given by the Porteous formula [34]:

$$\Delta_{E, F\otimes L^{\otimes k}, i} = \begin{vmatrix} c_{n-i} & c_{n-i+1} & \cdots & \\ c_{n-i-1} & c_{n-i} & \cdots & \\ & & \ddots & \\ c_{n-m+1} & & \cdots & c_{n-i} \end{vmatrix},$$

where $\mathrm{rank}\, E = m$, $\mathrm{rank}\, F = n$, and c_j is the j-th Chern class $c_j(F \otimes L^{\otimes k} - E)$ defined by the equality

$$\begin{aligned} &1 + c_1(F \otimes L^{\otimes k} - E) + c_2(F \otimes L^{\otimes k} - E) + \cdots \\ &\quad = \big(1 + c_1(F \otimes L^{\otimes k}) + c_2(F \otimes L^{\otimes k}) + \cdots\big) / \big(1 + c_1(E) + c_2(E) + \cdots\big). \end{aligned}$$

If the rank of E and F, and i are chosen so that $\Sigma^{i-1}(\tau_k)$ is empty, then $\Sigma^i(\tau_k)$ is a closed 2-calibrated submanifold.

Corollary 1.1. *Let (M, α), $\alpha \in \Omega^1(M)$, be an exact contact manifold of dimension $2n+1$. Let E, F be complex vector bundles and let i be a positive integer such that*

- *the codimension in* $\mathrm{Hom}(E, F)$ *of the strata of homomorphisms of rank i is not bigger than $2n+1$,*
- *the codimension in* $\mathrm{Hom}(E, F)$ *of the strata of homomorphisms of rank $i-1$ is bigger than $2n+1$.*

Then there exist contact submanifolds whose Poincaré dual is $\Delta_{E,F,i}$. In particular, for any even cohomology class which is a Chern class of some complex vector bundle over M, there exist a contact submanifold Poincaré dual to it.

Remark 1.1. One is expecting that the determinantal submanifolds coming Proposition 1.2 will be more general than the zeroes of vector bundles coming from Proposition 1.1. A more detailed discussion of this issue appears in Appendix B.

The next application is an analog for 2-calibrated manifolds of the embedding theorem for symplectic manifolds of [32] (Theorem 1.2), extending results of [31] for contact manifolds.

Corollary 1.2. *Let (M^{2n+1}, D, ω) be an integral 2-calibrated manifold. Then it is possible to find maps $\phi_k : M \to \mathbb{CP}^{2n}$ so that for all $k \in \mathbb{N}$ large enough one has:*

- *$d\phi_{k|D}$ is injective (ϕ_k is an immersion along D).*
- *$[\phi_k^* \omega_{FS}] = [k\omega]$, where ω_{FS} is the Fubini–Study 2-form of $\mathbb{CP}^{2n}$.*

In particular if (M^3, D) is a 3-manifold with a (smooth) taut confoliation, it is possible to find immersions along D in $\mathbb{CP}^2$.

The previous corollary can be improved in two directions.

Corollary 1.3 (see [32], Corollary 2.6). *Let $(M^{2n+1}, \mathcal{D}, \omega)$ be a manifold with an integral 2-calibrated foliation. Then the maps of Corollary 1.2 can be composed from the right with diffeomorphisms of M, so that for all $k \in \mathbb{N}$ large enough the equality $[\phi_k^* \omega_{FS}] = [k\omega]$ holds also at the level of foliated 2-forms, i.e., $\phi_k^* \omega_{FS|\mathcal{D}} = kw_{|\mathcal{D}}$.*

The second improvement is that the immersion along D can be perturbed to be transverse to any finite collection of complex submanifolds of projective space.

Another application is the existence of Lefschetz pencil structures, introduced in [23].

Definition 1.3 (see Section 1 in [13]). Let (M, D, ω) be a 2-calibrated manifold and $x \in M$. A chart $\varphi : (\mathbb{C}^n \times \mathbb{R}, 0) \to (M, x)$ is compatible with (D, ω) (at x) if at the origin it sends the foliation of $\mathbb{C}^n \times \mathbb{R}$ by complex hyperplanes into D, and $\varphi^*\omega(0)$ restricted to the subspace $\mathbb{C}^n \times \{0\}$ is of type $(1,1)$.

Definition 1.4 (see [35]). A Lefschetz pencil structure for (M, D, ω) is a triple (f, B, Δ), where $B \subset M$ is a codimension four 2-calibrated submanifold, and $f : M \backslash B \to \mathbb{CP}^1$ is a smooth map such that:

(1) f is a submersion along D away from Δ, a 1-dimensional manifold transverse to D where the restriction of the differential of f to D vanishes.

(2) For any $x \in \Delta$ there exist a chart φ compatible with (D, ω) at x and a complex coordinate ζ of $\mathbb{CP}^1$ defined about $f(x)$ such that

$$\zeta \circ f \circ \varphi(z, s) = (z^1)^2 + \cdots + (z^n)^2 + t(s), \tag{1}$$

where $t \in C^\infty(\mathbb{R}, \mathbb{C})$.

(3) For any $x \in B$ there exist a chart φ compatible with (D, ω) at x and a complex coordinate ζ of $\mathbb{CP}^1$ defined about $f(x)$ such that $B \equiv z^1 = z^2 = 0$ and $\zeta \circ f \circ \varphi(z, s) = z^1/z^2$.

(4) $f(\Delta)$ is an immersed curve with generic self intersections.

Theorem 1.1. *Let (M, D, ω) be an integral 2-calibrated manifold and let h be an integer lift of $[\omega]$. Then for all $k \in \mathbb{N}$ large enough there exist Lefschetz pencils (f_k, B_k, Δ_k) such that*

(1) *the regular fibers are Poincaré dual to kh,*

(2) *the inclusion $l : W_k \hookrightarrow M$ induces maps $l_* : \pi_j(W_k) \to \pi_j(M)$ (resp. $l_* : H_j(W_k; \mathbb{Z}) \to H_j(M; \mathbb{Z})$) which are isomorphisms for $j \leq n - 2$, and an epimorphism for $j = n - 1$.*

All the results stated follow mostly from a general principle of (estimated) transversality along D (Theorems 7.1 and 7.2).

In a problem $\mathscr{P}$ of transversality along D we have three ingredients: (i) the bundle $E \to (M, D, \omega)$, (ii) the submanifold or more generally the stratification $\mathscr{S} \subset E$, and (iii) the section $\tau : M \to E$ to be perturbed to become transverse along D to $\mathscr{S}$.

In Section 2 we will define the class of sections and bundles we will work with, the so-called sequences of very ample bundles and approximately holomorphic sections.

As in the approximately holomorphic theory for symplectic manifolds (see [11], [4]), transversality problems will be solved by patching local solutions. The right strategy to solve the corresponding local problems for sections is to turn them into local problems for approximately holomorphic functions. This will be done through the use of reference sections, which can be thought of as the bump functions of the theory. The necessary local analysis needed to construct such sections is developed in Section 3.

There is a second strategy to solve $\mathscr{P}$. It is not only true that the natural example of a 2-calibrated structure is a hypersurface inside a symplectic manifold, but every 2-calibrated manifold (D co-oriented) admits a symplectization $(M \times [-\varepsilon, \varepsilon], \Omega)$ (Lemma 3.4). We will introduce a new transversality problem $\overline{\mathscr{P}}$ for a stratification $\overline{\mathscr{S}}$ of a bundle $\overline{E} \to (M \times [-\varepsilon, \varepsilon], \Omega)$ so that a solution $\bar{\tau} : M \times [-\varepsilon, \varepsilon] \to \overline{E}$ to $\overline{\mathscr{P}}$ restricts to $\bar{\tau}_{|M}$ a solution to $\mathscr{P}$. The advantage of this point of view is that since we are in a symplectic manifold, as long as the extension $\overline{\mathscr{P}}$ falls in the right class of problems we can use the existing approximately holomorphic theory for symplectic manifolds to solve it. Still, the existing approximately holomorphic theory turns out not to be enough for our purposes, so we need to develop further the relative approximately holomorphic theory introduced by J. P. Mohsen [30]. We will make an exposition of both the intrinsic and the relative approximately holomorphic theories, and we will prove the main transversality theorem using the latter.

In Section 4 we give an account of the notion of estimated transversality of a section along a distribution. For the intrinsic theory (problem $\mathscr{P}$) the distribution will be D, whereas for the relative theory the problem $\overline{\mathscr{P}}$ will amount to achieving transversality over $M \subset (M \times [-\varepsilon, \varepsilon], \Omega)$. We will also introduce the right class of stratifications $\mathscr{S}$ (already defined in the symplectic setting in [4]), the so-called approximately holomorphic finite Whitney stratifications, whose strata roughly behave as the zero section of a vector bundle in the sense that locally they will be given by approximately holomorphic functions and they will be transverse enough to the fibers. The fundamental technical result (Lemma 4.5) is that locally estimated transversality along D (resp. over M) of an approximately holomorphic section to $\mathscr{S}$ (resp. $\overline{\mathscr{S}}$) is equivalent to estimated transversality along D (resp. over M) to $\mathbf{0}$ of a related $\mathbb{C}^l$-valued approximately holomorphic function.

Section 5 is devoted to the study of bundles of pseudo-holomorphic jets needed to obtain what we call generic approximately holomorphic maps to projective spaces, constructed by projectivizing $(m+1)$-tuples of approximately holomorphic sections of powers of the pre-quantum line bundle $L^{\otimes k}$ (i.e., analogs of generic linear systems in complex geometry); genericity will be defined as the solution of a uniform strong transversality problem to a stratification $\mathscr{S}$ in these bundles of pseudo-holomorphic jets. Several difficulties have to be overcome. Firstly, since we want to obtain a strong transversality result the jet of the section to be perturbed has to be itself an approximately holomorphic section, so that the transversality problem falls in the right class, something which fails to hold due to the uniform positivity along D of the sequence $L^{\otimes k}$. This is solved by introducing a new connection in the bundles of pseudo-holomorphic jets. Secondly, we need to define a stratification $\mathscr{S}$ of the right kind. This is done in Section 6 by introducing the bundles of pseudo-holomorphic jets for maps to projective spaces, and defining there $\mathbb{P}\mathscr{S}$—a "linear" analog of the Thom–Boardman stratification; $\mathscr{S}$ is then constructed by pulling back $\mathbb{P}\mathscr{S}$ by the corresponding jet extension of the projectivization map $\pi : \mathbb{C}^{m+1} \backslash \{0\} \to \mathbb{CP}^m$. The properties of both the map and of $\mathbb{P}\mathscr{S}$ are used to conclude that $\mathscr{S}$ is indeed of the right kind, and thus the transversality problem falls in the right class. The necessary modifications for the relative theory are also described.

In Section 7 we give the main strong transversality result.

The proofs of the theorems stated in this introduction are given in Section 8.

Our results are based on the existence of plenty of approximately holomorphic sections of very ample line bundles. In the integrable setting the existence of enough meromorphic functions/holomorphic sections has been used to prove results of similar nature to ours:

(i) In [16] E. Ghys gave conditions on a compact space laminated by Riemann surfaces for the existence of plenty of meromorphic functions. More generally, B. Deroin has extended those results to laminations by complex leaves without vanishing cycle, and endowed with positive Hermitian line bundles [10]. The work of Ghys and Deroin proves the existence of leafwise holomorphic embeddings into projective spaces of the aforementioned laminated spaces (compare with Corollary 1.2), although the maps—even in the case of smooth foliations—are in general only continuous in the transverse directions. The strategy they follow is working in the universal cover of the leaves of the lamination. Interestingly enough, Deroin's results are obtained by extending some techniques of approximately holomorphic geometry to the leaves, which are open Kähler manifolds with bounded geometry.

(ii) In [33] Ohsawa and Sibony gave a solution to the tangential Cauchy–Riemann equation with L^2-estimates for sections of a positive CR line bundle over a Levi-flat compact manifold. As a consequence they were able to produce CR embeddings into projective space of any prescribed order of regularity (though in general non-smooth).

Part of the results of the present article were announced in [22], [23] (Proposition 1.1, Corollary 1.2, Corollary 1.3, Theorem 1.1 and Theorem 7.1), where an account of the results available through an intrinsic approximately holomorphic theory was presented.

While a more detailed study of 2-calibrated structures is feasible, we do not think the results that could be obtained would be relevant enough to justify its undertaking.

There are two main reasons to develop an approximately holomorphic theory for 2-calibrated structures. The first one is because they contain contact structures and 2-calibrated foliations. Approximately holomorphic geometry has already been introduced in the contact setting [24], [35], [30], [31]. Its most important application has been the construction of compatible open book decompositions for contact manifolds of arbitrary dimension [17]. Our contribution in this article to contact geometry is the construction of a large class of contact submanifolds and the determination of their homology class (Corollary 1.1). We want to propose 2-calibrated foliations as an interesting higher-dimensional generalization of 3-dimensional taut foliations. In [26]—and building on the results of this article—it is shown that any such foliation $(M, \mathscr{D}, \omega)$ contains a 3-dimensional taut foliation $(W^3, \mathscr{D}_W) \hookrightarrow (M, \mathscr{D})$ so that the inclusion descends to a homeomorphism between leaf spaces. This is done by showing that W^3 can be chosen to intersect each leaf of $(M, \mathscr{D})$ in a unique connected component, which is somehow surprising since often the leaves are immersed submanifolds dense in M.

The second reason to develop an approximately holomorphic theory for 2-calibrated structures is that sometimes they appear as auxiliary structures. If M

is an odd-dimensional manifold and ω a maximally non-degenerate closed 2-form, any distribution D complementary to the kernel of ω endows M with a 2-calibrated structure. In [27] this idea was applied to almost contact manifolds to construct (via approximately holomorphic theory) open book decomposition with control on the topology of the leaves (see also [36]).

If $(M, D, J) \hookrightarrow (\mathbb{CP}^N, \omega_{FS})$ is a CR manifold of hypersurface type which has a CR embedding in projective space, then in [25] we show that the constructions of this article can be performed in the CR category. In particular CR Lefschetz pencils are constructed, yielding CR Morse functions defined away from a CR submanifold of base points.

All the applications outlined so far for contact manifolds, 2-calibrated foliations, and projective CR manifolds use at most pseudo-holomorphic 1-jets. If the CR manifold is Levi-flat then it makes sense to speak about r-generic CR functions. These are defined to be leafwise r-generic holomorphic functions, i.e., functions whose leafwise holomorphic r-jet is transverse to the Thom–Boardman stratification of the bundle of holomorphic r-jets over each leaf. In [25] we show that Levi-flat CR manifolds embedded in projective space admit for all $k \gg 1$ r-generic linear systems. These are (holomorphic) linear systems of $\mathcal{O}(k) \to \mathbb{CP}^N$ of rank $m(r)$ whose restriction to M define r-generic CR functions away from base points (Definition 5.1). Briefly, such functions are easily seen to be CR functions whose CR r-jet prolongation solve $\mathscr{P}_{\text{int}}$ a transversality problem over the leaves of the foliation $\mathscr{D}$ in the bundle of CR r-jets of CR maps from M to $\mathbb{CP}^{m(r)}$. One has to show that it can be "linearized" to a transversality problem $\mathscr{P}_{\text{lin}}$ (the bundle, the stratification, and the notion of CR r-jet all have to be replaced by "linear" analogs) that fits into the ones solved in Theorem 7.2; solutions are shown to exist among restrictions of holomorphic sections $\mathcal{O}(k)$. Finally, it has to be checked that the CR solution to $\mathscr{P}_{\text{lin}}$ is also a solution of $\mathscr{P}_{\text{int}}$.

We think that the existence of r-generic linear systems for projective Levi-flat CR manifolds is a relevant result by itself and justifies the extension of the approximately holomorphic theory to higher order jet bundles, which is technically awkward. We expect it to be useful to analyze such manifolds. For example, one can use it to define r-generic functions $f : (M^{2n+1}, \mathscr{D}, J) \to \mathbb{CP}^n$ (with no base points) for which the regular level sets are unions of circles (with variable number of components), and using the analysis of the singularities one can define a dynamical system transverse to $\mathscr{D}$ (at least for low values of $n \geq 2$); by iterating the Lefschetz pencil construction (the dimensional induction of [6], Section 5) one can also define maps to $\mathbb{CP}^{n-1}$ whose fibers (by [26]) are 3-manifolds intersecting each leaf of $\mathscr{D}$ in a connected Riemann surface.

We point out that the results in [25] do not include those of Ghys and Deroin [16], [10] and those of Ohsawa and Sibony [33]. Our results require starting with a CR embedding into projective space ([33] gives sufficient conditions to produce it).

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2. Ample bundles and approximately holomorphic sections

Let (M, D, ω) be an integral 2-calibrated manifold. Let us fix once and for all a compatible almost complex structure $J : D \to D$, and a metric g so that $g_{|D} = \omega(\cdot, J)$. The kernel of ω is required to be g-orthogonal to D, so as to make some of the computations in the local theory simpler. Notice that for any such metric the closed $2n$-form ω^n is a calibration for D [21].

If we forget about the 2-form what remains is the following structure.

Definition 2.1. An almost CR structure is a tuple (M, D, J, g) where D is a codimension 1 distribution, $J : D \to D$ an almost complex structure, and g a metric whose restriction to D is compatible with J (J is g-orthogonal and g-antisymmetric).

Let $(L, \nabla) \to M$ be any Hermitian line bundle—or more generally vector bundle—with compatible connection. Let $\hat{D}$ denote the pullback to L of D; let $\hat{J}$ and $\hat{g}$ be the almost complex structure and metric on L, which extend the Hermitian structure on the fibers and are defined on the horizontal distribution associated to ∇ by pulling back J and g, respectively. Then $(L, \hat{D}, \hat{J}, \hat{g})$ is an almost CR manifold.

Our goal is to be able to construct sections $\tau : M \to L$ which (i) are close enough to satisfying $\tau_* J = \hat{J} \tau_*$ (for which we use the adjective almost holomorphic instead of almost CR to be consistent with the terminology of [24] and [35]), and (ii) transverse to suitable submanifolds of the total space of L. In the almost complex setting we know that what ensures their existence is roughly speaking asking the curvature of the connection to be of type $(1, 1)$ and positive.

Definition 2.2 (see [4], Definition 2.1). Given $c > 0$, $\delta \geq 0$, a Hermitian line bundle with compatible connection $(L, \nabla) \to (M, D, J, g)$ is (c, δ)-ample (or just ample) if its curvature F satisfies $iF(v, Jv) \geq cg(v, v)$ for all $v \in D$, and $|F_{|D} - F^{1,1}_{|D}|_g \leq \delta$, where we use the supremum norm.

A sequence (L_k, ∇_k) of Hermitian line bundles with compatible connections is asymptotically very ample (or just very ample) if fixed constants $c > 0$, δ,

$(C_j)_{j\geq 0} \geq 0$ exist so that for all $k \gg 1$ the following inequalities for the curvatures F_k hold:

(1) $iF_k(v, Jv) \geq ckg(v, v)$ for all $v \in D$.

(2) $|F_{k|D} - F_{k|D}^{1,1}|_g \leq \delta k^{1/2}$.

(3) $|\nabla^j F_k|_g \leq C_j k$.

Another motivation for the previous definition is the case of Levi-flat CR manifolds, where according to the results of Ohsawa and Sibony [33] leafwise positivity grants the existence of plenty of CR sections (with an appropriate twisting by a line bundle).

The fundamental example of an ample bundle is the pre-quantum line bundle L of an integral 2-calibrated manifold (M, D, ω) (with $c = 2\pi$, $\delta = 0$). Its tensor powers $L^{\otimes k}$ define a very ample sequence of line bundles.

From now on we will only consider almost CR structures on 2-calibrated manifolds defined by compatible almost complex structures and metrics. Similarly, we will only consider the very ample sequence $L^{\otimes k}$.

For any $\tau_k \in \Gamma(L^{\otimes k})$ we use J to split the restriction of $\nabla\tau_k$ to D:

$$\nabla_D \tau_k = \partial\tau_k + \bar{\partial}\tau_k, \qquad \partial\tau_k \in \Gamma(D^{*1,0} \otimes L^{\otimes k}),\ \bar{\partial}\tau_k \in \Gamma(D^{*0,1} \otimes L^{\otimes k}).$$

We can see $\bar{\partial}\tau_k$ as a section of $T^*M \otimes L^{\otimes k}$ by declaring it to vanish on $D^{\perp}$, and then use the Levi-Civita connection on T^*M to define $\nabla^{r-1}\bar{\partial}\tau_k \in \Gamma(T^*M^{\otimes r} \otimes L^{\otimes k})$.

Let us denote the rescaled metric kg by g_k.

Definition 2.3. A sequence of sections τ_k of $L^{\otimes k}$ is approximately J-holomorphic (or approximately holomorphic or simply A.H.) if positive constants $(C_j)_{j\geq 0}$ exist such that

$$|\nabla^j \tau_k|_{g_k} \leq C_j, \qquad |\nabla^{j-1}\bar{\partial}\tau_k|_{g_k} \leq C_j k^{-1/2}.$$

If we want to make the bounds explicit speak of an A.H.(C_j) sequence.

Remark 2.1. The original notion of A.H. sequence introduced in [24], [35] is a bit more general than Definition 2.3. The difference—as well as the fact that only a finite number of derivatives were taken into account—is that the direction orthogonal to D had a different treatment. The main theorem of [24] produced appropriate A.H. sequences of sections with good control on any finite number of derivatives along D but little along $D^{\perp}$. Using the relative theory one can obtain solutions with control in all directions, so we can avoid using the technically more complicated definition of [24], [35].

3. The local approximately holomorphic theory

Perhaps the most important idea in Donaldson's work [11] was the construction of localized A.H. sections (inspired by the work of Tian [38]) by adopting a unitary point of view instead of a holomorphic one. The use of a unitary connection in a Darboux chart allowed him to find a model for the coupled Cauchy–Riemann equation invariant under rescaling—provided that one works in the appropriate tensor power of the pre-quantum line bundle—and to explicitly write down concentrated solutions giving rise to the so-called reference sections.

The local approximately holomorphic theory, using an intrinsic construction or the symplectization to be introduced in Section 3.1, is based on the choice of appropriate families of charts. In the intrinsic local theory we need as well a local model for the coupled Cauchy–Riemann equations and a good choice of explicit solution.

For 2-calibrated manifolds the local model for the intrinsic approximately holomorphic theory (that can only be achieved asymptotically when $k \to \infty$) is the following:

- The domain is $\mathbb{C}^n \times \mathbb{R}$, with coordinates $z^1, \dots, z^n$, s (sometimes we write them as $x^1, \dots, x^{2n+1}$ or $x^1, \dots, x^{2n}$, s).
- The distribution D_h is the tangent space to the level hyperplanes of the vertical or real coordinate s.
- The identification of each leaf with $\mathbb{C}^n$ means that we have fixed the leafwise standard almost complex structure J_0.
- The metric is the Euclidean one g_0 with Levi-Civita connection d (usual partial derivatives), and the distance is the Euclidean norm $|\cdot|$.
- The 2-form in the fixed coordinates is required to be

$$\omega_{\text{std}} = \frac{i}{2}\sum_{i=1}^{n} dz^i \wedge d\bar{z}^i. \tag{2}$$

- We ask for a choice of unitary trivialization of the line bundle whose connection form is

$$A = \frac{1}{4}\sum_{i=1}^{n} z^i\, d\bar{z}^i - \bar{z}^i\, dz^i. \tag{3}$$

In $\mathbb{R}^N$ with coordinates $x^1, \dots, x^N$ let $\mathbb{R}^p$ denote the distribution by p-planes $\langle \partial/\partial x^{i_1}, \dots, \partial/\partial x^{i_p} \rangle$, $1 \le i_1 < \cdots < i_p \le N$; its Euclidean orthogonal is denoted by $\mathbb{R}^{N-p}$. If we have a distribution D' of dimension p in $\mathbb{R}^N$ which is transverse to $\mathbb{R}^{N-p}$, we can measure its distance to $\mathbb{R}^p$ to order j with respect to the flat connection d as follows: D' can be identified with an element of $\mathrm{Hom}(\mathbb{R}^p, \mathbb{R}^{N-p})$. We let v^{i_l}, $l = 1, \dots, p$, be the vector field in $\mathbb{R}^{N-p}$ such that $\partial/\partial x^{i_l} + v^{i_l} \in D'$. Then we define

$$|\mathrm{d}^j(\mathbb{R}^p - D')|_{g_0} = \max\{|\mathrm{d}^j v^{i_1}|_{g_0}, \dots, |\mathrm{d}^j v^{i_p}|_{g_0}\},$$

which by definition is coordinate dependent.

In the previous local model let us denote the line field spanned by $\partial/\partial s$ by D_v. According to the previous paragraph we can measure the distance in $\mathbb{C}^n \times \mathbb{R}$ to D_h (resp. D_v) of any codimension 1 (resp. dimension 1) distribution transverse to D_v (resp. D_h).

Definition 3.1. Let $\varphi_{k,x} : (\mathbb{C}^n \times \mathbb{R}, 0) \to (U_{k,x}, x)$, for all $x \in M$ and all $k \gg 1$, be a family of charts with coordinates $z_k^1, \dots, z_k^n$, s_k. We call them a family of approximately holomorphic coordinates if there exist constants independent of k, x (uniform) so that the following estimates hold for all $k \gg 1$ at the points of $B(0, \rho k^{1/2})$, $\rho > 0$:

(1) The Euclidean and the induced metric are comparable to any order, i.e.,

$$\frac{1}{\gamma} g_0 \le g_k \le \gamma g_0, \qquad \gamma > 0, \qquad \text{and} \qquad |\nabla^j \varphi_{k,x}^{-1}|_{g_0} \le O(k^{-1/2}) \quad \text{for all } j \ge 2,$$

where ∇ denotes the Levi-Civita connection with respect to g.

(2) The kernel of ω, which is $D^\perp$, is sent to a line field $\varphi_{k,x}^* D^\perp$ transverse to D_h and such that

$$|\varphi_{k,x}^* D^\perp - D_v|_{g_0} \le |(z_k, s_k)| O(k^{-1/2}),$$
$$|\mathrm{d}^j(\varphi_{k,x}^* D^\perp - D_v)|_{g_0} \le O(k^{-1/2}) \quad \text{for all } j \ge 1.$$

The pullback of D is transverse to D_v and

$$|\varphi_{k,x}^* D - D_h|_{g_0} \le |(z_k, s_k)| O(k^{-1/2}),$$
$$|\mathrm{d}^j(\varphi_{k,x}^* D - D_h)|_{g_0} \le O(k^{-1/2}) \quad \text{for all } j \ge 1.$$

(3) Regarding the antiholomorphic components,

$$|\bar\partial \varphi_{k,x}^{-1}(z_k, s_k)|_{g_0} \le |(z_k, s_k)| O(k^{-1/2}),$$
$$|\nabla^j \bar\partial \varphi_{k,x}^{-1}(z_k, s_k)|_{g_0} \le O(k^{-1/2}) \quad \text{for all } j \ge 1,$$

where $\bar\partial \varphi_{k,x}^{-1}$ is the antiholomorphic component of $\nabla_D(\pi_{D_h} \circ \varphi_{k,x}^{-1})$, with $\pi_{D_h} : \mathbb{C}^n \times \mathbb{R} \to \mathbb{C}^n$ the projection onto the first factor.

We speak of Darboux coordinates when the additional condition $\varphi_{k,x}^* k\omega = \omega_0$ holds.

Remark 3.1. According to condition (2) (resp. (3)) we have $\varphi_{k,x}^* D = D_h$, $\varphi_{k,x}^* D^\perp = D_v$ (resp. $\varphi_{k,x}^* J = J_0$) at the origin. For most of our constructions it is enough to require the equality up to a summand of size $O(k^{-1/2})$ at most, but since these equalities are needed to prove results concerning pseudo-holomorphic jets (in particular the identities concerning local representations and subsets of transverse holonomy of Lemma 6.2) we decided to ask for them from the very beginning.

Remark 3.2. If we are in an almost complex manifold, then conditions (1) and (3) ((2) makes no sense) recover the notion of approximately holomorphic charts (resp. Darboux charts if we add the Darboux condition on the 2-form).

A chart centred at a point for which the Darboux condition holds can always be obtained: (M, D, ω) is a coisotropic submanifold of its symplectization, as defined in Lemma 3.4. The local normal form theorem for coisotropic submanifolds ([39], Theorem 3.4.10) provides such a chart. Families of Darboux charts can be constructed using the same local normal form. Since this would fall into the relative theory we prefer to give a different proof.

Lemma 3.1. *Let* (M, D, ω) *be a (compact) 2-calibrated manifold (with* J, g *already fixed). Then a family of Darboux charts can always be constructed.*

Proof. Let us fix a family of charts $\psi_x : \mathbb{R}^{2n+1} \to U_x$ depending smoothly on x, where $x \in M_1$ a small enough subset of M, so that $\psi_x^* D = D_h$, $\psi_x^* D^\perp = D_v$ at the origin. Denote by $x^1, \dots, x^{2n}, s$ the coordinates on $\mathbb{R}^{2n+1}$. We compose ψ_x with the diffeomorphism $\Theta_x : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ which is the identity on $\mathbb{R}^{2n} \times \{0\}$, preserves setwise the horizontal foliation D_h and sends $\operatorname{Ker} \psi_x^* \omega$ to D_v. The diffeomorphisms Θ_x depend smoothly on x.

Now we fix J_0 to identify $\mathbb{R}^{2n+1}$ with $\mathbb{C}^n \times \mathbb{R}$ and compose with an element of $\mathrm{GL}(2n, \mathbb{R}) \subset \mathrm{GL}(2n+1, \mathbb{R})$ (again depending smoothly on $x \in M_1$), so that we obtain charts φ_x for which the pullback of J at the origin equals J_0.

By compactness M can be covered with a finite number of subsets $M_1, \dots, M_h$ in which the above charts can be constructed. In this way we obtain charts centred at every $x \in M$ (we might have more than one chart for each $x \in M$, but that is not relevant) so that the bounds on tensors pulled back from M to a ball of fixed radius in the domain of the charts will not depend on x.

We define $\varphi_{k,x}$ to be the composition $\varphi_x \circ \gamma_{k^{-1/2}}$, where $\gamma_{k^{-1/2}} : \mathbb{C}^n \times \mathbb{R} \to \mathbb{C}^n \times \mathbb{R}$ is the homothety by factor $k^{-1/2}$. The equalities at the origin together with the smooth dependence on x of the constructions previous to the rescaling, imply that we have obtained approximately holomorphic coordinates.

To obtain Darboux charts we need to modify $\varphi_{k,x}$ as follows: we apply Darboux's lemma with estimates (Lemma 2.2 in [4]) to the almost complex manifolds $(\mathbb{C}^n \times \{0\}, \varphi_{k,x}^* J_{|\mathbb{C}^n \times \{0\}}, \varphi_{k,x}^* g_{|\mathbb{C}^n \times \{0\}})$ and the 2-forms $\varphi_{k,x}^* \omega_{|\mathbb{C}^n \times \{0\}}$. We get diffeomorphisms $\Psi_{k,x}$ on this leaf that are extended to $\mathbb{C}^n \times \mathbb{R}$ independently of the vertical coordinate s_k. The bounds on $\Psi_{k,x}$ and their derivatives coming from Lemma 2.2 in [4] imply that the compositions $\varphi_{k,x} \circ \Psi_{k,x} : (\mathbb{C}^n \times \mathbb{R}, 0) \to (U_{k,x}, x)$ still define approximately holomorphic coordinates. Moreover, we can assume $(\varphi_{k,x} \circ \Psi_{k,x})^* J = J_0$ at the origin.

Since $\partial/\partial s_k$ generates the kernel of

$$(\varphi_{k,x} \circ \Psi_{k,x})^* \omega = \sum_{1 \le i < l \le 2n} \omega_{il} \, dx_k^i \wedge dx_k^l + \sum_{1 \le i \le 2n} \omega_i \, dx_k^i \wedge ds_k$$

all ω_i vanish. Closedness implies that each function ω_{il} is independent of s_k. Therefore $(\varphi_{k,x} \circ \Psi_{k,x})^* \omega$ is determined by its restriction to $\mathbb{C}^n \times \{0\}$, which by construction is $\omega_{\mathrm{std}|\mathbb{C}^n \times \{0\}}$. Thus, ω is sent to ω_{std}. □

Darboux charts are useful because there local computations become simpler.

Let d_k denote the distance defined by the metric g_k.

Recall that in the domain of a Darboux chart we can always fix $\xi_{k,x}$ a unitary trivialization of $L^{\otimes k}$ whose connection form is A (equation (3)).

Lemma 3.2. *Let $\varphi_{k,x} : (\mathbb{C}^n \times \mathbb{R}, 0) \to (U_{k,x}, x)$ be a family of Darboux charts with coordinates $x_k^1, \dots, x_k^{2n}, x_k^{2n+1}$. Let F be a bundle associated to either TM or D and let $F_{k,x} \to B(0, \rho k^{1/2}) \subset \mathbb{C}^n \times \mathbb{R}$ denote the pullback of F by $\varphi_{k,x}$. Associated to the Darboux coordinates there is a canonical trivialization $\zeta_{k,x,j}$ of $F_{k,x}$. Let T_k be a sequence of sections of $F \otimes L^{\otimes k}$ and use the frames $\zeta_{k,x,j} \otimes \xi_{k,x}$ to write $\varphi_{k,x}^* T_k$ locally as a function $T'_{k,x}$. Let P_j be a polynomial such that for any multi-index α of length $j = 0, \dots, r$, at the points of $B(0, \rho k^{1/2})$ and for all $k \gg 1$ we have*

$$\left| \frac{\partial}{\partial x_k^\alpha} T'_{k,x} \right|_{g_0} \le P_j(|(z_k, s_k)|) O(k^{-1/2}).$$

Then $|\nabla^r T_k(y)|_{g_k} \le Q_r\big(d_k(x,y)\big) O(k^{-1/2})$, where the polynomial Q_r depends only on $P_1, \dots, P_r$. Conversely, from bounds using the global metric elements g_k, d_k, ∇ we obtain similar bounds for the local Euclidean elements.

Proof. This is a simple calculation based on items (1) and (2), and in the Darboux condition of Definition 3.1. Also notice that the presence of the connection form and its derivatives is absorbed by the polynomial, since $|A| \le O(|(z_k, s_k)|)$ and its derivatives are of order $O(1)$. □

Remark 3.3. Lemma 3.2 admits different modifications. It holds in a similar fashion for bounds of order $O(1)$ instead of order $O(k^{-1/2})$, and also for sections T_k of F instead of $F \otimes L^{\otimes k}$ (with $F_{k,x}$ locally trivialized by $\zeta_{k,x,j}$). It is also possible to consider the inequalities in the ball of (uniform) radius $\rho > 0$ rather than $\rho k^{1/2}$. There is also a version for symplectic manifolds.

Let $\bar{\partial}_0$ denote the $(0,1)$-component with respect to $J_0 : \mathbb{C}^n \times \mathbb{R} \to \mathbb{C}^n \times \mathbb{R}$ of the leafwise derivation operator d_{D_h}.

Lemma 3.3. *Let $\varphi_{k,x} : (\mathbb{C}^n \times \mathbb{R}, 0) \to (U_{k,x}, x)$ be a family of Darboux charts with coordinates $x_k^1, \dots, x_k^{2n}$, s_k. Let $L_{k,x} \to B(0, \rho k^{1/2}) \subset \mathbb{C}^n \times \mathbb{R}$ denote the pullback of $L^{\otimes k}$ by $\varphi_{k,x}$. Let τ_k be a sequence of sections of $L^{\otimes k}$ such that $\varphi_{k,x}^* \tau_k = f_{k,x} \xi_{k,x}$. Let P_j, $P_{j'}$ be polynomials such that for any multi-indices α, β of length $j = 0, \dots, r-1$, and $j' = 0, \dots, r$, respectively, at the points of $B(0, \rho k^{1/2})$ and for all $k \gg 1$ the following inequalities hold:*

$$\left| \frac{\partial}{\partial x_k^{\beta}} f_{k,x} \right|_{g_0} \leq P'_j(|(z_k, s_k)|) O(1), \tag{4}$$

$$\left| \frac{\partial}{\partial x_k^{\alpha}} (\bar{\partial}_0 + A^{0,1}) f_{k,x} \right|_{g_0} \leq P_j(|(z_k, s_k)|) O(k^{-1/2}). \tag{5}$$

Then we have

$$|\nabla^r \tau_k(y)|_{g_k} \leq Q'_r\big(d_k(x, y)\big) O(1), \tag{6}$$

$$|\nabla^{r-1} \bar{\partial} \tau_k(y)|_{g_k} \leq Q_{r-1}\big(d_k(x, y)\big) O(k^{-1/2}), \tag{7}$$

where the polynomial Q_{r-1} (resp. Q'_r) depends only on $P_1, \dots, P_{r-1}, P'_1, \dots, P'_r$ (resp. $P'_1, \dots, P'_r$). Conversely, from bounds using g_k, d_k, ∇, J we obtain similar bounds for g_0, $|\cdot|$, $\mathrm{d} + A$, J_0.

Proof. The equivalence between equations (4) and (6) is the content of Lemma 3.2, but for bounds of order $O(1)$ (see Remark 3.3). The equivalence of equations (4), (5) and equations (6), (7) follows again easily from the properties of Darboux charts. We sketch the case $r = 1$.

From now on $\varphi_{k,x}^* J$, $\varphi_{k,x}^* D$, $\varphi_{k,x}^* g_k$ and all the tensors and sections pulled back to the domain of the charts will be denoted by J, D, $g_k, \dots$ whenever there is no risk of confusion.

Let e_i be any of the local vector fields associated to the first $2n$ coordinates. By condition (2) in Definition 3.1 there exists u_i a local vector field such that $e_i + u_i$ is tangent to D and

$$|u_i|_{g_0} \leq |(z_k, s_k)| O(k^{-1/2}), \qquad |\mathrm{d}^j u_i|_{g_0} \leq O(k^{-1/2}), \qquad j \geq 1. \tag{8}$$

The endomorphism J is defined on D. We can use the orthogonal projection w.r.t g_0 onto D_h to induce out of J another almost complex structure $J_{D_h} : D_h \to D_h$.

Condition (3) in Definition 3.1 implies that

$$|J_0 - J_{D_h}|_{g_0} \leq |(z_k, s_k)|O(k^{-1/2}), \qquad |\mathrm{d}^j(J_0 - J_{D_h})|_{g_0} \leq O(k^{-1/2}), \qquad j \geq 1. \quad (9)$$

By definition $\bar{\partial}_{e_i+u_i}\tau_k = 1/2\nabla_{e_i+u_i}\tau_k + i/2\nabla_{J(e_i+u_i)}\tau_k$.

Equation (8) combined with Lemma 3.2 implies that

$$|\nabla_{u_i}\tau_k|_{g_k} \leq P_1'\big(d_k(x, y)\big)O(k^{-1/2}).$$

Again equations (8) and (6), condition (3) in Definition 3.1, and Lemma 3.2 imply that

$$|\nabla_{J(e_i+u_i)}\tau_k - \nabla_{J_h e_i}\tau_k|_{g_k} \leq P_1''\big(d_k(x, y)\big)O(k^{-1/2}).$$

Therefore the bounds in equation (7) we want for $\bar{\partial}_{e_i+u_i}\tau_k$ are equivalent to the same kind of bounds for

$$1/2\nabla_{e_i}\tau_k + i/2\nabla_{J_h e_i}\tau_k,$$

and by equation (9) for

$$1/2\nabla_{e_i}\tau_k + i/2\nabla_{J_0 e_i}\tau_k,$$

and by definition

$$1/2\nabla_{e_i}\tau_k + i/2\nabla_{J_0 e_i}\tau_k = \big((\bar{\partial}_0 + A^{0,1})_{e_i} f_{k,x}\big)\xi_{k,x}.$$

Bounds for higher order derivatives are proven similarly. □

Definition 3.2 (see [4], Definition 2.2). A sequence of sections of $L^{\otimes k}$ has Gaussian decay with respect to x if there exist polynomials $(P_j)_{j\geq 0}$ and a constant $\lambda > 0$ so that

$$|\nabla^j\tau_k(y)|_{g_k} \leq P_j\big(d_k(x, y)\big)\mathrm{e}^{-\lambda d_k(x,y)^2}.$$

for all $y \in M$ and for all $j \geq 0$.

The main purpose of the use of Darboux charts is the construction of reference sections $\tau_{k,x}^{\mathrm{ref}}$.

Corollary 3.1. *Let (M, D, ω) be a compact 2-calibrated manifold. Then for all $x \in M$ A.H. sections $\tau_{k,x}^{\mathrm{ref}}$ with Gaussian decay with respect to x can be constructed. The bounds are uniform on k, x and these sections have norm greater than some constant κ in $B_{g_k}(x, \rho)$, where $\kappa, \rho > 0$ are uniform on k, x.*

Proof. We follow Donaldson's ideas in [11], Section 2. Let us fix Darboux charts and $\xi_{k,x}$ trivializations of $L^{\otimes k}$ for which the connection form is A. Let β be a standard cut-off function of a single variable, with $\beta(t) = 1$ when $|t| \leq 1/2$ and $\beta(t) = 0$ when $|t| \geq 1$.

Define $\beta_k(z_k, s_k) = \beta(k^{-1/6}|(z_k, s_k)|)$.

In the points where the derivatives of β_k do not vanish we have $|(z_k, s_k)| \geq Ck^{1/6}$, with C uniform (on k, x). Using this inequality we deduce that

$$\begin{aligned} |\mathrm{d}\beta_k|_{g_0} &\leq |(z_k, s_k)|^2 O(k^{-1/2}), \\ |\mathrm{d}^2\beta_k|_{g_0} &\leq |(z_k, s_k)| O(k^{-1/2}), \\ |\mathrm{d}^j\beta_k|_{g_0} &\leq O(k^{-1/2}), \qquad j \geq 3. \end{aligned} \tag{10}$$

Consider the function $f(z_k, s_k) = \mathrm{e}^{-|(z_k, s_k)|^2/4}$. We have

$$\bar{\partial}_0 f + A^{0,1} f = 0. \tag{11}$$

The reference sections are

$$\tau^{\mathrm{ref}}_{k,x} := \beta_k f \xi_{k,x}. \tag{12}$$

Equation (10) implies that for any multi-index α of length $j \leq r$,

$$\left| \frac{\partial}{\partial x^\alpha} \beta_k f \right|_{g_0} \leq P_j(|(z_k, s_k)|)|f| O(1).$$

Therefore, Lemma 3.2 for bounds of type $\mathrm{e}^{-\lambda'|(x,y)|^2} O(1)$, $\lambda' > 0$, gives the Gaussian decay with respect to x:

$$|\nabla^r \tau^{\mathrm{ref}}_{k,x}(y)|_{g_k} \leq Q_r\big(d_k(x, y)\big)\mathrm{e}^{-\lambda d_k(x,y)^2} O(1), \qquad \lambda > 0,$$

where λ appears when relating the distance induced by g and g_0. The Gaussian decay also implies

$$|\nabla^r \tau^{\mathrm{ref}}_{k,x}|_{g_k} \leq O(1).$$

The bound for $|\nabla^{r-1}\bar{\partial}\tau^{\mathrm{ref}}_{k,x}|_{g_k}$ is obtained using the same ideas: from equations (10) and (11) it follows that

$$\left| \frac{\partial}{\partial x^\alpha} (\bar{\partial}_0 + A^{0,1})\beta_k f \right|_{g_0} \leq P_j(|(z_k, s_k)|)|f| O(k^{-1/2})$$

for any multi-index α of length $j \leq r - 1$.

Lemma 3.3 for bounds of type $e^{-\lambda'|(x,y)|^2}O(1)$, $e^{-\lambda'|(x,y)|^2}O(k^{-1/2})$, $\lambda' > 0$ (in equations (4) and (5) resp.) gives

$$|\nabla^{r-1}\bar{\partial}\tau^{\mathrm{ref}}_{k,x}|_{g_k} \leq Q_r(d_k(x,y))e^{-\lambda d_k(x,y)^2}O(k^{-1/2}) \leq O(k^{-1/2}).$$

for some $\lambda > 0$.

The existence of constants $\kappa, \rho > 0$ such that $|\tau^{\mathrm{ref}}_{k,x}| \geq \kappa$ in $B_{g_k}(x,\rho)$, can be easily checked. □

We observe that many of the inequalities we are using (for global tensors) have the same pattern. We will introduce a definition that will avoid the excessive appearance in the notation of such inequalities.

Let E be a Hermitian bundle with connection, F a bundle associated either to TM or to D, and let E_k denote the sequence $F \otimes E \otimes L^{\otimes k}$.

Definition 3.3. Let $T_{k,x}$, $x \in M$, be a family of sequences of sections of E_k. We say that $T_{k,x}$ is C^r-approximately vanishing (or that the sequence vanishes in the C^r-approximate sense), and denote it by $T_{k,x} \approxeq_r 0$, if positive constants $C_0, \dots, C_r$ exist so that

$$|\nabla^j T_{k,x}|_{g_k} \leq C_j k^{-1/2}, \qquad j = 0, \dots, r. \tag{13}$$

There is an analogous definition for sequences T_k of sections of E_k (i.e., without extra dependence on the point $x \in M$).

Using the above language one of the conditions for a sequence τ_k of $L^{\otimes k}$ to be A.H. (Definition 2.3) is that $\bar{\partial}\tau_k \in \Gamma(D^{*0,1} \otimes L^{\otimes k})$ has to be approximately vanishing.

Remark 3.4. Given τ_k an approximately holomorphic sequence of sections of $L^{\otimes k}$, we have defined $\nabla^{r-1}\bar{\partial}\tau_k \in T^*M^{\otimes r} \otimes L^{\otimes k}$ by taking covariant derivatives of $\bar{\partial}\tau_k$ thought of as a section of $T^*M \otimes L^{\otimes k}$. We might have equally defined $\nabla^{r-1}\bar{\partial}\tau_k$ as the image of $\nabla^r\tau_k$ by the projection $\bar{p}_r : T^*M^{\otimes r} \otimes L^{\otimes k} \to T^*M^{\otimes r-1} \otimes D^{*0,1} \otimes L^{\otimes k}$, for using Darboux charts and Lemmas 3.2 and 3.3 (with the inequalities $|\nabla^j\tau_k|_{g_k} \leq O(1)$, $j \geq 0$) one checks that $\bar{\partial}\tau_k \approxeq 0$ if and only if $|\bar{p}_j(\nabla^j\tau_k)|_{g_k} \leq O(k^{-1/2})$, $j \geq 1$.

3.1. Relative approximately holomorphic theory and symplectizations

Definition 3.4. Let (P, Ω) be a symplectic manifold and (M, D, ω) a 2-calibrated manifold. We say that $l : M \hookrightarrow P$ embeds M as a 2-calibrated submanifold of P if $l^*\Omega = \omega$.

Lemma 3.4. *Let (M, D, ω) be a compact co-oriented 2-calibrated manifold. Then it is possible to define a symplectization so that (M, D, ω) embeds as a 2-calibrated submanifold. Any fixed compatible almost complex structure and metric can be extended to a compatible almost complex structure and metric in the symplectization.*

Proof. Let J and g be fixed compatible almost complex structure and metric. The symplectization $(M \times [-\varepsilon, \varepsilon], J, g, \Omega)$ is constructed as follows: let t be the coordinate of the interval. Let α be the unique 1-form of pointwise norm 1 (and positively oriented) whose kernel is D. The closed 2-form Ω is defined to be $\omega + d(t\alpha)$, where α and ω represent the pullback of the corresponding forms to $M \times [-\varepsilon, \varepsilon]$. If ε is chosen small enough then Ω is symplectic.

In the points of M the almost complex structure is extended by sending the positively oriented g-unitary vector in $D^{\perp}$ to $\partial/\partial t$; in those points $\partial/\partial t$ is also defined to have norm 1 and to be orthogonal to TM. It is routine to further extend J to a compatible almost complex structure on the symplectization. The metric defined by Ω and the almost complex structure also extends g. We will not use different notation for the extension of the almost complex structure and metric if there is no risk of confusion.

We also fix G a J-complex distribution on the symplectization restricting to D at the points of M. To do that we choose any line field that at the points of M contains $\partial/\partial t$; this line field spans a complex line field. Its orthogonal with respect to g is by construction J-complex and extends D. □

Remark 3.5. We want to work out a relative theory for embeddings in arbitrary symplectic manifolds—not just in symplectizations—because of our applications to CR manifolds, where we need an ambient complex manifold with plenty of holomorphic sections.

Let (M, D, ω) be a 2-calibrated submanifold of (P, Ω). Let us fix J a compatible almost complex structure on (P, Ω) so that D is J-invariant, and let us define $g = \Omega(\cdot, J\cdot)$. The restriction of (J, g) to (M, D) induces an almost CR structure. We also choose G a J complex distribution that coincides with D at the points of M. The main example to have in mind is the symplectization of (M, D, ω) with an almost complex structure as defined in Lemma 3.4.

We have at our disposal the approximately holomorphic theory for symplectic manifolds [4]. At this point we pause to warn the reader that throughout this section and the rest of the article we will be using A.H. sequences of sections defined in both 2-calibrated (Definition 2.3) and symplectic manifolds (see definitions in [4] or Definition 2.3 for an almost complex base space). Whenever there is no risk of confusion about the base space we will just speak about A.H. sequences of sections.

Let $(L_\Omega, \nabla) \to (P, \Omega)$ be the pre-quantum line bundle. Its powers $(L_\Omega^{\otimes k}, \nabla_k)$ define a very ample sequence of line bundles (in the sense of [4]), which restricts to a very ample sequence of line bundles $(L^{\otimes k}, \nabla_k) \to (M, D, J, g_k)$ (Definition 2.2).

One expects that if $\tau_k \in \Gamma(L_\Omega^{\otimes k})$ is a (symplectic) A.H. sequence of sections, then $\tau_{k|M} : M \to L^{\otimes k}$ is also an A.H. sequence of sections (Definition 2.3). Even more, we will see that it is possible to construct reference sections by restricting (symplectic) reference sections centred at points of M. The key point to prove these results is the choice of appropriate charts.

Recall that in $\mathbb{C}^p = \mathbb{R}^{2p}$ we denote the foliation whose leaves are associated to g distinguished complex coordinates (resp. d distinguished real coordinates) by $\mathbb{C}^g$ (resp. $\mathbb{R}^d$); its Euclidean orthogonal is denoted by $\mathbb{C}^{p-g}$ (resp. $\mathbb{R}^{2p-d}$). From now on if we compare the distance of $\mathbb{C}^g$ to any distribution of the same dimension, we will assume the latter to be transverse to $\mathbb{C}^{p-g}$.

Definition 3.5. Let (P, Ω) be a compact symplectic manifold and G a J-complex distribution of complex dimension g. A family of (symplectic) approximately holomorphic coordinates (resp. Darboux charts) $\varphi_{k,x} : (\mathbb{C}^p, 0) \to (U_{k,x}, x)$ is said to be adapted to G if

$$
\begin{aligned}
|\mathbb{C}^g - G|_{g_0} &\le |(z_k, s_k)| O(k^{-1/2}), & |\mathrm{d}^j(\mathbb{C}^g - G)|_{g_0} &\le O(k^{-1/2}),\\
|\mathbb{C}^{p-g} - G^\perp|_{g_0} &\le |(z_k, s_k)| O(k^{-1/2}), & |\mathrm{d}^j(\mathbb{C}^{p-g} - G^\perp)|_{g_0} &\le O(k^{-1/2}),
\end{aligned}
$$

for all $j \ge 1$.

The existence of approximately holomorphic (resp. Darboux) charts adapted to G is straightforward: once we have approximately holomorphic (resp. Darboux) charts, we compose with a unitary transformation sending G to $\mathbb{C}^g$ at the origin.

Given a 2-calibrated submanifold $(M, D) \hookrightarrow (P, \Omega)$, in order to select coordinate charts adapted to M we fix a distribution $T^{\|}M$ defined in a tubular neighborhood of M as follows: the neighborhood is defined by flowing a little bit the geodesics normal to M. For each point y in the neighborhood, let $x \in M$ be the starting point of the unique geodesic normal to M through y. Then $T^{\|}_y M$ is the result of parallel transport of $T_x M$ along that geodesic.

Definition 3.6. Let $(M, D) \hookrightarrow (P, \Omega)$ be a 2-calibrated submanifold, G a J-complex distribution which extends D (perhaps defined in a tubular neighborhood of M), and $T^{\|}M$ a distribution constructed as above. A family of (symplectic) A.H. coordinates $\varphi_{k,x} : (\mathbb{C}^p, 0) \to (U_{k,x}, x)$ (centred at every point of P) is adapted to (M, G) if it is adapted to G and for the charts centred at points of M the following conditions hold:

(1) M sits in each chart as a fixed linear subspace $\mathbb{R}^{2n+1} \times \{0\} \subset \mathbb{C}^p$ and at the origin $D = \mathbb{R}^{2n} \times \{0\} \subset \mathbb{R}^{2n+1} \times \{0\}$, $D^\perp = \{0\} \times \mathbb{R} \subset \mathbb{R}^{2n+1} \times \{0\}$.

(2) $|\mathbb{R}^{2n+1} - T^{\|}M|_{g_0} \leq |(z_k, s_k)| O(k^{-1/2})$, $|\mathrm{d}^j(\mathbb{R}^{2n+1} - T^{\|}M)|_{g_0} \leq O(k^{-1/2})$ for all $j \geq 1$.

We speak of A.H. charts adapted to (M, G) and Darboux over M if

$$\varphi^*_{k,x}\omega_{|M} = \omega_0. \tag{14}$$

Lemma 3.5. *Let $(M, D) \hookrightarrow (P, \Omega)$ be a 2-calibrated submanifold. Then approximately holomorphic charts adapted to (M, G) and Darboux over M can always be constructed.*

Proof. We start by fixing approximately holomorphic coordinates adapted to G. Then we forget about the ones centred at points of M, that are going to be substituted by new ones. For every $x \in M$ we fix initial charts φ_x depending smoothly on the center—at least in a small neighborhood about each point—with $(\varphi_x J^*, \varphi_x^* g) = (J_0, g_0)$ at the origin. Then we compose with maps $\Theta_x : (\mathbb{C}^p, 0) \to (\mathbb{C}^p, 0)$ that are tangent to the identity map at the origin and send M to a vector space in $\mathbb{C}^p$. The image of the distribution D is J_0-complex at the origin. By composing with a unitary transformation $(D_x, T_x M)$ can be assumed to be sent to $(\mathbb{C}^n \times \{0\}, \mathbb{R}^{2n+1} \times \{0\}) \subset \mathbb{R}^{2p}$.

Next we essentially apply Lemma 3.1 on the leaf $\mathbb{R}^{2n+1} \times \{0\} \subset \mathbb{R}^{2p}$ to get Darboux charts for M: let $\Xi_x : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ be the map which is the identity on $\mathbb{C}^n \times \{0\}$, preserves the foliation by complex hyperplanes, and sends the kernel of ω to the "vertical" or real line field in $\mathbb{R}^{2n+1} \times \{0\}$. We extend it to a diffeomorphism of $\mathbb{R}^{2p}$ independently of the coordinates $x^{2n+2}, \ldots, x^{2p}$. Since the map is by construction tangent to the identity at the origin, we keep the properties at the origin described in the previous paragraph.

We now apply Darboux' lemma on $\mathbb{R}^{2n} \times \{0\}$ for each x. The result is a diffeomorphism on $\mathbb{R}^{2n}$ that can be assumed to preserve J_0 at the origin. We extend it independently of $x^{2n+1}, \ldots, x^{2p}$ to a diffeomorphism of $\mathbb{C}^p$. Notice that $(D_x, T_x M)$ goes to $(\mathbb{R}^{2n} \times \{0\}, \mathbb{R}^{2n+1} \times \{0\})$, J_x to J_0, $G_x \oplus G_x^\perp$ to $\mathbb{C}^n \oplus \mathbb{C}^{p-n}$, and $\operatorname{Ker} \omega_{|D_x}$ to the Euclidean orthogonal of $\mathbb{R}^{2n} \times \{0\} \subset \mathbb{R}^{2n+1} \times \{0\}$. Hence if we apply the homothety $\gamma_{k^{-1/2}} : \mathbb{R}^{2p} \to \mathbb{R}^{2p}$ we obtain a family of charts with the desired properties. □

Lemma 3.6. *A family of A.H. charts $\varphi_{k,x} : (\mathbb{C}^p, 0) \to (U_{k,x}, x)$ adapted to (M, G) and Darboux over M constructed as in Lemma 3.5 restricts to M to Darboux charts.*

Proof. It follows because the charts in Lemma 3.5 are obtained by applying a construction depending smoothly on the center of the chart to obtain a number of equalities for tensors and distributions at the origin, and then rescaling. Therefore when we restrict the charts to M condition (1) in Definition 3.1 holds. Conditions (2) and (3) follow because before rescaling $D_x \oplus D_x^{\perp}$ is sent to $\mathbb{R}^{2n} \oplus \mathbb{R}$ and J_x to J_0. The Darboux condition (equation (14)) holds by construction. □

Lemma 3.7. *Let* $\varphi_{k,x} : (\mathbb{C}^p, 0) \to (P, x)$ *be charts coming from Lemma* 3.5. *Then in* $B(0, \rho k^{1/2}) \subset \mathbb{C}^p$ *it is possible to fix a family of unitary trivializations of* $\varphi_{k,x}^* L_\Omega^{\otimes k}$ *with connection forms* $A_{k,x}$ *such that for all* $k \gg 1$:

(1) $|A_{k,x}|_{g_0} \leq O(|z_k|)$, $|\mathrm{d}A_{k,x}|_{g_0} \leq O(1)$, $|\mathrm{d}^j A_{k,x}|_{g_0} \leq O(k^{-1/2})$, $j \geq 2$.

(2) $A_{k,x|M} = \frac{1}{2}\sum_{i=1}^{n}(x_k^{2i-1} \wedge dx_k^{2i} - x_k^{2i} \wedge dx_k^{2i-1})$.

Proof. By construction $|\varphi_{k,x}^* k\omega|_{g_0} \leq O(1)$, $|\mathrm{d}^j \varphi_{k,x}^* k\omega|_{g_0} \leq O(k^{-1/2})$, $j \geq 1$, on $B(0, \rho k^{1/2})$. Hence, we deduce the existence unitary trivializations with connection forms $A'_{k,x}$ satisfying the bounds of condition (1).

When we restrict the connection forms to M they coincide with A up to a exact 1-form $dF_{k,x}$ defined on $\mathbb{R}^{2n+1} \times \{0\}$; its bounds are as in item (1) above, but on $\mathbb{R}^{2n+1} \times \{0\}$ instead of on $\mathbb{C}^p$. We extend it to $\mathbb{C}^p$ independently of the remaining coordinates and still denote it by $F_{k,x}$. It is always possible to find a unitary trivialization $\xi_{k,x}$ of $\varphi_{k,x}^* L_\Omega^{\otimes k}$ whose connection form is $A'_{k,x} + dF_{k,x}$. These trivializations give the desired result. For simplicity we will denote the family by A when there is no risk of confusion. □

Let G be the J-complex distribution on P that extends D. Given $\tau_k \in \Gamma(L_\Omega^{\otimes k})$, the restriction of the covariant derivative of τ_k to G will be denoted by $\nabla_G \tau_k \in \Gamma(G^* \otimes L_\Omega^{\otimes k})$. Since G is J-complex we can write

$$\nabla_G \tau_k = \bar{\partial}_G \tau_k + \partial_G \tau_k, \qquad \bar{\partial}_G \tau_k \in \Gamma(G^{*0,1} \otimes L_\Omega^{\otimes k}),\ \partial_G \tau_k \in \Gamma(G^{*1,0} \otimes L_\Omega^{\otimes k}).$$

Lemma 3.8. (1) *If* $\tau_k : P \to L_\Omega^{\otimes k}$ *is an A.H. sequence then* $\tau_{k|M} : M \to L^{\otimes k}$ *is also an A.H. sequence.*

(2) *Moreover, the restriction of a family of reference sections of* $(L_\Omega^{\otimes k}, \nabla_k) \to (P, \Omega)$ *centred at the points of* M (*as defined in* [4]) *is a family of reference sections of* $(L^{\otimes k}, \nabla_k) \to (M, D, \omega)$.

(3) *If* $\tau_k : P \to L_\Omega^{\otimes k}$ *is an A.H. sequence then* $\bar{\partial}_G \tau_k \approxeq 0$.

Proof. We fix a family of A.H. charts adapted to (M, G) and Darboux over M, and trivialize the bundles $L_\Omega^{\otimes k}$ as in Lemma 3.7. Let $x_k^1, \ldots, x_k^{2p}$ be the coordinates and write $\tau_{k,x} = f_{k,x}\xi_{k,x}$.

We first observe that Lemmas 3.2 and 3.3 for symplectic manifolds also hold for the connection forms $A_{k,x}$ provided by Lemma 3.7. By Lemma 3.6 the restriction of the coordinates to M are Darboux charts. We can apply Lemma 3.2 for almost complex manifolds, bounds of order $O(1)$, and the connection forms provided by Lemma 3.7, to conclude that the partial derivatives of $f_{k,x}$ are bounded by $O(1)$ in the ball $B(0,\rho) \subset \mathbb{R}^{2p}$. In particular we get the same bounds if we only take into account the partial derivatives with respect to the variables $x_k^1, \ldots, x_k^{2n+1}$ and restrict our attention to $B(0,\rho) \subset \mathbb{R}^{2n+1}$. Now if we apply Lemma 3.2 (this time for almost CR manifolds) we conclude that $|\nabla^j(\tau_{k|M})|_{g_0} \leq O(1)$ in $B(0,\rho) \subset \mathbb{R}^{2n+1}$, for all $j \geq 0$ and for all $x \in M$, the constants being independent of x. Therefore $|\nabla^j(\tau_{k|M})|_{g_k} \leq O(1)$, for all $j \geq 0$, in all points of M.

Lemma 3.3 for symplectic manifolds and the connections of Lemma 3.7 gives

$$\left| \frac{\partial}{\partial x_k^\alpha} (\bar{\partial}_0 + A_{k,x}^{0,1}) f_{k,x} \right|_{g_0} \leq O(k^{-1/2}) \tag{15}$$

in $B(0,\rho) \subset \mathbb{R}^{2p}$. Let us consider the splitting $\mathbb{C}^n \times \mathbb{C}^{p-n}$. The operator $\bar{\partial}_0 + A_{k,x}^{0,1}$ and its derivatives can be split into two pieces using it. We consider the part involving $d\bar{z}_k^1, \ldots, d\bar{z}_k^n$, for which the above inequalities also hold, but now in $B(0,\rho) \subset \mathbb{R}^{2n+1}$. Since the restriction of $A_{k,x}$ to $\mathbb{C}^n \times \mathbb{R}$ is A, the restriction to M of the piece of $\bar{\partial}_0 + A_{k,x}^{0,1}$ involving $d\bar{z}_k^1, \ldots, d\bar{z}_k^n$ is the operator $\bar{\partial}_0 + A^{0,1}$ of Lemma 3.3. Thus we can apply this lemma (we already have the required bounds for the partial derivatives of $f_{k,x}$) to conclude $\bar{\partial}(\tau_{k|M}) \approxeq 0$, and this proves item (1).

It is also easy to check that reference sections for $L_\Omega^{\otimes k}$ centred at the points of M restrict to reference sections for $L^{\otimes k}$, and hence item (2) also holds.

To prove $\bar{\partial}_G \tau_k \approxeq 0$ we use the previous ideas: equation (11) and Lemma 3.3 give, for all $j \geq 0$,

$$|\nabla^j \bar{\partial}_{\mathbb{C}^n} \tau_k|_{g_0} \leq O(k^{-1/2})$$

in $B(0,\rho) \subset \mathbb{R}^{2p}$, where $\bar{\partial}_{\mathbb{C}^n}$ is the part of $\bar{\partial}_0 + A_{k,x}$ involving $d\bar{z}_k^1, \ldots, d\bar{z}_k^n$. The choice of A.H. charts adapted to G and the bounds $|\nabla^j \tau_k|_{g_0} \leq O(1)$, for all $j \geq 0$, easily imply that

$$|\nabla^j(\bar{\partial}_{\mathbb{C}^n} \tau_k - \bar{\partial}_G \tau_k)|_{g_0} \leq O(k^{-1/2})$$

for all $j \geq 0$ and therefore $\bar{\partial}_G \tau_k \approxeq 0$. □

Remark 3.6. Notice that item (3) in Lemma 3.8 is an assertion about a section defined on P, and not on M unlike in item (1).

3.2. Higher rank ample bundles. So far we have only considered approximately holomorphic theory for the sequence of line bundles $(L^{\otimes k}, \nabla_k) \to (M, D, \omega)$, but there are obvious extensions for sequences of the form $E \otimes L^{\otimes k}$, where E is any Hermitian bundle of rank m with compatible connection. Regarding the local theory the role of the reference sections is played by the reference frames $\tau^{\mathrm{ref}}_{k,x,1}, \ldots, \tau^{\mathrm{ref}}_{k,x,m}$, where each $\tau^{\mathrm{ref}}_{k,x,j}$ is an A.H. sequence with Gaussian decay with respect to x and they are a frame of E comparable to a unitary one in $B_{g_k}(x, \rho)$, $\rho > 0$. Reference frames are constructed by tensoring reference sections for $L^{\otimes k}$ with local unitary frames of E.

4. Estimated transversality and finite, Whitney (A), approximate holomorphic stratifications

Let τ_k be an A.H. sequence of sections of $L^{\otimes k} \to (M, D, \omega)$. Proposition 1.1 for codimension 2 submanifolds is proved by pulling back the $\mathbf{0}$ section of $L^{\otimes k}$. To obtain that W_k is a 2-calibrated submanifold, τ_k has to be transverse along D so that $TW_k \cap D$ defines a codimension 1 distribution on W_k. Next, to make sure that $W_k \cap D$ is a symplectic distribution, the ratio $|\bar{\partial}\tau_k(x)|/|\partial\tau_k(x)|$ has to be smaller than 1; since $\nabla_D = \bar{\partial} + \partial$, $\nabla_D \tau_k(x)$ is required not only surjective but also to have norm greater than $O(k^{-1/2})$ (estimated transversality).

For each point x we can use the reference sections to turn the local estimated transversality problem along D on $B_{g_k}(x, \rho)$ into an estimated transversality problem along D_h for an A.H. sequence of functions

$$F_{k,x} : B(0, \rho') \subset \mathbb{C}^n \times \mathbb{R} \to \mathbb{C},$$

where $\tau_k \circ \varphi_{k,x} = F_{k,x} \cdot (\tau^{\mathrm{ref}}_{k,x} \circ \varphi_{k,x})$ (more generally $\mathbb{C}^m$-valued functions for bundles of rank m). Equivalently, we have to solve an estimated transversality problem for a 1 real parameter family of A.H. functions

$$F_{k,x}(\cdot, s_k) : B(0, \rho') \subset \mathbb{C}^n \to \mathbb{C}.$$

This problem is known to have a solution [5], [24]. The solution of the local transversality problem along D_h will produce a new function $F_{k,x} - u_{k,x}$, and therefore a perturbation

$$\chi_{k,x} := (-u_{k,x} \circ \varphi^{-1}_{k,x}) \cdot \tau^{\mathrm{ref}}_{k,x}$$

so that we obtain estimated transversality along D for $\tau_k + \chi_{k,x}$ over the ball $B_{g_k}(x, \rho)$. But the reference section is supported in $B_{g_k}(x, \rho'' k^{1/6})$, being the consequence that there will be interference among different local solutions. However, unlike transversality, estimated transversality does behave well under addition, and in the presence of "enough" local estimated transversality, Donaldson's globalization procedure gives global estimated transversality (see the proof of Theorem 7.2).

Definition 4.1. Let (P, g) be a Riemannian manifold, (E, ∇) a Hermitian bundle over it, and Q_x a subspace of T_xP. We say that $\tau : P \to E$ is η-transverse to $\mathbf{0}$ at x along Q_x if either $|\tau(x)| \geq \eta$ or $\nabla_{Q_x}\tau(x)$ has a right inverse with norm bounded by η^{-1}.

If Q is a distribution we say that τ is η-transverse along Q to $\mathbf{0}$ if the above condition holds at all the points where Q is defined. When Q is the tangent bundle of a submanifold we also say that τ is η-transverse over the submanifold to $\mathbf{0}$.

Let (M, D, ω) be a 2-calibrated manifold, $E_k := E \otimes L^{\otimes k}$, and $\tau_k : (M, g_k) \to (E_k, \nabla_k)$ a sequence of sections. We say that the sequence τ_k is uniformly transverse along D to $\mathbf{0}$ if $\eta > 0$ exist such that τ_k is η-transverse along D to $\mathbf{0}$ for all $k \gg 1$.

For a symplectic manifold the definition of uniform transversality along a distribution Q (possibly the tangent bundle to a 2-calibrated submanifold) is analogous.

It is possible to attain estimated transversality along D using both the intrinsic and the relative point of view. Using the former what we do is (locally) solving transversality problems for 1-parameter families of A.H. functions from $\mathbb{C}^n$ to $\mathbb{C}^m$. Regarding the latter we follow the ideas of J.-P. Mohsen developed for contact manifolds, working in the symplectization $(M \times [-\varepsilon, \varepsilon], \Omega)$ and solving the estimated transversality problem for A.H. sections, but this time over M. Then we can use the following

Lemma 4.1 ([30], second lemma in Section 6.1). *Let (M, D, ω) be a 2-calibrated manifold. If in the symplectization $(M \times [-\varepsilon, \varepsilon], \Omega)$ we are able to find an A.H. sequence τ_k, η-transverse over M to $\mathbf{0}$, then for any constant C, $0 < C < \sqrt{2}/2$, there exists $k_0(C)$ such that for any $k \geq k_0$ the section $\tau_{k|M}$ is $C\eta$-transverse along D to $\mathbf{0}$.*

The proof is just an estimated version of the following elementary fact: if a J_0-complex linear function $l : \mathbb{C}^n \times \mathbb{R} \to \mathbb{C}^m$ is surjective, then it has a surjective restriction to each complex hyperplane. Otherwise the kernel of the restriction, being complex, would have real dimension bigger than $2(n - m) + 2$, and l could not be surjective.

4.1. Geometric reformulation of estimated transversality. We recall that in this section we deal with estimated transversality along D in a 2-calibrated manifold (intrinsic theory), or with estimated transversality over a 2-calibrated submanifold M inside a symplectic manifold P (relative theory). Sometimes we might refer to both situations as transversality along a distribution Q in the Riemannian manifold P.

As remarked in the previous subsection, for sequences of 1-parameter families of A.H. functions $F_{k,x}(\cdot, s_k) : B(0,\rho) \subset \mathbb{C}^n \to \mathbb{C}^m$ one can achieve estimated transversality, and thus the use of reference frames allows us to get local estimated transversality along D to the $\mathbf{0}$ section of very ample vector bundles E_k. More generally, one expects to be able to attain estimated transversality along D to sequences of submanifolds $S_k \subset E_k$ of very ample vector bundles, where the S_k locally look like the zero section of a trivial vector bundle: more precisely, the sequence of submanifolds should be locally defined by functions $f_k : U_k \subset E_k \to \mathbb{C}^l$, $S_k \cap U_k = f_k^{-1}(\mathbf{0})$, which are approximately holomorphic with respect to the almost CR structure in the total space of the bundles $(E_k, \nabla_k) \to (M, D, J, g_k)$ induced by the one on M, the connection, and the Hermitian metric on E_k, so that

$$f_k \circ \tau_k \circ \varphi_{k,x} : B(0,\rho) \subset \mathbb{C}^n \times \mathbb{R} \to \mathbb{C}^m$$

is an A.H. sequence of functions (or a weaker property that ensures this last condition). That should allow us to find an A.H. perturbation

$$\chi_{k,x} : B_{g_k}(x, \rho'' k^{1/6}) \to E_k$$

so that the A.H. sequence

$$f_k \circ (\tau_k + \chi_{k,x}) : B_{g_k}(x,\rho) \to \mathbb{C}^m$$

is uniformly transverse along D to $\mathbf{0}$. Finally, we should make sure that this implies enough estimated transversality along D to S_k for the sequence of sections

$$\tau_k + \chi_{k,x} : B_{g_k}(x,\rho) \to E_k$$

to make Donaldson's globalization procedure work.

In the relative context $\tau_k : P \to E_k$ the estimated transversality problem over $M \subset P$ to the $\mathbf{0}$ section has the same difficulty as the usual estimated transversality problem to the $\mathbf{0}$ section (this is the work of J.-P. Mohsen [30], Section 5). Thus, one expects this principle to be valid in the case of relative estimated transversality to more complicated strata S_k.

To give a global definition of what transversality to a submanifold $S \subset E$ is, we need to recall a more geometric definition of estimated transversality along a distribution Q, together with the following concepts.

Definition 4.2. Let W be a vector space with non-degenerate inner product so that for any $u, v \in W$ we can compute the (unoriented) angle $\angle(u,v)$. Given

$U \in \mathrm{Gr}(p, W)$ and $V \in \mathrm{Gr}(q, W)$, $p, q > 0$, the maximal angle of U and V, $\angle_{\mathrm{M}}(U, V)$, is defined as follows:

$$\angle_{\mathrm{M}}(U, V) := \max_{u \in U \setminus \{0\}} \min_{v \in V \setminus \{0\}} \angle(u, v).$$

In general the maximal angle is not symmetric, but when $p = q$ it has symmetry and defines a distance in the corresponding Grassmannian (see [32]).

The minimum angle between transverse complementary subspaces is defined as the minimum angle between two non-zero vectors, one on each subspace. An extension of this notion for transverse subspaces with non-trivial intersection is:

Definition 4.3 (Definition 3.3 in [32]). Using the notation of Definition 4.2, $\angle_{\mathrm{m}}(U, V)$, the minimum angle between U and V non-void subspaces of W, is defined as follows:

- If $\dim U + \dim V < \dim W$, then $\angle_{\mathrm{m}}(U, V) := 0$.
- If the intersection is non-transverse, then $\angle_{\mathrm{m}}(U, V) := 0$.
- If the intersection is transverse, we consider the orthogonal to the intersection and its intersections U_c and V_c with U and V, respectively. We define $\angle_{\mathrm{m}}(U, V) := \min_{u \in U_c \setminus \{0\}} \min_{v \in V_c \setminus \{0\}} \angle(u, v)$.

The minimum angle is symmetric.

The most important property relating maximal and minimal angle is:

Proposition 4.1 (Proposition 3.5 in [32]). *For non-void subspaces U, V, W of $\mathbb{R}^n$ the following inequality holds:*

$$\angle_{\mathrm{m}}(U, V) \leq \angle_{\mathrm{M}}(U, W) + \angle_{\mathrm{m}}(W, V).$$

We will also be using the following

Lemma 4.2 (Lemma 3.8 in [32]). *Let U, V be non-zero subspaces of $\mathbb{R}^n$ and let $h : U \to V^{\perp}$ be the projection from U with respect to the decomposition $\mathbb{R}^n = V \oplus V^{\perp}$. If h has a right inverse θ satisfying $|\theta| < \eta^{-1}$ then $\angle_{\mathrm{m}}(U, V) > \eta$.*

Let $\tau : P \to E$ be a section of a Hermitian bundle with connection and Q a distribution on P. Let us denote the pullback of Q to E by $\hat{Q}$. Let $\mathscr{H}$ be the horizontal distribution associated to the linear connection and let $\mathscr{H}_Q$ denote its intersection with $\hat{Q}$. Finally let $T_Q\tau$ denote the intersection of the tangent bundle of the graph of τ with $\hat{Q}$.

Lemma 4.3. *There exists a constant $C > 0$ determined by upper bounds on $|\nabla_Q \tau(x)|$, $|\tau(x)|$ such that:*

(1) *If $\nabla_Q \tau(x)$ has a right inverse with norm bounded by η^{-1} then $\angle_{\mathrm{m}}(\mathscr{H}_Q, T_Q\tau) \geq C^{-1}\eta$ (the angle measured in $\hat{Q}_{\tau(x)}$).*

(2) *If $\angle_{\mathrm{m}}(\mathscr{H}_Q, T_Q\tau) \geq \eta$ then $\nabla_Q \tau(x)$ has a right inverse with norm bounded by $(C \sin \eta)^{-1}$.*

Proof. Let us assume that $Q = TP$. The vector space $T_{\tau(x)}E = \mathscr{H}_{\tau(x)} \oplus T^v E_x$ is endowed with the direct sum metric. We compose with an isometry preserving the direct sum structure so that $\mathscr{H}_{\tau(x)} \oplus T^v E_x$ becomes $\mathbb{R}^a \oplus \mathbb{R}^b$ with the Euclidean metric. Let $h : T\tau(x) \to \mathbb{R}^b$ be the orthogonal projection. By Lemma 4.2 applied to $U = T\tau(x)$ and $V = \mathbb{R}^a \times \{0\} = \mathscr{H}_{\tau(x)}$, if h has a right inverse θ with $|\theta| \leq \eta^{-1}$ then $\angle_{\mathrm{m}}\big(\mathscr{H}_{\tau(x)}, T\tau(x)\big) \geq \eta$.

By definition $\nabla\tau(x) : T_xP \to T^v E_x = E_x$ is the composition $h \circ d\tau(x)$, with the differential $d\tau(x) : T_xP \to T\tau(x)$, which is an isomorphism. Now if θ' is a right inverse for $\nabla\tau(x)$, $|\theta'| \leq \eta^{-1}$, then $d\tau(x) \circ \theta'$ is a right inverse for h with norm bounded by $|d\tau(x)|\eta^{-1}$. Thus, by Lemma 4.2 $\angle_{\mathrm{m}}\big(\mathscr{H}_{\tau(x)}, T\tau(x)\big) \geq |d\tau(x)|^{-1}\eta$.

Conversely, the projection h has always a right inverse θ of minimum norm. Let us define $W := T\tau(x) \cap \mathscr{H}_{\tau(x)}$ and $U_c := T\tau(x) \cap W^{\perp}$. If we compose θ with the orthogonal projection $T\tau(x) \to U_c$, we obtain a right inverse $\hat{\theta}$ for $h_{|U_c}$ such that $|\theta| = |\hat{\theta}|$. If now $\angle_{\mathrm{m}}\big(\mathscr{H}_{\tau(x)}, T\tau(x)\big) \geq \eta$ then the equation involving inequalities of Lemma 3.8 in [32] implies that

$$|\hat{\theta}| \leq (\sin \eta)^{-1}, \tag{16}$$

and therefore $d\tau(x)^{-1} \circ \theta$ is a right inverse for $\nabla\tau(x)$ with norm bounded by $|d\tau(x)|^{-1}(\sin \eta)^{-1}$.

In the case $Q \neq TP$ we fix an isometry sending $(\mathscr{H}_Q, \mathscr{H})$ at $\tau(x)$ to $(\mathbb{R}^{a'} \times \{0\}, \mathbb{R}^a)$ with the Euclidean metric, and apply the above arguments to $\mathbb{R}^{a'} \oplus \mathbb{R}^b$.

Note that we have $C = |d_Q\tau(x)|$, with $d_Q\tau(x)$ the restriction of $d\tau(x)$ to Q_x. Observe that a bound for $|d_Q\tau(x)|$ can be obtained from upper bounds for $|\tau(x)|$ and $|\nabla_Q\tau(x)|$. □

Remark 4.1. In the definition of minimum angle $\angle_{\mathrm{m}}(U, V)$, when U, V are not complementary we work with the intersections in $(U \cap V)^{\perp}$ where we can apply the usual notion of minimum angle for complementary subspaces. Instead of $(U \cap V)^{\perp}$ one might choose any other subspace W complementary to $U \cap V$ to give a different notion of minimum angle. In certain situations this is a good strategy because there are natural complementary subspaces available. It is easy to see that the new notion of minimum angle is comparable to the one of Definition 4.3, and the comparison is given by multiplying by a constant depending only on

$\angle_{\mathrm{m}}(U \cap V, W)$ (there is no ambiguity since these are complementary subspaces). Actually, those new notions depending on the complementary coincide with the one given in 4.3, but for a new metric which is comparable to the Euclidean one in terms of $\angle_{\mathrm{m}}(U \cap V, W)$ (very much as it happened with the isomorphism $d_Q\tau(x)$ in the previous lemma).

We need a second result relating angles and intersections.

Lemma 4.4. *Let U, V, W be linear subspaces of $\mathbb{R}^n$ such that $\angle_{\mathrm{m}}(V, W) \geq \gamma > 0$. Let $\angle_{\mathrm{M}}(U, V) \leq \delta$. Then there exists $C(\gamma, \dim V, n) > 0$ such that*

$$\angle_{\mathrm{M}}(U \cap W, V \cap W) \leq C\delta.$$

Proof. For each $u \in U \backslash \{0\}$, we have $\angle(u, V) = \angle(u, h(u))$, where $h : \mathbb{R}^n \to V$ is the orthogonal projection. We consider a complementary space to V possibly different from $V^{\perp}$: because $\angle_{\mathrm{m}}(V, W) \geq \gamma > 0$ the dimension of W is greater or equal than the codimension of V, and the intersection of V and W is transverse. As a consequence any subspace (of W) complementary to $V \cap W$ in W is also complementary to V in $\mathbb{R}^n$. We let V_W be the orthogonal to $V \cap W$ in W, and we define $h_W : \mathbb{R}^n \to V$ to be the projection along V_W (whose restriction to W is the orthogonal projection onto $V \cap W$). It follows that $\angle(u, h_W(u)) \leq C\angle(u, h(u)) = C\angle(u, V)$, and by construction if $u \in U \cap W$ then $\angle(u, h_W(u)) = \angle(u, V \cap W)$. □

Let $S \subset E$ be a submanifold in the total space of the vector bundle E over either a 2-calibrated or a symplectic manifold, transverse to the fibers. Let $\hat{g}$ be the metric in E induced by the connection, the bundle metric, and the metric g in the base. The submanifold might not have a tubular neighborhood of positive radius. If we assume S to be in a compact region—as it will be the case in our applications—then the problem comes from the behavior near its boundary $\partial S = \bar{S} \backslash S$. Thus a reasonable extension of Definition 4.1 to our non-linear setting must deal separately with points close to ∂S and with the other points of S.

Definition 4.4. Given $\bar{\eta} > 0$ the points of S $\bar{\eta}$-far from (resp. $\bar{\eta}$-close to) the boundary are those points in S at $\hat{g}$-distance of ∂S greater or equal (resp. smaller) than $\bar{\eta} > 0$. For any $\eta > 0$—typically much smaller than $\bar{\eta}$—we define $\mathscr{N}_S(\eta, \bar{\eta})$ to be those points that can be joined to a point $\bar{\eta}$-far from the boundary by a geodesic arc normal to S and of length smaller or equal than η.

We now define the distribution $T^{\|}S$ at the points of $\mathscr{N}_S(\eta, \bar{\eta})$ by parallel transport of TS along the geodesics normal to S, starting at the points $\bar{\eta}$-far from the boundary of S.

$T^{\|}S$ plays the role of $\mathscr{H}$. We use the notation $T^{\|}_Q S := T^{\|}S \cap \hat{Q}$.

Definition 4.5. τ is $(\eta, \bar{\eta})$-transverse along Q to S at x if either (i) $\tau(x)$ misses the union of S with $\mathcal{N}_S(\eta, \bar{\eta})$, or (ii) $\tau(x)$ enters in $\mathcal{N}_S(\eta, \bar{\eta})$ so that $\angle_{\mathrm{m}}(T_Q\tau, T_Q^{\|}S) \geq \eta$ at $\tau(x)$, or (iii) $\tau(x)$ intersects S at the points $\bar{\eta}$-close to the boundary with $\angle_{\mathrm{m}}(T_Q\tau, T_QS) \geq \bar{\eta}$ at $\tau(x)$.

Uniform transversality of τ_k along Q to S_k is defined as $(\eta, \bar{\eta})$-transversality for some $\eta, \bar{\eta} > 0$ and for all $k \gg 1$.

Conditions on a sequence of submanifolds S_k of complex codimension l (or more generally on stratifications) can be imposed, so that local estimated transversality along Q of $\tau_{k,x}$ at the points of $B_{g_k}(x, \rho)$ to the points of S_k far from ∂S_k is equivalent to estimated transversality along Q of a related $\mathbb{C}^l$-valued function to $\mathbf{0}$ (Lemma 4.5).

We will consider stratifications $\mathcal{S} = (S_k^a)$, $a \in A_k$, which are (i) finite in the sense that $\#(A_k)$ must be bounded independently of k, and (ii) the boundary of each strata $\partial S_k^b = \bar{S}_k^b \backslash S_k^b$ will be the union of the strata of smaller dimension

$$\partial S_k^b = \bigcup_{a<b} S_k^b.$$

Definition 4.6. Let $E_k = E \otimes L^{\otimes k} \to (M, D, J, g_k)$ and let $(S_k^a)_{a \in A_k}$ be finite stratifications of E_k whose strata are transverse to the fibers. Let $r \in \mathbb{N}$, $r \geq 2$. The sequence of strata is Whitney C^r-approximately holomorphic (C^r-A.H.) if for any bounded open set U_k of the total space of E_k and any $\varepsilon > 0$, constants $C_\varepsilon, \rho_\varepsilon > 0$ only depending on ε and on the size of U_k, but not on k, can be found, so that for any point $y \in U_k$ in a strata S_k^a for which $d_{\hat{g}_k}(y, \partial S_k^a) > \varepsilon$, there exist complex valued functions $f_1, \ldots, f_l$ such that $B_{\hat{g}_k}(y, \rho_\varepsilon) \cap S_k^a$ is given $f_1 = \cdots = f_l = 0$, and the following properties hold:

(1) (Uniform transversality to the fibers + transverse comparison) The restriction of $df_1 \wedge \cdots \wedge df_l$ to T^vE_k is bounded from below by ρ_ε.

(2) (Approximate holomorphicity along the fibers) The restriction of the function $f = (f_1, \ldots, f_l)$ to each fiber is C^r-A.H.(C_ε).

(3) (Horizontal approximate holomorphicity + holomorphic variation of the restriction to the fiber + estimated variation of the restriction to the fiber) For any λ, k and τ C^r-A.H.(λ) local section of E_k with image cutting $B_{\hat{g}_k}(y, \rho_\varepsilon)$, $f_j \circ \tau$ is C^r-A.H.(λC_ε). Moreover, if θ is a local C^r-A.H.(λ) section of $\tau^*T^vE_k$, then $df_\tau(\theta)$ is C^r-A.H.(λC_ε).

(4) (Estimated Whitney condition (A)) For each $\eta > 0$ small enough, there exists $\delta(\eta) > 0$ such that for all $y \in S_k^b$ at distance smaller than δ of $S_k^a \subset \partial S_k^b$, $\angle_{\mathrm{M}}(T^{\|}S_k^a, TS_k^b)$ at y is bounded by η.

Remark 4.2. If we give the corresponding definition using as base space an almost complex manifold instead of an almost CR manifold, we almost recover the Definition 3.2 in [4] (our condition (4) is a bit weaker).

Condition (1) is equivalent to the strata have minimum angle with the fibers bounded from below. We just try to mimic the picture of the **0** section with respect to the fibers of a vector bundle, in which case we even have orthogonality.

Conditions (2) and (3) guarantee that if $\tau_k : M \to E_k$ is A.H., then the corresponding $\mathbb{C}^l$-valued function to be made transverse to **0** is A.H.

Recall that for a stratification $\mathscr{S}$ of some $\mathbb{R}^N$, a stratum S^b satisfies Whitney's condition (A) if for every converging sequence $x_n \to x$, $x_n \in S^b$, $x \in S^a \subset \partial S^b$, so that $T_{x_n}S^b$ is converging, the limit contains $T_x S^a$. Condition (4) is an estimated Whitney condition (A).

Definition 4.7. Let $\mathscr{S}$ be as in Definition 4.6 (over either a 2-calibrated or a symplectic manifold). Then τ_k is uniformly transverse along Q to $\mathscr{S}$ if there exists strictly positive numbers $(\eta_a, \bar{\eta}_a)$ for all $a \in A_k$ such that:

(1) For all $a \in A_k$ and for all $k \gg 1$ τ_k is $(\eta_a, \bar{\eta}_a)$-transverse along Q to S_k^a.

(2) For each b, $\bigcup_{a<b} \mathscr{N}_{S_k^a}(\eta_a, \bar{\eta}_a)$ contains the points of S_k^b $\overline{\eta_b}$-close to ∂S_k^b.

Now that we have the notion of uniform transversality of a sequence of sections to an appropriate stratification, we need tools to relate it with local uniform transversality for sequences of (related) functions.

Lemma 4.5. *Let S_k^a be a sequence of strata as those in the stratifications of Definition* 4.6 *for the base space P either an almost CR manifold (intrinsic theory) or an almost complex manifold (relative theory). Let $\varepsilon > 0$ and $0 < \eta \ll \varepsilon$. Let $y \in E_k$ be a point in the stratum ε-far from the boundary, and let $f = (f_1, \dots, f_l)$ be the corresponding local $\mathbb{C}^l$-valued function defining the stratum in $B_{\hat{g}_k}(y, \rho_\varepsilon)$. Let τ_k be a section of E_k whose graph enters in $B_{\hat{g}_k}(y, \rho_\varepsilon)$. Then there exist constants $\rho'(\varepsilon, \eta, |\tau_k|)$, $C(\varepsilon, |\nabla_Q \tau_k|, |\tau_k|)$, $C'(\varepsilon, |\nabla_Q \tau_k|, |\tau_k|) > 0$ such that:*

(1) *If $\angle_{\mathrm{m}}(T_Q\tau, T_Q^{\|}S^a) \geq \eta$ in $B_{\hat{g}_k}(y, \rho_\varepsilon)$, then $d_Q(f \circ \tau)$ has a right inverse with norm*
bounded by $\left(C \sin(\eta/2)\right)^{-1}$ in $B_{\hat{g}_k}(y, \rho')$.

(2) *If $d_Q(f \circ \tau)$ has a right inverse with norm bounded by η^{-1} in $B_{\hat{g}_k}(y, \rho_\varepsilon)$, then $\angle_{\mathrm{m}}(T_Q\tau, T_Q^{\|}S^a) \geq C'^{-1}\eta$ in $B_{\hat{g}_k}(y, \rho')$.*

Proof. By simplicity we omit the subindices for the sections τ_k, the bundles, and the strata.

Let us assume that $\angle_{\mathrm{m}}(T_Q\tau, T_Q^{\|}S^a) \geq \eta$.

Step 1: Show the existence of $\rho'(\varepsilon, \eta, |\tau|) > 0$ such that $\angle_{\mathrm{m}}(T_Q\tau, \operatorname{Ker} df \cap \hat{Q}) \geq \eta/2$ in $B_{\hat{g}_k}(y, \rho')$.

According to Proposition 4.1 (Proposition 3.5 in [32])

$$\angle_{\mathrm{m}}(T_Q^{\|}S^a, T_Q\tau) \leq \angle_{\mathrm{M}}(T_Q^{\|}S^a, \operatorname{Ker} df \cap \hat{Q}) + \angle_{\mathrm{m}}(\operatorname{Ker} df \cap \hat{Q}, T_Q\tau),$$

and therefore we need to prove the existence of $\rho' > 0$ so that in $B_{\hat{g}_k}(y, \rho')$

$$\angle_{\mathrm{M}}(T_Q^{\|}S^a, \operatorname{Ker} df \cap \hat{Q}) \leq \eta/2. \tag{17}$$

Condition (1) in Definition 4.6 implies that $\angle_{\mathrm{m}}(\operatorname{Ker} df, \hat{Q}) \geq \gamma(\varepsilon)$. If we find $\rho' > 0$ such that in $B_{\hat{g}_k}(y, \rho')$

$$\angle_{\mathrm{M}}(T^{\|}S^a, \operatorname{Ker} df) \leq C\big(\gamma(\varepsilon)\big)^{-1}\eta/2, \tag{18}$$

we can apply Lemma 4.4, where $U = T^{\|}S^a$, $V = \operatorname{Ker} df$, $W = \hat{Q}$ to conclude that equation (17) holds.

Equation (18) is proven using appropriate charts. The situation we are trying to mimic is that of a locally trivialized vector bundle and we measure the maximal angle between the parallel copies of the $\mathbf{0}$ section (here the leaves of $\ker df$) and $\mathscr{H}$ (here $T^{\|}S^a$).

Due to the bounds in Definition 4.6 we can find a chart $\Phi_y : \mathbb{R}^a \to B_{\hat{g}_k}(y, \rho_\varepsilon)$ such that in $B(0, \rho'') \subset \mathbb{R}^a$ (i) the metrics g_0 and $\Phi_y^*\hat{g}_k$ (we write $\hat{g}_k$ if it is clear that we work in the chart) are comparable, and the Christoffel symbols of $\hat{g}_k$ are bounded by $O(1)$ (the bounds being uniform on k, y), and (ii) the foliation $\operatorname{Ker} df$ is sent to the foliation $\mathbb{R}^{a-2l}$. In $B(0, \rho'') \subset \mathbb{R}^a$ the stratum S becomes $\mathbb{R}^{a-2l} \times \{0\}$ and tubular neighborhoods for $\hat{g}_k$ and g_0 are comparable. At any point q in the neighborhood, a vector in $u \in T^{\|}S$ is the result of parallel translating (with $\hat{g}_k$) a vector v in $\mathbb{R}^{a-2l} \times \{0\}$ over $y' \in \mathbb{R}^{a-2l} \times \{0\}$ along the corresponding $\hat{g}_k$-geodesic. Since the Christoffel symbols are bounded, $\angle(u, v)$ is bounded by $e^{\Gamma t} - 1$, $\Gamma > 0$. So by decreasing t, the distance of q to S, we bound the maximal angle by $C(\gamma)^{-1}\eta/2$. Therefore the final radius ρ' depends on η, on ε (because $C(\gamma)$ depends on ε), and on how g_0 and $\hat{g}_k$ are related (to order one). This final relation depends on f (and hence on ε) and on the metric $\hat{g}_k$ (and hence on $|\tau|$).

Step 2: Show that $\angle_{\mathrm{m}}(T_Q\tau, \operatorname{Ker} df \cap \hat{Q}) \geq \eta/2$ implies that $d_Q(f \circ \tau)$ has a right inverse with norm bounded by $\big(C(\varepsilon, |\nabla_Q\tau|, |\tau|)\sin(\eta/2)\big)^{-1}$.

The proof of item (2) in Lemma 4.3 implies that the orthogonal projection $h : T_Q\tau \to (\operatorname{Ker} df \cap \hat{Q})^{\perp}$ has a right inverse with norm bounded by $\big(\sin(\eta/2)\big)^{-1}$ (equation (16)). Let V_E denote the orthogonal in the fiber T^vE of $(\operatorname{Ker} df \cap \hat{Q}) \cap T^vE$. Due to condition (1) in Definition 4.6, this is a subspace complementary to

$\operatorname{Ker} df \cap \hat{Q}$ and such that $\angle_{\mathrm{m}}(V_E, \operatorname{Ker} df \cap \hat{Q})$ is bounded from below in terms of ρ_ε, and hence in terms of ε.

Let $h_E : T_Q\tau \to V_E$ be the projection along $\operatorname{Ker} df \cap \hat{Q}$. It follows that there is a constant $C_1(\varepsilon)^{-1} > 0$ and a right inverse for h_E with norm bounded by $C_1(\varepsilon)^{-1}(\sin \eta/2)^{-1}$. We now define

$$h'' = df \circ h_E \circ d_Q\tau : Q \to \mathbb{C}^l.$$

By construction, $h'' = d_Q(f \circ \tau)$. Condition (1) about the restriction of df to the fiber implies the existence of a right inverse for h'' with norm bounded by $|d_Q\tau|^{-1} C_2(\varepsilon)^{-1} C_1(\varepsilon)^{-1}(\sin \eta/2)^{-1}$. Therefore, $d_Q(f \circ \tau)$ has a right inverse with norm bounded by $C(\varepsilon, |d_Q\tau|)\sin(\eta/2)^{-1}$ in $B_{\hat{g}_k}(y, \rho'(\varepsilon, \eta, |\tau|))$, which proves item (1).

Conversely, if $d_Q(f \circ \tau)$ has a right inverse $B_{\hat{g}_k}(y, \rho_\varepsilon)$ with norm bounded by η^{-1}, Step 2 above implies that $h_E \circ d_Q\tau$ has a right inverse with norm bounded by $(C_1'(\varepsilon)2\eta)^{-1}$.

Item (1) in Lemma 4.3 gives

$$\angle_{\mathrm{m}}(T_Q\tau, \ker df \cap \hat{Q}) \geq C'(\varepsilon, |\nabla_Q\tau|, |\tau|)2\eta,$$

and combined with Step 1 we conclude that

$$\angle_{\mathrm{m}}(T_Q\tau, T_Q^{\|}S^a) \geq C'(\varepsilon, |d_Q\tau|)\eta \qquad \text{in } B_{\hat{g}_k}(y, \rho(\varepsilon, \eta, |\tau|)).$$

Observe that the constants C, C' grow very large as ε and η tend to zero. □

Remark 4.3. The previous lemma does not involve almost complex structures at all. Hence it also holds for arbitrary Hermitian bundles, sections, and strata which fulfill condition (1) in Definition 4.6.

Using appropriate choices of complementary subspaces to get a bound from below for certain minimal angles, as noticed in Remark 4.1, we can prove the following

Lemma 4.6. *Let $\mathscr{S} = (S_k^a)_{a \in A}$ be a sequence of approximately holomorphic stratifications as in Definition 4.6. Assume that the sequence τ_k is uniformly transverse to $\mathscr{S}$ along a distribution Q whose dimension is greater of equal than the codimension of the strata, and that the uniform bounds $|\tau_k|, |\nabla\tau_k|_{g_k} \leq O(1)$ hold. Then $\tau_k^{-1}(S_k^a)$ is a subvariety of M uniformly transverse to Q for each $a \in A$.*

Proof. We must prove that for a sequence of points $x(k)$ in $\tau_k^{-1}(S_k^a)$ we have

$$\angle_{\mathrm{m}}(T_x\tau_k^{-1}(S_k^a), Q) \geq \gamma > 0 \tag{19}$$

for all $k \gg 1$ independently of the points.

Denote $\tau_k(x) = q$. We claim that equation (19) will follow from

$$\angle_{\mathrm{m}}\big(\tau_{k*}T_x\tau_k^{-1}(S_k^a), \tau_{k*}Q\big) \geq \gamma' > 0, \tag{20}$$

where the angle is measured in $T\tau_k(x)$ with the induced metric. The reason is that the bound on $|\tau_k|$ implies that the metric in E_k is comparable to the product metric given by any trivialization by reference frames (and using on each factor the Hermitian metric in the fiber and g_k coming from the base). Then we use the bound on $|\nabla\tau_k|$ to conclude that in this product metric $\angle_{\mathrm{m}}\big(T\tau_k(x), T^vE_k(q)\big) \geq \delta_1 > 0$, where T^vE_k is the tangent space to the fiber. Hence, our claim follows.

We can rewrite equation (20) as

$$\angle_{\mathrm{m}}\big(T_Q\tau_k(x), T\tau_k \cap TS_k^a(q)\big) \geq \gamma' > 0. \tag{21}$$

Our second claim is that

$$\angle_{\mathrm{m}}\big(TS_k^a(q), T^vE_k(q)\big) \geq \delta_2 > 0. \tag{22}$$

Indeed, this follows from condition (1) in Definition 4.6 if we are in a point $\bar{\eta}$-far from the boundary of S_k^a. For points $\bar{\eta}$-close, we use the estimated Whitney condition (A) together with Proposition 4.1 to prove equation (22). Since $T^vE_k \subset \hat{Q}$, we also conclude that

$$\angle_{\mathrm{m}}\big(TS_k^a(q), \hat{Q}\big) \geq \delta_3 > 0. \tag{23}$$

We will reinterpret equation (23) by choosing a suitable complementary space to $TS_k^a \cap \hat{Q}(q)$ which is not its orthogonal W (see remark 4.1). Let $W_1 \subset \hat{Q}$ (resp. $W_2 \subset \hat{Q}$) be the intersection of $T\tau_k(x)$ (resp. $TS_k^a(q)$) with the orthogonal of $TS_k^a \cap T\tau_k \cap \hat{Q}(q)$ inside $\hat{Q}$, and let W_3 be the intersection of $T\tau_k(x)$ with the orthogonal of $\hat{Q}$. From $\angle_{\mathrm{m}}\big(T\tau_k(x), T^vE_k(q)\big) \geq \delta_1$ we obtain that $\angle_{\mathrm{m}}\big(T\tau_k(x), \hat{Q}\big) \geq \delta_1$, and by hypothesis $\angle_{\mathrm{m}}\big(T_QT\tau_k(x), T_QS_k^a(q)\big) \geq \delta_4 > 0$. Both inequalities imply that $W' := W_1 \oplus W_3$ can be used instead of W. By construction $W' \cap \hat{Q} = W_1$, so from equation (23) we conclude that

$$\angle_{\mathrm{m}}\big(W' \cap TS_k^a(q), W_1\big) \geq \delta_5 > 0. \tag{24}$$

Notice as well that to compute equation (21) we have to intersect the corresponding vector subspaces with the orthogonal of $TS_k^a \cap T\tau_k \cap \hat{Q}(q)$ inside $T\tau_k(x)$. From what we have seen, we can rather choose as complementary space W'. Since $W' \cap T_Q\tau_k(x) = W_1$ and $W' \cap \big(T\tau_k \cap TS_k^a(q)\big) = W' \cap TS_k^a(x)$, we have to compute the left-hand side of equation (23), so the result follows. □

In particular the following corollary is deduced:

Corollary 4.1. *Let $\mathscr{S} = (S_k^a)_{a \in A}$ be a sequence of A.H. stratifications over the 2-calibrated manifold (M, D, ω) as in Definition* 4.6. *Assume that the A.H. sequence τ_k is uniformly transverse to $\mathscr{S}$ along D. Then for each $a \in A_k$, $\tau_k^{-1}(S_k^a)$ is either empty if the codimension of S_k^a is bigger than the dimension of D (or M), or a subvariety uniformly transverse to D.*

For a symplectic manifold, transversality along the directions of a (compact) subvariety N implies that either (i) *$\tau_k^{-1}(S_k^a)$ is at g_k-distance of N bounded from below or* (ii) *it is a subvariety (at least defined in a g_k-neighborhood of N) uniformly transverse to N.*

If we analyze the proof of Lemma 4.6, Corollary 4.1 for 2-calibrated manifolds is equivalent to saying that uniform transversality along D implies uniform transversality over M (along TM). The converse is also true, extending therefore Mohsen's relative transversality result to appropriate sequences of stratifications.

Corollary 4.2. *Let $\mathscr{S} = (S_k^a)_{a \in A_k}$ be a sequence of A.H. stratifications over the 2-calibrated manifold (M, D, ω) as in Definition* 4.6. *Assume that the A.H. sequence τ_k is uniformly transverse to $\mathscr{S}$ (over M) for suitable constants $(\eta_a, \bar{\eta}_a)$, $a \in A_k$. Then τ_k is also uniformly transverse along D to $\mathscr{S}$.*

Proof. By induction we can assume that τ_k is uniformly transverse along D to S_k^a for every $a < b$. Let $q \in S_k^b$, with $\tau_k(x) = q$, $\bar{\eta}'$-close to ∂S_k^b. We want to show that

$$\angle_{\mathrm{m}}\big(T_D\tau_k(x), T_D S_k^b(q),\big) \geq \bar{\eta}'$$

and will do it by applying for some index $a \in A_k$ the inequality

$$\angle_{\mathrm{m}}\big(T_D\tau_k(x), T_D^{\|} S_k^a(q)\big) \leq \angle_{\mathrm{M}}\big(T_D^{\|} S_k^a(q), T_D S_k^b(q)\big) + \angle_{\mathrm{m}}\big(T_D\tau_k(x), T_D S_k^b(q)\big). \quad (25)$$

If $\bar{\eta}'$ is small enough, condition (2) in Definition 4.7 implies the existence of an index $a \in A_k$ such that $q \in \mathscr{N}_{S_k^a}(\eta_a, \bar{\eta}_a)$. If we apply induction we conclude $\angle_{\mathrm{m}}\big(T_D\tau_k(x), T_D^{\|} S_k^a(q)\big) \geq \eta_a$, so we only need to make

$$\angle_{\mathrm{M}}\big(T_D^{\|} S_k^a(q), T_D S_k^b(q)\big) \ll \eta_a.$$

This is done using Lemma 4.4 with $U = T^{\|} S^a(q)$, $V = TS^b(q)$, $W = \hat{D}$. We need to check that

$$\angle_{\mathrm{M}}\big(T^{\|} S_k^a(q), TS_k^b(q)\big) \ll \eta_a, \quad (26)$$

$$\angle_{\mathrm{m}}\big(TS_k^b(q), \hat{D}\big) \geq \gamma. \quad (27)$$

Equation (26) follows by the estimated Whitney condition by taking $\bar{\eta}'$ small enough; equation (27) uses again the inequality of Proposition 4.1,

$$\angle_{\mathrm{m}}(\hat{D}, T^{\|}S_k^a(q)) \leq \angle_{\mathrm{M}}(T^{\|}S_k^a(q), TS_k^b(q)) + \angle_{\mathrm{m}}(\hat{D}, TS_k^b(q)),$$

together with $\angle_{\mathrm{m}}(\hat{D}, T^{\|}S_k^a(q)) \geq 2\gamma$ (by condition (1) in Definition 4.6) and equation (26).

So far we deduced some $\bar{\eta}'$-transversality only at the points $\bar{\eta}'$-close to the boundary of S_k^b. Now let us assume that for some $\eta > 0$, $\angle_{\mathrm{m}}(T\tau_k(x), T^{\|}S_k^b(q)) \geq \eta$ in the tubular neighborhood $\mathcal{N}_{S_k^b}(\eta, \bar{\eta}')$ (here comes the requirement on the constants controlling the transversality, i.e., in those points $\bar{\eta}'$-far from the boundary we need to make sure that $\angle_{\mathrm{m}}(T\tau_k(x), T^{\|}S_k^b(q))$ is uniformly bounded from below). If $\tau_k(x) \in \mathcal{N}_{S_k^b}(\eta, \bar{\eta}')$, then, by Lemma 4.5, η-transversality implies η'-transversality to $\mathbf{0}$ of the function $f \circ \tau_k : B_{g_k}(x, \rho') \to \mathbb{C}^l$. From the approximate holomorphicity of the composition $f \circ \tau_k$, for all $k \gg 1$ a result analogous to Lemma 4.1 yields $\frac{\sqrt{2}}{3}\eta'$-transversality along D, which again by Lemma 4.5 gives η''-transversality along D to S_k^b (we suppose that $\eta'' \leq \eta$).

Therefore, it follows that τ_k is $(\eta'', \bar{\eta}')$-transverse along D to S_k^b. □

5. Pseudo-holomorphic jets

The main applications of the theory of approximately holomorphic geometry for 2-calibrated manifolds are deduced from the existence of generic rank m linear systems.

Let us assume that $(M, \mathscr{D}, J)$ is a Levi-flat CR manifold and $L \to M$ a positive CR line bundle. Let $\underline{\mathbb{C}}^m \to M$ denote the trivial (and trivialized) bundle of rank m endowed with the trivial connection.

Definition 5.1. A CR section $\tau : M \to \underline{\mathbb{C}}^{m+1} \otimes L$ (or a rank m linear system of L) is r-generic if its zero set B is a CR submanifold of the expected dimension, and the projectivization $\phi : M \backslash B \to \mathbb{CP}^m$ is a leafwise r-generic holomorphic map, i.e., when restricted to each leaf it is transverse to the Thom–Boardman stratification of the bundle of holomorphic r-jets of holomorphic maps from the leaf to $\mathbb{CP}^m$.

The proof of the existence of r-generic linear systems (possibly of large enough powers of L) is the main subject of [25].

The strong transversality property for a CR function $\phi : M \to \mathbb{CP}^m$ to be r-generic is as follows: we consider $\mathscr{J}_{CR}^r(M, \mathbb{CP}^m)$ the bundle of CR r-jets (of foliated holomorphic r-jets) of CR maps from M to $\mathbb{CP}^m$. This bundle admits a CR

Thom–Boardman stratification $\mathbb{P}\Sigma$, which restricts to each leaf to the corresponding holomorphic Thom–Boardman stratification. A CR function ϕ is r-generic if and only if its CR r-jet $j^r_{CR}\phi : M \to \mathscr{J}^r_{CR}(M, \mathbb{CP}^m)$ (which by definition is the foliated holomorphic r-jet) is transverse along $\mathscr{D}$ to $\mathbb{P}\Sigma$.

Assume that our CR submanifold embeds holomorphically in some complex manifold P and that $\mathscr{D}$ extends to a holomorphic foliation integrating the complex distribution G. There is a canonical submersion $p_G : \mathscr{J}^r(P, \mathbb{CP}^m) \to \mathscr{J}^r_G(P, \mathbb{CP}^m)$ from holomorphic r-jets to foliated ones. The foliated Thom–Boardman stratification $\mathbb{P}\Sigma \subset \mathscr{J}^r_G(P, \mathbb{CP}^m)$ restricts over M to the CR Thom–Boardman stratification $\mathbb{P}\Sigma$ of $\mathscr{J}^r_{CR}(M, \mathbb{CP}^m)$. Let us denote the pullback ${p_G}^{-1}(\mathbb{P}\Sigma)$ by $\mathbb{P}\Sigma^G$.

For any holomorphic function $\phi : P \to \mathbb{CP}^m$ it is an elementary fact that $j^r_G\phi \in \Gamma\big(\mathscr{J}^r_G(P, \mathbb{CP}^m)\big)$—the holomorphic r-jet along G—is transverse along G to $\mathbb{P}\Sigma$ at the points of M, if and only if $j^r\phi \in \Gamma\big(\mathscr{J}^r(P, \mathbb{CP}^m)\big)$ is transverse along G to $\mathbb{P}\Sigma^G$ at the points of M. By the results of the previous section, this is equivalent to being transverse over M to $\mathbb{P}\Sigma^G$.

To obtain an r-generic linear system there is an additional complication coming from the base locus. We first need to make sure that $\tau : P \to \underline{\mathbb{C}}^{m+1} \otimes L$ is transverse over M to the zero section, and then solve the r-genericity problem for the projectivization (in a compact region of $P \backslash \tau^{-1}(\mathbf{0})$). Instead of working first with the section τ and then with the projectivization, following ideas of D. Auroux [4] we restate the whole issue as a unique transversality problem over M for the pseudo-holomorphic r-jet extension of τ, a section of a vector bundle $\mathscr{J}^r(\underline{\mathbb{C}}^{m+1} \otimes L)$. The advantage is that we work with vector bundles and we can use the module structure of sections.

5.1. The integrable case. Let $E \to P$ be a Hermitian bundle over a complex manifold with compatible connection ∇, whose curvature satisfies $F^{0,2}_\nabla = 0$. The total space of the bundle is a complex manifold (Theorem 2.1.53 in [14]) and there is a notion of holomorphic section and hence of holomorphic r-jet. The space of r-jets has natural charts obtained out of holomorphic coordinates in the base and a holomorphic trivialization of the bundle. They provide a local identification of the holomorphic r-jets with $\mathscr{J}^r_{n,m}$, the usual r-jets for holomorphic maps from $\mathbb{C}^n$ to $\mathbb{C}^m$.

Let ∂_0 be the Cauchy–Riemann operator defined (locally) using the canonical structure J_0 in the base (the chart) and the trivial connection d in $\underline{\mathbb{C}}^m$. The connection on the fiber bundle can be used to give a different notion of local holomorphic r-jet (in principle chart dependent) by just considering the operator ∂_∇: if the connection matrix in the trivialization is $A_x = A^{1,0}_x$, then the coupled 1-jet of a holomorphic section τ is defined to be $(\tau, \partial_0\tau + A_x\tau))$. Higher order coupled jets are constructed by induction using the connection induced by the flat metric and ∇.

Observe that locally for the above choice of coordinates and trivialization of the bundle, both the usual r-jets and coupled r-jets fill the bundle

$$\Big(\sum_{j=0}^{r}(T^{*1,0}\mathbb{C}^n)^{\odot j}\Big)\otimes \underline{\mathbb{C}}^m = \mathscr{J}^r_{m,n},$$

where $\odot$ stands for the symmetric part of the tensor product and $(T^{*1,0}\mathbb{C}^n)^{\odot 0}\otimes \underline{\mathbb{C}}^m$ for $\underline{\mathbb{C}}^m$. This is due to the existence through any point of E of holomorphic frames tangent to the horizontal distribution of the connection, together with the vanishing $F^{2,0}_{\nabla}$ (the latter implying that $\mathrm{d}A$ and its derivatives are symmetric tensors when evaluated on $(1,0)$-vectors).

For Levi-flat CR manifolds the local model for the pseudo-holomorphic jets to be introduced is the following: the base space is $(\mathbb{C}^n\times\mathbb{R}, J_0, g_0)$ (or rather a ball of Euclidean radius $\rho>0$), the bundle is assumed to be trivialized by a CR frame and the curvature is of type $(1,1)$. The bundle of CR r-jets is denoted by $\mathscr{J}^r_{D_h,n,m}$ (foliated holomorphic r-jets along D_h); its fiber over each point is that of $\mathscr{J}^r_{n,m}$. There is an obvious notion of CR coupled r-jet. The hypothesis on the trivialization and on the curvature imply that they are also symmetric, so they fill the bundle $\mathscr{J}^r_{D_h,n,m}=\mathscr{J}^r_{n,m}\times\mathbb{R}$.

Using Darboux charts and suitable trivializations this model will be achieved in an approximate way in the theory for 2-calibrated manifolds.

There is a final local model we wish to introduce that would appear in Kähler manifolds P with a holomorphic foliation integrating a complex distribution G. Locally, we have holomorphic coordinates $\mathbb{C}^g\times\mathbb{C}^{p-g}$ with G sent to $\mathbb{C}^g$ (which integrates into the foliation with leaves $\mathbb{C}^g\times\{\cdot\}$), and we work with foliated coupled jets along the leaves of $\mathbb{C}^g$. The corresponding bundle of coupled foliated r-jets is denoted by $\mathscr{J}^r_{\mathbb{C}^g,p,m}$. It coincides with $\mathscr{J}^r_{g,m}\times\mathbb{C}^{p-g}$. Transversality problems for this bundle will be transferred to transversality problems in $\mathscr{J}^r_{p,m}$, so we need no further analysis of its properties, though we will be interested at some point in studying the natural submersion $\mathscr{J}^r_{p,m}\to\mathscr{J}^r_{\mathbb{C}^g,p,m}$. This local model is achieved in an approximate way in a symplectic manifold (with compatible almost complex structure and metric) with a J-complex distribution G—not necessarily integrable—, by using approximate holomorphic charts adapted to G.

5.2. Pseudo-holomorphic jets. Denote sequence $E\otimes L^{\otimes k}\to(M,D,\omega)$ by E_k. We define the bundles

$$\mathscr{J}^r_D E_k := \Big(\sum_{j=0}^{r}(D^{*1,0})^{\odot j}\Big)\otimes E_k,$$

where $\odot$ stands for the symmetric part of the tensor product of complex vector bundles. They carry Hermitian vector bundle metrics induced by $g_{k|D}$, the one on E_k, and the symmetrization map

$$\mathrm{sym}_j : (D^{*1,0})^{\otimes j} \to (D^{*1,0})^{\odot j}. \tag{28}$$

The Levi-Civita connection induces a connection on D^* (using the metric to see $D^* \hookrightarrow T^*M$ and then projecting $T^*M \to D^*$) and therefore in $D^{*1,0}$ (using the splitting $D^{*1,0} + D^{*0,1}$); combined with the connection on E_k and the symmetrization map they define connections $\nabla_{k,r}$. The total spaces $\mathscr{J}_D^r E_k$ also carry metrics constructed in the usual fashion out of the metric in the base, the connection, and the vector bundle Hermitian metric.

The definition of pseudo-holomorphic r-jets along D (or just pseudo-holomorphic r-jets) for a sequence E_k of Hermitian vector bundles is given by induction (see [4]). Let τ_k be a sequence of A.H. sections of E_k. By definition $j_D^0 \tau_k = \tau_k$. Let $j_D^{r-1}\tau_k \in \mathscr{J}_D^{r-1} E_k$ be the $(r-1)$-jet of τ_k. It has homogeneous components of degrees $0, 1, \ldots, r-1$. We will denote the homogeneous component of degree $j \in \{0, \ldots, r-1\}$ by $\partial_{\mathrm{sym}}^j \tau_k \in \Gamma\big((D^{*1,0})^{\odot j} \otimes E_k\big)$. The connection $\nabla_{k,r-1}$ is actually a direct sum of connections defined on the direct summands $(D^{*1,0})^{\odot j} \otimes E_k$, $j = 0, \ldots, r-1$. For simplicity and if there is no risk of confusion we will use the same notation for the restriction of $\nabla_{k,r-1}$ to each of the summands. The restriction of $\nabla_{k,r-1}\partial_{\mathrm{sym}}^{r-1}\tau_k$ to D defines a section $\nabla_{k,r-1,D}\partial_{\mathrm{sym}}^{r-1}\tau_k \in \Gamma\big(D^* \otimes (D^{*1,0})^{\odot r-1} \otimes E_k\big)$. For each $x \in M$ it is a form on D with values in the complex vector space $(D^{*1,0})^{\odot r-1} \otimes E_k$. Therefore we can consider its $(1,0)$-component $\partial\partial_{\mathrm{sym}}^{r-1}\tau_k \in \Gamma\big(D^{*1,0} \otimes (D^{*1,0})^{\odot r-1} \otimes E_k\big)$. By applying the symmetrization map sym_r of equation (28) we obtain $\partial_{\mathrm{sym}}^r \tau_k \in \Gamma\big((D^{*1,0})^{\odot r} \otimes E_k\big)$.

Definition 5.2. Let τ_k be a section of (E_k, ∇_k). The pseudo-holomorphic r-jet $j_D^r \tau_k$ is a section of the bundle $\mathscr{J}_D^r E_k = \big(\sum_r^{j=0} (D^{*1,0})^{\odot j}\big) \otimes E_k$ defined out of the $(r-1)$-jet by the formula $j_D^r \tau_k := (j_D^{r-1}\tau_k, \partial_{\mathrm{sym}}^r \tau_k)$.

Remark 5.1. The previous definition incorporates the fact that the degree r and $(r-1)$ homogeneous components of the r-jet are symmetrization of the pseudo-holomorphic 1-jet of $\partial_{\mathrm{sym}}^{r-1}\tau_k$; then we have to add the homogeneous components of lower degree. Actually, we could have equally defined $j_D^r \tau_k$ by taking the symmetrization of the pseudo-holomorphic 1-jet of $j_D^{r-1}\tau_k$ (because this gives the homogeneous components of degree $1, \ldots, r$) and then adding τ_k, the degree zero homogeneous component.

Remark 5.2. The pseudo-holomorphic r-jets are useless for our purposes for low values of k. We are interested in having a notion of r-jet of an A.H. sequence which in approximately holomorphic coordinates and for suitable local trivializations of E_k, is as close as possible to the local coupled holomorphic r-jet defined in $\mathbb{C}^n \times \mathbb{R}$ using J_0 and the flat metric (introduced in Section 5.1). As k grows large and due to the proximity between g_k, J and J_0, g_0 in $B(0,\rho) \subset \mathbb{C}^n \times \mathbb{R}$ we will see that the norm of the difference at any order between the two notions of r-jet is bounded by $O(k^{-1/2})$.

For a symplectic manifold (P,Ω) with a J-complex distribution G the bundle of pseudo-holomorphic r-jets along G will be defined to be

$$\mathscr{J}_G^r E_k := \Big(\sum_{j=0}^{r} (G^{*1,0})^{\odot j}\Big) \otimes E_k.$$

We have a canonical projection $p_G : \mathscr{J}^r E_k \to \mathscr{J}_G^r E_k$. We also use the splitting $TP = G \oplus G^\perp$ to see $\mathscr{J}_G^r E_k$ as a subbundle of $\mathscr{J}^r E_k$; hence every section of $\mathscr{J}_G^r E_k$ can be seen as a section of $\mathscr{J}^r E_k$. To define the pseudo-holomorphic r-jet along G we use the same induction procedure as in the definition of pseudo-holomorphic r-jets along D, but either before or after symmetrizing we project $T^{*1,0}P \to G^{*1,0}$ (or even before taking the $(1,0)$-component we project $T^*P_{\mathbb{C}} \to G^*_{\mathbb{C}}$); the result of either choice is the same.

Once approximately holomorphic coordinates have been fixed we have a canonical pointwise $(J_0 - J)$-complex linear identification

$$T\mathbb{C}^n \to D, \qquad \frac{\partial}{\partial x_k^i} \mapsto \frac{\partial}{\partial x_k^i} + a_i \frac{\partial}{\partial s_k}, \qquad \frac{\partial}{\partial y_k^i} \mapsto J\Big(\frac{\partial}{\partial x_k^i} + a_i \frac{\partial}{\partial s_k}\Big). \tag{29}$$

The inverse of its dual is a $(J_0 - J)$-complex bundle map

$$\varpi_{k,x} : T^{*1,0}\mathbb{C}^n \to D^{*1,0}. \tag{30}$$

It should be stressed that this identification is only important in the ball of some g_k radius $\rho > 0$, the region where our computations have to be more accurate (in order to obtain local estimated transversality). There, for some constant $\gamma > 0$,

$$|\varpi_{k,x}|_{g_0} \le \gamma, \qquad |\varpi_{k,x}^{-1}|_{g_0} \le \gamma \quad \text{and} \quad |\mathrm{d}^j \varpi_{k,x}|_{g_0} \le O(k^{-1/2}) \tag{31}$$

for all $j \ge 1$. The Gaussian decay of the reference sections will take care of what happens out of these balls. We also notice that by writing dz_k^i we will mean $\varpi_{k,x}(dz_k^i)$.

Let us assume that we have also fixed a family of reference sections of $\tau_{k,x}^{\mathrm{ref}} \in \Gamma(L^{\otimes k})$. Using any local unitary basis of E (with bounds uniform on x) together with the reference sections, we have a family of trivializations $\tau_{k,x,j}^{\mathrm{ref}}$, $j = 1,\dots,m$, of E_k in the balls $B_{g_k}(x,\rho)$ for all x and for all k large enough. The A.H. coordinates and the associated bundle maps $\varpi_{k,x}$ provide a local basis

$dz_k^1, \ldots, dz_k^n$ of $D^{*1,0}$. We obtain a family of trivializations of $\mathscr{J}_D^r E_k$ about any point as follows: for $I = (i_0, i_1, \ldots, i_n)$ with $1 \le i_0 \le m$, $0 \le i_1 + \cdots + i_n \le r$ we set

$$\mu_{k,x,I} := dz_k^{1\,\odot i_1} \odot \cdots \odot dz_k^{n\,\odot i_n} \otimes \tau_{k,x,i_0}^{\mathrm{ref}}. \tag{32}$$

Definition 5.3. A family of sequences $\tau_{k,x,I} : M \to E_k$ is called a family of holonomic frames if

(1) they are A.H. sections with Gaussian decay w.r.t to x,

(2) there exist $\rho, \gamma > 0$ such that in the balls $B_{g_k}(x, \rho)$ and for all points and all k large enough the sequences $j_D^r \tau_{k,x,I} : M \to \mathscr{J}_D^r E_k$ define a frame which is γ-comparable to $\mu_{k,x,I}$ in the following sense: if we write $j_D^r \tau_{k,x,I}$ in the basis $\mu_{k,x,I}$, for the corresponding matrix $M_{k,x}$ we have

$$|M_{k,x}|_{g_0} \le \gamma, \qquad |M_{k,x}^{-1}|_{g_0} \le \gamma.$$

One checks that the notion of holonomic reference frame does not depend either on the fixed approximately holomorphic coordinates, or in the chosen reference sections of E_k to define $\mu_{k,x,I}$. Only the constants involved in the definition change.

In this situation there is still a weak point. The main goal is to construct sections whose pseudo-holomorphic r-jets are transverse to certain stratifications. For that we need the pseudo-holomorphic r-jets to be A.H. sections of the bundles $\mathscr{J}_D^r E_k$ (resp. $\mathscr{J}^r E_k$ for symplectic manifolds with J-complex distribution G), so that we can apply the transversality results from approximately holomorphic theory (to be proved in section 7). We intend to use holonomic reference frames defined as follows: if I is one of the $(n+1)$-tuples introduced before we set

$$\nu_{k,x,I} := j_D^r \tau_{k,x,I}^{\mathrm{ref}}, \qquad \text{where } \tau_{k,x,I}^{\mathrm{ref}} := (z_k^1)^{i_1} \ldots (z_k^n)^{i_n} \tau_{k,x,i_0}^{\mathrm{ref}} \in \Gamma(E_k). \tag{33}$$

In the Kähler case and due to the presence of curvature (see [5]), the coupled jets are not anymore holomorphic sections of $\mathscr{J}_{n,m}^r$ with respect to the complex structure induced by the connection. Similarly, the frames $\nu_{k,x,I}$ fail to be families of holonomic frames because the sections are not approximately holomorphic if $r \ge 1$. This difficulty is overcome by introducing a new almost complex structure (a new connection) in $\mathscr{J}_D^r E_k$ (resp. $\mathscr{J}^r E_k$). This is the content of the following proposition whose proof is given in Appendix A.

Proposition 5.1. *The sequence $\mathscr{J}_D^r E_k \to (M, D, J, g_k)$, which is very ample for the connections $\nabla_{k,r}$ previously described, admits new connections ∇_{k,H_r} such that:*

(1) $\nabla_{k,r} - \nabla_{k,H_r} \in D^{*0,1} \otimes (\mathscr{J}_D^r E_k)$. *Hence, if in order to compute the pseudo-holomorphic jets* (*Definition* 5.2) *we use the connections ∇_{k,H_r} instead of $\nabla_{k,r}$, then the result is the same.*

(2) *Let us denote the curvatures of* ∇_{k,H_r} *and* $\nabla_{k,r}$ *by* F_{k,H_r} *and* $F_{k,r}$*, respectively. Then* $F_{k,H_r} \approxeq F_{k,r}$ *and hence* $(\mathscr{J}_D^r E_k, \nabla_{k,H_r})$ *is a very ample sequence.*

(3) *If* $\tau_k : M \to E_k$ *is a* C^{r+h}*-A.H. sequence of sections, then* $j_D^r \tau_k : M \to \mathscr{J}_D^r E_k$ *is a* C^h*-A.H. sequence of sections for the connections* ∇_{k,H_r}*.*

In the integrable model $(E,\nabla) \to (\mathbb{C}^n \times \mathbb{R}, D_h, J_0, g_0)$*, with* $E = L_1 \oplus \cdots \oplus L_m$*, we can introduce new connections* ∇_{H_r} *(here there is no dependence in k, since distribution, (almost) complex structure, and metric are the standard ones). If the curvature* F_i *of each line bundle* L_i*,* $i = 1, \dots, m$*, restricted to the leaves is of type* $(1,1)$ *and has constant components with respect to the coordinates* $z_1, \dots, z_n$*, then the restrictions to each leaf of the curvatures* F_{H_r} *and* F_r *(item (2) above) coincide. As a consequence the new almost CR structure in the total space of* $\mathscr{J}^r_{D_h,n,m}$ *induced by* ∇_{H_r} *is also integrable (the foliation does not vary, just the leafwise complex structure). Also if* τ *is a CR section (*$\mathbb{C}^m$*-valued function), then the coupled CR jet is a CR section of* $(\mathscr{J}^r_{D_h,n,m}, \nabla_{H_r})$*.*

In the case of (P,Ω) *symplectic with a J-complex distribution G, analogous results hold for* $\mathscr{J}^r E_k$ *and for the integrable model.*

As we said we postpone the proof until Appendix A, but we introduce the formula for the connection.

Let $\sigma_k = (\sigma_{k,0}, \sigma_{k,1})$ be a section (maybe local) of $\mathscr{J}_D^1 E_k$. We define

$$\nabla_{H_1}(\sigma_{k,0}, \sigma_{k,1}) = (\nabla\sigma_{k,0}, \nabla\sigma_{k,1}) + (0, -F_D^{1,1}\sigma_{k,0}),$$

where $F_D^{1,1}\sigma_{k,0} \in D^{*0,1} \otimes D^{*1,0} \otimes E_k$ (see [5]).

Remark 5.3. The approximate equality $F_{H_1,k} \approxeq F_k$ has useful consequences. Assume for simplicity that $E_k = L^{\otimes k}$. Fix approximately holomorphic coordinates and trivialize the line bundle so that the connection form is A (equation (3)). Then in the local frame $(1,0) \otimes \tau_k$, $(0, dz_k^1) \otimes \tau_k, \dots, (0, dz_k^n) \otimes \tau_k$ $\mathscr{J}_D^1 L_k$ and over $B(0,\rho) \subset \mathbb{C}^n \times \mathbb{R}$ the connection matrix of ∇_{k,H_1} is up to summands bounded (at any order) by $O(k^{-1/2})$

$$\begin{vmatrix} A & -\frac{1}{2}d\bar{z}_k^1 & \cdots & -\frac{1}{2}d\bar{z}_k^n \\ 0 & A & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & A \end{vmatrix}.$$

In particular we have a uniform control on the new metric of the total space of the bundles $\mathscr{J}_D^1 L_k$ (resp. $\mathscr{J}^1 L_k$). In a similar manner this uniform control also holds for the bundles $\mathscr{J}_D^r E_k$ (resp. $\mathscr{J}^r E_k$). A useful outcome is that if we have a sequence of stratifications $\mathscr{S}$ such that for a choice of approximate holomorphic coordinates and reference frames, in the associated local basis $\mu_{k,x,I}$ of equation (32) the strata S_k^a are given by equations (functions) that do not depend neither on k nor on x, then the different bounds associated to the strata (basically those of the local functions defining them) will not depend on k and x (because we can compute them for the corresponding model with the Euclidean metric elements).

6. The linearized Thom–Boardman stratification

For the very ample sequences E_k there is an easy sufficient condition for a sequence of stratifications to be finite, Whitney (A), and approximately holomorphic.

Let us denote by T the group of translations of $\mathbb{C}^n \times \mathbb{R}$ (resp. $\mathbb{C}^p$ in the relative case).

Lemma 6.1. *Let $(S_k^a)_{a \in A}$ be a sequence of stratifications of $E_k \to (M, D, \omega)$ such that for a choice of approximately holomorphic coordinates and approximately holomorphic trivialization it is sent to $(S^a)_{a \in A}$, a fixed CR finite, Whitney (A) stratification of $\underline{\mathbb{C}}^m \to \mathbb{C}^n \times \mathbb{R}$ transverse to the fibers. Then the sequence $(S_k^a)_{a \in A}$ is as in Definition* 4.6.

Conversely, from a Whitney (A) CR stratification of $\underline{\mathbb{C}}^m \to \mathbb{C}^n \times \mathbb{R}$ transverse to the fibers and invariant under the action of $\mathsf{T} \times \mathrm{GL}(m, \mathbb{C})$ (or $\mathsf{T} \times \mathbb{C}^*$), using the local identifications of E_k with $\underline{\mathbb{C}}^m$ furnished by A.H. coordinates and A.H. trivializations, it is possible to induce an approximately holomorphic sequence of finite, Whitney (A) stratifications of E_k.

Proof. Recall that we are interested in constructing A.H. sequences of sections transverse to $(S_k^a)_{a \in A}$; in particular this sections will be uniformly bounded. Therefore, for each k, x we can work in the subset $B(0, \rho) \times B(0, R) \subset (\mathbb{C}^n \times \mathbb{R}) \times \mathbb{C}^m = \underline{\mathbb{C}}^m$, for some $R > 0$. Let f be a function defining locally a stratum S^a, which by hypothesis can be chosen to be CR. Condition (1) in Definition 4.6 holds trivially for the model S and therefore for $(S_k^a)_{a \in A}$, because when we compare the Euclidean metric and $\hat{g}_k$ we get the same inequalities as in condition (1) in Definition 3.1.

Since the model stratification is Whitney (A) and we work in a compact region, Whitney's condition (A) implies the estimated Whitney condition (A) for the Euclidean metric and hence for $\hat{g}_k$.

Let $\hat{J}_0$ be the leafwise holomorphic structure associated to the canonical CR structure of $\underline{\mathbb{C}}^m = (\mathbb{C}^n \times \mathbb{R}) \times \mathbb{C}^m$ and let $\hat{D}_h$ denote the foliation by complex hyperplanes. Since the local function f defining S^a is CR, it is in particular fiberwise holomorphic, and this proves condition (2) in Definition 4.6.

Let $(\hat{D}, \hat{J}, \hat{g}_k)$ be the almost CR structure on $B(0, \rho) \times B(0, R)$ induced by the one on E_k. In order to prove condition (3) it suffices to check that f is A.H. with respect to that almost CR structure. We are going to slightly modify the induced almost CR structure: instead of $\hat{D}$ we select $\hat{D}_h$. By using the Euclidean orthogonal projection, we can push $\hat{J} : \hat{D} \to \hat{D}$ into an almost complex structure $J' : \hat{D}_h \to \hat{D}_h$.

Since $|\mathrm{d}^j(\hat{D} - \hat{D}_h)|_{g_0} \leq O(k^{-1/2})$ for all $j \geq 0$, it follows that f is A.H. with respect to $(\hat{D}, \hat{J}, \hat{g}_k)$ if and only if it is A.H. with respect to $(\hat{D}_h, J', g_0)$ (this appears in the proof of Lemma 3.3).

In $\underline{\mathbb{C}}^m = (\mathbb{C}^n \times \mathbb{R}) \times \mathbb{C}^m$ we have canonical coordinates $z_k^1, \ldots, z_k^n$, s_k, $u_k^1, \ldots, u_k^m$. These are CR coordinates with respect to $(\hat{D}_h, \hat{J}_0)$. By hypothesis

$$\frac{\partial f}{\partial \bar{z}_k^1} = \cdots \frac{\partial f}{\partial \bar{z}_k^n} = \frac{\partial f}{\partial \bar{u}_k^1} = \cdots \frac{\partial f}{\partial \bar{u}_k^m} = 0.$$

If we show that $z_k^1, \ldots, z_k^n$, s_k, $u_k^1, \ldots, u_k^m$ are A.H. coordinates for $(\hat{D}_h, J', g_0)$ then we are done (this is again Lemma 3.3 in the absence of connection form). But this follows from the fact that the trivialization of $\underline{\mathbb{C}}^m$ is given by an A.H. frame and therefore the induced distribution (by the connection form) $\mathscr{H}$ on $\hat{D}_h$ is such that $|\mathrm{d}^j(\mathscr{H} - \hat{J}_0\mathscr{H})|_{g_0} \leq O(k^{-1/2})$ for all $j \geq 0$.

To prove the result in the other direction we fix A.H. coordinates and A.H. frames for E_k. The $\mathsf{T} \times \mathrm{GL}(m, \mathbb{C})$-invariance of $(S^a)_{a \in A} \subset \underline{\mathbb{C}}^m$ means that the local identifications define a sequence of global stratifications, and that these do not depend either on the A.H. coordinates or on the A.H. trivializations. It is an approximately holomorphic sequence of finite, Whitney (A) stratifications by the first part of the proof. □

In contrast to what happens for 0-jets, it is not easy to find non-trivial approximately holomorphic stratifications for higher order jets. The difficulty comes from the fact that the modification of the connection of Proposition 5.1 that makes the r-jets of A.H. sequences of sections of E_k into A.H. sequences of sections of $\mathscr{J}_D^r E_k$, makes it very complicated to guarantee that the strata are given by functions whose composition with an A.H. section is an A.H. function.

Example 6.1. Let $L_\Omega^{\otimes k}$ be the sequence of powers of the pre-quantum line bundle of a symplectic manifold of dimension $2p$. Let us consider the following sequence of strata in $\mathscr{J}^1 L_\Omega^{\otimes k}$:

$$\Sigma_{k,p} = \{(\sigma_0, \sigma_1) \,|\, \sigma_1 = 0\}.$$

The second subindex in our notation indicates the complex dimension of the kernel of the degree one homogeneous component of the 1-jet (see equation (54)).

Using the local sections $\mu_{k,x,I}$ of equation (32), where $I = 1, \dots, p$, and taking reference sections in Darboux charts, we get coordinates $z_k^1, \dots, z_k^p$, $v_k^0, v_k^1, \dots, v_k^p$ for the total space. $\Sigma_{k,p}$ is then defined by the zeros of the function $f = (v_k^1, \dots, v_k^p) : \mathbb{C}^{2p+1} \to \mathbb{C}^p$, which is not holomorphic (or A.H.) with respect to the modified almost complex structure of the total space. Otherwise, the composition $f \circ j^1(z_k^1 \tau_{k,x}^{\mathrm{ref}})$ would be A.H., but that composition is $(1 + z_k^1 \bar{z}_k^1, z_k^1 \bar{z}_k^2, \dots, z_k^1 \bar{z}_k^p)$.

Actually, we cannot find A.H. functions f defining $\Sigma_{k,p}$: let us work in Darboux coordinates with the canonical complex structure J_0 in the base. Assume that $\mu_{k,x,I}$ is built out of the reference section $\mathrm{e}^{-|z_k|^2/4}\xi$, where ξ is a unitary trivialization of L_Ω whose connection form is A in equation (3). Then $\mathscr{J}^1 L_\Omega^{\otimes k}$ becomes locally $\underline{\mathbb{C}}^{p+1}$ with diagonal connection matrix $A\mathrm{I}_{p+1\times p+1}$. Proposition 5.1 for complex manifolds implies that the modified almost complex structure on $\underline{\mathbb{C}}^{p+1}$ is integrable. The submanifold $z_k^2 = \cdots = z_k^p = v_k^2 = \cdots = v_k^p = 0$ is complex with respect to the modified almost complex structure. Therefore, we can restrict our attention to the case $p = 1$. The sections $j_{\mathrm{hol}}^1 \mathrm{e}^{-|z_k|^2/4}\xi$, $j_{\mathrm{hol}}^1 z_k \mathrm{e}^{-|z_k|^2/4}\xi$ are by Proposition 5.1 holomorphic. If we use them to trivialize $\mathscr{J}^1 L_\Omega^{\otimes k}$ in a neighborhood of the origin, then we obtain a new identification with $\underline{\mathbb{C}}^3$ with its canonical complex structure. Let z_k, t_k, s_k be the new complex coordinates. A short computation shows that

$$v_k^0 = t_k + z_k s_k, \qquad v_k^1 = -\bar{z}_k/2 t_k + (1 - z_k \bar{z}_k/2) s_k.$$

Hence away from the origin $\Sigma_{k,p}$ admits the parametrization

$$(z_k, s_k) \mapsto \big(z_k, s_k, s_k(2/\bar{z}_k - z_k)\big).$$

Therefore, $\Sigma_{k,p}$ is not holomorphic with respect to the modified almost complex structure, and it follows that we cannot find f A.H. defining $\Sigma_{k,p}$ locally.

6.1. Quasi-stratifications. For the applications we have in mind the notion of stratification has to be weakened. We start doing it for the local model (endowed with the trivial connection).

Let $\sigma \in S$, S a submanifold of $\mathscr{J}_{D_h,n,m}^{r+1}$. We say that $\alpha \in \Gamma(\mathscr{J}_{D_h,n,m}^r)$ is a local representation for σ if (i) $\alpha(0) = \pi_r^{r+1}\sigma$, and (ii) $\sigma = j_{D_h}^1\alpha(0) \in \mathscr{J}_{D_h,n,m}^{r+1}$, where $\pi_r^{r+1} : \mathscr{J}_{D_h,n,m}^{r+1} \to \mathscr{J}_{D_h,n,m}^r$ is the natural projection and $j_{D_h}^1\alpha$ denotes the CR 1-jet of α. The equality in (ii) should be understood in the following sense: the degree 1 component of the 1-jet should give an element of $\mathscr{J}_{D_h,n,m}^{r+1}$ (with vanishing degree 0 homogeneous component) and whose homogeneous components of degree $1, \dots, r+1$ coincide with those of σ.

Definition 6.1 (see [5]). Let S be a submanifold of $\mathscr{J}_{D_h,n,m}^r$ (resp. $\mathscr{J}_{\mathbb{C}^g,p,m}^r$). We define Θ_S to be the set of points $\sigma \in S$ for which there exists an $(r+1)$-jet $\tilde{\sigma}$ (resp.

$(r+1)$-jet along G) such that $\pi_r^{r+1}\tilde{\sigma} = \sigma$ and with a local representation α intersecting S at σ transversely along D_h (resp. along $\mathbb{C}^g$). We refer to Θ_S as the holonomic transverse subset of S.

It can be checked that if S is invariant under the action of $\mathsf{T} \times (\mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(m,\mathbb{C}))$, the second factor $\mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(m,\mathbb{C})$ acting fiberwise, (resp. $\mathsf{T} \times (\mathrm{GL}(g,\mathbb{C}) \times \mathrm{GL}(m,\mathbb{C}))$), then Θ_S has the same invariance property.

When an $(r+1)$-jet σ is represented by a local section of $\mathscr{J}^r_{D_h,n,m}$, in order to check whether $\pi_r^{r+1}\sigma \in S$ belongs to Θ_S the local representation is essentially unique: regarding transversality, it is enough to consider the degree 1 part of the Taylor expansion in the coordinates $z_k^1, \bar{z}_k^1, \ldots, z_k^n, \bar{z}_k^n$ (we turn the section into a function using the basis μ_I). The degree 0 part is determined by the r-jet, the hypothesis implies that the antiholomorphic part is vanishing and the holomorphic part is determined by the $(r+1)$-jet. That means in particular that we can restrict our attention to CR representations if necessary.

The importance of Θ_S is twofold: on the one hand it will be used to define the stratifications we are interested in. On the other hand it is a very relevant subset when we study transversality to the strata: indeed, if τ is a CR section of $\underline{\mathbb{C}}^m$ and $\alpha := j^r_{D_h}\tau$ is such that $\alpha(0) = \sigma$ and $\sigma \notin \Theta_S$, then α cannot be transverse along D_h to S at σ (notice that $\tilde{\sigma} := (\tau(0), \mathrm{d}_{D_h}\alpha(0)) = j^{r+1}_{D_h}\tau(0) \in \mathscr{J}^{r+1}_{D_h,n,m}$ and therefore α is a local representation of $\tilde{\sigma}$). The consequence is that if $S \backslash \Theta_S \subset \partial S'$, transversality of τ to S implies that τ misses a neighborhood of $S \backslash \Theta_S$ in S'.

Definition 6.1 extends to strata $S_k \subset \mathscr{J}^r_D E_k$ (resp. $\mathscr{J}^r_G E_k$): we have a notion of pseudo-holomorphic 1-jet of a section of $\mathscr{J}^r_D E_k$ (resp. pseudo-holomorphic 1-jet along G of a section of $\mathscr{J}^r_G E_k$)—because we have a connection $\nabla_{H,D}$ (resp. a connection on $\mathscr{J}^r_G E_k$ defined out of ∇_H and the projection $p_G : \mathscr{J}^r E_k \to \mathscr{J}^r_G E_k$)—and hence the notion of local representation. Then Θ_{S_k} are those points σ with lifts $\tilde{\sigma}$ having a local representation transverse along D (resp. G) to S_k at σ.

Recall that once a family of A.H. charts has been fixed we have identifications $\varpi_{k,x} : T^{*1,0}\mathbb{C}^n \to D^{*1,0}$. If we also fix a family of A.H. trivializations of E_k over the charts there is an induced identification

$$\Pi_{k,x} : \mathscr{J}^r_D E_k \to \mathscr{J}^r_{D_h,n,m}. \tag{34}$$

Lemma 6.2. *Let S_k be a sequence of strata of $\mathscr{J}^r_D E_k$, where either $r = 0$ and $E_k = E \otimes L^{\otimes k}$, or $E_k = \underline{\mathbb{C}}^m$ and $r \in \mathbb{N}$.*

(1) *If $E_k = \underline{\mathbb{C}}^m$ assume that for a choice of A.H. charts $\Pi_{k,x}(S_k) = S$, where $S \subset \mathscr{J}^r_{D_h,n,m}$ is invariant under the action $\mathsf{T} \times \mathrm{GL}(n,\mathbb{C})$, then $\Pi_{k,x}(\Theta_{S_k}) = \Theta_S$.*

(2) *The same result holds for $E \otimes L^{\otimes k}$ and $r = 0$; we need to fix A.H. trivializations of E_k (so $\Pi_{k,x}$ is defined) and require invariance of $S \subset \underline{\mathbb{C}}^m$ under the action of $\mathsf{T} \times \mathrm{GL}(m,\mathbb{C})$.*

For jets along G we have analogous results, but we need A.H. charts adapted to G and we ask for $\mathsf{T} \times \mathrm{GL}(g, \mathbb{C})$-invariance of S instead of $\mathsf{T} \times \mathrm{GL}(n, \mathbb{C})$-invariance.

Proof. Since S is $\mathrm{GL}(n, \mathbb{C})$-invariant, so is Θ_S. We have the local identifications $\Pi_{k,x} : \mathscr{J}^r_D E_k \to \mathscr{J}^r_{D_h,n,m}$. Let $y \in M$ belong to $B(0, \rho)$ in the domain of the charts centred at x_1 and x_2, for some k. Then there is a fiber bundle isomorphism

$$\Phi_{k,x_1,x_2} : \mathscr{J}^r_{D_h,n,m} \to \mathscr{J}^r_{D_h,n,m} \tag{35}$$

defined as follows: for each point y in the intersection of the domains of the charts, the restriction of the differential to D is a complex J-linear map L_y. Consider the linear map $\varpi_{k,x_2} \circ L^*_y \circ \varpi^{-1}_{k,x_2} : T^{*1,0}\mathbb{C}^n \to T^{*1,0}\mathbb{C}^n$, which belongs to $\mathrm{GL}(n, \mathbb{C})$. Φ_{k,x_1,x_2} in the fiber over y (or over the origin in both charts due to the T-invariance) is the vector space isomorphism induced by $\varpi_{k,x_2} \circ L^*_y \circ \varpi^{-1}_{k,x_2}$ (and the identity acting on the $\mathbb{C}^m$ factor of the tensor product). Since S is invariant under the $\mathsf{T} \times \mathrm{GL}(n, \mathbb{C})$-action, it follows that $\Phi_{k,x_1,x_2}(\Theta_S, S) = (\Theta_S, S)$. In particular the pair (Θ_S, S) does not depend on the chosen family of A.H. charts. We construct an appropriate family of A.H. charts (there is no Darboux condition involved here) by the usual rescaling procedure, but starting from normal coordinates composed with a linear transformation so that $(D, J) = (D_h, J_0)$ at the origin. Recall that since $E_k = \underline{\mathbb{C}}^m$, the connection $\nabla_{k,r}$ on $\mathscr{J}^r_D E_k$ is just induced by the Levi-Civita connection (in the $\underline{\mathbb{C}}^m$ factor we use the trivial connection d). Hence the pushforward of $\nabla_{k,r}$ by $\Pi_{k,x}$ to $\mathscr{J}^r_{D_h,n,m}$ has vanishing connection form at the origin. Since we also have $(D \oplus D^\perp, J) = (D_h \oplus D_v, J_0)$ at the origin, for any section α of $\mathscr{J}^r_{D_h,n,m}$ we have $j^1_D\alpha(0) = j^1_{D_h}\alpha(0)$. Therefore, the local representations at the origin for the canonical CR structure and the induced one coincide. From that and $D = D_h$ at the origin, we conclude that $\Pi_{k,x}(\Theta_{S_k}) = \Theta_S$.

Item (2) is proven in the same fashion. The $\mathrm{GL}(m, \mathbb{C})$-invariance implies that we can choose any arbitrary family of A.H. trivializations. What we do is selecting trivializations such that the connection form over the origin is vanishing (here we deal with the connection ∇_k on E_k).

Notice that we cannot state item (2) for higher order jets because the action of $\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(m, \mathbb{C})$ does not allow us to kill at the origin of each chart the connection form of the modified connection ∇_{k,H_r}.

For the relative results we start by modifying a bit the vector bundle isomorphism $\varpi_{k,x} : T^{*1,0}\mathbb{C}^p \to T^{*1,0}P$; the original (J_0, J)-complex map $T\mathbb{C}^p \to TP$ can be easily arranged to be compatible with the splittings $T\mathbb{C}^g \oplus T\mathbb{C}^{p-g}$ and $G \oplus G^\perp$. Due to the $\mathsf{T} \times (\mathrm{GL}(g, \mathbb{C}))$-invariance we are free to pick any family of A.H. charts adapted to G. The ones we need come from rescaling normal coordinates composed with a linear transformation sending $(G \oplus G^\perp, J)$ to $(\mathbb{C}^g \oplus \mathbb{C}^{p-g}, J_0)$ at the origin. In these coordinates the connection form on $T^{*1,0}\mathbb{C}^g$ is vanishing, because we project the Levi-Civita connection which is already vanishing at the origin. Hence the 1-jets along G and $\mathbb{C}^g$ at the origin coincide (also because $(G \oplus G^{\perp}, J) = (\mathbb{C}^g \oplus \mathbb{C}^{p-g}, J_0)$ at the origin), which proves the result. $\square$

The only relevant strata $S_k \subset \mathscr{J}_D^r E_k$ for which we have to consider the subsets Θ_{S_k} are the zero sections Z_k. In that case (see [3]) the subsets Θ_{Z_k} are those r-jets whose degree 1 component is onto.

Definition 6.2. An approximately holomorphic quasi-stratification of $\mathscr{J}_D^r E_k$ is an approximately holomorphic stratification in which the partial order condition is relaxed in the following way: Z_k are strata of the quasi-stratification, and for any other strata $S_k \neq Z_k$ when we approach Z_k, it accumulates into points of $Z_k \backslash \Theta_{Z_k}$ (so in particular Z_k is not in the closure of S_k).

6.2. The Thom–Boardman–Auroux stratification for maps to projective spaces. Let $E_k = \underline{\mathbb{C}}^{m+1} \otimes L^{\otimes k}$. Let $Z^0, \dots, Z^m$ be the complex coordinates associated to the trivialization of $\mathbb{C}^{m+1}$ (at any fiber) and let $\pi : \mathbb{C}^{m+1} \backslash \{0\} \to \mathbb{CP}^m$ be the canonical projection. Consider the canonical affine coordinates

$$\varphi_i^{-1} : U_i \to \mathbb{C}^m, \qquad [Z_0 : \dots : Z_m] \mapsto \left(\frac{Z^0}{Z^i}, \dots, \frac{Z^{i-1}}{Z^i}, \frac{Z^{i+1}}{Z^i}, \dots, \frac{Z^m}{Z^i} \right).$$

For each chart φ_i we consider the bundle

$$\mathscr{J}_D^r(M, \mathbb{C}^m)_i := \left(\sum_{j=0}^{r} (D^{*1,0})^{\odot j} \right) \otimes \underline{\mathbb{C}}^m. \tag{36}$$

We now return to the discussion at the beginning of Section 5. Assume for the moment that M is a Levi-flat CR manifold and fix a family of CR charts. Over each of the balls $B_{g_k}(x, \rho)$ we have the bundles $\mathscr{J}_{D_h,n,m}^r$ of CR r-jets. Notice that if we use the frames $\mu_{k,x,I}$ of equation (32) they are vector bundles.

The local bundles $\mathscr{J}_{D_h,n,m}^r$ glue into the non-linear bundle $\mathscr{J}_{CR}^r(M, \mathbb{C}^m)_i$: let $y \in M$ be a point belonging to two different charts centred at x_0 and x_1, respectively. If we send y in both charts to the origin via a translation, then the change of coordinates restricts to the leaf through the origin to a holomorphic map fixing the origin. The fibers over y are related by the action of the holomorphic r-jet of the bi-holomorphism. If we only take the linear part of the action, which is the vector bundle map Φ_{k,x_1,x_2} of equation (35), we are equally defining a bundle, for the cocycle condition still holds. Moreover, it is a vector bundle. Besides, since we only use the linear part we do not need either D or J to be integrable. This bundle is $\mathscr{J}_D^r(M, \mathbb{C}^m)_i$ as defined in equation (36) (what we defined there is rather a sequence in which the metric in the $D^{*1,0}$ factors is induced from g_k). Thus for Levi-flat manifolds the vector bundles $\mathscr{J}_D^r(M, \mathbb{C}^m)_i$ are "linear approximations" of the non-linear bundles $\mathscr{J}_{CR}^r(M, \mathbb{C}^m)_i$.

Proposition 6.1. (1) *The vector bundles* $\mathscr{J}^r_D(M,\mathbb{C}^m)_i$ *can be glued to define the almost complex fiber bundles* $\mathscr{J}^r_D(M,\mathbb{CP}^m)$ *of pseudo-holomorphic r-jets of maps from* M *to* $\mathbb{CP}^m$*, so that their fibers inherit a canonical holomorphic structure.*

(2) *Given* $\phi_k : M \to \mathbb{CP}^m$ *there is a notion of pseudo-holomorphic r-jet extension* $j^r_D\phi_k : M \to \mathscr{J}^r_D(M,\mathbb{CP}^m)$ *which is compatible with the notion of pseudo-holomorphic r-jet for the sections* $\varphi_i^{-1}\circ\phi_k : M \to \mathbb{C}^m$ *of Definition 5.2. If* $\phi_k : M \to \mathbb{CP}^m$ *is an A.H. sequence then* $j^r_D\phi_k : M \to \mathscr{J}^r_D(M,\mathbb{CP}^m)$ *is also A.H.*

Analogous results hold in the relative setting for the bundles $\mathscr{J}^r(P,\mathbb{CP}^m)$ *and* $\mathscr{J}^r_G(P,\mathbb{CP}^m)$*. Also there is an approximately holomorphic sequence of canonical submersions* $p_G : \mathscr{J}^r(P,\mathbb{CP}^m) \to \mathscr{J}^r_G(P,\mathbb{CP}^m)$*. These submersions are left inverses of the natural inclusions* $l_G : \mathscr{J}^r_G(P,\mathbb{CP}^m) \hookrightarrow \mathscr{J}^r(P,\mathbb{CP}^m)$ *so that for* $\phi_k : P \to \mathbb{CP}^m$ *an A.H. sequence,* $j^r_G\phi_k : P \to \mathscr{J}^r_G(P,\mathbb{CP}^m) \hookrightarrow \mathscr{J}^r(P,\mathbb{CP}^m)$ *is A.H.*

Proof. Let us denote the change of coordinates $\varphi_j^{-1}\circ\varphi_i$ by Ψ_{ji}. For any $y \in M$ the points in $\{y\}\times(U_i\cap U_j) \subset \mathscr{J}^r_D(M,\mathbb{C}^m)_i$ are identified with points in $\{y\}\times(U_i\cap U_j) \subset \mathscr{J}^r_D(M,\mathbb{C}^m)_j$ using the same transformation $j^r\Psi_{ji}$ in $\mathscr{J}^r_{n,m}$ induced by the fiberwise holomorphic change of coordinates Ψ_{ji}. In other words, if we take an approximately holomorphic chart centred at x say and containing y, we get as in equation (34) a vector bundle isomorphism $\Pi_{k,x,i} : \mathscr{J}^r_D(M,\mathbb{C}^m)_i \to \mathscr{J}^r_{D_h,n,m}$. Thus for $\sigma \in \mathscr{J}^r_D(M,\mathbb{C}^m)_i$ there exists $F : \mathbb{C}^n \to \mathbb{C}^m$ a CR function such that $\Pi_{k,x,i}(\sigma) = j^r_{D_h}F(x)$.

The bundle map we define is

$$j^r\Psi_{ji} : \mathscr{J}^r_D(M,\mathbb{C}^m)_i \to \mathscr{J}^r_D(M,\mathbb{C}^m)_j, \qquad \sigma \mapsto \Pi^{-1}_{k,x,j}\big(j^r_{D_h}(\Psi_{ji}\circ F)(x)\big). \tag{37}$$

This map does not depend on the charts either: if we have a point y in two different charts centred at x_1 and x_2, then we saw in the proof of Lemma 6.2 that the vector space isomorphism $\Phi_{k,x_1,x_2} : \mathscr{J}^r_{D_h,n,m} \to \mathscr{J}^r_{D_h,n,m}$ was induced by $T \in \mathrm{GL}(n,\mathbb{C})$. The bundle map of equation (37) is equivariant with respect to this action, because in the CR setting it is equivariant with respect to the action in the base of CR transformations. Hence, the result follows by considering the affine CR transformation sending y in the first chart to its image in the second and whose linear part is $T^*\times \mathrm{I} : \mathbb{C}^n\times\mathbb{R} \to \mathbb{C}^n\times\mathbb{R}$.

Equivalently, the r-jet of $\Psi_{ji}\circ F$ admits a coordinate free expression only in terms of the r-jet of F.

Therefore the identifications $j^r\Psi_{ji}$ give rise to a well defined locally trivial fiber bundle $\mathscr{J}^r_D(M,\mathbb{CP}^m)$.

Remark 6.1. If our manifold is CR and we have x belonging to two different CR charts, then there is a natural induced identification $\mathscr{J}^r_{D_h,n,m} \to \mathscr{J}^r_{D_h,n,m}$ over the points belonging to both charts. This identification is just the action of the CR r-jet of the change of coordinates. We observe that this is not the action of Φ_{k,x_1,x_2}, which is just the action induced by the 1-jet of the change of coordinates (the only one available for all almost CR structures!).

The fibers of $\mathscr{J}_D^r(M, \mathbb{CP}^m)$ admit a canonical holomorphic structure because using the local identifications $\Pi_{k,x,i}$ the fiber is some $\mathbb{C}^N$ and the change of coordinates is a fiberwise holomorphic map (because it is the holomorphic r-jet of Ψ_{ji}), and this proves item (1).

Let $\phi : (M, J, D) \to \mathbb{CP}^m$. Its pseudo-holomorphic r-jet $j_D^r\phi$ is defined as follows: the affine charts of projective space induce maps $\phi_i := \varphi_i^{-1} \circ \phi : M \to \mathbb{C}^m$. Using the trivial connection d in this trivial vector bundle and the induced connection on $D^{*1,0}$, we can define the corresponding pseudo-holomorphic r-jet $j_D^r\phi_i$ (Definition 5.2). We must check that

$$j_D^r\phi_j = j^r\Psi_{ji}(j_D^r\phi_i). \tag{38}$$

More generally let $H : \mathbb{C}^{m_1} \to \mathbb{C}^{m_2}$ be any holomorphic map. Then use the local identifications $\Pi_{k,x,s} : \mathscr{J}_D^r(M, \mathbb{C}^{m_s}) \to \mathscr{J}_{D_h,n,m_s}^r$, $s = 1, 2$, to induce the map $j^rH : \mathscr{J}_D^r(M, \mathbb{C}^{m_1}) \to \mathscr{J}_D^r(M, \mathbb{C}^{m_2})$. We claim that for any function $\phi : M \to \mathbb{C}^{m_1}$ we have

$$j_D^r(H \circ \phi) = j^rH(j_D^r\phi). \tag{39}$$

Equation (38) follows from the claim by taking $H = \Psi_{ji}$.

The proof of the claim take the next two and a half pages, and it is by induction on r. Firstly we notice that from the proof of the claim for $m_2 = 1$, the proof for any m_2 follows immediately. Therefore we assume $m_2 = 1$. Secondly we observe that it is enough to check the equality in (39) for the degree r homogeneous component of the r-jet.

We shall denote the degree r homogeneous component of j^rH by d^rH; recall that $\mathrm{d}^rH\big(j_D^r\phi(x)\big)$ depends on the components of every order of $j_D^r\phi(x)$. Let $F = (F^1, \dots, F^{m_1}) : \mathbb{C}^n \times \mathbb{R} \to \mathbb{C}^{m_1}$ be a CR function such that

$$j_D^r\phi(x) = j_{D_h}^rF(x).$$

Also the degree r homogeneous component of $j_{D_h}^rF$ is denoted by ∂_0^rF. By definition

$$\partial_{\mathrm{sym}}^j\phi(x) = \partial_0^jF(x), \qquad j = 0, \dots, r. \tag{40}$$

We start the proof of the claim for 1-jets. Once we use the identification $\partial\phi(x) = \partial_0F(x)$, we have

$$\mathrm{d}H\big(\partial\phi(x)\big) := \mathrm{d}H\big(\partial_0F(x)\big) = \sum_{a=1}^{m_1} \frac{\partial_0 H}{\partial_0 z^a}\partial_0F^a(x), \tag{41}$$

and using the identification of equation (40) above we get the following formula for the right-hand side of equation (39) for 1-jets:

$$\mathrm{d}H\big(\partial\phi(x)\big) = \mathrm{d}H\big(\partial_0 F(x)\big) = \sum_{a=1}^{m_1} \frac{\partial_0 H}{\partial_0 z^a} \partial\phi^a(x), \tag{42}$$

where the partial derivatives of H are evaluated on $\phi(x) = F(x)$, but we omit it in the notation.

Regarding the left-hand side of equation (39), the computation of $\partial(H \circ \phi)(x)$ is done by firstly taking in $\nabla(H \circ \phi)(x)$ its projection over D^* (or restricting the differential to D). Since

$$\nabla(H \circ \phi)(x) = \sum_{a=1}^{m_1} \frac{\partial_0 H}{\partial_0 z^a} \nabla\phi^a(x) \tag{43}$$

is the sum of partial derivatives of H multiplied by the components $\nabla\phi_a(x)$ of $\nabla\phi(x)$, taking $\nabla_D(H \circ \phi)(x)$ amounts to substituting in equation (43) the factors $\nabla\phi^a(x)$ by $\nabla_D\phi^a(x)$.

Next the holomorphic component is singled out; since H is holomorphic $\partial(H \circ \phi)(x)$ is computed by taking the component $\partial\phi^a(x)$ of $\nabla_D\phi^a(x)$ in equation (43). Thus we obtain the same result as in equation (42), and this proves the claim for 1-jets.

We need to prove the claim for 2-jets before going to the induction step. The reason is that for 1-jets the symmetrization step is not present, unlike the case of higher order jets.

By definition

$$\mathrm{d}^2H\big(j^2_{D_h}F(x)\big) = \sum_{b,a=1}^{m_1} \frac{\partial_0^2 H}{\partial_0 z^a \partial_0 z^b} \partial_0 F^a(x) \otimes \partial_0 F^b(x) + \sum_{c=1}^{m_1} \frac{\partial_0 H}{\partial_0 z^c} \partial_0^2 F^c(x), \tag{44}$$

so using equation (40) we get for the right-hand side of equation (39)

$$\mathrm{d}^2H\big(j^2_D\phi(x)\big) = \sum_{b,a=1}^{m_1} \frac{\partial_0^2 H}{\partial_0 z^a \partial_0 z^b} \partial\phi^a(x) \otimes \partial\phi^b(x) + \sum_{c=1}^{m_1} \frac{\partial_0 H}{\partial_0 z^c} \partial^2_{\mathrm{sym}}\phi^c(x). \tag{45}$$

To compute $\partial^2_{\mathrm{sym}}(H \circ \phi)(x)$ we first differentiate $\partial(H \circ \phi)$ at x:

$$\nabla\partial(H \circ \phi)(x) = \sum_{b,a=1}^{m_1} \frac{\partial_0^2 H}{\partial_0 z^a \partial_0 z^b} \nabla\phi^a(x) \otimes \partial\phi^b(x) + \sum_{c=1}^{m_1} \frac{\partial_0 H}{\partial_0 z^c} \nabla\partial\phi^c(x). \tag{46}$$

Taking the component along D and then the holomorphic part amounts to substituting in equation (46) $\nabla\phi^a(x)$ by $\partial\phi^a(x)$, and $\nabla\partial\phi^c(x)$ by $\partial^2\phi^c(x)$:

$$\partial^2(H\circ\phi)(x)=\sum_{b,a=1}^{m_1}\frac{\partial_0^2 H}{\partial_0 z^a\partial_0 z^b}\partial\phi^a(x)\otimes\partial\phi^b(x)+\sum_{c=1}^{m_1}\frac{\partial_0 H}{\partial_0 z^c}\partial^2\phi^c(x). \qquad (47)$$

We need to show that symmetrizing equation (47) amounts to writing $\partial^2_{\mathrm{sym}}\phi^c(x)$ instead of $\partial^2\phi^c(x)$.

In equation (47) we have terms of "type" 2—those containing a second derivative of ϕ—and terms of "type" $(1,1)$ which contain the tensor product of two derivatives of ϕ. Terms of "type" $(1,1)$ are already symmetric (just exchange the indices a,b); the symmetrization, being a linear projection, does not alter them. Now one checks that the symmetrization of each summand $\frac{\partial_0 H}{\partial_0 z^c}\partial^2\phi^c(x)$ is exactly $\frac{\partial_0 H}{\partial_0 z^c}\partial^2_{\mathrm{sym}}\phi^c(x)$, which proves the claim for 2-jets.

We now move onto the induction step. We assume $\mathrm{d}^r H(j^r_D\phi(x))=\partial^r_{\mathrm{sym}}(H\circ\phi)(x)$ and we want to prove the claim for $(r+1)$-jets. By a partition of r of degree s we understand any (ordered) s-tuple $(r_1,\dots,r_s)$, $1\le s\le r$, $1\le r_j\le r$, $r_1+\cdots+r_s=r$. In the computation of $\mathrm{d}^r H(j^r_D\phi(x)):=\partial^r_0(H\circ F)(x)$ we get an algebraic expression whose summands are of the form

$$\frac{\partial_0^{r_1+\cdots+r_s}H}{\partial_0^{r_1}z^{i_1}\dots\partial_0^{r_s}z^{i_s}}\partial_0^{r_1}F^{i_1}(x)\otimes\cdots\otimes\partial_0^{r}F^{i_s}(x), \qquad (48)$$

each belonging to a partition $(r_1,\dots,r_s)$. Notice that to some partitions correspond summands that are originated from different partitions of $r-1$. For example, in degree 3 we have $(1,2)$-terms coming from the derivation of the terms of "type" 2 and others obtained from the derivation of the $(1,1)$-terms. We do not add summands of the same "type" but keep them distinguished. By induction we assume that $\partial^r_{\mathrm{sym}}(H\circ\phi)(x)$ is computed by the same algebraic expression as $\mathrm{d}^r H(j^r_{D_h}F(x))$, but writing in the summands of equation (48) $\partial^{r_j}_{\mathrm{sym}}\phi^{i_j}$ in place of $\partial_0^{r_j}F^{i_j}(x)$, and then evaluating at x.

To compute $\partial^{r+1}_{\mathrm{sym}}(H\circ\phi)(x)$ we have to firstly differentiate the algebraic expression that computes $\partial^r_{\mathrm{sym}}(H\circ\phi)(x)$. From the previous assumption a one-to-one correspondence compatible with the partitions between the summands of $\mathrm{d}^{r+1}H(j^{r+1}_{D_h}F(x))$ and of $\nabla\partial^r_{\mathrm{sym}}(H\circ\phi)(x)$ can be established. It is clear that restricting to D and taking the $(1,0)$-component does not affect the identification.

In each summand of $\partial\partial^r_{\mathrm{sym}}(H\circ\phi)(x)$ all the factors but possibly one in the tensor product are of the form $\partial^{r_j}_{\mathrm{sym}}\phi^{i_j}$ and hence already symmetric; the different one is of the form $\partial\partial^{r'_j}_{\mathrm{sym}}\phi^{i'_j}$. Observe that the symmetrization of each summand in $\partial\partial^r_{\mathrm{sym}}(H\circ\phi)(x)$ amounts to putting instead of $\partial\partial^{r'_j}_{\mathrm{sym}}\phi^{r'_j}$ its symmetrization $\partial^{r'_j+1}_{\mathrm{sym}}\phi^{r'_j}$ and then symmetrizing the resulting expression (this is an elementary result concerning symmetric products which is proved by suitably re-grouping the permutations). Thus we have proven that

$$\partial^{r+1}_{\mathrm{sym}}(H \circ \phi)(x) = \mathrm{sym}_{r+1}\big(\mathrm{d}^{r+1} H\big(j^{r+1}_D \phi(x)\big)\big),$$

but $\mathrm{d}^{r+1} H\big(j^{r+1}_{D_h} F(x)\big)$ is already symmetric. Therefore we conclude that

$$\partial^{r+1}_{\mathrm{sym}}(H \circ \phi)(x) = \mathrm{d}^{r+1} H\big(j^{r+1}_D \phi(x)\big),$$

where the equality also holds for each summand in the algebraic expression computing both quantities.

Therefore we conclude that the pseudo-holomorphic r-jet of a map to $\mathbb{CP}^m$ is well defined.

To be able to say when a sequence of functions of $\mathscr{J}^r_D(M, \mathbb{CP}^m)$ is A.H. we need to introduce an almost CR structure in the total space of the r-jets. This can be done using a connection (for example out of the Levi-Civita connection associated to the Fubini–Study metric in the projective space and of the connection on D^*). In our case we choose to do something different but equivalent: we use the identifications with $\mathscr{J}^r_D(M, \mathbb{C}^m)_i$. Each of these trivial vector bundles with trivial connection has a natural almost CR structure. Let $K_i \subset U_i$ be compact sets whose interiors cover $\mathbb{CP}^m$. We have the corresponding subsets $\mathscr{J}^r_D\big(M, \varphi_i^{-1}(K_i)\big) \subset \mathscr{J}^r_D(M, \mathbb{C}^m)_i$.

We say that $\sigma_k : M \to \mathscr{J}^r_D(M, \mathbb{CP}^m)$ is A.H. if there exist constants $(C_j)_{j \geq 0}$ such that

$$\max_{i \in \{0,\dots,m\}} |\nabla^j (j^r \varphi_i^{-1} \circ \sigma_k)(x)|_{g_k} \leq C_j,$$

$$\max_{i \in \{0,\dots,m\}} |\nabla^{j-1} \bar{\partial} (j^r \varphi_i^{-1} \circ \sigma_k)(x)|_{g_k} \leq C_j k^{-1/2},$$

for all $x \in M$, $j \geq 1$, and $k \in \mathbb{N}$, where for each x we only take into account those indices for which $\sigma_k(x)$ belongs to the interior of $\mathscr{J}^r_D(M, K_i)$.

Notice that in the local models the identifications $j^r \Psi_{ji}$ are holomorphic, therefore when restricted to subsets associated to compact regions of $\mathbb{C}^m_i$ and $\mathbb{C}^m_j$ the sequence of maps $j^r \Psi_{ji} : \mathscr{J}^r_D(M, \mathbb{C}^m)_i \to \mathscr{J}^r_D(M, \mathbb{C}^m)_j$ is A.H. In particular the notion of a sequence $\sigma_k : M \to \mathscr{J}^r_D(M, \mathbb{CP}^m)$ being A.H. does not depend on the covering K_i. It is also clear that if a sequence of functions ϕ_k is A.H. then $j^r_D \phi_k$ is also A.H. This proves item (2) of the proposition.

If (P, Ω) is symplectic the definition of $\mathscr{J}^r(P, \mathbb{CP}^m)$ is the same (we just do not need to project the full derivative into the subspace D^*). When we have a J-complex distribution G there is an analogous definition of the bundle of pseudo-holomorphic r-jets along G. Using the previous affine coordinates of projective space we consider the sub-bundles

$$\mathscr{J}^r_G(P, \mathbb{C}^m)_i := \Big(\sum_{j=0}^r (G^{*1,0})^{\odot j}\Big) \otimes \underline{\mathbb{C}}^m,$$

where $\mathscr{J}^r_G(P, \mathbb{C}^m)_i \subset \mathscr{J}^r(P, \mathbb{C}^m)_i$ via the splitting $G \oplus G^\perp = TP$.

It is easily checked using the local identification between $\mathscr{J}^r_{p,m}$ and $\mathscr{J}(P, \mathbb{C}^m)$ coming from approximately holomorphic coordinates adapted to G that the diffeomorphisms $j^r\Psi_{ji} : \mathscr{J}^r(P, \mathbb{C}^m)_i \to \mathscr{J}^r(P, \mathbb{C}^m)_j$ preserve these sub-bundles.

The proof that shows that the $j^r\phi$ is well defined is exactly the same we gave for 2-calibrated manifolds; a small modification shows that $j^r_G\phi$ is well defined (instead of keeping the component ∇_D of the odd-dimensional case, we project over G^*).

Going to the models furnished by approximately holomorphic coordinates adapted to G, the submersion $p_G : \mathscr{J}^r_{p,m} \to \mathscr{J}^r_{\mathbb{C}^g,p,m}$ is just a projection on some of the holomorphic coordinates, and therefore it is an approximately holomorphic sequence of maps.

Using approximately holomorphic coordinates adapted to G it is straightforward to check that if $\phi_k : P \to \mathbb{CP}^m$ is A.H., then both $j^r_G\phi_k$ and $j^r\phi_k$ are A.H. sequences of $\mathscr{J}^r(P, \mathbb{CP}^m)$. □

We recall that Z_k denotes the sequence of strata of $\mathscr{J}^r_D E_k$ (resp. $\mathscr{J}^r E_k$, $\mathscr{J}^r_G E_k$) of r-jets whose degree 0-component vanishes. We define $\mathscr{J}^r_D E^*_k := \mathscr{J}^r_D E_k \backslash Z_k$ (resp. $\mathscr{J}^r E^*_k := \mathscr{J}^r E_k \backslash Z_k$, $\mathscr{J}^r_G E^*_k := \mathscr{J}^r_G E_k \backslash Z_k$).

Proposition 6.2. (1) *There exists a bundle map $j^r\pi : \mathscr{J}^r_D E^*_k \to \mathscr{J}^r_D(M, \mathbb{CP}^m)$ which is a fiberwise holomorphic submersion.*

(2) *Let τ_k be a section of E_k, and let $\phi_k = \pi \circ \tau_k : M \backslash Z(\tau_k) \to \mathbb{CP}^m$ be its projectivization defined away from the zero subset of τ_k. Then the following equation holds:*

$$j^r\pi(j^r_D\tau_k) = j^r_D\phi_k. \tag{49}$$

In the almost complex case we have an analogous map $j^r\pi$, and for $\tau_k : P \to E_k$ and its projectivization ϕ_k the equality

$$j^r\pi(j^r\tau_k) = j^r\phi_k \tag{50}$$

holds where defined.

Given G a J-complex distribution we have the following commutative square of submersions:

$$\begin{array}{ccc} \mathscr{J}^r E^*_k & \xrightarrow{p_G} & \mathscr{J}^r_G E^*_k \\ \big\downarrow{\scriptstyle j^r\pi} & & \big\downarrow{\scriptstyle j^r\pi} \\ \mathscr{J}^r(P, \mathbb{CP}^m) & \xrightarrow{p_G} & \mathscr{J}^r_G(P, \mathbb{CP}^m). \end{array} \tag{51}$$

If $j^r_G\tau_k$ is a section of $\mathscr{J}^r_G E^*_k$ the equality

$$j^r\pi(j^r_G\tau_k) = j^r_G\phi_k \tag{52}$$

holds where defined.

Proof. We define $j^r\pi$ to have the same expression as in the integrable case. This means that we fix approximately holomorphic coordinates and a section trivializing $L^{\otimes k}$ and a local frame of $E = \underline{\mathbb{C}}^{m+1}$, so that the r-jet σ in question is identified with the usual CR r-jet at a point x of a CR function F. Then $j^r\pi(\sigma)$ is defined to be the CR r-jet of $\pi \circ F$. Notice that for an appropriate chart φ_i^{-1} of projective space,

$$j^r\pi(\sigma) := \Pi^{-1}_{k,x,i}\big(j^r_{D_h}(\varphi_i^{-1} \circ \pi \circ F)(x)\big) \in \mathscr{J}^r_D(M, \mathbb{C}^m)_i. \tag{53}$$

The arguments in Proposition 6.1 that showed that the bundles $\mathscr{J}^r_D(M, \mathbb{CP}^m)$ are well defined, also prove that $j^r\pi(\sigma)$ is well defined independently of the approximately holomorphic coordinates and of the chart of $\mathbb{CP}^m$ we used; it is as well independent of the local frame of E_k, because the map is equivariant with respect to the action of $\mathrm{GL}(m+1, \mathbb{C})$ on the fibers of E_k and on $\mathbb{CP}^m$.

It is clear that $j^r\pi$ is a submersion, and it is fiberwise holomorphic because in each fiber we have a map from some $\mathbb{C}^{m_1}$ to some $\mathbb{C}^{m_2}$ (after composing with a chart φ_i), whose formula is that of the integrable case which is holomorphic, so item (1) holds.

We now prove the equality $j^r_D(\pi \circ \tau_k) = j^r\pi(j^r_D\tau_k)$: let φ_i^{-1} be any chart whose domain contains $\pi \circ \tau_k(x)$. Then by the definition given in Proposition 6.1,

$$j^r_D(\pi \circ \tau_k)(x) := j^r_D(\varphi_i^{-1} \circ \pi \circ \tau_k)(x).$$

We just defined in equation (53)

$$j^r\pi\big(j^r_D\tau_k(x)\big) := \Pi^{-1}_{k,x,i}\big(j^r_{D_h}(\varphi_i^{-1} \circ \pi \circ F)(x)\big).$$

By Proposition 6.1 the right-hand side of the two previous equalities coincides, i.e.,

$$\Pi^{-1}_{k,x,i}\big(j^r_{D_h}(\varphi_i^{-1} \circ \pi \circ F)(x)\big) = j^r_D(\varphi_i^{-1} \circ \pi \circ \tau_k)(x).$$

Here the holomorphic function $\varphi_i^{-1} \circ \pi : \mathbb{C}^{m+1}\backslash\{0\} \to \mathbb{C}^m$ plays the role of H in Proposition 6.1. Also observe that the proposition is in principle only valid when $\underline{\mathbb{C}}^{m+1}$ has the trivial connection. In the current situation $\underline{\mathbb{C}}^{m+1}$ is endowed with a diagonal connection coming from the one in $L^{\otimes k}$. The key point is that the composition $\varphi_i^{-1} \circ \pi \circ \phi_k$ is a section of $\underline{\mathbb{C}}^m \otimes L^{\otimes k} \otimes L^{-\otimes k}$ and hence a $\mathbb{C}^m$-valued function independently of the trivialization of $L^{\otimes k}$. Therefore the flat connection d on $\underline{\mathbb{C}}^m$ is induced from $\mathrm{d} \otimes \mathrm{I} + \mathrm{I} \otimes \nabla_k$ in $\underline{\mathbb{C}}^{m+1} \otimes L^{\otimes k}$, where ∇_k is any Hermitian connection on $L^{\otimes k}$. In other words, the equations of Proposition 6.1 involving the connection $\nabla_g \otimes \mathrm{I} + \mathrm{I} \otimes \mathrm{d}$ on $(T^{*1,0}\mathbb{C}^{n^{\odot r}}) \otimes \underline{\mathbb{C}}^{m+1}$ are also valid in this setting for the connection $\nabla_g \otimes \mathrm{I} + \mathrm{I} \otimes (\mathrm{d} \otimes \mathrm{I} + \mathrm{I} \otimes \nabla_k)$, and this finishes the proof of item (2).

The previous ideas work word by word to show that for symplectic manifolds $j^r\pi : \mathscr{J}^r E_k^* \to \mathscr{J}^r(P, \mathbb{CP}^m)$ is a well defined submersion and that equation (50) holds.

If we have a distribution G, once we use the local identification coming from approximately holomorphic coordinates adapted to G, the commutativity of the diagram (51) follows from the commutativity in the holomorphic case. It is also clear that $j^r\pi : \mathscr{J}_G^r E_k^* \to \mathscr{J}_G^r(P, \mathbb{CP}^m)$ is a submersion and that equation (52) holds. □

In order to describe the linearized Thom–Boardman stratification we need to define, at least for certain kinds of strata $\mathbb{P}S_k^a$ of $\mathscr{J}_D^r(M, \mathbb{CP}^m)$, the corresponding subsets of transverse holonomy $\Theta_{\mathbb{P}S_k^a}$.

Definition 6.3. Let $\mathbb{P}S_k$ be a sequence of strata of $\mathscr{J}_D^r(M, \mathbb{CP}^m)$ so that in canonical affine charts of $\mathbb{CP}^m$ and approximately holomorphic coordinates it is identified with a stratum $\mathbb{P}S$ of $\mathscr{J}_{D_h,n,m}^r$ invariant under the action of $\mathsf{T} \times \mathrm{GL}(n, \mathbb{C})$. We let $\mathbb{P}S_{k,i} := \mathbb{P}S_k \cap \mathscr{J}_D^r(M, \mathbb{C}^m)_i$ and then define

$$\Theta_{\mathbb{P}S_k} := \bigcup_{i \in \{0,\dots,m\}} \Theta_{\mathbb{P}S_{k,i}}.$$

For $S_k := j^r\pi^{-1}(\mathbb{P}S_k)$, with $j^r\pi : \mathscr{J}_D^r E_k^* \to \mathscr{J}_D^r(M, \mathbb{CP}^m)$ the submersion of Proposition 6.2, we define $\check{\Theta}_{S_k} := j^r\pi^{-1}(\Theta_{\mathbb{P}S_k})$.

In the relative theory we assume that for a choice of approximately holomorphic coordinates adapted to G and canonical affine charts of projective space, the sequence $\mathbb{P}S_{k,i} \subset \mathscr{J}_G^r(P, \mathbb{C}_i^m)$ is identified with a stratum $\mathbb{P}S$ of $\mathscr{J}_{\mathbb{C}^g,p,m}^r = \mathscr{J}_{g,m}^r \times \mathbb{C}^{p-g}$ invariant under the action of $\mathsf{T} \times \mathrm{GL}(g, \mathbb{C})$. Then we define

$$\Theta_{\mathbb{P}S_k} := \bigcup_{i \in \{0,\dots,m\}} \Theta_{\mathbb{P}S_{k,i}}.$$

For $S_k := j^r\pi^{-1}(\mathbb{P}S_k) \subset \mathscr{J}_G^r E_k^*$, $S_k^G := p_G{}^{-1}(S_k) \subset \mathscr{J}^r E_k^*$, we define the subset $\check{\Theta}_{S_k^G} \subset S_k^G$ by pulling back $\Theta_{\mathbb{P}S_k}$ to $\mathscr{J}^r E_k^*$ using either of the sides of the commutative diagram (51).

Notice that by item (1) of Lemma 6.2 the subsets $\Theta_{\mathbb{P}S_{k,i}}$ are well defined, so Definition 6.3 makes sense. It is also satisfactory because of the following result:

Lemma 6.3. *We have*

$$\Theta_{\mathbb{P}S_k} \cap \mathscr{J}^r_D(M, \mathbb{C}^m_i) = \Theta_{\mathbb{P}S_{k,i}},$$
$$\Theta_{\mathbb{P}S_k} \cap \mathscr{J}^r_G(P, \mathbb{C}^m_i) = \Theta_{\mathbb{P}S_{k,i}}.$$

Proof. Fix approximately holomorphic coordinates and canonical affine charts of $\mathbb{CP}^m$, so that $\Pi_{k,x,i}(\mathbb{P}S_{k,i}) = \mathbb{P}S$, for all k, x, i. We need to show is that

$$j^r\Psi_{ji}(\Theta_{\mathbb{P}S}) = \Theta_{\mathbb{P}S}$$

in the domain of definition of $j^r\Psi_{ji}$, where Ψ_{ji} is a change of canonical affine coordinates.

Let ψ be an r-jet in $\Theta_{\mathbb{P}S}$. Then we have a lift $\tilde{\psi}$ to $\mathscr{J}^{r+1}_{D_h,n,m}$ and a local representation α of the lift cutting $\mathbb{P}S$ transversally along D_h at ψ. As we mentioned regarding transversality the local representation is essentially unique. That means in particular that any other representation α' will also share the transversality property. By definition $\tilde{\psi}$ is the $(r+1)$-jet of a local CR function F. Then $j^r_{D_h}F(0) = \psi$ and $\big(F(0), \mathrm{d}_{D_h} j^r_{D_h}F(0)\big) = \big(F(0), \partial_0 j^r_{D_h}F(0)\big) = j^{r+1}_{D_h}F(0) = \tilde{\psi}$. Thus, $j^r_{D_h}F$ is a local representation of $\tilde{\psi}$ which is transverse to $\mathbb{P}S$ along D_h at ψ.

Since $j^{r+1}\Psi_{ji}(j^{r+1}_{D_h}F) = j^{r+1}_{D_h}(\Psi_{ij} \circ F)$, we deduce that $j^{r+1}\Psi_{ji}(\tilde{\psi})$ is a lift of $j^r\Psi_{ji}(\psi)$ with local representation $j^r_{D_h}(\Psi_{ij} \circ F)$, which is obviously transverse along D_h to $j^r\Psi_{ji}(\mathbb{P}S) = \mathbb{P}S$ because $j^r\Psi_{ji}$ is a diffeomorphism that preserves the pullback of D_h to $\mathscr{J}^r_{D_h,n,m}$. We just checked one inclusion, but that suffices because Ψ_{ji} is a diffeomorphism, thus the result for jets along D follows.

An analogous proof shows the desired result for jets along G. □

The linearized Thom–Boardman stratification is the pullback to $\mathscr{J}^r_D E^*_k$ by $j^r\pi$ of the analog of the Thom–Boardman stratification of $\mathscr{J}^r_D(M, \mathbb{CP}^m)$ (see for example [8]), together with the strata Z_k. The definition is the natural extension of the one given for symplectic manifolds by D. Auroux in [4].

A first rough definition of the stratification of $\mathscr{J}^r_D(M, \mathbb{CP}^m)$ is the following: we fix approximately holomorphic coordinates and canonical affine charts of projective space, so we have charts $\Pi^{-1}_{k,x,i} : \mathscr{J}^r_{D_h,n,m} \to \mathscr{J}^r_D(M, \mathbb{C}^m)_i$. In each $\mathscr{J}^r_{D_h,n,m}$ there is a CR Thom–Boardman stratification which is $\mathsf{T} \times (\mathscr{H}^r_n \times \mathscr{H}^r_m)$-invariant, where $\mathscr{H}^r_l$ is the group of r-jets of germs of bi-holomorphic transformations from $\mathbb{C}^l$ to $\mathbb{C}^l$; in particular it is $\mathsf{T} \times \mathrm{GL}(n, \mathbb{C})$-invariant, so it defines a stratification on each $\mathscr{J}^r_D(M, \mathbb{C}^m)_i$. The $\mathscr{H}^r_m$-invariance implies that the identifications that define $\mathscr{J}^r_D(M, \mathbb{CP}^m)$ are compatible with the aforementioned stratifications on $\mathscr{J}^r_D(M, \mathbb{C}^m)_i$.

Once we pullback the stratification to $\mathscr{J}^r_D E^*_k$ the behavior of the strata when they approach Z_k needs to be clarified. To do that we re-define the stratification as follows (see [4]):

Given $\sigma \in \mathscr{J}_D^r E_k^*$ let us denote its image in $\mathscr{J}_D^r(M, \mathbb{CP}^m)$ by $\phi = (\phi_0, \ldots, \phi_r)$. Let us define

$$\Sigma_{k,i} = \{\sigma \in \mathscr{J}_D^r E_k^* \,|\, \dim_{\mathbb{C}} \ker \phi_1 = i\}. \tag{54}$$

If $\max(0, n-m) < i \leq n$, the strata $\Sigma_{k,i}$ are smooth submanifolds whose boundary is the union $\bigcup_{j>i} \Sigma_{k,j}$ together with a subset of $Z_k \backslash \Theta_{Z_k}$.

Each $\Sigma_{k,i}$ is the pullback of a stratum $\mathbb{P}\Sigma_{k,i} \subset \mathscr{J}_D^r(M, \mathbb{CP}^m)$, and the given description of their closure is easy to check.

For $r \geq 2$, define $\check{\Theta}_{\Sigma_{k,i}}$ as the subset of r-jets $\sigma = (\sigma_0, \ldots, \sigma_r) \in \Sigma_{k,i}$ so that

$$\Xi_{k,i;\sigma} = \{u \in D \,|\, (i_u\phi, 0) \in T_\phi \mathbb{P}\Sigma_{k,i}\} \tag{55}$$

has the expected (complex) codimension in D, which is the (complex) codimension of $\Sigma_{k,i}$ in $\mathscr{J}_D^r E_k$, which equals the codimension of $\mathbb{P}\Sigma_{k,i}$ in $\mathscr{J}_D^r(M, \mathbb{CP}^m)$.

The subset $\check{\Theta}_{\Sigma_{k,i}}$ is also the one coming from Definition 6.3: observe that $\Theta_{\mathbb{P}\Sigma_{k,i}}$ are exactly those points of $\mathbb{P}\Sigma_{k,i}$ which have a lift with a transverse local representation. Since the term that we add to the r-jet to define the lift is of order $r+1 > 2$, the transversality of the local representation does not depend on the lift that can be chosen to have vanishing component of order $r+1$.

Fix as in the proof of Lemma 6.2 A.H. coordinates so that at the origin $(D \oplus D^\perp, J) = (D_h \oplus D_v, J_0)$ and the induced connection form (on $\mathscr{J}_{D_h,n,m}^r$) is vanishing; fix also the canonical affine charts of $\mathbb{CP}^m$. Then the strata $\mathbb{P}\Sigma_{k,i}$ are sent to the Thom–Boardmman stratum $\mathbb{P}\Sigma_i$ of $\mathscr{J}_{D_h,n,m}^r$. The local representation of $(\phi, 0)$ can be taken to be a CR section α of $\mathscr{J}_{D_h,n,m}^r$. The stratum $\mathbb{P}\Sigma_i$ is CR, therefore

$$T_{D_h} j_{D_h}^1 \alpha(0) \cap (T\mathbb{P}\Sigma_i \cap \hat{D}_h)$$

is a complex subspace of $T\mathbb{C}^n$. Undoing the identifications the previous subspace goes to the subspace in equation (55). By definition of transversality along D, $\Theta_{\mathbb{P}\Sigma_{k,i}}$ are exactly those ϕ for which $\Xi_{k,i;\sigma}$ has the codimension of $\mathbb{P}\Sigma_{k,i}$ in $\mathscr{J}_D^r(M, \mathbb{CP}^m)$. By construction (equation (55))

$$\check{\Theta}_{\Sigma_{k,i}} = j^r \pi^{-1}(\Theta_{\mathbb{P}\Sigma_{k,i}}).$$

Hence $\check{\Theta}_{\Sigma_{k,i}}$ is the same subset introduced in Definition 6.3.

If $p + 1 \leq r$, we define inductively

$$\Sigma_{k,i_1,\ldots,i_p,i_{p+1}} = \{\sigma \in \Theta_{\Sigma_{k,i_1,\ldots,i_p}} \,|\, \dim_{\mathbb{C}}(\ker \phi_1 \cap \Xi_{k,i_1,\ldots,i_p;\sigma}) = i_{p+1}\},$$

with

$$\Xi_{k,I;\sigma} = \{u \in D \,|\, (i_u\phi, 0) \in T_\phi \mathbb{P}\Sigma_{k,I}\}.$$

As in the previous case we define $\check{\Theta}_{\Sigma_{k,I}}$ either as the points such that the complex codimension of $\Xi_{k,I;\sigma}$ in D is the same as the codimension of $\Sigma_{k,I}$ in $\mathscr{J}^r_D E_k$, or as the pullback of $\Theta_{\mathbb{P}\Sigma_{k,I}}$.

If $i_1 \geq \cdots \geq i_{p+1} \geq 1$, $\Sigma_{k,i_1,\dots,i_{p+1}}$ is—in the local model—a smooth CR submanifold whose closure in $\Sigma_{k,i_1,\dots,i_p}$ is the union of the $\Sigma_{k,i_1,\dots,i_p,j}$, $j > i_{p+1}$, and a subset of $\Sigma_{k,i_1,\dots,i_p}\backslash\check{\Theta}_{\Sigma_{k,i_1,\dots,i_p}}$ [8]. The problem is that for large values of r, n, m, the closure of the strata in $\mathscr{J}^r_{D_h,n,m}$ is hard to understand, and what we have defined, once Z_k has been added, might very well not be a Whitney (A) quasi-stratification. More precisely, let $\Sigma_{m+1;q} := \Sigma_{m+1,1,\overset{(q)}{\dots},1} \subset \mathscr{J}^r_{D_h,n,m}$ be a so-called Morin stratum. Then in [40] it is shown that

$$\overline{\Sigma_{m+1;q}} \cap \Sigma_{m+2,0} \neq \emptyset,$$

but for q large enough $\dim \Sigma_{m+1;q} < \dim \Sigma_{m+2,0}$, thus Whitney's condition (A) can never hold. It is known that $\mathscr{J}^r_{D_h,n,m}$ admits a Whitney (A) stratification containing the Morin strata. If the dimensions satisfy $n < 4$ or $2n > 3m - 4$, then a generic function will avoid $\Sigma_{m+2,0}$ and $\Sigma_{m+1,2}$ and therefore will only intersect the Morin strata, so the aforementioned previous stratification suffices (also because the strata $\Sigma_{k,I}$ do not accumulate in points of Θ_{Z_k}). In general one must refine the Thom–Boardman stratification.

Recall that using the local identifications the stratification we have defined (minus Z_k) is the union running over the affine charts of the pullback by $j^r(\varphi_i^{-1} \circ \pi) : \mathscr{J}^r_{D_h,n,m+1}\backslash Z \to \mathscr{J}^r_{D_h,n,m}$ of the CR Thom–Boardman stratification $\mathbb{P}\Sigma$ of $\mathscr{J}^r_{D_h,n,m}$. The latter is CR and $\mathsf{T} \times \left(\mathrm{GL}(n,\mathbb{C}) \times \mathscr{H}^r_m\right)$-invariant.

On the domain of each chart $\mathscr{J}^r_{D_h,n,m}$ we can use the results of Mather [28] to refine $\mathbb{P}\Sigma$ into a CR finite, Whitney (A) stratification transverse to the fibers and invariant under the action of $\mathsf{T} \times \left(\mathrm{GL}(n,\mathbb{C}) \times \mathscr{H}^r_m\right)$, and such that the submanifolds $\mathbb{P}\Sigma_I$ are unions of strata of the refinement. Due to the required invariance properties for the refinements, they can be glued to give a refinement of the stratification $\mathbb{P}\Sigma_k \subset \mathscr{J}^r_D(M,\mathbb{CP}^m)$, which is independent of the choice of approximately holomorphic coordinates. Thus, its pullback is a finite, Whitney (A) stratification of $\mathscr{J}^r_D E^*_k$ and such that the $\Sigma_{k,I}$ are union of strata. It is by construction invariant by the action of $\mathrm{GL}(m+1,\mathbb{C})$ on the fiber.

It is important to notice that since all the strata are contained in the closure of $\Sigma_{k,\max(0,n-m)+1}$, they accumulate near Z_k in points of $Z_k\backslash\Theta_{Z_k}$. Therefore, by adding Z_k we obtain a quasi-stratification of $\mathscr{J}^r_D E_k$.

If we have a distribution G we use exactly the same definitions but in the subbundles $\mathscr{J}^r_G E_k$ and $\mathscr{J}^r_G(P,\mathbb{CP}^m)$. That is, we have the strata

$$\mathbb{P}\Sigma_{k,i} = \{\phi \in \mathscr{J}^r_G(P,\mathbb{CP}^m) \mid \dim_{\mathbb{C}} \ker \phi_1 = i\}$$

and for $r \geq 2$, $\check{\Theta}_{\mathbb{P}\Sigma_{k,i}} \subset \mathbb{P}\Sigma_{k,i}$ is the subset of r-jets along G, $\phi = (\phi_0,\dots,\phi_r)$ so that

$$\Xi_{k,i;\sigma} = \{u \in G \mid (i_u\phi, 0) \in T_\phi \mathbb{P}\Sigma_{k,i}\} \tag{56}$$

has the expected (complex) codimension in G, which is the (complex) codimension of $\mathbb{P}\Sigma_{k,i}$ in $\mathscr{J}^r_G(P, \mathbb{CP}^m)$.

The subsets $\mathbb{P}\Sigma_{k,I}$ are defined similarly. The result is a stratification $\mathbb{P}\Sigma_k$ of $\mathscr{J}^r_G(P, \mathbb{CP}^m)$. In charts adapted to G as in the proof of Lemma 6.2 and affine charts—for which $\mathscr{J}^r_{\mathbb{C}^g,p,m} = \mathscr{J}^r_{g,m} \times \mathbb{C}^{p-g}$—, the induced stratification $\mathbb{P}\Sigma$ is seen to be the leafwise Thom–Boardman stratification, i.e., the Thom–Boardman stratification of $\mathscr{J}^r_{g,m}$ multiplied by $\mathbb{C}^{p-g}$.

Using the lower part of the commutative diagram (51), we pull back $\mathbb{P}\Sigma_k$ to $\mathbb{P}\Sigma_k^G \subset \mathscr{J}^r(P, \mathbb{CP}^m)$. Let Σ_k^G be the pullback of $\mathbb{P}\Sigma_k^G$ to $\mathscr{J}^r E_k^*$. To refine it we first locally refine $\mathbb{P}\Sigma_k$ as follows: we go the leafwise Thom–Boardman stratification furnished by the previous A.H. coordinates and affine charts and construct a holomorphic $\mathsf{T} \times \big(\mathrm{GL}(g, \mathbb{C}) \times \mathscr{H}^r_m\big)$-invariant refinement in one of the leaves of $\mathscr{J}^r_{\mathbb{C}^g,p,m}$ (which is identified with $\mathscr{J}^r_{g,m}$). Next we extend it independently of the remaining $p-g$ complex coordinates $z_k^{g+1}, \ldots, z_k^p$. The local refinements of the leafwise Thom–Boardman stratification glue well and thus define a sequence of Whitney (A) stratifications $\mathscr{J}^r_G(P, \mathbb{CP}^m)$, which does not depend either on the A.H. coordinates adapted to G or in the chosen affine charts of $\mathbb{CP}^m$. Its pullback to $\mathscr{J}^r E_k^*$ refines Σ_k^G to a sequence of Whitney (A) stratifications.

Definition 6.4 (see [4]). (1) Given (M, D, J, g_k) and $E_k = \underline{\mathbb{C}}^{m+1} \otimes L^{\otimes k}$, the Thom–Boardman–Auroux stratification of $\mathscr{J}^r_D(M, \mathbb{CP}^m)$, denoted by $\mathbb{P}\Sigma_k$, is the stratification (or rather its refinement) built out of the pieces of the Thom–Boardman stratifications of $\mathscr{J}^r_{D_h,n,m}$. The Thom–Boardman–Auroux quasi-stratification of $\mathscr{J}^r_D E_k$ is the pullback of the Thom–Boardman–Auroux stratification of $\mathscr{J}^r_D(M, \mathbb{CP}^m)$ together with the zero section. We denote it by Σ_k.

(2) Given (P, J, G, g_k) and $E_k = \underline{\mathbb{C}}^{m+1} \otimes L^{\otimes k}$, the Thom–Boardman–Auroux stratification of $\mathscr{J}^r(M, \mathbb{CP}^m)$ along G, denoted by $\mathbb{P}\Sigma_k^G$, is the stratification (or rather its refinement) built out of the pieces of the Thom–Boardman stratifications of $\mathscr{J}^r_{\mathbb{C}^g,p,m}$. The Thom–Boardman–Auroux quasi-stratification of $\mathscr{J}^r E_k$ along G, that we denote by Σ_k^G, is the pullback of the Thom–Boardman–Auroux stratification of $\mathscr{J}^r(M, \mathbb{CP}^m)$ along G together with Z_k.

Lemma 6.4. *The Thom–Boardman–Auroux quasi-stratification of $\mathscr{J}^r_D E_k$ and the Thom–Boardman–Auroux quasi-stratification of $\mathscr{J}^r E_k$ along G are finite, Whitney (A), and approximately holomorphic.*

Proof. We start with jets along D. The description of the closure of the strata inside Z_k implies that the quasi-stratification condition holds.

The delicate point is checking that the strata are approximately holomorphic (for the modified connection).

First we study the sequence Z_k. Though for this sequence the approximate holomorphicity is obvious, we will give a proof that works for other sequences of strata. Indeed, by Lemma 6.1 the sequence of zero sections $Z_k \subset E_k$ is as required. If we prove that the natural projections

$$\pi^r : \mathscr{J}_D^r E_k \to E_k$$

are an A.H. sequence of maps which is also ε-transverse for some $\varepsilon > 0$, then the composition of the local maps defining $Z_k \subset E_k$ with the projection π^r are local functions for $(\pi^r)^{-1}(Z_k) = Z_k \subset \mathscr{J}_D^r E_k$ meeting the conditions of Definition 4.6.

More generally we prove that the natural projection $\pi^r_{r-h} : \mathscr{J}_D^r E_k \to \mathscr{J}_D^{r-h} E_k$ is approximately holomorphic: we fix A.H. coordinates and A.H. reference frames $j_D^r \tau^{\text{ref}}_{k,x,I}$ of $\mathscr{J}_D^r E_k$ (resp. $j_D^{r-h} \tau^{\text{ref}}_{k,x,I'}$ of $\mathscr{J}_D^{r-h} E_k$) as in equation (33). Recall that Proposition 5.1 implies that the sequences are indeed A.H. Using these frames we obtain A.H. coordinates $z_k^1, \ldots, z_k^n$, u_k^I, s_k (resp. $z_k^1, \ldots, z_k^n$, $v_k^{I'}$, s_k) for the total space of $\mathscr{J}_D^r E_k$ (resp. $\mathscr{J}_D^{r-h} E_k$). From

$$\pi^r_{r-h}(j_D^r \tau^{\text{ref}}_{k,x,I}) = j_D^{r-h} \tau^{\text{ref}}_{k,x,I} \tag{57}$$

we deduce $\pi^r_{r-h}(j_D^r \tau^{\text{ref}}_{k,x,I}) = W_I(z_k, v_k^{I'})$, where $W_I(z_k, v_k^{I'})$ is A.H. with respect to the canonical CR structures associated to the coordinates. This, together with the fiberwise linearity of π^r_{r-h} imply that in these coordinates π^r_{r-h} is A.H., and hence it is A.H. with respect to the almost CR structures of the total spaces. It is also straightforward from equation (57) that the projections are ε-transverse (another way is to use rather than holonomic frames the frames $\mu_{k,x,I}$ of equation (32). They are also frames for the modified metric because of for example remark 5.3, therefore one can check estimated transvesality using them, something which is straightforward).

We would like to do something similar with the strata $\Sigma_{k,I}$ and the projection $j^r\pi : \mathscr{J}_D^r E_k^* \to \mathscr{J}_D^r(M, \mathbb{CP}^m)$ (away from a uniform tubular neighborhood of the zero section, where the differential goes to infinity). The image of a trivialization $j_D^r \tau^{\text{ref}}_{k,x,I}$ is $j_D^r(\pi \circ \tau^{\text{ref}}_{k,x,I})$, also approximately holomorphic. The map is equally fiberwise holomorphic, but the difference is the non-linearity of the restriction to the fibers.

We adopt a different strategy that amounts to perturbing the almost CR structures into integrable ones and then checking that $j^r\pi$ is CR with respect to them: we take Darboux charts and trivialize $L^{\otimes k}$ with a unitary section ξ_k whose associated connection form in the domain of Darboux charts is A. Next we trivialize $\mathscr{J}_D^r E_k$ with the frames $\mu_{k,x,I}$ of equation (32), but using ξ_k tensored with a basis of $\mathbb{C}^{m+1}$ to trivialize $\underline{\mathbb{C}}^{m+1} \otimes L^{\otimes k}$. In this way $\mathscr{J}_D^r E_k$ becomes the trivial bundle $\mathscr{J}^r_{D_h,n,m+1}$ (which is canonical trivialization constructed out of $dz_k^1, \ldots, dz_k^n$). Let us use in the base the canonical CR structure (D_h, J_0). Proposition 5.1 in the integrable case (and for curvature of type $(1,1)$ and with trivial derivative, as it is the case in Darboux coordinates) implies that the modified connection defines a new CR structure in the total space of $\mathscr{J}^r_{D_h,n,m}$; let $(\hat{D}_h, \bar{J}_0)$ be the corresponding distribution and almost complex structure, and let $(\hat{D}, \hat{J})$ be the distribution and almost complex structure induced by the almost CR structure of $\mathscr{J}^r_D E_k$. If in the fiber of $\mathscr{J}^r_{D_h,n,m+1}$ we fix a ball $B(\sigma, R)$, then in $B(0,\rho) \times B(\sigma, R)$ the Euclidean metric is comparable with the metric carried by $\mathscr{J}^r_D E_k$. More important

$$|\mathrm{d}^j(\hat{D} - \hat{D}_h)|_{g_0} \leq O(k^{-1/2}), \qquad j \geq 0. \tag{58}$$

If we use the orthogonal projection to push $\hat{J}$ into $\hat{J}_h : \hat{D}_h \to \hat{D}_h$ we also have

$$|\mathrm{d}^j(\hat{J}_h - \bar{J}_0)|_{g_0} \leq O(k^{-1/2}), \qquad j \geq 0. \tag{59}$$

We use the same Darboux charts for $\mathscr{J}^r_{D_h}(\mathbb{C}^n \times \mathbb{R}, \mathbb{CP}^m)$, so locally and using canonical affine charts we have identifications with $\mathscr{J}^r_{D_h,n,m}$. This is a trivial vector bundle (again using the basis induced by $dz^1_k, \ldots, dz^n_k$ and the basis of $\mathbb{C}^m$). We fix the product CR structure and denote by $(\tilde{D}_h, \tilde{J}_0)$ the distribution and almost complex structure. Let $(\tilde{D}, \tilde{J})$ be the distribution and almost complex structure induced by the almost CR structure of $\mathscr{J}^r_D(M, \mathbb{CP}^m)$. By construction,

$$|\mathrm{d}^j(\tilde{D} - \tilde{D}_h)|_{g_0}, |\mathrm{d}^j(\tilde{J}_h - \tilde{J}_0)|_{g_0} \leq O(k^{-1/2}), \qquad j \geq 0, \tag{60}$$

where $\tilde{J}_h$ is the almost complex structure on $\tilde{D}_h$ defined out of $\tilde{J}$ and the orthogonal projection.

Equations (58), (59), (60) imply that if $j^r(\varphi_i^{-1} \circ \pi) : \mathscr{J}^r_{D_h,n,m+1} \to \mathscr{J}^r_{D_h,n,m}$ is CR with respect to $(\hat{D}_h, \bar{J}_0)$ and $(\tilde{D}_h, \tilde{J}_0)$, then it is almost CR with respect to the global almost CR structures.

The map $j^r(\varphi_i^{-1} \circ \pi) : \mathscr{J}^r_{D_h,n,m+1} \to \mathscr{J}^r_{D_h,n,m}$ is exactly the same as in the holomorphic (or rather CR) models. It is CR with respect to the aforementioned CR structures because it preserves the foliations, it is fiberwise holomorphic and sends "enough" CR sections of $\mathscr{J}^r_{D_h,n,m+1}$ to CR sections of $\mathscr{J}^r_{D_h,n,m}$. To be more precise, for any point $\sigma \in \mathscr{J}^r_{D_h,n,m+1}$ and any vector v in its tangent space along the leaf and not tangent to the fiber, we can find a CR section F whose CR r-jet in x is σ and such that the tangent space to its graph contains v. Since $j^r(\varphi_i^{-1} \circ \pi)(j^r_{D_h} F) = j^r_{D_h}(\varphi_i^{-1} \circ \pi \circ F)$ is also a CR section, we deduce that $j^r(\varphi_i^{-1} \circ \pi)_*(\bar{J}v) = \tilde{J}_0\big(j^r(\varphi_i^{-1} \circ \pi)_*(v)\big)$.

The strata $\mathbb{P}\Sigma_k$ (or rather of its refinement), once we choose A.H. coordinates and affine charts of projective space, are identified with the strata of (the refinement of) the CR Thom–Boardman stratification of $\mathscr{J}^r_{D_h,n,m}$, which are CR. The comparison between the $(\hat{D}_h, \bar{J}_0, g_0)$ and the original almost CR structure implies that the strata of $\mathbb{P}\Sigma_k$ are A.H., and hence $\Sigma_k = j^r \pi^{-1}(\mathbb{P}\Sigma_k)$ is A.H. That the projections are ε-transverse is also clear, therefore the desired result follows.

In the almost complex setting $j^r\pi : \mathcal{J}^r E_k^* \to \mathcal{J}^r(P, \mathbb{CP}^m)$ is equally shown to be approximately holomorphic away from a uniform neighborhood of the zero section. In the relative case, and for a sequence of A.H. strata $\mathbb{P}S_k$ fulfilling the conditions of Definition 6.3, the approximate holomorphicity of $p_G{}^{-1} j^r\pi^{-1} S_k$ follows from the commutativity of the diagram 51, and from the approximate holomorphicity of $j^r\pi : \mathcal{J}^r E_k^* \to \mathcal{J}^r(P, \mathbb{CP}^m)$ and of $p_G : \mathcal{J}^r(P, \mathbb{CP}^m) \to \mathcal{J}_G^r(P, \mathbb{CP}^m)$. Recall that the strata $\mathbb{P}\Sigma_k$ come from holomorphic models (the refinement of the strata of the leafwise Thom–Boardman stratification), so they are A.H. But Σ_k^G is not truly a quasi-stratification of $\mathcal{J}^r E_k$. To be more precise it is not true that the strata only accumulate in points of $Z_k \backslash \Theta_{Z_k} \subset Z_k$, but it is still true that the points of Z_k in which the other strata accumulate are never hit by a section transverse to Z_k along G. Thus, the Whitney type reasoning can be applied as long as we work with r-jets along G (see the proof of Theorem 7.2). □

Remark 6.2. Notice that we only conclude that the strata different form the zero section are approximately holomorphic uniformly far from Z_k. This is enough for our purposes, for once we obtain transversality to Z_k our r-jet will be uniformly far from $Z_k \backslash \Theta_{Z_k}$. All the remaining strata approach Z_k accumulating only on points of $Z_k \backslash \Theta_{Z_k}$. Therefore, the r-jet will only hit them outside of a uniform tubular neighborhood of Z_k, where the approximate holomorphicity holds.

Definition 6.5. (1) An A.H. sequence of sections of $E_k \to (M, D, J, g_k)$ is said to be r-generic if its pseudo-holomorphic r-jet is uniformly transverse along D to the Thom–Boardman–Auroux quasi-stratification of $\mathcal{J}_D^r E_k$.

(2) An A.H. sequence of sections of $E_k \to (P, J, G, g_k)$ is said to be r-G-generic over M if its pseudo-holomorphic r-jet is uniformly transverse over M to $\Sigma_k^G \subset \mathcal{J}^r E_k$.

(3) Let $\phi_k : M \backslash B_k \to \mathbb{CP}^m$ be sequence of functions which is A.H. outside of a uniform tubular neighborhood of g_k-radius $\eta > 0$ of B_k. It is said to be r-generic if for k large enough B_k is a codimension $2(m+1)$ calibrated submanifold and $j_D^r \phi_k : M \backslash B_k \to \mathcal{J}_D^r(M \backslash B_k, \mathbb{CP}^m)$ is uniformly transverse along D to the Thom–Boardman–Auroux stratification. Moreover, it is required to intersect the strata of strictly positive codimension out of a tubular neighborhood of B_k of g_k-radius η.

Lemma 6.5. *Let τ_k be an A.H. sequence of sections of $E_k \to (M, D, J, g_k)$. Then if τ_k is r-generic its projectivization $\phi_k : M \backslash \tau_k^{-1}(Z_k) \to \mathbb{CP}^m$ is also r-generic.*

Proof. It is elementary from the construction of the Thom–Boardman–Auroux (quasi)-stratifications of $\mathcal{J}_D^r E_k$ and $\mathcal{J}_D^r(M, \mathbb{CP}^m)$, Proposition 6.2 relating $j_D^r \tau_k$, and $j_D^r \phi_k$ and Lemma 6.4.

Uniform transversality of τ_k to Z_k implies by Remark 6.2 that ϕ_k intersects the remaining strata uniformly away from the zero set. Estimated transversality along D is also preserved when composed with $j^r\pi$ uniformly away from Z; the key point is selecting appropriate local A.H. defining functions for the strata: in A.H. coordinates and affine charts $\mathbb{P}\Sigma_{k,I}$ corresponds to a CR stratum $\mathbb{P}\Sigma_I$. Let f be a local CR function defining it. Then $f \circ \Pi_{k,x,i} \circ j^r(\varphi_i^{-1} \circ \pi)$ are local defining functions for $\Sigma_{k,I}$. Now Lemma 4.5 implies that local uniform estimated transversality along D of $j^r_D\tau_k$ to $\Sigma_{k,I}$ is equivalent to uniform transversality along D to $\mathbf{0}$ of $f \circ j^r(\varphi_i^{-1} \circ \pi) \circ j^r_D\tau_k = f \circ j^r_D(\varphi_i^{-1} \circ \phi_k)$. Again by the same lemma this is equivalent to uniform transversality along D of $j^r_D\phi_k$ to $\mathbb{P}\Sigma_{k,I}$. The case of the points close to the boundary of the strata is just a problem in a vector space; it follows from $j^r(\varphi_i^{-1} \circ \pi) : \mathscr{J}^r_{D_h,n,m+1}\backslash Z \to \mathscr{J}^r_{D_h,n,m}$ being a submersion which amounts to suppressing coordinates of the fiber of $\mathscr{J}^r_{D_h,n,m+1}$ (and because the metrics in these coordinates are comparable with the ambient metric, so the projection is ε-transverse). □

Let (P, Ω) be a symplectic manifold with $(M, D, \omega := \Omega_{|M})$ 2-calibrated and G a local J-complex distribution extending D. Let τ_k be an A.H. sequence of sections of E_k and denote by ϕ_k its projectivization away from its zero set.

Proposition 6.3. *Using the above notation, if $j^r\tau_k : P \to \mathscr{J}^r E_k$ is uniformly transverse over M to $\Sigma_k^G \subset \mathscr{J}^r_G E_k$ then $\phi_{k|M}$ is r-generic.*

Proof. We will make extensive use of diagram (51):

$$\begin{array}{ccc} \mathscr{J}^r E_k^* & \xrightarrow{p_G} & \mathscr{J}^r_G E_k^* \\ \big\downarrow j^r\pi & & \big\downarrow j^r\pi \\ \mathscr{J}^r(P, \mathbb{CP}^m) & \xrightarrow{p_G} & \mathscr{J}^r_G(P, \mathbb{CP}^m). \end{array}$$

Step 1: Study the compatibility of the Thom–Boardman–Auroux stratifications with the identification of $\mathscr{J}^r_D(M, \mathbb{CP}^m)$ with $\mathscr{J}^r_G(P, \mathbb{CP}^m)_{|M}$.

At the points of M there is a canonical J-complex identification between D and G, inducing isometries

$$\Lambda_{k,i} : \mathscr{J}^r_D(M, \mathbb{CP}^m) \to \mathscr{J}^r_G(P, \mathbb{CP}^m)_{|M}.$$

Let $z_k^1, \ldots, z_k^p$ be A.H. coordinates adapted to (M, G). We can rewrite them as $z_k^1, \ldots, z_k^n$, x_k^{2n+1}, x_k^{2n+2}, $z_k^{n+2}, \ldots, z_k^p$, where $z_k^1, \ldots, z_k^n$, x_k^{2n+1} are by Lemma 3.6 A.H. coordinates for M. Using also the canonical affine charts of projective space we have

$$\Pi^D_{k,x,i} : \mathscr{J}^r_D(M,\mathbb{C}^m)_i \to \mathscr{J}^r_{D_h,n,m} = \mathscr{J}^r_{n,m} \times \mathbb{R},$$
$$\Pi^G_{k,x,i} : \mathscr{J}^r_G(P,\mathbb{C}^m)_i \to \mathscr{J}^r_{\mathbb{C}^n,p,m} = \mathscr{J}^r_{n,m} \times \mathbb{C}^{p-n},$$

and a canonical identification in $\mathbb{C}^n \times \mathbb{R} \subset \mathbb{C}^p$

$$\Lambda : \mathscr{J}^r_{D_h,n,m} \to \mathscr{J}^r_{\mathbb{C}^n,p,m|\mathbb{C}^n\times\mathbb{R}}.$$

The construction of $\Pi^D_{k,x,i}$, $\Pi^G_{k,x,i}$ (see equation (29) and the last paragraph in the proof of Lemma 6.2) implies the commutativity of

$$\begin{array}{ccc} \mathscr{J}^r_D(M,\mathbb{CP}^m) & \xrightarrow{\Lambda_k} & \mathscr{J}^r_G(P,\mathbb{CP}^m)_{|M} \\ \Big\downarrow \Pi^D_{k,x,i} & & \Big\downarrow \Pi^G_{k,x,i} \\ \mathscr{J}^r_{D_h,n,m} & \xrightarrow{\Lambda} & \mathscr{J}^r_{\mathbb{C}^n,p,m|\mathbb{C}^n\times\mathbb{R}}. \end{array} \tag{61}$$

The restriction of $\mathscr{J}^r_{\mathbb{C}^n,p,m}$ to $\mathbb{C}^n \times \mathbb{R} \approx M$ coincides with $\mathscr{J}^r_{n,m} \times \mathbb{R} = \mathscr{J}^r_{D_h,n,m}$.

The identification Λ obviously preserves the Thom–Boardman–Auroux stratifications (and even the refinements), and hence so Λ_k does.

Step 2: Check that $\Lambda_k^{-1} \circ (j^r_G\phi_k)_{|M} \approxeq j^r_D(\phi_{k|M})$.

Since Λ_k are J-complex isometries preserving the Thom–Boardman–Auroux stratifications we omit them from now on.

By using the charts $\Pi^D_{k,x,i}$, $\Pi^G_{k,x,i}$ it is easy to see that for any $j \in \{1,\dots,r\}$, the degree j homogeneous component of $j^r_D(\phi_{k|M})$ approximately coincides with $\nabla^j_D(\phi_{k|M})$. Similarly, the degree j homogeneous component of $j^r_G\phi_k$ approximately coincides with $\nabla^j_G\phi_k$. The result follows because we also have

$$(\nabla^j_G\phi_k)_{|M} \approxeq \nabla^j_D(\phi_{k|M}).$$

Step 3: Analyze the behavior of $j^r_D(\phi_{k|M})$ near the set of base points B_k.

Since $Z_k \subset \mathscr{J}^rE_k$ is an A.H. sequence of submanifolds and $j^r\tau_k$ an A.H. sequence of sections, by Corollary 4.2 uniform transversality over M is equivalent to uniform transversality along G at the points of M. In A.H. coordinates adapted to G, we are saying that the matrix of partial derivatives of τ_k with respect to $z^1_k,\dots,z^g_k$ has maximum rank and norm greater than some $\eta > 0$. But this is equivalent to saying that is uniformly transverse to Z^G_k, the pullback of the zero section of $\mathscr{J}^r_GE_k$.

By construction $\Sigma^G_k\backslash Z_k = p_G^{-1}j^r\pi^{-1}(\mathbb{P}\Sigma_k) = p_G^{-1}(\Sigma_k\backslash Z_k)$, and the strata of $\Sigma^G_k\backslash Z_k$ when approaching the zero section accumulate into $p_G^{-1}(\Theta_{Z_k})$, where here $\Theta_{Z_k} \subset \mathscr{J}^r_GE_k$. Therefore $j^r\tau_k$ intersects the strata of $\Sigma^G_k\backslash Z_k$ away from a tubular neighborhood in P (and hence in M) of radius η' of B_k, the zero set of $j^r\tau_k$. Thus $(j^r\phi_k)_{|M} = \big(j^r\pi(j^r\tau_k)\big)_{|M}$ intersects the strata of $\mathbb{P}\Sigma^G_k$ away from a tubular neighborhood in M of radius η' of B_k.

In general $p_G(j^r\phi_k) \neq j^r_G\phi_k$ but using A.H. coordinates it is easy to check that $p_G(j^r\phi_k) \approxeq j^r_G\phi_k$. Hence, $j^r_G\phi_k$ intersects the strata of $\mathbb{P}\Sigma_k \subset \mathscr{J}^r_G(P, \mathbb{CP}^m)$ away from a tubular neighborhood in M of radius η' of B_k for all $k \gg 1$.

By Steps 1 and 2 we deduce that $j^r_D(\phi_{k|M})$ intersects the strata of $\mathbb{P}\Sigma_k \subset \mathscr{J}^r_D(P, \mathbb{CP}^m)$ away from a tubular neighborhood in M of radius η' of B_k for all $k \gg 1$.

Step 4: Relate uniform transversality over M of $j^r\tau_k$ to $\Sigma^G_k \backslash Z_k$ with uniform transversality along D of $j^r_D(\phi_{k|M})$ to $\mathbb{P}\Sigma_k \subset \mathscr{J}^r_D(M, \mathbb{CP}^m)$.

The same ideas used in the proof of Lemma 6.5 combined with $p_G(j^r\phi_k) \approxeq j^r_G\phi_k$, show that uniform transversality over M of $j^r\tau_k$ to $\Sigma^G_k \backslash Z_k$ is equivalent to uniform transversality over M of $j^r_G\phi_k$ to $\mathbb{P}\Sigma_k \subset \mathscr{J}^r_G(P, \mathbb{CP}^m)$.

Uniform transversality over M of $j^r_G\phi_k$ to $\mathbb{P}\Sigma_k \subset \mathscr{J}^r_G(P, \mathbb{CP}^m)$ is comparable to uniform transversality of $(j^r_G\phi_k)_{|M}$ to $\mathbb{P}\Sigma_{k|M} \subset \mathscr{J}^r_G(P, \mathbb{CP}^m)_{|M}$ (it can be easily proven in the charts $\Pi^D_{k,x,i}$, $\Pi^G_{k,x,i}$).

By Steps 1 and 2, $j^r_D(\phi_{k|M})$ is uniformly transverse to $\mathbb{P}\Sigma_k \subset \mathscr{J}^r_D(M, \mathbb{CP}^m)$.

If the hypothesis on the amount of transversality over M of Corollary 4.2 are met, then $j^r_D(\phi_{k|M})$ is uniformly transverse along D to $\mathbb{P}\Sigma_k \subset \mathscr{J}^r_D(M, \mathbb{CP}^m)$. Observe that this requirement is not a problem, since the induction construction to obtain uniform transversality over M for $j^r\tau_k$ to $\Sigma^G_k \backslash Z_k$ can guarantee that. □

The vector bundles $\mathscr{J}^r_G E_k$ are endowed with hermitian metrics $\hat{g}_k$ and connections $\nabla_{k,H}$ (or just ∇_H), which are induced by the metrics and connections on $\mathscr{J}^r E_k$ via the projection p_G. We do not know whether $\mathscr{J}^r_G E_k$ is an almost CR submanifold of $\mathscr{J}^r E_k$, but in any case we are not interested in doing almost complex geometry on $\mathscr{J}^r_G E_k$.

Let σ_k be a sequence of sections of $\mathscr{J}^r_G E_k$ with $|\nabla^j \sigma_k|_{g_k} \leq O(1)$ for all $j \geq 0$. Using the metric $\hat{g}_k$ we have a well defined notion of uniform transversality of σ_k to the Thom–Boardman–Auroux stratification $\Sigma_k \subset \mathscr{J}^r_G E_k$ (Definition 4.5); notice that we have no notion of approximate holomorphicity neither for the sequence of sections nor for the strata.

Remark 6.3. If $\tau_k : P \to E_k$ is A.H. then $|\nabla^j j^r_G \tau_k|_{g_k} \leq O(1)$ for all $j \geq 0$. Having into account remark 4.3, it can also be shown that if $j^r\tau_k : P \to \mathscr{J}^r E_k$ is uniformly transverse over M to Σ^G_k, then $j^r_G\tau_k : P \to \mathscr{J}^r_G E_k$ is uniformly transverse over M to Σ_k.

We finish this section by proving the following

Lemma 6.6. (1) *Let $\mathscr{S} = (S^a_k)_{a \in A_k}$ be an approximately holomorphic finite invariant stratification of E_k such that in approximately holomorphic coordinates and A.H. frames each sequence of strata has a CR model transverse to the fibers.*

Let $\tau_k : M \to E_k$ be an A.H. sequence uniformly transverse along D to $\mathscr{S}$. Then $\tau_k^{-1}(\mathscr{S})$ is a stratification of (M, D, ω) by 2-calibrated submanifolds for all $k \gg 1$.

(2) *Let $\tau_k : M \to E_k$ be an A.H. uniformly transverse to Z_k and whose projectivization ϕ_k is r-generic. Then $B_k \cup \phi_k^{-1}(\mathbb{P}\Sigma_k)$ is a stratification by 2-calibrated submanifolds of (M, D, ω) for all $k \gg 1$.*

Proof. Let $S_k^a \subset E_k$. Corollary 4.1 implies that $\tau_k^{-1}(S_k^a)$ is uniformly transverse to D. Hence, if we check that for each $x \in \tau_k^{-1}(S_k^a)$ the sequence of linear subspaces $T_D\tau_k^{-1}(S_k^a) \subset D$ is A.H., i.e.,

$$\angle_M\big(T_D\tau_k^{-1}(S_k^a), JT_D\tau_k^{-1}(S_k^a)\big) \leq O(k^{-1/2})$$

(uniformly on the point), we are done.

Let $\hat{J}$ denote the induced the almost complex structure on E_k. In approximately holomorphic coordinates and A.H. frames, the strata $S_k \subset E_k$ have a CR model $S \subset \underline{\mathbb{C}}^m$ with respect to the canonical product CR structure. Recall that any almost CR structure defined out of J_0 in the base and the fiber, and a connection form with vanishing $(0,1)$-component, coincides with the product CR structure (this appears also in the proof of Lemma 6.1). Hence the linear subspaces $T_D S = T_D S_k$ verify $\angle_M(T_D S, \hat{J} T_D S) \leq O(k^{-1/2})$, the bounds being uniform on the points of $\underline{\mathbb{C}}^m$, and hence uniform on the points of E_k.

The approximate holomorphicity of τ_k implies $\angle_M(T_D\tau_k, \hat{J}T_D\tau_k) \leq O(k^{-1/2})$. Since $\angle_{\mathrm{m}}(T_D\tau_k, T_D S_k) \geq \eta$, by Proposition 3.7 in [32] for all $k \gg 1$ the intersection $T_D\tau_k \cap T_D S_k$ is an A.H. sequence and thus also its projection to M, which proves item (1).

Regarding item (2), $B_k := {\tau_k}^{-1}(Z_k)$. Therefore item (1) applies.

The strata $\Sigma_{k,I}$ are intersected uniformly away from B_k. Therefore it is equivalent to work with the projectivizations ϕ_k and the Thom–Boardman–Auroux stratification of $\mathscr{J}_D^r(M, \mathbb{CP}^m)$, because $j_D^r\tau_k^{-1}(\Sigma_{k,I}) = j_D^r\phi_k^{-1}(\mathbb{P}\Sigma_{k,I})$. Since for each canonical chart of projective space the strata have CR models in $\mathscr{J}_D^r(M, \mathbb{C}^m)_i$, everything reduces to item (1). □

We would like the pullback of any regular value of ϕ_k to be a 2-calibrated submanifold, which forces us to study the behavior of an r-generic function near its base locus and near the pullback of the Thom–Boardman–Auroux strata. In our applications we would only need this analysis for the Lefschetz pencils $\phi_k : M \backslash B_k \to \mathbb{CP}^1$: the same ideas used in [35] show that indeed near the base locus $|\partial\phi_k| > |\bar{\partial}\phi_k|$ and thus the regular "fibers" are 2-calibrated submanifolds. On the other hand, near the strata of the Thom–Boardman–Auroux stratification there is no such inequality between the holomorphic and antiholomorphic component of the derivative, and ad hoc modifications are needed to obtain 2-calibrated regular fibers.

In [25] the approximately holomorphic theory is appropriately modified to construct generic CR sections for a Levi-flat CR manifold. The complication near the base locus and degeneration loci of the leafwise differential does not occur (over each complex leaf the CR-Thom–Boardmann stratification is holomorphic and the restriction of the CR-r-jet holomorphic as well, therefore the former is pulled back to the leaf to a stratification by holomorphic strata).

7. The main theorem

It is possible to perturb A.H. sections of $E_k = E \otimes L^{\otimes k} \to (M, D, \omega)$ so that their r-jets are transverse to an A.H. quasi-stratification of $\mathscr{J}_D^r E_k$.

Theorem 7.1. *Let $E_k \to (M, D, \omega)$, $E_k = E \otimes L^{\otimes k}$, and $\mathscr{S} = (S_k^a)_{a \in A_k}$ an A.H. sequence of finite, Whitney (A) quasi-stratifications of $\mathscr{J}_D^r E_k$ transverse to the fibers. Let us fix $h \in \mathbb{N}$. Let δ be a strictly positive constant. Then a constant $\eta > 0$ exists such that for any A.H. sequence τ_k of E_k, it is possible to find an A.H. sequence σ_k of E_k so that for every k bigger than some k_0,*

(1) $|\nabla_D^j(\tau_k - \sigma_k)|_{g_k} < \delta$, $j = 0, \dots, r + h$,

(2) $j_D^r \sigma_k$ *is η-transverse along D to $\mathscr{S}$.*

Theorem 7.2 below suffices for our applications; the proof of Theorem 7.1, which is left to the interested reader, is a suitable modification of the proof of Theorem 1.1 in [4]. The main difference is the use of a result on local estimated transversality along D_h to $\mathbf{0}$ for A.H. functions $f_k : \mathbb{C}^n \times \mathbb{R} \to \mathbb{C}^m$.

Observe in Theorem 7.1 that while for any $h \in \mathbb{N}$ we can bound $|\nabla_D^j(\tau_k - \sigma_k)|_{g_k}$, $j = 0, \dots, r + h$, by any arbitrarily small δ, we cannot do the same for the full derivative. For the latter it can be proven that $|\nabla^j(\tau_k - \sigma_k)|_{g_k} \le C_j$ for all $j \in \mathbb{N}$, where C_j are constants independent of k whose value we cannot control. Moreover the non-integrability of D also forces us to work with sequences of A.H. functions all whose derivatives are controlled (even if we want to control the size of the perturbation along D up to a finite order h); basically the derivatives along the directions of D (up to some finite order h) will be arbitrarily small only if we have control for the full derivative of all the orders, and k is chosen to be very large.

We can prove a strong transversality result for symplectic manifolds with distribution G along compact 2-calibrated subvarieties.

Theorem 7.2. *Let $E_k \to (P, \Omega)$ and let (M, D) be a compact 2-calibrated submanifold of the symplectic manifold (P, Ω) and G a J-complex distribution extending D. Let us consider $\mathscr{S}^G$ a C^h-A.H. sequence of finite, Whitney (A) quasi-stratifications* *of $\mathscr{J}^r E_k$ $(h \geq 2)$. Let δ be a positive constant. Then a constant $\eta > 0$ and a natural number k_0 exist such that for any C^{r+h}-A.H.(C) sequence τ_k of E_k it is possible to find a C^{r+h}-A.H. sequence σ_k of E_k so that for any k bigger than k_0,*

(1) $|\nabla^j(\tau_k - \sigma_k)|_{g_k} < \delta$, $j = 0, \dots, r+h$ *($\tau_k - \sigma_k$ is C^{r+h}-A.H.(δ)),*

(2) *$j^r \sigma_k$ is η-transverse over M to $\mathscr{S}^G$.*

Proof. We will closely follow the pattern of the proof of Theorem 1.1 in [4] but introduce appropriate modifications.

The very basic strategy of the proof is to add a perturbation for each sequence of strata $S^{G^b}_k$, so that a sequence of strata is dealt with only if all the preceding ones have been already dealt with. The solution σ_k will be the result of adding all the perturbations. To achieve our goal in this way we must make sure that at a stage corresponding to the strata $S^{G^b}_k$, the perturbation added is such that:

(i) uniform transversality to preceding strata is not destroyed,

(ii) uniform transversality to $S^{G^b}_k$ is attained.

To make sure that item (i) above holds, we start by adapting the definition of local open condition of [3] to our setting:

Definition 7.1. Let $\eta, \bar{\eta} > 0$. A family of properties $\mathscr{P}(\eta, \bar{\eta}, x)_{x \in M}$ of sections of bundles over P is local and C^q-open if given a section τ that satisfies $\mathscr{P}(\eta, \bar{\eta}, x)$ and a section σ so that $|\tau - \sigma|_{C^q(P,g)} \leq \varepsilon$, there exists $L > 0$ only depending on the C^q-norm of τ so that $\tau - \sigma$ satisfies $\mathscr{P}(\eta - L\varepsilon, \bar{\eta} - L\varepsilon, x)$.

The advantage of a local open property is that we have an estimate on how much it varies according to the size of the perturbation.

In our specific problem we say that a C^{r+2}-A.H. sequence of sections τ_k of E_k satisfies $\mathscr{P}_k(\eta, \bar{\eta}, x)$, $x \in M$, if $j^r \tau_k$ is $(\eta, \bar{\eta})$-transverse over M to $S^{G^b}_k$ at x. We want to show that this is a local C^{r+2}-open condition, because if that is the case we know that if at a given stage we add a perturbation with small enough C^{r+2}-norm we will still have a sequence of sections uniformly transverse over M to $S^{G^b}_k$.

This is proven in Theorem 1.1 [4] for full transversality. For estimated transversality over M the theorem is equally true because a perturbation χ_k with C^{r+2}-size bounded by C gives rise to an r-jet such that (i) $|j^r \chi_k|_{g_k} \leq L'C$, (ii) $|\nabla_{TM} j^r \chi_k|_{g_k} \leq L'C$, and (iii) $|\nabla \nabla_{TM} j^r \chi_k|_{g_k} \leq L'C$ for some $L' > 0$. Therefore small perturbations of a given section give rise to an r-jet that remains within controlled distance of the one for the initial section and whose derivative along TM varies in a controlled way. Similarly for a given r-jet we can control in a ball of uniform radius its variation up to order 2, and hence the variation of its derivative along TM in the ball.

Next we have to make sure that the perturbation added at each stage fulfills condition (ii). We will split the problem of achieving transversality over M to $S^{G^b_k}$ into doing it for points close to the boundary and far from the boundary. Actually, the former problem turns out to be already solved. To show this we must check that $(\eta_a, \bar{\eta}_a)$-transversality over M of $j^r\tau_k$ to $S^{G^a_k}$, for all $a < b$, implies the existence of $\bar{\eta}_b > 0$ such that $j^r\tau_k$ is $\bar{\eta}_b$-transverse over M to $S^{G^b_k}$ at the points $\bar{\eta}_b$-close to its boundary.

In Theorem 1.1 [4] it is shown that the quasi-stratification condition together with full uniform transversality can be used to show that $j^r\tau_k$ stays uniformly away from $S^{G^a_k}\backslash\Theta_{S^{G^a_k}}$, say at distance greater than some $\eta' > 0$; since uniform transversality over M is stronger than uniform transversality we deduce the same result.

We now make use of the estimated Whitney condition (A) as in Corollary 4.2. We have the inequality

$$\angle_{\mathrm{m}}(T_M j^r\tau_k, T^{\|}_M S^{G^a_k}) \le \angle_{\mathrm{M}}(T^{\|}_M S^{G^a_k}, T_M S^{G^b_k}) + \angle_{\mathrm{m}}(T_M j^r\tau_k, T_M S^{G^b_k}). \quad (62)$$

For $\eta'' > 0$ small enough the induction hypothesis implies that for points η''-close to $\bar{\partial} S^{G^b_k}$ there is some index $a \in A_k$ such that

$$\angle_{\mathrm{m}}(T_M j^r\tau_k, T^{\|}_M S^{G^a_k}) \ge \eta_a.$$

Let $\hat{M}$ denote the pullback of TM to $\mathscr{J}^r E_k$. In order to make

$$\angle_{\mathrm{M}}(T^{\|}_M S^{G^a_k}, T_M S^{G^b_k}) < \eta_a/2$$

we use the estimated Whitney condition (A) that gives $\angle_{\mathrm{m}}(\hat{M}, TS^{G^b_k}) > \gamma > 0$ and $\angle_{\mathrm{M}}(T^{\|} S^{G^a_k}, TS^{G^b_k}) < C(\gamma)^{-1}\eta_a/2$ (see the proof of Corollary 4.2), for η'' small enough. Then the desired result holds for

$$\bar{\eta}_b := \min\big(\eta', \eta'', \min_{a<b}(\eta_a/2)\big).$$

Therefore our task is reduced to constructing arbitrarily small perturbations which solve the uniform transversality problem in points $\bar{\eta}_b$-far from the boundary. We will construct such a perturbation following Donaldson's globalization method. The key point is the following.

Proposition 7.1. *Let $\mathscr{P}_k(\eta, \bar{\eta}, x)_{x \in M, \eta, \bar{\eta} > 0}$ be a family of C^q-open properties of sections of $E_k \to (P, g_k)$. Assume that there exist (uniform) constants ρ, c', c'', p such that given any $\delta > 0$ small enough, any $x \in M$, and any sequence τ_k with uniform C^q-bound C, there exist C^q-bounded sections $\chi_{k,x}$ for all $k \gg 1$ with the following properties:*

(1) $|\nabla^j \chi_{k,x}|_{g_k} < c''\delta$, $j = 0, \dots, q$.

(2) *The sections* $\frac{1}{\delta}\chi_{k,x}$ *have Gaussian decay away from* x *in* C^q*-norm.*

(3) $\tau_k + \chi_{k,x}$ *satisfy the property* $\mathscr{P}_k(\eta, \bar\eta - c'\delta, y)$ *for all* $y \in B_{g_k}(x, \rho) \cap M$, *with* $\eta = c'\delta\big(\log(\delta^{-1})\big)^{-p}$.

Then given any $\alpha > 0$ *and* C^q*-bounded sections* τ_k *of* E_k, *there exist, for* $k \gg 1$, C^q*-bounded sections* σ_k *of* E_k *such that*

(i) $|\nabla^j(\tau_k - \sigma_k)|_{g_k} < \alpha$, $j = 0, \dots, q$,

(ii) *the sections* σ_k *satisfy* $\mathscr{P}_k(\varepsilon, \bar\eta - L\delta, x)$ *for some uniform* $\varepsilon, L > 0$ *at any* $x \in M$.

We do not give the proof of this proposition, since it is a repetition step by step of Donaldson's globalization procedure [11].

Hence we must check that the hypothesis of Proposition 7.1 hold. We will use the following local transversality result, which is a reformulation of Lemma 5.2 and Theorem 5.4 in [30].

Proposition 7.2. *Let* F *be a function with values in* $\mathbb{C}^l$ *defined over the ball of radius* $11/10$ *in* $\mathbb{C}^l$. *Let* V *be a vector subspace of* $\mathbb{C}^l$. *Let* δ *be a constant* $0 < \delta < 1/2$. *Let* $\eta = \delta\big(P\big(\log(\delta^{-1})\big)\big)^{-1}$, *where* P *is a real monomial depending on* n, l, V. *If in the ball of radius* $11/10$ *we have*

$$|F|_{g_0} \le 1, \qquad |\bar\partial F|_{g_0} \le \eta, \qquad |\mathrm{d}\bar\partial F|_{g_0} \le \eta,$$

then there exists $u \in \mathbb{C}^p$ *such that* $F - u$ *is* η*-transverse over* V *to* $\mathbf{0}$ *in the interior of* $B(0,1) \cap V$.

We assume that τ_k is already $\bar\eta_b$-transverse over M at the points $\bar\eta_b$-close to the boundary. Let $0 < \varepsilon < \bar\eta_b/4$ small enough. If $x \in M$ such that $j^r\tau_k(x) \notin \mathscr{N}_{S^{G^b_k}}(\varepsilon/2, \bar\eta_b)$ then $\chi_{k,x}$ is chosen to be the zero perturbation. If $j^r\tau_k(x) = p \in \mathscr{N}_{S^{G^b_k}}(\varepsilon/2, \bar\eta_b)$ then there exists ρ_1 such that $j^r\tau_k\big(B_{g_k}(x, \rho_1)\big) \subset B_{\hat g_k}(p, \rho_\varepsilon) \subset \mathscr{N}_{S^{G^b_k}}(\varepsilon, 3\bar\eta_b/4)$. We consider the composition $f \circ j^r\tau_k$ pulled back to the domain of an A.H. chart adapted to (M, G) and centred at x. In this way we obtain a function $H_k : B(0, \rho_2) \subset \mathbb{C}^p \to \mathbb{C}^l$. If we apply Proposition 7.2 directly to H_k, with $V = TM$, and for $\delta \ll \bar\eta_b/6$, we will obtain $\delta\big(P\big(\log(\delta^{-1})\big)\big)^{-1}$-transversality over M to $\mathbf{0}$ for $H_k - u_k$ in $B_{g_k}(x, \rho_3)$. The problem is how to associate u_k to a perturbation of τ_k (the difficulty coming from the non-linearity of the strata). Instead, we consider for each index I the $\mathbb{C}^l$-valued function such that for each $y \in B_{g_k}(x, \rho_4)$,

$$\Theta_I(y) = \big(df_1\big(j^r\tau_k(y)\big)j^r\tau^{\mathrm{ref}}_{k,x,I}, \dots, df_l\big(j^r\tau_k(y)\big)j^r\tau^{\mathrm{ref}}_{k,x,I}\big),$$

with $\tau_{k,x,I}^{\mathrm{ref}}$ as defined in equation (33). There is a choice of l indices $I_1, \ldots, I_l$ such that the corresponding A.H. sections $j^r \tau_{k,x,I_j}^{\mathrm{ref}}$ are a frame for a distribution complementary to $\operatorname{Ker} df$ (and with minimal angle bounded from below). Then $\Theta_{I_1}, \ldots, \Theta_{I_l}$ is a frame (depending on y) of $\mathbb{C}^l$ comparable to the canonical one. We can write

$$H_k = h_k^1 \Theta_{I_1} + \cdots + h_k^l \Theta_{I_l}.$$

We apply Proposition 7.2 (after suitable rescalings) to the $\mathbb{C}^l$-valued function $h_k = (h_k^1, \ldots, h_k^l)$, with $V = TM$, for some δ small enough, so we get $u_k \in \mathbb{C}^l$ such that $h_k - u_k$ is $c_1 \delta \big(P\big(\log(\delta^{-1})\big)\big)^{-1}$-transverse over M to $\mathbf{0}$ in $B_{g_k}(x, \rho_5)$. If we multiply by the functions $\Theta_{I_1}, \ldots, \Theta_{I_l}$ we obtain $c_2 \delta \big(P\big(\log(\delta^{-1})\big)\big)^{-1}$-transversality over M to $\mathbf{0}$ for $H_k - u_k^1 \Theta_{I_1} - \cdots - u_k^l \Theta_{I_l}$. Our perturbation is the section

$$\chi_{k,x} := -u_k^1 \tau_{k,x,I_1}^{\mathrm{ref}} - \cdots - u_k^l \tau_{k,x,I_l}^{\mathrm{ref}}.$$

The key point is that in view of the norm of u_k and the bounds on the second derivatives of f, the C^1-norm of

$$H_k - u_k^1 \Theta_{I_1} - \cdots - u_k^l \Theta_{I_l} - f \circ j^r(\tau_k + s_{k,x})$$

is bounded by $O(\delta^2)$. Since the C^1-norm majorates the C^1-norm along TM we conclude that for δ small enough $f \circ j^r(\tau_k + s_{k,x})$ is $c_3 \delta \big(P\big(\log(\delta^{-1})\big)\big)^{-1}$-transverse over M to $\mathbf{0}$. By Lemma 4.5 we get $\mathscr{P}_k\big(c_4 \delta \big(P\big(\log(\delta^{-1})\big)\big)^{-1}, \bar{\eta}_b - L\delta, y\big)$ for all $y \in B_{g_k}(x, \rho_5)$. Since $\mathscr{P}_k(\eta, \bar{\eta}, x)$ is C^{r+2}-open, if δ is small enough compared to $\bar{\eta}_b$ and η_a, $\bar{\eta}_a$, we still get uniform transversality to the previous strata and $5\bar{\eta}_b/6$-transversality over M at the points $3\bar{\eta}_b/4$-close to the boundary of $S_k^{G^b}$.

So we can apply Proposition 7.1 to obtain $\mathscr{P}_k(\eta_b, 3\bar{\eta}_b/4, x)$ (with respect to $S_k^{G^b}$) in all the points of M.

Hence we deduce the existence of a C^{r+2}-A.H. sequence σ_k such that:

(1) $|\nabla^j(\tau_k - \sigma_k)|_{g_k} < \delta$, $j = 0, \ldots, r+h$ (σ_k is C^{r+2}-A.H.(δ)).

(2) $j^r \sigma_k$ is η-transverse over M to $\mathscr{S}^G$. □

8. Applications

We begin by proving Proposition 1.1, which can be also obtained as a simple corollary of the work of J.-P. Mohsen [30] together with some extra local work borrowed from [27].

Proof of Proposition 1.1. We consider a more general situation than that of the statement of Proposition 1.1. Let E be any rank m Hermitian vector bundle over (M^{2n+1}, D, ω), and let $E_k = E \otimes L_\Omega^{\otimes k}$ (L_Ω is the pre-quantum line bundle of the symplectization and E is meant to be the pullback of the initial E to the symplectization). We want to apply Theorem 7.2 to the sequence of zero sections Z_k, but with some changes. Basically we want to start with an A.H. sequence which vanishes at y and is uniformly transverse on $B_{g_k}(\rho, y)$, and then add perturbations not destroying these properties. We fix A.H. coordinates adapted to (M, G) and reference sections $\tau^{\mathrm{ref}}_{k,x,j}$ centred at the points of $M \subset M \times [-\varepsilon, \varepsilon]$. In A.H. coordinates adapted to (M, G) we take the sections $z_k^j \tau^{\mathrm{ref}}_{k,y,j}$, $j = 1, \dots, m \leq n+1$, and consider their direct sum, a section of E_k. This sequence of sections $\tau_{k,y}$ vanishes at y and is η-transverse over M to Z_k in $B_{g_k}(y, \rho)$. The key point is to keep on adding local perturbations, as described in the proof of Theorem 7.2, which vanish at y and with C^1-norm small enough compared to η. For that we need new reference sections vanishing at y. Notice that if $d_k(x, y) \geq O(k^{1/6})$ then $\tau^{\mathrm{ref}}_{k,x,j}$ is already vanishing at y, so we do not need to change the reference section. Assuming $d_k(x, y) \leq O(k^{1/6})$ once we go to A.H. coordinates adapted to (M, G) and centred at x, the point y belongs to $B(0, \rho' k^{1/6}) \subset \mathbb{C}^{n+1}$. Consider the polynomial $P(z_k^1, \dots, z_k^{n+1}) = 1 - z_k^1$. Let $L_{k,y,x} \in \mathrm{GL}(n+1, \mathbb{C})$ be the composition of homothety and of a rotation sending y to $(1, 0, \dots, 0)$. We define $P_{k,y,x} = P \circ L_{k,y,x}$ and $\xi^{\mathrm{ref}}_{k,x,j} := P_{k,y,x} \tau^{\mathrm{ref}}_{k,x,j}$. For any $\gamma > 0$, if we suppose $d_k(x, y) \geq \gamma$ then $\xi^{\mathrm{ref}}_{k,x,j}$ becomes an A.H. sequence (with bounds independent of x) that vanishes at y and so that $\xi^{\mathrm{ref}}_{k,x,j}$, $j = 1, \dots, m$, fits into a local frame of E_k over $B_{g_k}\big(x, \rho(\gamma)\big)$ (we chose the linear map to arrange that the vanishing (affine) hyperplane of $P_{k,y,x}$ is at distance of the origin bounded from below). Since $\tau_{k,y}$ is η-transverse over M to Z_k in $B_{g_k}(y, \rho)$, we only need to add perturbations centred at points away from $B_{g_k}(y, \rho/2)$, and thus the globalization procedure can be applied with reference sections vanishing at y.

Thus it is possible to find sequences of A.H. sections τ_k of E_k uniformly transverse over M to Z_k and vanishing at y. Hence $\tau_{k|M}$ are uniformly transverse to Z_k and vanishing at y. Let $W_k = \tau_{k|M}^{-1}(Z_k)$. For all $k \gg 1$, by Corollary 4.1, W_k is uniformly transverse to D, and by Lemma 6.6 approximately almost complex and therefore 2-calibrated.

The study of its topology is done very much as in the symplectic and contact cases (see the proofs in [11], [2], [24]). □

The next result we want to prove is the existence of determinantal submanifolds (Proposition 1.2), which is still a transversality result for 0-jets (vector bundles E_k), but not anymore to the **0** section but to a sequence of non-linear approximately holomorphic stratifications.

Proof of Proposition 1.2. Let $E, F \to M$ be Hermitian bundles with connection and let us define the sequence of very ample vector bundles $I_k := E^* \otimes F \otimes L^{\otimes k}$.

In the total space of I_k we consider the sequence of stratifications S_k whose strata are $S_{k,i} = \{A \in I_k \mid \operatorname{rank}(A) = i\}$, where $A \in \operatorname{Hom}(E, F \otimes L^{\otimes k})$.

Let E, F still denote the pullback of E, F to the symplectization $(M \times [-\varepsilon, \varepsilon], \Omega)$. Let $I_{k,\Omega} \to M \times [-\varepsilon, \varepsilon]$ be $E^* \otimes F \otimes L_\Omega^{\otimes k} = \operatorname{Hom}(E, F \otimes L_\Omega^{\otimes k})$. Let G be as usual a J-complex distribution defined on $M \times [-\varepsilon, \varepsilon]$ that extends D, and let

$$S_{k,i}^G = \{A \in I_{k,\Omega} \mid \operatorname{rank}(A) = i\}, \qquad A \in \operatorname{Hom}(E, F \otimes L_\Omega^{\otimes k}).$$

By Lemma 6.1 (applied to almost complex manifolds) $S_{k,i}^G$ is an approximately holomorphic sequence of finite, Whitney (A) stratifications. Therefore we can apply Theorem 7.2 to construct an A.H. sequence of sections τ_k of $I_{k,\Omega}$ uniformly transverse over M to S_k^G, and thus along D.

Hence M is stratified by the submanifolds $S_i(\tau_k) = \{x \in M \mid \operatorname{rank}(\tau_k(x)) = i\}$ for all k large enough, which are uniformly transverse to D and 2-calibrated by Lemma 6.6. □

Corollary 1.1 follows from the fact that in the contact case the 2-form is exact and hence the cohomological computations are those of the bundle $E^* \otimes F$.

Theorem 8.1. *Let (M, D, ω) be a closed integral 2-calibrated manifold, set $E_k = \underline{\mathbb{C}}^{m+1} \otimes L^{\otimes k}$, and let r be any natural number. Any A.H. sequence of sections of $\underline{\mathbb{C}}^{m+1} \otimes L_\Omega^{\otimes k} \to (M \times [-\varepsilon, \varepsilon], \Omega, G)$ admits an arbitrarily small C^{r+h}-perturbation such that $\phi_{k|M} : M \backslash B_k \to \mathbb{CP}^m$, the restriction to M of its projectivization, is an r-generic A.H. sequence.*

Proof. The proof is just Theorem 7.2 applied to the Thom–Boardman–Auroux quasi-stratification along G of $\mathscr{J}^r E_k \to (M \times [-\varepsilon, \varepsilon], J, G, g_k)$, combined with Proposition 6.3. □

It must be pointed out that the behavior of A.H. functions at the points close to the degeneration loci is more complicated than that of the leafwise holomorphic model: firstly, and similarly to what happens for even-dimensional almost complex manifolds, to obtain normal forms it is necessary to add perturbations so that the function becomes holomorphic (at least in certain directions); otherwise the approximate holomorphicity is not significative due to the vanishing (degeneracy) of the holomorphic part. Secondly, we have an extra non-holomorphic direction that we do not control. At most, we can apply the usual genericity results to that direction (the perturbations at most of size $O(k^{-1/2})$ so as not to destroy the other properties).

One instance of the preceding theorem is when the target space has large dimension, so that the generic map is an immersion along the directions of D.

Proof of Corollary 1.2. Set $E_k = \underline{\mathbb{C}}^{m+1} \otimes L^{\otimes k}$, where $m \geq 2n$. Theorem 7.2 is applied to the Thom–Boardman–Auroux quasi-stratification along G of $\mathscr{J}^1 E_k \to (M \times [-\varepsilon, \varepsilon], J, G, g_k)$ to obtain 1-generic A.H. maps $\phi_k : M \to \mathbb{CP}^m$. From the choice of m it follows that the set of base points and of points where $\partial\phi_k$ is not injective is empty. It is clear that by construction that $\phi_k^*[\omega_{FS}] = [\omega_k]$. □

This is a non-trivial result because the property of being an immersion along D is not generic (for smooth maps to $\mathbb{CP}^{2n}$). Notice that if for example D is integrable the property is generic for each leaf (locally), but not for the 1-parameter family.

As mentioned in the introduction, the previous corollary can be improved in two different ways.

Proof of Corollary 1.3. Let us assume that any 2-form in the path $\rho_{k,t} = (1-t)\omega_k + t\phi_k^*\omega_{FS}$ is non-degenerate over $\mathscr{D}$, where ω_{FS} is be the Fubini–Study 2-form. Then Moser's trick can be applied leafwise: if α is a 1-form such that $d\alpha = -(\phi_k^*\omega_{FS} - \omega_k)$, the vector fields tangent to $\mathscr{D}$ defined by the condition $-i_{X_t}\rho_{k,t} = -\alpha$ generate a 1-parameter family of diffeomorphisms preserving each leaf and sending $\rho_{k,t}$ to ω_k.

The non-degeneracy over $\mathscr{D}$ of ρ_t follows from the estimated transversality of ϕ_k together with the approximate holomorphicity. For any $v \in D_x$ of g_k-norm 1,

$$\rho_{k,t}(v, Jv) = (1-t)\omega_k(v, Jv) + t\omega_{FS}(\phi_{k*}v, \phi_{k*}Jv) \geq (1-t) + t\eta > 0. \qquad \square$$

In general a closed Poisson manifold with codimension 1 leaves does not admit a lift to a 2-calibrated structure (for example any non-taut smooth foliation in M^3). The previous corollary can be used to state the following result:

Corollary 8.1. *Let* $(M^{2n+1}, \mathscr{D}, \omega_{\mathscr{D}})$ *be a closed Poisson manifold with co-oriented codimension* 1 *leaves. Then the Poisson structure admits a lift to a* (*rational*) 2-*calibrated structure if and only if a multiple of* $\omega_{\mathscr{D}}$ *is induced by a leafwise immersion in* $\mathbb{CP}^{2n}$ (*by pulling back* ω_{FS}).

It is worth mentioning that it is possible to obtain uniform transversality to a finite number of quasi-stratifications of the same sequences of bundles. For example, and this leads to the second improvement of Corollary 1.2, we can obtain the 1-genericity result that gives rise to embeddings in $\mathbb{CP}^m$ transverse to a finite number of complex submanifolds of $\mathbb{CP}^m$.

We just need to consider for each submanifold the sequence of stratifications $\mathbb{P}\mathscr{S}$ of $\mathscr{J}_G^1(M, \mathbb{CP}^m)$, whose unique stratum (for each k) is defined to be the 1-jets along G whose degree 0 component is a point of the submanifold; next we pull it back to a stratification $\mathscr{S}$ of $\mathscr{J}_G^1 E_k^*$ and finally to a stratification $\mathscr{S}^G$ of

$\mathscr{J}^1 E_k^*$ (the structure near Z_k is not relevant because transversality to the Thom–Boardman–Auroux quasi-stratification along G implies that the sections stay away from Z_k). Therefore, we have defined a stratification of $\mathscr{J}^1 E_k$ which is trivially approximately holomorphic because it is the pullback by A.H. maps of an initial approximately holomorphic stratification of $\mathscr{J}_G^0(M, \mathbb{CP}^m)$. Any 1-generic sequence of A.H. sections of E_k uniformly transverse to $\mathscr{S}^G$, when restricted to M gives rise to maps $\phi_k : M \hookrightarrow \mathbb{CP}^m$ uniformly transverse along D to the submanifold.

Proof of Theorem 1.1. We first apply Theorem 8.1 to obtain $\phi_{k|M} : M \backslash B_k \to \mathbb{CP}^1$ 1-generic.

Near the base points and the points where $\nabla_D \phi_{k|M}$ vanishes, we apply the perturbations defined in [35] to obtain the required local models. □

Another possible application is, as proposed by D. Auroux for symplectic manifolds [3], [4], to obtain r-generic applications to $\mathbb{CP}^m$ whose composition with certain projections $\mathbb{CP}^m \to \mathbb{CP}^{m-h}$ are still r-generic (the corresponding stratifications are approximately holomorphic because they are pullback of approximately holomorphic stratifications by A.H. maps; the structure near Z_k is also seen to be appropriate).

It is also possible to develop an analogous construction but for A.H. maps to Grassmannians $\mathrm{Gr}(r, m)$, starting from sections of $\underline{\mathbb{C}}^r \otimes E_k$, E_k of rank m (see [32], [5]).

Our techniques can be applied to any closed 2-calibrated manifold to give a finer topological description of the 2-calibrated structure. It is possible to apply the same idea to manifolds for which the 2-calibrated structure enters as an auxiliary tool. This point of view has already been adopted in [27].

We recall the following result.

Theorem 8.2 (Gromov). *Let M^{2n+1} be a closed manifold whose structural group reduces to $U(n)$, and let $a \in H^2(M; \mathbb{Z})$. Then there exists ω a closed maximally non-degenerate 2-form such that $[\omega] = a$.*

Proof. The structural group of the open manifold $M \times \mathbb{R}$ reduces to $U(n+1)$. Then by [19] it carries a symplectic form representing any given cohomology class, in particular the pullback of a to $M \times \mathbb{R}$. Its restriction to $M \times \{0\}$ is ω. □

So by selecting any codimension 1 distribution transverse to the kernel of ω, we have:

Corollary 8.2. *Let M^{2n+1} be a closed manifold whose structural group reduces to $U(n)$, and let $a \in H^2(M; \mathbb{Z})$. Then M admits 2-calibrated structures (D, ω) for which $[\omega] = a$.*

Notice that if we apply any of the previous constructions to (M, D, w), we obtain submanifolds and more generally stratifications of M by 2-calibrated submanifolds. Regarding the initial structure, which was just a reduction of the structural group to $U(n)$, we can conclude that the corresponding strata also admit such a reduction.

Appendix A. Proof of Proposition 5.1

We write down the proof for the bundle $\mathscr{J}^r E_k$ because it is a necessary ingredient in the proof of Theorem 7.2. The case of $\mathscr{J}^r_D E_k$ bears no further complications and it is left to the interested reader.

We omit the subindices k and r for the connections whenever there is no risk of confusion.

Recall that in coordinates the curvature can be computed as follows: in a chart where T^*P is trivialized using the derivatives of the coordinates, we have the corresponding flat connection d on T^*P. We have the operator

$$\nabla^1 : T^*P \otimes E_k \to T^*P \otimes T^*P \otimes E_k, \qquad \nabla^1 := \mathrm{d} \otimes \mathrm{I} - \mathrm{I} \otimes \nabla,$$

and the antisymmetrization map

$$\mathrm{asym}_2 : T^*P \otimes T^*P \to \bigwedge^2 T^*P, \qquad \alpha \otimes \beta \mapsto \alpha \wedge \beta,$$
$$\alpha \wedge \beta(u, v) := \alpha(u)\beta(v) - \alpha(v)\beta(u).$$

The curvature is the composition $\mathrm{asym}_2(\nabla^1 \circ \nabla)$.

Let $\sigma_k = (\sigma_{k,0}, \sigma_{k,1})$ be a section (maybe local) of $\mathscr{J}^1 E_k$. The modified connection is $\nabla_{H_1}(\sigma_{k,0}, \sigma_{k,1}) = (\nabla\sigma_{k,0}, \nabla\sigma_{k,1}) + (0, -F^{1,1}\sigma_{k,0})$, where $-F^{1,1}\sigma_{k,0} \in T^{*0,1}P \otimes T^{*1,0}P \otimes E_k$ (see [5]). For jets along D we add $-F^{1,1}_D$.

The previous formula defines a connection.

Lemma A.1. *Let $\underline{\mathbb{C}}^m \to \mathbb{C}^p$ be the trivial bundle endowed with a connection ∇ whose curvature is of type $(1,1)$ with respect to the canonical complex structure J_0; the connection splits into $\partial_\nabla + \bar{\partial}_\nabla$. Let τ be a holomorphic section of $\underline{\mathbb{C}}^m$ (with respect to the holomorphic structure induced by ∇). Then*

$$\nabla_H(\tau, \partial_\nabla\tau) = \nabla(\tau, \partial_\nabla\tau) - (0, \bar{\partial}_\nabla\partial_\nabla\tau) \tag{63}$$

and $\bar{\partial}_{\nabla_H}(\tau, \partial_\nabla\tau) = 0$.

Proof. By definition

$$F\tau = \mathrm{asym}_2(\nabla^1\nabla\tau).$$

Let us denote the trivialization of the bundle that identifies it with $\underline{\mathbb{C}}^m$ by $\xi_1, \dots, \xi_m$. Since τ is holomorphic

$$F\tau = \mathrm{asym}_2\big((\mathrm{d} \otimes \mathrm{I} - \mathrm{I} \otimes \nabla)\partial_\nabla \tau\big).$$

If we write $\partial_\nabla \tau = dz^i h_i^j \xi_j$ then

$$F\tau = \mathrm{asym}_2\big(-(\mathrm{I} \otimes \nabla)\, dz^i h_i^j \xi_j\big).$$

But being the curvature of type $(1,1)$ we can write

$$F\tau = \mathrm{asym}_2\big(-(\mathrm{I} \otimes \bar{\partial}_\nabla)\, dz^i h_i^j \xi_j\big). \tag{64}$$

Recall that $F\tau$ has to be understood as an element of $T^{*0,1}\mathbb{C}^p \otimes T^{*1,0}\mathbb{C}^p \otimes \underline{\mathbb{C}}^m$. That amounts to switch the $d\bar{z}^l$'s with the dz^l's, which cancels the negative sign on the right-hand side of equation (64). Thus what we obtain is

$$F\tau := (\mathrm{I} \otimes \bar{\partial}_\nabla)\, dz^i h_i^j \xi_j \in \Gamma(T^{*0,1}\mathbb{C}^p \otimes T^{*1,0}\mathbb{C}^p \otimes \underline{\mathbb{C}}^m). \tag{65}$$

But equation (65) equals

$$(\bar{\partial}_0 \otimes \mathrm{I} + \mathrm{I} \otimes \bar{\partial}_\nabla)\, dz^i h_i^j \xi_j,$$

which by definition is

$$\bar{\partial}_\nabla \partial_\nabla \tau. \tag{66}$$

By equation (66)

$$\bar{\partial}_{\nabla_H}(\tau, \partial_\nabla \tau) = (\bar{\partial}_\nabla \tau, \bar{\partial}_\nabla \partial_\nabla \tau - \bar{\partial}_\nabla \partial_\nabla \tau) = 0. \qquad \square$$

It is also clear that $\partial_\nabla = \partial_{\nabla_H}$ and therefore they define the same coupled holomorphic jets.

Lemma A.1 has an obvious approximately holomorphic version: if we have a very ample sequence of rank m vector bundles by definition the sequences of curvatures is approximately of type $(1,1)$. Then we can fix approximately holomorphic coordinates and the first part of Lemma A.1 implies that for τ_k a sequence of A.H. sections of E_k, one has

$$F\tau_k \approxeq \bar{\partial}\partial\tau_k,$$

and by the second part

$$\bar{\partial}_H j^1 \tau_k \approxeq 0.$$

We now move into computing the curvature of the modified connection in the integrable case. We will denote the coupled holomorphic r-jet in the integrable model by $j^r_{\mathrm{hol}}\tau$.

Lemma A.2. *Let* $\underline{\mathbb{C}}^m \to \mathbb{C}^p$ *be the trivial bundle as in Lemma* A.1. *Assume also that for the fixed trivialization* $\xi_1, \dots, \xi_m$ *the curvature is a matrix with constant coefficients and that we have a frame given by holomorphic sections* $\tau_1, \dots, \tau_m$. *Then* $F_\nabla = F_{\nabla_H}$.

Proof. If the holomorphic sections $\tau_1, \dots, \tau_m$ generate the bundle, then the holomorphic 1-jets of $z^l\tau_j$, τ_j, $1 \le l \le p$, $1 \le j \le m$, are a basis of $\mathscr{J}^1_{p,m}$ (at least on $B(0,\rho)$). By Lemma A.1, they are a holomorphic basis.

$$\nabla_H j^1_{\mathrm{hol}} z^l\tau_j = \big(\partial_\nabla(z^l\tau_j), \nabla\partial_\nabla(z^l\tau_j)\big) - (0, Fz^l\tau_j) = \nabla j^1_{\mathrm{hol}} z^l\tau_j - (0, Fz^l\tau_j). \tag{67}$$

Let us write again $\partial_\nabla\tau_j = dz^i h^s_{i,j}\xi_s$, and $F = a_{ts}\, d\bar{z}^t\, dz^s \in \Gamma(T^{*0,1}\mathbb{C}^p \otimes T^{*1,0}\mathbb{C}^p)$. If we apply to $\nabla j^1_{\mathrm{hol}} z^l\tau_j$ the operator $\mathrm{asym}_2\, \nabla^1_H$, $\nabla^1_H := \mathrm{d} \otimes \mathrm{I} - \mathrm{I} \otimes \nabla_H$, we get:

$$F_\nabla j^1_{\mathrm{hol}} z^l\tau_j + \big(0, \mathrm{asym}_2(dz^l a_{ts}\, d\bar{z}^t\, dz^s\tau_j + z^l\, dz^i a_{ts}\, d\bar{z}^t\, dz^s h^s_{i,j}\xi_s)\big). \tag{68}$$

When we apply the same operator to $(0, Fz^l\tau_j)$, if recall that the a_{ts} are constant and that $z^l\tau_j$ is a holomorphic section, we get

$$\mathrm{asym}_2\, \nabla^1_H(0, Fz^l\tau_j) = \big(0, \mathrm{asym}_2(-a_{ts}\, d\bar{z}^t\, dz^l\, dz^s\tau_j - a_{ts}\, d\bar{z}^t z^l\, dz^i h^s_{i,j}\xi_s)\big), \tag{69}$$

and the right-hand side of equation (69) equals

$$\big(0, \mathrm{asym}_2(dz^l F\tau_j + z^l\, dz^i F h^s_{i,j}\xi_s)\big). \tag{70}$$

If we put together equations (67), (68), and (70) we obtain

$$F_{\nabla_H}\tau_j = F_\nabla\tau_j. \qquad \square$$

We want to use a recursive construction based on Lemmas A.1 and A.2 to introduce the desired connection on $\mathscr{J}^r_{p,m}$.

Before doing that we recall that the coupled holomorphic jets are sections of $\mathscr{J}^r_{p,m}$. We now prove how to modify the connection on $\mathscr{J}^2_{p,m}$.

Step 1: We identify $\mathscr{J}^2_{p,m}$ with the subbundle of $\mathscr{J}^1\mathscr{J}^1_{p,m}$ spanned by holonomic sections, i.e., sections of the form $j^1_{\mathrm{hol}} j^1_{\mathrm{hol}}\tau$, where τ is any holomorphic section of $\underline{\mathbb{C}}^m$. Pointwise, an element γ of the fiber of $\mathscr{J}^1\mathscr{J}^1_{p,m}$ is of the form

$$(\gamma_{0,0}, \gamma_{0,1}, \gamma_{1,0}, \gamma_{1,1}) \in \big(\mathbb{C} \oplus T^{*1,0}\mathbb{C}^p \oplus T^{*1,0}\mathbb{C}^p \oplus (T^{*1,0}\mathbb{C}^p \otimes T^{*1,0}\mathbb{C}^p)\big) \otimes \mathbb{C}^m,$$

and belongs to $\mathscr{J}^2_{p,m}$ if and only if $\gamma_{1,1} \in T^{*1,0}\mathbb{C}^p \odot T^{*1,0}\mathbb{C}^p \otimes \mathbb{C}^m$ and $\gamma_{1,0} = \gamma_{0,1}$.

Using the metric induced by the Euclidean one on the base and fiber and the connection, we have a orthogonal projection $r : \mathscr{J}^1\mathscr{J}^1_{p,m} \to \mathscr{J}^2_{p,m}$.

Step 2: We introduce a new connection on $\mathscr{J}^1\mathscr{J}^1_{p,m}$.

On $\mathscr{J}^1_{p,m}$ we use the modified connection ∇_{H_1}. This, together with the flat connection d on $T^*\mathbb{C}^p$ defines a connection $\nabla_{H_{1,1}}$ on $\mathscr{J}^1\mathscr{J}^1_{p,m}$. Notice that on $\mathscr{J}^1\mathscr{J}^1_{p,m}$ we also have a connection ∇_2 coming from d and ∇_1.

We consider the trivialization of $\mathscr{J}^1_{p,m}$ furnished by the sections ξ_j, $dz^i\xi_j$, $1 \le j \le m$, $1 \le i \le p$, so we can identify the bundle with $\underline{\mathbb{C}}^{mp+m}$. This is a trivial bundle with connection ∇_{H_1}. By Lemma A.2 $F_{\nabla_{H_1}} = F_{\nabla_1}$. Recall also that in the basis ξ_j, $dz^i\xi_j$ the curvature F_{∇_1} is a matrix that decomposes into $p+1$ blocks corresponding to $\xi_1, \dots, \xi_m$ and to $dz^i\xi_1, \dots, dz^i\xi_m$, $1 \le i \le p$. For each such block the corresponding matrix is the one for F_∇ in the basis ξ_j. Therefore $F_{\nabla_{H_1}}$ is still of type $(1,1)$ and has constant entries in the aforementioned basis.

Let ∇_{H_2} be the result of modifying $\nabla_{H_{1,1}}$. Since ∇_{H_1} is of type $(1,1)$ by Lemma A.1 applied to $(\underline{\mathbb{C}}^{mp+m}, \nabla_{H_1})$, if $\tau^1 \in \Gamma(\mathscr{J}^1_{p,m})$ is holomorphic with respect to ∇_{H_1}, then $j^1_{\text{hol}}\tau^1$ is holomorphic with respect to ∇_{H_2}. In particular $j^1_{\text{hol}}(j^1_{\text{hol}}z^i\tau_j)$, $j^1_{\text{hol}}(z^l j^1_{\text{hol}}z^i\tau_j)$ are a local holomorphic frame of $(\mathscr{J}^1\mathscr{J}^1_{p,m}, \nabla_{H_2})$ (recall that τ_j was a local holomorphic frame of $\underline{\mathbb{C}}^m$).

Taking into account that the curvature of $(\underline{\mathbb{C}}^{mp+m}, \nabla_{H_1})$ is of type $(1,1)$ and with constant entries, and that $(\underline{\mathbb{C}}^{mp+m}, \nabla_{H_1})$ has a local holomorphic basis, Lemma A.2 gives $F_{\nabla_{H_2}} = F_{\nabla_{H_{1,1}}}$. From $F_{\nabla_{H_1}} = F_{\nabla_1}$ it follows that $F_{\nabla_{H_{1,1}}} = F_{\nabla_2}$. Therefore

$$F_{\nabla_{H_2}} = F_{\nabla_2} \qquad \text{on } \mathscr{J}^1\mathscr{J}^1_{p,m}.$$

Step 3: Check that ∇_{H_2} restricts to $\mathscr{J}^2_{p,m} \hookrightarrow \mathscr{J}^1\mathscr{J}^1_{p,m}$ with the desired properties.

Let $I = (i_0, i_1, \dots, i_p)$ with $1 \le i_0 \le m$, $0 \le i_j \le 2$, $i_1 + \dots + i_p \le 2$, and let $\tau_I := z_1^{i_1} \dots z_p^{i_p}\tau_{i_0}$. We consider the sections $j^1_{\text{hol}}j^1_{\text{hol}}\tau_I$, which are a local holomorphic frame $\mathscr{J}^2_{p,m}$ (using the identification described in Step 1). We will see that $\nabla_{H_2}j^1_{\text{hol}}j^1_{\text{hol}}\tau_I \in \Gamma(T^{*1,0}\mathbb{C}^p \otimes \mathscr{J}^2_{p,m})$, and therefore that the connection ∇_{H_2} preserves $\mathscr{J}^2_{p,m}$.

We just proved in Step 2 that $j^1_{\text{hol}}j^1_{\text{hol}}\tau_I$ is holomorphic with respect to ∇_{H_2} and that $\partial_{\nabla_{H_2}} = \partial_{\nabla_{H_{1,1}}} = \partial_{\nabla_2}$. Let us write $j^1_{\text{hol}}j^1_{\text{hol}}\tau_I = (\tau_I, \partial_\nabla\tau_I, \partial_\nabla\tau_I, \partial^2_\nabla\tau_I)$. Then

$$\begin{aligned}\nabla_{H_2}j^1_{\text{hol}}j^1_{\text{hol}}\tau_I &= \partial_{\nabla_{H_2}}j^1_{\text{hol}}j^1_{\text{hol}}\tau_I \\ &= \partial_{\nabla_2}(\tau_I, \partial_\nabla\tau_I, \partial_\nabla\tau_I, \partial^2_\nabla\tau_I) \\ &= (\partial_\nabla\tau_I, \partial_\nabla\partial_\nabla\tau_I, \partial_\nabla\partial_\nabla\tau_I, \partial_\nabla\partial^2_\nabla\tau_I),\end{aligned}$$

which belongs to $\Gamma(T^{*1,0}\mathbb{C}^p \otimes \mathscr{J}^2_{p,m})$.

Therefore, the curvature of the restriction of ∇_{H_2} to $\mathscr{J}^2_{p,m}$ is of course of type $(1,1)$. The last observation is its expression in a suitable basis. The curvature of ∇_2 on $\mathscr{J}^1\mathscr{J}^1_{p,m}$ splits on blocks corresponding to the basis $\xi_1,\dots,\xi_m$, $dz^i\xi_1,\dots,dz^i\xi_m$, $dz^l\xi_1,\dots,dz^l\xi_m$, $dz^r\otimes dz^t\xi_1,\dots,dz^r\otimes dz^t\xi_m$, $1\le i,l,r,t\le p$. Each submatrix is F_∇. If we use the basis $\xi_1,\dots,\xi_m$, $dz^i\xi_1,\dots,dz^i\xi_m$, $dz^r\odot dz^t\xi_1,\dots,dz^r\odot dz^t\xi_m$, $1\le i,r,t\le p$, the curvature equally splits into blocks each matching F_∇.

The general case uses the following induction step: on $\mathscr{J}^r_{p,m}$ there exists a connection ∇_{H_r} with the following properties:

(1) $\partial_{H_r}=\partial_r$.

(2) $F_{\nabla_{H_r}}=F_{\nabla_r}$ and therefore $F_{\nabla_{H_r}}$ is of type $(1,1)$.

(3) If $\bar\partial_\nabla\tau=0$ then $\bar\partial_{H_r}j^r_{\mathrm{hol}}\tau=0$.

(4) In the basis $\xi_I:=(dz^1_k)^{\odot i_1}\dots(dz^n_k)^{\odot i_n}\xi_{i_0}$ the curvature splits into blocks each matching F_∇.

To define $\nabla_{H_{r+1}}$ on $\mathscr{J}^{r+1}_{p,m}$ we reproduce the previous three steps.

Firstly we consider the identification of $\mathscr{J}^{r+1}_{p,m}$ with the subbundle of $\mathscr{J}^1\mathscr{J}^r_{p,m}$ spanned by sections of the form $j^1_{\mathrm{hol}}j^r_{\mathrm{hol}}\tau$, τ a holomorphic section of $\underline{\mathbb{C}}^m$.

Secondly we consider the connection $\nabla_{H_{1,r}}$ on $\mathscr{J}^{r+1}_{p,m}$ constructed out of d and ∇_{H_r} and modify it to $\nabla_{H_{r+1}}$. By the induction hypothesis using the basis ξ_I we are in the situation of Lemma A.2, for $\mathscr{J}^r_{p,m}$ identifies with $\underline{\mathbb{C}}^{N_r}$ with a connection whose curvature is of type $(1,1)$ and with constant coefficients, and with a frame of holomorphic sections. Therefore $F_{\nabla_{H_{r+1}}}=F_{\nabla_{H_{1,r}}}=F_{\nabla_{r+1}}$. Since we can also apply Lemma A.1, for any $\tau^r\in\Gamma(\mathscr{J}^r_{p,m})$ the 1-jet $j^1_{\mathrm{hol}}\tau^r$ is holomorphic with respect to $\nabla_{H_{r+1}}$.

The third step is to check that the modified connection restricts to $\mathscr{J}^{r+1}_{p,m}\hookrightarrow\mathscr{J}^1\mathscr{J}^r_{p,m}$. Using that $\partial_{\nabla_{H_{r+1}}}=\partial_{\nabla_{r+1}}$, any frame of sections of the form $j^1_{\mathrm{hol}}j^r_{\mathrm{hol}}\tau_I$, τ_I holomorphic, is sent by the connection to sections of $\mathscr{J}^{r+1}_{p,m}$.

It is also routine to check that in the basis ξ_I the curvature matrix is made of blocks of the form F_∇.

The almost complex counterpart of the result we just proved is done exactly in the same way. The only modification is that the connection on $\mathscr{J}^1\mathscr{J}^rE_k$ does not descend automatically to a connection on $\mathscr{J}^{r+1}E_k\hookrightarrow\mathscr{J}^1\mathscr{J}^rE_k$. We have to project via $r:\mathscr{J}^1\mathscr{J}^rE_k\to\mathscr{J}^{r+1}E_k$, but this is seen to introduce an error which is approximately vanishing. It might happen that the resulting connection amounts to adding also a pseudo-holomorphic part. If that is the case we forget about this contribution (which again would be approximately vanishing). Therefore, we obtain a connection with all the desired properties.

Using similar considerations to the ones for 1-jets, it can be deduced that the $(r+1)$-jet of a C^{r+1+h}-A.H. sequence of sections of E_k is a C^h-A.H. sequence of sections of $(\mathscr{J}^{r+1}E_k,\nabla_{H_{r+1}})$.

Appendix B. Chern classes and top Chern classes

Corollary 1.1 proves the existence of contact determinantal submanifolds, which we expect to be more general than those coming from zeroes of vector bundles constructed in [24]. To support this we recall that it is known that in the algebro-geometric setting that determinantal varieties are more general that zeroes of vector bundles (see for example [20], [1]), and a similar result should be expected to hold in the smooth category. A way to prove it would be exhibiting a manifold in which there exist a cohomology class a which is the Chern class of a complex vector bundle F but it is not the top Chern class of any complex vector bundle (i.e., showing that Chern classes are more general than top Chern classes), the reason being that if we choose as E the trivial complex vector bundle of the appropriate rank and the appropriate determinantal locus, we have

$$\Delta_{E,F,i} = a.$$

As far as the author knows such a question has not been addressed. A lot is known about cohomology classes which can be Chern classes, mainly because for a given finite CW complex of dimension n there is a rather clear picture of complex vector bundles of rank $\geq [n/2]$ (the so-called stable rank) [7]; much less is known about lower ranks and that is what makes it difficult to discard a Chern class as a top Chern class (besides, according to Thom [37], Theorem II.25, in a (compact, oriented) manifold any $a \in H^{2k}(M;\mathbb{Z})$ has a multiple which is a top Chern class). In any case, finding manifolds with certain cohomological properties would prove that Chern classes are more general than top Chern classes. For example, according to [7] for a (compact, oriented) manifold X of dimension ≤ 7, any $a \in H^4(X;\mathbb{Z})$ is the second Chern class of a rank 3 complex vector bundle. If it were the top Chern class of some F, then Corollary 2.2 in [7] applied to the direct sum of F with the trivial line bundle would imply that

$$c_1(F)a + \mathrm{Sq}^2 a \equiv 0 \quad \text{in } H^6(X;\mathbb{Z}_2). \tag{71}$$

Therefore, if $H^2(X;\mathbb{Z}_2) = 0$ and there exists a class a with non-vanishing second Steenrod square, equation (71) could not hold and hence a would not be a top Chern class.

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