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Asymptotic distribution of certain statistics relevant to the fitting of max-semistable models

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Abstract. According to the results in Canto e Castro *et al.* [1], max-semistable distribution functions can be characterized by a parameter $r \geq 1$, by the extreme value index γ and by a real function w defined in $[0, \log r]$. The estimation of the parameters r and γ based on ratios of differences of order statistics, or appropriate functions of these sequences, was treated in Dias and Canto e Castro [4]. In this work we study the asymptotic distribution of these sequences of statistics.

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1. Introduction

For a long time many efforts have been made in order to extend the class of max-stable distributions which was considered too restrictive for some important statistical applications. In fact, this class, formed by all distribution functions (d.f.'s) G such that there exist real constants a_n positive and b_n satisfying $G^n(a_n x + b_n) = G(x)$, for all positive integer n, is sometimes inadequate to model some non traditional phenomena, in particular those concerned with discrete distributions. It is well known that a distribution function (d.f.) F belongs to the domain of attraction of a max-stable d.f. G, and we write $F \in \mathrm{MS}(G)$, if and only if there exist normalizing real sequences $\{a_n\}$ positive and $\{b_n\}$ such that

$$\lim_{n \to +\infty} F^n(a_n x + b_n) = G(x) \qquad \text{for all } x \in \mathbb{R}. \tag{1}$$

The most common continuous d.f.'s belong to the domain of attraction of some max-stable distribution, but the same does not happen for a long range of discrete

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and continuous multi-modal distributions. For instance, there are no normalizing real constants a_n positive and b_n such that the limit (1) occurs for $F(x) = (1 - e^{-[x]})\mathbb{1}_{[0,+\infty]}(x)$, $x \in \mathbb{R}$, and for the Von Mises d.f. given by $F(x) = (1 - e^{-x-(1/2)\sin x})\mathbb{1}_{[0,+\infty]}(x)$, $x \in \mathbb{R}$. However, some of those d.f.'s can be included in a new class, which is characterized by the following limiting behavior

$$\lim_{n\to+\infty} F^{k_n}(a_n x + b_n) = G(x) \qquad \text{for all } x \in C_G, \tag{2}$$

where C_G denotes the set of continuity points of the non-degenerate d.f. G. As we will see later, max-semistable distribution functions present a log-periodic component that makes them attractive in areas like seismology, turbulence and finance. In the previous alternative limit, $\{a_n\}$ and $\{b_n\}$, with $a_n > 0$, are suitable real sequences and $\{k_n\}$ is a non decreasing positive sequence verifying the geometric growing condition

$$\lim_{n\to+\infty} \frac{k_{n+1}}{k_n} = r \geq 1 \qquad (r < \infty). \tag{3}$$

In this case, we obtain a larger class of possible limiting distributions for the normalized maximum known in the literature of extremes as the class MSS of max-semistable distributions. Moreover, in this new context, if (2) holds, we say that the d.f. F belongs to the domain of attraction of the max-semistable d.f. G, and we write $F \in \mathrm{MSS}(G)$. When $r = 1$, even for $k_n \neq n$, we obtain the particular case of the max-stable class. In the next examples the d.f's F are such that $F \in \mathrm{MSS}(G)$ but $F \notin \mathrm{MS}(G)$.

Example 1.1. The geometric d.f. $F(x) = \big(1 - \exp(-[x])\big)\mathbb{1}_{[0,+\infty[}(x)$, $x \in \mathbb{R}$, verifies (2) with $k_n = [e^n]$, $a_n = 1$, $b_n = n$ and $G(x) = \exp\big(-\exp(-[x])\big)$, $x \in \mathbb{R}$.

Example 1.2. Consider the d.f. $F(x) = \big(1 - x^{-1}\big(27 + \cos(8\pi \log x)\big)\big)\mathbb{1}_{[x_0,+\infty[}(x)$, $x \in \mathbb{R}$, where x_0 is solution of the equation $1 = x^{-1}\big(27 + \cos(8\pi \log x)\big)$. Choosing $k_n = [e^{n/4}]$, $a_n = e^{-n/4}$ and $b_n = 0$, the limit (2) occurs with $G(x) = \exp\big(-x^{-1}\big(27 + \cos(8\pi \log x)\big)\big)\mathbb{1}_{]0,+\infty[}(x)$, $x \in \mathbb{R}$.

During the last fifteen years there has been much interest in this topic and several works have appeared concerning max-semistable laws. In the genesis of this class are the papers of Pancheva [8] and Grinevich [6], [7]. These two authors have established that a d.f. G is max-semistable if and only if is solution of the functional equation

$$G(x) = G^r(ax + b) \qquad \text{for all } x \in \mathbb{R} \tag{4}$$

for some $a > 0$, $b \in \mathbb{R}$ and $r > 1$. More precisely, a d.f. G is max-semistable if there exist a d.f. F, k_n, a_n and b_n as above, such that (2) holds or, equivalently, if G is a solution of (4). The characterization of max-semistable domains of attraction can be found in Grinevich [7] and in Canto e Castro *et al.* [1]. Recently, some studies on the estimation of the unknown parameters of this new class of d.f.'s arises in the literature, for instance Temido [10], Canto e Castro *et al.* [2] and Dias and Canto e Castro [4]. Furthermore, we notice that if (4) is verified then, for each $n \in \mathbb{N}$, there exist reals $c_n > 0$ and d_n such that

$$G^{r^n}(c_n x + d_n) = G^r(ax + b) = G(x),$$

which enables us to prove that the class $G_{r,a,b} = \{G : G(x) = G^r(ax + b)\}$ coincides with the class G_{r^n, c_n, d_n}, where $c_n = a^n$ and $d_n = b(a^{n-1} + \cdots + a^2 + a + 1)$. Then the constant r in (4) is not unique and can be replaced by an integer power of itself. Grinevich [6] solve the functional equation (4) proving that there are three main families of max-semistable laws. An unifying standard expression for these families, analogous to the generalized form of Von Mises–Jenkinson for the max-stable f.d.'s, is given by

$$G_{\gamma,\nu}(x) = \begin{cases} \exp\{-(1+\gamma x)^{-1/\gamma}\nu(\log(1+\gamma x))\}, & x \in \mathbb{R},\ 1+\gamma x > 0 \text{ and } \gamma \neq 0, \\ \exp\{-e^{-x}\nu(x)\}, & x \in \mathbb{R} \text{ and } \gamma = 0, \end{cases}$$

where ν is a positive, bounded and periodic function. The parameters γ and the period p of the function ν are related with the parameters a, b and r in (4) in the following way:

- $p = \log a = \gamma \log r$, for $\gamma \neq 0$;
- $p = b = \log r$, for $\gamma = 0$.

Observe that the p-quantile of $G_{\gamma,\nu}$ and the p-quantile of $G_{\gamma,1}(x) = \exp\{-(1+\gamma x)^{-1/\gamma}\}$ are related through $y = (x + 1/\gamma)(\nu(1+\gamma x))^{-\gamma} - 1$. So, a max-semistable distribution can be a reasonable choice if a qq-plot fit to a max-stable model shows a log-periodic oscillation along a straight line.

A characterization of max-semistable laws involving generalized inverse functions was established in Canto e Castro *et al.* [1]. Supposing, without loss of generality, that

$$\begin{cases} G(0) = e^{-1}, \\ G(1) = \exp(-r^{-1}), \\ G \text{ is continuous at } x = 0, \end{cases}$$

the generalized inverse function of $-\log(-\log G)$ verifies

$$\big(-\log(-\log G)\big)^{\leftarrow}(m\log r + x) = s_m + a^m w(x) \qquad \text{for all } x \in [0, \log r],\ m \in \mathbb{Z} \quad (5)$$

where the function $w : [0, \log r] \to [0, 1]$ is non decreasing, left continuous and continuous at $x = 0$, and $s_m = \frac{a^m - 1}{a - 1}$ if $a \neq 1$ and $a > 0$ or $s_m = m$ if $a = 1$. This representation allowed those authors to prove that the following conditions are necessary and sufficient for (2) to be verified for some sequence $\{k_n\}$ satisfying (3):

$$\lim_{n\to+\infty} \frac{V(\log k_{n+1}) - V(\log k_n)}{V(\log k_n) - V(\log k_{n-1})} = a \tag{6}$$

and

$$\lim_{n\to+\infty} \frac{V(\log k_n + x) - V(\log k_n)}{V(\log k_{n+1}) - V(\log k_n)} = w(x), \qquad x \in [0, \log r] \tag{7}$$

where $V(x) := \big(-\log(-\log F)\big)^{\leftarrow}(x)$. A max-semistable d.f. G can be completely characterized using the parameters r and γ, and also the function w. As we have already said, the problem of the parameters estimation was firstly studied in Temido [10] and, later on, in Dias and Canto e Castro [4]. Temido [10] proposed that, in the estimation of the parameters, convenient functions of the sequences of statistics

$$Z_s(m_N) := \frac{X_{(m_N/s)} - X_{(m_N)}}{X_{(m_N)} - X_{(m_N s)}},$$

should be used. Here $X_{(m_N)} := X_{N-[m_N]+1:N}$ represents the order statistics of a sample of size N from any random variable X and $m := m_N$ is an intermediate sequence, that is, m is an integer sequence verifying $\lim_{N\to+\infty} m = +\infty$ and $\lim_{N\to+\infty} m/N = 0$. Dias and Canto e Castro [4] analyzed the asymptotic behavior of this sequence of statistics and proved that $Z_s(m)$ converges in probability to a^c if and only if $s = r^c$, $c \in \mathbb{N}$. Those authors also proposed some methods to estimate the parameter r involving the sequence of statistics

$$R_s(m) := \frac{Z_{s^2}(m)}{\big(Z_s(m)\big)^2}$$

which converges in probability to 1 if $s = r^c$, $c \in \mathbb{N}$. The main goal of this paper is to establish the asymptotic distribution of $Z_s(m)$ and $R_s(m)$ for $s = r^c$, $c \in \mathbb{N}$. In the class of max-stable laws, the study of the limit distribution of estimators for γ can be done using results obtained by Cooil [3]. These results concern the joint asymptotic distributions of intermediate order statistics, when F is in the first order differentiable domain of attraction of a limit law G, and we write

$F \in \mathrm{MS}_{dif}(G)$, that is, F is differentiable in a left neighborhood of $x_0 := \sup\{x : F(x) < 1\}$ and there exist real sequences $\{\alpha_n\}$ and $\{\beta_n\}$, with $\alpha_n > 0$, such that

$$\frac{dF^n}{dx}(\alpha_n x + \beta_n) \xrightarrow[n\to+\infty]{} G'(x)$$

locally uniformly for all $x \in S_G$, where S_G denotes the support of G. Indeed, Cooil [3] proved that if $F \in \mathrm{MS}_{dif}(G)$ then there are real functions a_y positive and b_y, for $y > 0$, such that for all intermediate sequences m and for $t > 0$, the stochastic process

$$\sqrt{m}(X_{(mt)} - b_{n/mt})/a_{n/m}$$

converges (in the sense of the convergence of all finite marginal distributions) to a Gaussian process $\{W(t)\}_{t>0}$ characterized by

$$E\big(W(t)\big) = 0, \qquad t > 0,$$
$$\operatorname{cov}\big(W(t_1), W(t_2)\big) = t_1^{-\gamma} t_2^{-\gamma-1}, \qquad 0 < t_1 \le t_2.$$

2. The asymptotic distribution of the sequences of statistics

In the sequel we need the following lemma.

Lemma 2.1. *Let $Y_1, Y_2, \ldots, Y_N$ be independent and identically distributed random variables with standard Gumbel d.f.. Suppose that $\{m_N\}$ is an intermediate sequence. Then, with $m := m_N$ and for all $\theta > 0$, the stochastic process*

$$Q_{N,m}(\theta) := \sqrt{m}\left(\frac{Y_{(m\theta)} - \log\left(\frac{N}{m\theta}\right)}{1 + \frac{m}{N}}\right)$$

converges (in the sense of the convergence of all finite marginal distributions) to a Gaussian process $Q(\theta)$ with mean zero and covariance structure given by

$$\operatorname{cov}\big(Q(\theta_1), Q(\theta_2)\big) = \theta_1^{-\gamma} \theta_2^{-\gamma-1}, \qquad 0 < \theta_1 \le \theta_2. \tag{8}$$

Proof. According to, for instance Draisma [5], for a d.f. $F \in \mathrm{MS}_{dif}(G)$, the constants a_n and b_n can be chosen as $a_n = nU'(n)$ and $b_n = U(n)$, where $U(t) := \big(1/(1-F)\big)^{\leftarrow}(t)$. As we said before, according to Cooil [3],

$$\sqrt{m}\left(\frac{Y_{(m\theta)} - U\left(\frac{N}{m\theta}\right)}{\frac{N}{m} U'\left(\frac{N}{m}\right)}\right) \tag{9}$$

converges to a Gaussian process with covariances given by (8). Considering now that F denotes the standard Gumbel distribution function, we have

$$U(t) = -\log\bigl(-\log(1 - t^{-1})\bigr) \tag{10}$$

and then

$$U(t) = \log t + O(t^{-1}) = \log t + o(1), \qquad \text{as } t \to +\infty.$$

By (10), we can obtain

$$U'(t) = \frac{1}{t^2(t^{-1} - 1)\log(1 - t^{-1})},$$

and so, as $t \to +\infty$,

$$tU'(t) = \frac{1}{t(t^{-1} - 1)\log(1 - t^{-1})} = 1 + t^{-1} + O(t^{-1}) = 1 + t^{-1} + o(1).$$

Developing (9) we get the desired result since

$$\sqrt{m}\left(\frac{Y_{(m\theta)} - U\left(\frac{N}{m\theta}\right)}{\frac{N}{m}U'\left(\frac{N}{m}\right)}\right) = \sqrt{m}\left(\frac{Y_{(m\theta)} - \log\left(\frac{N}{m\theta}\right) + o(1)}{1 + \frac{m}{N} + o(1)}\right)$$
$$= \sqrt{m}\left(\frac{Y_{(m\theta)} - \log\left(\frac{N}{m\theta}\right)}{1 + \frac{m}{N} + o(1)}\right) + o(1). \qquad \square$$

The next theorems establish the asymptotic normality of the sequences $Z_s(m)$ and $R_s(m)$ when $s = r^c$, $c \in \mathbb{N}$.

Theorem 2.2. *Let $\{X_i\}_{i\geq 1}$ be a sequence of independent and identically distributed random variables with continuous d.f. F. Suppose that, for some sequence $\{k_n\}$ satisfying* (3)*, F verifies* (2) *with G differentiable. Let $\{N_n\}$ be an integer sequence such that $N_n = [k_n]l_n$ where $1 \leq l_n < [k_{n+1}]/[k_n]$ and $\lim_{n\to+\infty} l_n = l \in [1, r]$. Let $\{t_n\}$ and $\{m_n\}$ be integer sequences such that $\lim_{n\to+\infty} m_n = \lim_{n\to+\infty} t_n = \lim_{n\to+\infty}(n - t_n) = +\infty$. With $m := m_n = [k_n/k_{n-t_n}]$, $N := N_n$ and $X_{(m)} := X_{N-[m]+1}$, consider the sequence of statistics*

$$Z_{r^c}(m) := \frac{X_{(m/r^c)} - X_{(m)}}{X_{(m)} - X_{(mr^c)}}.$$

Then there exist normalizing real sequences $\{\xi_m\}$ and $\{\upsilon_m\}$, with $\upsilon_m > 0$ and $\lim_{n\to+\infty} \xi_m = a^c$, such that the asymptotic distribution of $\upsilon_m^{-1}\sqrt{m}\bigl(Z_{r^c}(m) - \xi_m\bigr)$ is standard normal.

Proof. We only prove the case $a \neq 1$. The case $a = 1$ can be easily obtained in a similar way. In a first step we will prove that

$$Z_{r^c}(m) \stackrel{d}{=} a^c \frac{\frac{a^c-1}{a-1} + a^c w(q_{m/r^c,n} T_{(m/r^c)} + \delta_n) - w(q_{m,n} T_{(m)} + \beta_n) + o_P(1)}{\frac{a^c-1}{a-1} + a^c w(q_{m,n} T_{(m)} + \beta_n) - w(q_{mr^c,n} T_{(mr^c)} + \alpha_n) + o_P(1)}, \quad (11)$$

where $q_{m,n} = 1 + m/N$, $T_{(m)} = \left(Y_{(m)} - \log(N/m)\right)/q_{m,n}$, $\beta_n := \log\left(\frac{N}{k_{n-t_n}[k_n/k_{n-t_n}]}\right) = \log l_n + o(1)$, $\delta_n := \beta_n + \log\left(\frac{r^c k_{n-t_n}}{k_{n-t_n+c}}\right) = \beta_n + o(1)$ and $\alpha_n := \beta_n + \log\left(\frac{k_{n-t_n}}{r^c k_{n-t_n-c}}\right) = \beta_n + o(1)$. Taking into account that

$$X_{i:N} \stackrel{d}{=} F^{\leftarrow}(U_{i:N}) \stackrel{d}{=} F^{\leftarrow}\left(\Lambda(Y_{i:N})\right) = \left(-\log(-\log F)\right)^{\leftarrow}(Y_{i:N}) := V(Y_{i:N}), \quad (12)$$

where $Y_{i:N}$ are order statistics of a sample of size N from a Gumbel d.f. Λ, we get

$$Z_{r^c}(m) \stackrel{d}{=} \frac{V(Y_{(m/r^c)}) - V(Y_{(m)})}{V(Y_{(m)}) - V(Y_{(mr^c)})}.$$

This can be rewritten using $T_{(m)}$, as

$$Z_{r^c}(m) \stackrel{d}{=} \frac{V(\log k_{n-t_n+c} + q_{m/r^c,n} T_{(m/r^c)} + \delta_n) - V(\log k_{n-t_n} + q_{m,n} T_{(m)} + \beta_n)}{V(\log k_{n-t_n} + q_{m,n} T_{(m)} + \beta_n) - V(\log k_{n-t_n-c} + q_{mr^c,n} T_{(mr^c)} + \alpha_n)}. \quad (13)$$

Taking into account that the convergence in (7) is uniform, we obtain

$$V(\log k_n + x_n) = V(\log k_n) + w(x_n) d_n^{(1)} + o_P(d_n^{(1)}), \quad (14)$$

for all sequences x_n of elements in $[0, \log r]$ and where $d_n^{(i)} := V(\log k_{n+i}) - V(\log k_n)$, $i \in \mathbb{N}$. Using this in the developments of the numerator and denominator in (13) and normalizing conveniently we get, after some calculations

$$Z_{r^c}(m) \stackrel{d}{=} \frac{d_{n-t_n}^{(c)}}{d_{n-t_n-c}^{(c)}} \frac{1 + \frac{w(q_{m/r^c,n} T_{(m/r^c)} + \delta_n) d_{n-t_n+c}^{(1)}}{d_{n-t_n}^{(c)}} - \frac{w(q_{m,n} T_{(m)} + \beta_n) d_{n-t_n}^{(1)}}{d_{n-t_n}^{(c)}} + \frac{o_P(d_{n-t_n+c}^{(1)}) - o_P(d_{n-t_n}^{(1)})}{d_{n-t_n}^{(c)}}}{1 + \frac{w(q_{m,n} T_{(m)} + \beta_n) d_{n-t_n}^{(1)}}{d_{n-t_n-c}^{(c)}} - \frac{w(q_{mr^c,n} T_{(mr^c)} + \alpha_n) d_{n-t_n-c}^{(1)}}{d_{n-t_n-c}^{(c)}} + \frac{o_P(d_{n-t_n}^{(1)}) - o_P(d_{n-t_n-c}^{(1)})}{d_{n-t_n-c}^{(c)}}}.$$

Applying the results from Lemma 4.2 in [4] we obtain (11). By Lemma 2.1 we know that the asymptotic distribution of $(\sqrt{m} T_{(mr^c)}, \sqrt{m} T_{(m)}, \sqrt{m} T_{(m/r^c)})$ is 3-variate normal with mean zero and covariance matrix $[\sigma_{ij}]$ with $\sigma_{ij} = r^{(i+j-4)c\gamma + (i-2)c}$, $i \leq j$, $i, j = 1, 2, 3$. We can obtain the desired result using the delta method. This method allow us to prove that the asymptotic distribution of a function $g(T_{1,n}, T_{2,n}, \ldots, T_{k,n}, n)$ involving n explicitly, with suitable normalizing factors, is standard normal when the asymptotic distribution of the vector of statistics $\left(\sqrt{n}(T_{1,n} - \theta_1), \sqrt{n}(T_{2,n} - \theta_2), \ldots, \sqrt{n}(T_{k,n} - \theta_k)\right)$ is multidimensional normal, with mean zero and known covariance matrix, and the first order derivatives of g in order to x_i, $i = 1, \ldots, k$, exist and are finite, when $x_i \to \theta_i$ and $n \to +\infty$ (see for instance Rao [9]). Let $u = (a^c - 1)/(a - 1)$. Due to (11) we consider the function

$$g(x_1, x_2, x_3, m) = a^c \frac{u + a^c w(q_{m/r^c,n} x_3 + \delta_n) - w(q_{m,n} x_2 + \beta_n)}{u + a^c w(q_{m,n} x_2 + \beta_n) - w(q_{mr^c,n} x_1 + \alpha_n)},$$

with first order partial derivatives

$$\frac{\partial g}{\partial x_1} = a^c q_{mr^c,n} w'(q_{mr^c,n} x_1 + \alpha_n) \frac{u + a^c w(q_{m/r^c,n} x_3 + \delta_n) - w(q_{m,n} x_2 + \beta_n)}{[u + a^c w(q_{m,n} x_2 + \beta_n) - w(q_{mr^c,n} x_1 + \alpha_n)]^2},$$

$$\frac{\partial g}{\partial x_2} = -a^c q_{m,n} w'(q_{m,n} x_2 + \beta_n) \frac{u + a^c u + a^{2c} w(q_{m/r^c,n} x_3 + \delta_n) - w(q_{mr^c,n} x_1 + \alpha_n)}{[u + a^c w(q_{m,n} x_2 + \beta_n) - w(q_{mr^c,n} x_1 + \alpha_n)]^2},$$

$$\frac{\partial g}{\partial x_3} = \frac{a^{2c} q_{m/r^c,n} w'(q_{m/r^c,n} x_3 + \delta_n)}{u + a^c w(q_{m,n} x_2 + \beta_n) - w(q_{mr^c,n} x_1 + \alpha_n)}.$$

Defining $h_n := 1/[u + a^c w(\beta_n) - w(\alpha_n)]$, we have

$$\xi_m := g(0,0,0,m) = a^c \frac{u + a^c w(\delta_n) - w(\beta_n)}{u + a^c w(\beta_n) - w(\alpha_n)},$$

$$\frac{\partial g}{\partial x_1}(0,0,0,m) = a^c q_{mr^c,n} w'(\alpha_n) \big(u + a^c w(\delta_n) - w(\beta_n)\big) h_n^2,$$

$$\frac{\partial g}{\partial x_2}(0,0,0,m) = -a^c q_{m,n} w'(\beta_n) \big(u + a^c u + a^{2c} w(\delta_n) - w(\alpha_n)\big) h_n^2,$$

$$\frac{\partial g}{\partial x_3}(0,0,0,m) = a^{2c} q_{m/r^c,n} w'(\delta_n) h_n.$$

Therefore we can consider

$$\begin{aligned}
\upsilon_m &= \sum_{i=1}^{3} \sum_{j=1}^{3} \sigma_{ij} = \frac{\partial g}{\partial x_i}(0,0,0,m) \frac{\partial g}{\partial x_j}(0,0,0,m) \\
&= r^{-c} h_n^4 \big(q_{mr^c,n} w'(\alpha_n)\big)^2 \big(u + a^c w(\delta_n) - w(\beta_n)\big)^2 + r^{6c\gamma + c} h_n^2 \big(q_{m/r^c,n} w'(\delta_n)\big)^2 \\
&\quad + r^{2c\gamma} h_n^4 \big(q_{m,n} w'(\beta_n)\big)^2 \big(u + a^c u + a^{2c} w(\delta_n) - w(\alpha_n)\big)^2 - 2 r^{c\gamma - c} h_n^4 q_{mr^c,n} q_{m,n} \\
&\quad \times w'(\alpha_n) w'(\beta_n) \big(u + a^c w(\delta_n) - w(\beta_n)\big)\big(u + a^c u + a^{2c} w(\delta_n) - w(\alpha_n)\big) \\
&\quad + 2 r^{3c\gamma - c} h_n^3 q_{mr^c,n} q_{m/r^c,n} w'(\alpha_n) w'(\delta_n) \big(u + a^c w(\delta_n) - w(\beta_n)\big) \\
&\quad - 2 r^{4c\gamma} h_n^3 q_{m,n} q_{m/r^c,n} w'(\beta_n) w'(\delta_n) \big(u + a^c u + a^{2c} w(\delta_n) - w(\alpha_n)\big). \qquad \square
\end{aligned}$$

Theorem 2.3. *Let $\{X_i\}_{i\geq 1}$ be a sequence of independent and identically distributed random variables with continuous d.f. F. Suppose that, for some sequence $\{k_n\}$ satisfying* (3), *F verifies* (2) *with G differentiable. Let $\{N_n\}$ be an integer sequence such that $N_n = [k_n]l_n$ where $1 \leq l_n < [k_{n+1}]/[k_n]$ and $\lim_{n\to+\infty} l_n = l \in [1, r]$. Let $\{m_n\}$ and $\{t_n\}$ be integer sequences such that $\lim_{n\to+\infty} m_n = \lim_{n\to+\infty} t_n = \lim_{n\to+\infty}(n - t_n) = +\infty$. Consider the sequence of statistics, with $m := m_n = [k_n/k_{n-t_n}]$ and $N := N_n$,*

$$R_{r^c}(m) := \frac{Z_{r^{2c}}(m)}{\left(Z_{r^c}(m)\right)^2} = \frac{X_{(m/r^{2c})} - X_{(m)}}{X_{(m)} - X_{(mr^{2c})}} \left(\frac{X_{(m)} - X_{(mr^c)}}{X_{(m/r^c)} - X_{(m)}}\right)^2.$$

Then there exist normalizing real sequences $\{\xi_m\}$ and $\{\upsilon_m\}$, with $\upsilon_m > 0$ and $\lim_{n\to+\infty} \xi_m = 1$, such that the asymptotic distribution of $\upsilon_m^{-1}\sqrt{m}\left(R_{r^c}(m) - \xi_m\right)$ is standard normal.

Proof. Again we only prove the case $a \neq 1$. The case $a = 1$ can be easily obtained in a similar way. Using the same arguments as in the previous proof, by (12), we can write

$$\begin{aligned} Z_{r^{2c}}(m) &\stackrel{d}{=} \frac{V(Y_{(m/r^{2c})}) - V(Y_{(m)})}{V(Y_{(m)}) - V(Y_{(mr^{2c})})} \\ &= \frac{V(\log k_{n-t_n+2c} + q_{m/r^{2c},n}T_{(m/r^{2c})} + \varepsilon_n) - V(\log k_{n-t_n} + q_{m,n}T_{(m)} + \beta_n)}{V(\log k_{n-t_n} + q_{m,n}T_{(m)} + \beta_n) - V(\log k_{n-t_n-2c} + q_{mr^{2c},n}T_{(mr^{2c})} + \mu_n)}, \end{aligned}$$

where $q_{m,n} = 1 + m/N$, $T_{(m)} = \left(Y_{(m)} - \log(N/m)\right)/q_{m,n}$, $\beta_n := \log\left(\frac{N}{k_{n-t_n}[k_n/k_{n-t_n}]}\right) = \log l_n + o(1)$, $\varepsilon_n := \beta_n + \log\left(\frac{r^{2c}k_{n-t_n}}{k_{n-t_n+2c}}\right) = \beta_n + o(1)$ and $\mu_n := \beta_n + \log\left(\frac{k_{n-t_n}}{r^{2c}k_{n-t_n-2c}}\right) = \beta_n + o(1)$. Taking into account (14), we get

$$Z_{r^{2c}}(m) \stackrel{d}{=} \frac{d^{(2c)}_{n-t_n}}{d^{(2c)}_{n-t_n-2c}} \frac{1 + \frac{w(q_{m/r^{2c},n}T_{(m/r^{2c})}+\varepsilon_n)d^{(1)}_{n-t_n+2c} - w(q_{m,n}T_{(m)}+\beta_n)d^{(1)}_{n-t_n}}{d^{(2c)}_{n-t_n}} + \frac{o_P(d^{(1)}_{n-t_n+2c}) - o_P(d^{(1)}_{n-t_n})}{d^{(2c)}_{n-t_n}}}{1 + \frac{w(q_{m,n}T_{(m)}+\beta_n)d^{(1)}_{n-t_n} - w(q_{mr^{2c},n}T_{(mr^{2c})}+\mu_n)d^{(1)}_{n-t_n-2c}}{d^{(2c)}_{n-t_n-2c}} + \frac{o_P(d^{(1)}_{n-t_n}) - o_P(d^{(1)}_{n-t_n-2c})}{d^{(2c)}_{n-t_n-2c}}}. \tag{15}$$

Attending once again to Lemma 4.2 in [4], from (15) we deduce

$$Z_{r^{2c}}(m) \stackrel{d}{=} a^{2c}\, \frac{\frac{a^{2c}-1}{a-1} + a^{2c}w(q_{m/r^{2c},n}T_{(m/r^{2c})} + \varepsilon_n) - w(q_{m,n}T_{(m)} + \beta_n) + o_P(1)}{\frac{a^{2c}-1}{a-1} + a^{2c}w(q_{m,n}T_{(m)} + \beta_n) - w(q_{mr^{2c},n}T_{(mr^{2c})} + \mu_n) + o_P(1)}. \tag{16}$$

Due to (11) and (16) we obtain

$$R_{r^c}(m)$$

$$\stackrel{d}{=} a^{2c}\,\frac{\frac{a^{2c}-1}{a-1}+a^{2c}w(q_{m/r^{2c},n}T_{(m/r^{2c})}+\varepsilon_n)-w(q_{m,n}T_{(m)}+\beta_n)+o_P(1)}{\frac{a^{2c}-1}{a-1}+a^{2c}w(q_{m,n}T_{(m)}+\beta_n)-w(q_{mr^{2c},n}T_{(mr^{2c})}+\mu_n)+o_P(1)}$$

$$\times\left[a^{-c}\,\frac{\frac{a^{c}-1}{a-1}+a^{c}w(q_{m,n}T_{(m)}+\beta_n)-w(q_{mr^{c},n}T_{(mr^{c})}+\alpha_n)+o_P(1)}{\frac{a^{c}-1}{a-1}+a^{c}w(q_{m/r^{c},n}T_{(m/r^{c})}+\delta_n)-w(q_{m,n}T_{(m)}+\beta_n)+o_P(1)}\right]^2. \quad (17)$$

Using again Lemma 2.1 we can establish that the asymptotic distribution of

$$(\sqrt{m}T_{(mr^{2c})},\sqrt{m}T_{(mr^{c})},\sqrt{m}T_{(m)},\sqrt{m}T_{(m/r^{c})},\sqrt{m}T_{(m/r^{2c})})$$

is 5-variated normal with mean zero and covariance matrix given by $\sigma_{ij} = r^{(i+j-6)c\gamma+(i-3)c}$, $i \le j$, $i,j = 1,2,3,4,5$. We apply again the delta method and, due to (17), we choose the function

$$g(x_1,x_2,x_3,x_4,x_5,m) = \frac{u_2+a^{2c}w(q_{m/r^{2c},n}x_5+\varepsilon_n)-w(q_{m,n}x_3+\beta_n)}{u_2+a^{2c}w(q_{m,n}x_3+\beta_n)-w(q_{mr^{2c},n}x_1+\mu_n)}$$

$$\times\left[\frac{u_1+a^{c}w(q_{m,n}x_3+\beta_n)-w(q_{mr^{c},n}x_2+\alpha_n)}{u_1+a^{c}w(q_{m/r^{c},n}x_4+\delta_n)-w(q_{m,n}x_3+\beta_n)}\right]^2$$

$$:= \frac{\lambda}{\theta}\frac{\eta^2}{\tau^2},$$

where $u_i = \frac{a^{ic}-1}{a-1}$, $i = 1,2$, $\eta := u_1 + a^c w(q_{m,n}x_3+\beta_n) - w(q_{mr^c,n}x_2+\alpha_n)$, $\tau := u_1 + a^c w(q_{m/r^c,n}x_4+\delta_n) - w(q_{m,n}x_3+\beta_n)$, $\theta := u_2 + a^{2c}w(q_{m,n}x_3+\beta_n) - w(q_{mr^{2c},n}x_1+\mu_n)$ and $\lambda := u_2 + a^{2c}w(q_{m/r^{2c},n}x_5+\varepsilon_n) - w(q_{m,n}x_3+\beta_n)$. The first order partial derivatives of g are

$$\frac{\partial g}{\partial x_1} = q_{mr^{2c},n}w'(q_{mr^{2c},n}x_1+\mu_n)\frac{\lambda}{\theta^2}\frac{\eta^2}{\tau^2},$$

$$\frac{\partial g}{\partial x_2} = -2q_{mr^{c},n}w'(q_{mr^{c},n}x_2+\alpha_n)\frac{\lambda}{\theta}\frac{\eta}{\tau^2},$$

$$\frac{\partial g}{\partial x_3} = q_{m,n}w'(q_{m,n}x_3+\beta_n)\frac{\eta}{\tau^2\theta}\left[2a^c\lambda+2\lambda\frac{\eta}{\tau}-\eta-a^{2c}\eta\frac{\lambda}{\theta}\right],$$

$$\frac{\partial g}{\partial x_4} = -2a^c q_{m/r^{c},n}w'(q_{m/r^{c},n}x_4+\delta_n)\frac{\lambda}{\theta}\frac{\eta^2}{\tau^3},$$

$$\frac{\partial g}{\partial x_5} = a^{2c}q_{m/r^{2c},n}w'(q_{m/r^{2c},n}x_5+\varepsilon_n)\frac{1}{\theta}\frac{\eta^2}{\tau^2}.$$

Define $\eta_0 := u_1 + a^c w(\beta_n) - w(\alpha_n)$, $\tau_0 := u_1 + a^c w(\delta_n) - w(\beta_n)$, $\theta_0 := u_2 + a^{2c} w(\beta_n) - w(\mu_n)$, $\lambda_0 := u_2 + a^{2c} w(\varepsilon_n) - w(\beta_n)$ and $\varphi_n := 2a^c \lambda_0 + 2\lambda_0 \eta_0 / \tau_0 - \eta_0 - a^{2c} \eta_0 \lambda_0 / \theta_0$. Thus, with $\underline{0} = (0,0,0,0,0)$, we get

$$\xi_m := g(\underline{0}, m) = \frac{\lambda_0}{\theta_0} \frac{\eta_0^2}{\tau_0^2}, \qquad \frac{\partial g}{\partial x_1}(\underline{0}, m) = q_{mr^{2c},n} w'(\mu_n) \frac{\lambda_0}{\theta_0^2} \frac{\eta_0^2}{\tau_0^2},$$

$$\frac{\partial g}{\partial x_2}(\underline{0}, m) = -2 q_{mr^c,n} w'(\alpha_n) \frac{\lambda_0}{\theta_0} \frac{\eta_0}{\tau_0^2}, \qquad \frac{\partial g}{\partial x_3}(\underline{0}, m) = q_{m,n} \frac{w'(\beta_n)\eta_0}{\tau_0^2 \theta_0} \varphi_n,$$

$$\frac{\partial g}{\partial x_4}(\underline{0}, m) = -2a^c q_{m/r^c,n} w'(\delta_n) \frac{\lambda_0}{\theta_0} \frac{\eta_0^2}{\tau_0^3}, \qquad \frac{\partial g}{\partial x_5}(\underline{0}, m) = a^{2c} q_{m/r^{2c},n} w'(\varepsilon_n) \frac{1}{\theta_0} \frac{\eta_0^2}{\tau_0^2}.$$

Therefore

$$\upsilon_m := \sum_{i=1}^{5} \sum_{j=1}^{5} \sigma_{ij} \frac{\partial g}{\partial x_i}(\underline{0}, m) \frac{\partial g}{\partial x_j}(\underline{0}, m). \qquad \square$$

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