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The best constant for an almost critical Sobolev imbedding

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Abstract. The inequality,

$$\|u\|_\infty \le \frac{C}{\sqrt{\varepsilon}}\|u\|_{H^{N/2+\varepsilon}} \qquad \text{for all } u \in H^{N/2+\varepsilon}(\Omega),$$

is shown for Ω a sufficiently smooth domain in $\mathbb{R}^N$ or a product of N bounded open intervals. This inequality is optimal for any open domain Ω.

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1. Introduction

This paper is motivated by a question of C. Bernardi and A. Blouza. Inspired by the optimal estimate of the imbedding from $H^1(\Omega)$ into $L^p(\Omega)$

$$\|u\|_p \le C\sqrt{p}\|u\|_{H^1}$$

when Ω is a domain in $\mathbb{R}^2$ (see, e.g., [4]), C. Bernardi conjectured that for reasonably regular 2-dimensional domains one should have for ε small enough

$$\|u\|_\infty \le \frac{C}{\sqrt{\varepsilon}}\|u\|_{H^{1+\varepsilon}} \qquad \text{for all } u \in H^{1+\varepsilon}(\Omega).$$

The purpose of this paper is to show that the conjecture is true and moreover a similar property is valid in any dimension. In addition the inequality is optimal for all domains. This property has been used for two-dimensional rectangles in the recent paper of C. Bernardi and A. Blouza [2]. For general results on Sobolev spaces and lifting operators we refer to [1], [3].

2. Main results

In this work we shall establish the following results:

Theorem 2.1. *Let $N \in \mathbb{N}^*$ and let Π be a product of N open intervals. There exists a constant $C = C(\Pi)$ for which*

$$\|u\|_\infty \le \frac{C}{\sqrt{\varepsilon}}\|u\|_{H^{N/2+\varepsilon}} \quad \text{for all } \varepsilon \in (0, 1/2) \text{ and all } u \in H^{N/2+\varepsilon}(\Pi).$$

Theorem 2.2. *Let Ω be an open subset of $\mathbb{R}^N$, and assume that there exists a lifting operator $P \in \mathscr{L}\big(H^{N/2}(\Omega), H^{N/2}(\mathbb{R}^N)\big) \cap \mathscr{L}\big(H^{N/2+\eta}(\Omega), H^{N/2+\eta}(\mathbb{R}^N)\big)$ for some $\eta > 0$ satisfying*

$$Pu|_\Omega = u \quad \text{for all } u \in H^{N/2}(\Omega).$$

Then there exists a constant $C = C(\Omega)$ for which

$$\|u\|_\infty \le \frac{C}{\sqrt{\varepsilon}}\|u\|_{H^{N/2+\varepsilon}} \quad \text{for all } \varepsilon \in (0, \eta) \text{ and all } u \in H^{N/2+\varepsilon}(\Omega).$$

Although Theorem 2.1 is obviously a special case of Theorem 2.2 we state two results and give two proofs, because our proof of Theorem 2.1 provides a way to compute an explicit constant C as soon as we choose a specific formula for the norm in $H^{N/2+\varepsilon}(\Pi)$ (there are infinitely many relevant choices). In addition the structure which appears in the proof of Theorem 2.1 turns out to be fundamental for the optimality result at the end of the paper.

3. Proof of Theorem 2.1

1) We consider first the special case $N = 1$. By scaling, we may obviously assume $\Pi = (0, \pi)$. In this case, for any $u \in H^{1/2+\varepsilon}(\Pi)$ we can write

$$u(x) = \sum_{n=0}^{\infty} u_n \cos nx,$$

where

$$\sum_{n=0}^{\infty} (1 + n^2)^{1/2+\varepsilon} u_n^2 < \infty,$$

and we can select as an equivalent norm on $H^{1/2+\varepsilon}$ the expression

$$p_\varepsilon(u) = \left\{ \sum_{n=0}^{\infty} (1 + n)^{1+2\varepsilon} u_n^2 \right\}^{1/2}.$$

Now we have obviously

$$\|u\|_\infty \le \sum_{n=0}^{\infty} |u_n|$$

and by the Cauchy–Schwarz inequality for series we deduce that

$$\|u\|_\infty \le \left\{ \sum_{n=0}^{\infty} (1+n)^{-(1+2\varepsilon)} \right\}^{1/2} p_\varepsilon(u).$$

The result follows immediately since

$$\sum_{n=0}^{\infty} (1+n)^{-(1+2\varepsilon)} \le 1 + \int_1^\infty x^{-(1+2\varepsilon)}\, dx = 1 + \frac{1}{2\varepsilon}.$$

2) In the general case we are reduced, by an independent scaling of the N variables, to the case $\Pi = (0,\pi)^N$. For any $u \in H^{N/2+\varepsilon}(\Pi)$ we write

$$u = \sum_\alpha u_\alpha \phi_\alpha,$$

where

$$\sum_\alpha (1 + \|\alpha\|^2)^{N/2+\varepsilon} u_\alpha^2 < \infty$$

with

$$\alpha = (m_1, m_2, \ldots, m_N) \in \mathbb{N}^N, \qquad \|\alpha\|^2 = \sum_{j=1}^{N} m_j^2,$$

$$\phi_\alpha(x) = \phi_\alpha(x_1, x_2, \ldots, x_N) = \prod_{j=1}^{N} \cos m_j x_j,$$

and we can select as an equivalent norm on $H^{N/2+\varepsilon}(\Pi)$ the expression

$$p_\varepsilon(u) = \left\{ \sum_\alpha (N + |\alpha|)^{N+2\varepsilon} u_\alpha^2 \right\}^{1/2}$$

with

$$|\alpha| = \sum_{j=1}^{N} m_j.$$

Now we have

$$\|u\|_{\infty} \leq \sum_{\alpha} |u_\alpha|$$

and by the Cauchy–Schwarz inequality for series we deduce that

$$\|u\|_{\infty} \leq \left\{ \sum_{\alpha} (N + |\alpha|)^{-(N+2\varepsilon)} \right\}^{1/2} p_\varepsilon(u).$$

By summing first in m_N for $\beta = (m_1, m_2, \ldots, m_{N-1})$ fixed and using the inequality

$$\begin{aligned}\sum_{n=1}^{\infty} (N - 1 + |\beta| + n)^{-(N+2\varepsilon)} &\leq \int_0^{\infty} (N - 1 + |\beta| + x)^{-(N+2\varepsilon)}\, dx \\ &= \frac{1}{N - 1 + 2\varepsilon} (N - 1 + |\beta|)^{-(N-1+2\varepsilon)} \\ &\leq (N - 1 + |\beta|)^{-(N-1+2\varepsilon)},\end{aligned}$$

we obtain that

$$\begin{aligned}\sum_{\alpha} (N + |\alpha|)^{-(N+2\varepsilon)} &= \sum_{\beta} \sum_{n=1}^{\infty} (N - 1 + |\beta| + n)^{-(N+2\varepsilon)} \\ &\leq \sum_{\beta} (N - 1 + |\beta|)^{-(N-1+2\varepsilon)}\end{aligned}$$

as long as $N \geq 2$. By induction we are reduced after $N - 1$ steps to the case $N = 1$ and find that

$$\sum_{\alpha} (N + |\alpha|)^{-(N+2\varepsilon)} \leq 1 + \frac{1}{2\varepsilon},$$

concluding the proof of Theorem 2.1 for $N \geq 2$.

4. Proof of Theorem 2.2

We start with the simple case $\Omega = \mathbb{R}^N$. In this case we have

$$u \in H^{N/2+\varepsilon} \Leftrightarrow f_\varepsilon = (1+|\xi|^2)^{N/4+\varepsilon/2}\hat{u} \in L^2$$

and

$$\|u\|_{H^{N/2+\varepsilon}} = \|f_\varepsilon\|_2.$$

Then

$$\hat{u} = \frac{f_\varepsilon}{(1+|\xi|^2)^{N/4+\varepsilon/2}}$$

and by Cauchy–Schwarz we deduce that

$$\|\hat{u}\|_1 \le \|f_\varepsilon\|_2 \left\{ \int \frac{d\xi}{(1+|\xi|^2)^{N/2+\varepsilon}} \right\}^{1/2}.$$

On the other hand,

$$\begin{aligned}
\int \frac{d\xi}{(1+|\xi|^2)^{N/2+\varepsilon}} &= C_N \int_0^\infty \frac{\rho^{N-1}\,d\rho}{(1+\rho^2)^{N/2+\varepsilon}} \\
&\le C_N \int_0^\infty \frac{(1+\rho^2)^{(N-1)/2}\,d\rho}{(1+\rho^2)^{N/2+\varepsilon}} \\
&= C_N \int_0^\infty \frac{d\rho}{(1+\rho^2)^{1/2+\varepsilon}} \\
&\le C_1 + C_2 \int_1^\infty \frac{d\rho}{\rho^{1+2\varepsilon}} \\
&\le C\left(1+\frac{1}{\varepsilon}\right)
\end{aligned}$$

and finally

$$\|\hat{u}\|_1 \le \frac{C}{\sqrt{\varepsilon}} \|u\|_{H^{N/2+\varepsilon}}$$

for ε in a bounded interval. By inverting the Fourier transform the result follows at once for any $\eta > 0$ in the case $\Omega = \mathbb{R}^N$.

In the general case it suffices to observe that by interpolation theory we have

$$P \in \mathscr{L}\big(H^{N/2+\varepsilon}(\Omega), H^{N/2+\varepsilon}(\mathbb{R}^N)\big)$$

with uniformly bounded norm for $0 \le \varepsilon \le \eta$. Then we have

$$\|u\|_\infty \le \|Pu\|_\infty \le \frac{C}{\sqrt{\varepsilon}}\|Pu\|_{H^{N/2+\varepsilon}} \le \frac{C'}{\sqrt{\varepsilon}}\|u\|_{H^{N/2+\varepsilon}},$$

concluding the proof of Theorem 2.2.

5. Optimality of the results

Theorem 5.1. *Let $N \in \mathbb{N}^*$ and let Ω be any open subset of $\mathbb{R}^N$. Then there exists a constant $c = c(\Omega) > 0$ such that for all $\varepsilon \in (0, 1/2)$ there exists $u_\varepsilon \in H^{N/2+\varepsilon}(\Omega)$, $u_\varepsilon \not\equiv 0$, with*

$$\|u_\varepsilon\|_\infty \ge \frac{c}{\sqrt{\varepsilon}}\|u_\varepsilon\|_{H^{N/2+\varepsilon}}.$$

Proof. We may assume, up to translation and scaling, that

$$\overline{\Pi} = [0, \pi]^N \subset \Omega.$$

Let $\varphi \in \mathscr{D}(\Omega)$ be a fixed function with $\varphi = 1$ on $\overline{\Pi}$ and define a bounded continuous function on $\mathbb{R}^N$ by the uniformly convergent N-periodic series

$$v_\varepsilon = \sum_\alpha (N + |\alpha|)^{-(N+2\varepsilon)}\phi_\alpha$$

and then a function on Ω by

$$u_\varepsilon = \varphi v_\varepsilon.$$

We have

$$\|u_\varepsilon\|_\infty \ge u_\varepsilon(0) = v_\varepsilon(0) = \sum_\alpha (N + |\alpha|)^{-(N+2\varepsilon)}.$$

On the other hand,

$$\|u_\varepsilon\|_{H^{N/2+\varepsilon}} \le M\|v_\varepsilon\|_{H^{N/2+\varepsilon}}$$

and

$$\|v_\varepsilon\|_{H^{N/2+\varepsilon}} \le M' p_\varepsilon(v_\varepsilon) = M'\Big\{\sum_\alpha (N + |\alpha|)^{-(N+2\varepsilon)}\Big\}^{1/2}.$$

Finally

$$\|u_\varepsilon\|_{H^{N/2+\varepsilon}} \le MM'\Big\{\sum_\alpha (N+|\alpha|)^{-(N+2\varepsilon)}\Big\}^{-1/2}\|u_\varepsilon\|_\infty.$$

It suffices to check that

$$\sum_\alpha (N+|\alpha|)^{-(N+2\varepsilon)} \ge M''\varepsilon^{-1}$$

for all $\varepsilon \in (0, 1/2)$. An easy calculation similar to the argument of Theorem 2.1 shows that in fact

$$\begin{aligned}\sum_\alpha (N+|\alpha|)^{-(N+2\varepsilon)} &\ge \int_1^\infty \int_1^\infty \dots \int_1^\infty (N+|\alpha|)^{-(N+2\varepsilon)}\, dm_1\, dm_2 \dots dm_N \\ &= \frac{1}{(2N)^{2\varepsilon}} \prod_{k=0}^{N-1} \frac{1}{k+2\varepsilon} \ge \frac{1}{2N.N^{N-1}.2\varepsilon} = \frac{1}{4N^N\varepsilon}\end{aligned}$$

since $2\varepsilon \le 1$, and this is clearly enough to conclude the proof. □

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References

[1] R. A. Adams, *Sobolev spaces*. Pure Appl. Math. 65, Academic Press, New York 1975. Zbl 0314.46030 MR 0450957

[2] C. Bernardi and A. Blouza, Spectral discretization of a Naghdi shell model. *SIAM J. Numer. Anal.* **45** (2007), 2653–2670. Zbl 1154.74048 MR 2361906

[3] J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications*. Dunod, Paris 1968. Zbl 0165.10801 MR 0247243

[4] G. Talenti, Best constant in Sobolev inequality. *Ann. Mat. Pura Appl.* (4) **110** (1976), 353–372. Zbl 0353.46018 MR 0463908

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