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Nonlinear reaction diffusion systems of degenerate parabolic type

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Abstract. In this paper we study the following parabolic problem

$$\begin{cases} \partial_t(u_i) - \Delta(|u_i|^{\sigma_i}u_i) = g_i(u) + \overrightarrow{b_i}\nabla(|u_i|^{m_i-1}u_i) & \text{in }]0,\infty[\times\Omega, \\ u_i = 0 & \text{on }]0,\infty[\times\partial\Omega, \\ u_i(0,.) = u_{i0} & \text{in } \Omega, \end{cases}$$

where Ω is a bounded domain with smooth boundary and $i = 1, 2, \dots d$. Our aim is to study existence of globally bounded weak solutions or blow-up, depending on the relations between the parameters that appear in the problem.

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1. Introduction

The purpose of this paper is to study a reaction-diffusion system of the following type:

$$\partial_t(u_i) - \Delta(|u_i|^{\sigma_i}u_i) = f_i(u, \nabla u_i) \quad \text{in } (0,\infty)\times\Omega \quad i = 1,\dots,d, \tag{1.1}$$

where u is the vector $u = (u_1, \dots, u_d)$, d is an integer ≥ 1, $\sigma_i > 0$ and the reacting functions f_i have the following model form

$$f_i(u, \nabla u_i) = g_i(u) + \overrightarrow{b_i}\nabla(|u_i|^{m_i-1}u_i) \quad i = 1,\dots,d, \tag{1.2}$$

with $\overrightarrow{b_i} = \overrightarrow{b_i}(t,x) \in \mathbb{R}^N$, $m_i > 0$. We supplement this system with boundary conditions

$$u_i = 0 \quad \text{in } (0,\infty)\times\partial\Omega \quad i = 1,\dots,d, \tag{1.3}$$

and the initial data

$$u_i(0,.) = u_{i0} \quad \text{in } \Omega,\ i = 1,\dots,d, \tag{1.4}$$

Throughout this paper we use the following notations.

Let i and j be positive integers such that $1 \leq i, j \leq d$, T and τ be positive real numbers such that $T > \tau$, η is arbitrary positive real number, Ω is a bounded open set in $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$, $x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N$, $\Delta := \sum_{k=1}^{N} \partial_k^2$ denotes the Laplace operator in Euclidean coordinates, ∇ is the gradient with respect to x and the outer normal on $\partial\Omega$ is denoted by $\nu = (\nu_1, \nu_2, \dots \nu_N)$, finally $\mathrm{Hess}(u)$ is the hessian of u. In the following we will denote $(0,T) \times \Omega$ by Q_T, and $(\tau, T) \times \Omega$ by $Q_{\tau,T}$. The norm in $L^p(\Omega)$, $p > 1$, will be written $\|.\|_p$ and we also make use of the Sobolev spaces, especially of

$$W^{1,p}(\Omega) := \{u : \Omega \to \mathbb{R} \,|\, u \in L^p(\Omega) \text{ and } \nabla u \in \left(L^p(\Omega)\right)^N\}$$
$$W_p^{1,2}(\Omega) := \{u : \Omega \to \mathbb{R} \,|\, u \in L^p(\Omega) \text{ and } \nabla u \in \left(L^2(\Omega)\right)^N\}$$
$$W^{2,p}(\Omega) := \{u \in W^{1,p}(\Omega) \,|\, \mathrm{Hess}(u) \in \left(L^p(\Omega)\right)^{N\times N}\}$$
$$W_p^{1,2}(Q_T) := \{u : Q_T \to \mathbb{R} \,|\, u \in L^p\left(]0,T[, W^{2,p}(\Omega)\right) \text{ and } u_t \in L^p\left(]0,T], L^p(\Omega)\right)\}$$

and

$$H(\Omega) := \{u = (u_1, u_2, \dots u_d)\ v_i : \Omega \to \mathbb{R} \,|\, \|\nabla(|u_i|_i^{\sigma_i} u)\|^2 \in L^2(Q_T), i = 1, d\}.$$

Once for all, we notice that the different constants (independent of ε) are denoted by the same latter C.

System (1.1)–(1.4), in the case $\overrightarrow{b_i} = \vec{0}$ has been studied extensively under various types of initial and boundary conditions by a large number of authors, see among others [3], [2], [4], [10], [11], [13], [15] and the literature therein.

This problem describes (in the case $\overrightarrow{b_i} = \vec{0}$) many phenomena, for example it describes non-stationary gas filtration in a porous medium (where u represents the density of the gas) or the diffusion in an biological population (u represents the density of the population) see [15]. Finally in [18] u can be treated as a temperature vector of interacting components of a combustible mixture. In the case $\overrightarrow{b_i} \neq \vec{0}$ the system (1.1)–(1.4) arises in:

1. *Population dynamics.* In the following system

$$\begin{cases} S_t - \Delta S^m = -I(\gamma S - \delta) + \vec{b}\nabla S, \\ I_t - \Delta I^n = I(\gamma S - \delta) + \alpha\vec{b}\nabla I \end{cases} \quad \text{in } (0,\infty) \times \Omega,$$

S and I represent, respectively, (as cited in [4] in the case when $\vec{b} = \vec{0}$) the densities of susceptibles and infectives under the effect of certain natural mechanism represented by $\vec{b}$, γIS is the force of infection or incidence term; it represents the number of susceptible individuals S infected by contact with infective individuals I per time unit; and δI is the number of infectives who become susceptibles after recovery.

2. *Environmental purification.* Suppose that a polluted river contains d suspensions with concentration u_i, $i = 1, 2, \dots d$. Then we obtain the following equations

$$\frac{\partial u_i}{\partial t} - \gamma_i \Delta u_i = F_i(u) - \operatorname{div}(Vu_i)$$

where V is the velocity of water flow.

The following results are well known. First, in the work of Galaktionov [10], it is proved that the global existence of nonnegative solutions of the boundary value problem (1.1)–(1.4) in the case when $d = 1$ and $f(u, \nabla u) = u^{\beta}$, depends on a relation between σ (the power in diffusion term), β, N and the data u_0, where $u_0 \geq 0$.

In [11] the authors considered the system (1.1)–(1.4) with: $d = 2$, $g_1(u_1, u_2) = (u_2)^p$; $g_2(u_1, u_2) = (u_1)^q$; $\overrightarrow{b_i} = \vec{0}$. They proved that the above system has a global nonnegative solution, for arbitrary nonnegative initial functions $u_{i0} \in L^{\sigma i+2}(\Omega)$, if $1 \leq p < \sigma_2 + 1$ and $1 \leq q < \sigma_1 + 1$. For the limit cases $p = \sigma_2 + 1$ or $q = \sigma_1 + 1$ they established that the global solvability of the system depends on the spatial structure of Ω.

In [15] Madallena generalized the preceding work by proving the existence of global nonnegative weak solutions for a reaction-diffusion system (1.1)–(1.4), for arbitrary nonnegative initial functions $u_{i0} \in L^{\infty}(\Omega)$, such that the functions f_i satisfy in the domain $u_j \geq 0$ the following conditions

- $f_i(0) = 0$,
- $f_i(u) \geq 0$ for every $u = (u_1, u_2, \dots u_d)$ such that $u_i = 0$ that is f_i is quasi-positive,
- $f_i(u) \leq \sum_{1 \leq j \leq d} c_{ij} u_j^{\alpha_{ij}} + c_i$ where c_{ij}, $c_i > 0$ and $0 < \alpha_{ij} < \sigma_j + 1$.

Moreover, existence of nonnegative mild solution for nonnegative initial data in $L^{\sigma_i+2}(\Omega)$, when $f_i(u) = \sum_{1 \leq j \leq d} c_{ij} u_j^{\alpha_{ij}}$ and $\alpha_{ij} < \sigma_j + 1$, is studied in [13], and it is proved also that if $\alpha_{ij} = \sigma_j + 1$ solutions may blow-up in finite time.

In this paper we generalize the preceding works, by supposing dependence on the gradient in the reacting terms, that is namely the system (1.1)–(1.4). The paper is organized as follows. In the next section we introduce a weak solution concept and we state our main results on existence, uniqueness and blow-up. In Section 3, which is the core of the remainder, we prove that one can pass from L^{σ_i+1} bounds to an L^∞ one, under various boundary conditions. To derive the L^∞ bounds we use the Moser-type iteration technique of Alikakos (see [1]), for a single equation (in the case $\overrightarrow{b_i} = \vec{0}$) and developed by Dung (see [9]), in the case $0 < \sigma_i < 1$, see also the method developed in [16]. It should be noted that this section has the advantage that, generally speaking, it is hard or almost impossible to establish L^∞ bounds directly from the equation.

Moreover we prove that the solution is more regular than the initial data (to be more precise, we prove that if $u_{i0} \in L^{\sigma_i+2}(\Omega)$ and $\|u_i(t,.)\|_{L^{\sigma_i+1}(\Omega)} \le C(\xi)$ for all $t \ge \xi > 0$ where C is an independent constant of the initial conditions, then $\|u_i(t,.)\|_{L^\infty(\Omega)} \le C(\xi)$ for all $t \ge \xi > 0$) thus we obtain uniform estimates with respect to the initial data u_0.

In Section 5, it will be established that if the initial data belongs to $\prod_{i=1}^d L^{\sigma_i+2}(\Omega)$ then under appropriate growth conditions on g_i, problem (1.1)–(1.4) has a global weak solution $u(t) = \big(u_1(t), u_2(t), \ldots u_d(t)\big)$ $\big(u_i(t) = u_i(t,x)\big)$, which belongs to $\big(L^\infty(\Omega)\big)^d$ for each $t \ge \xi > 0$ and we prove that if the initial data is bounded, problem (1.1)–(1.4) has a unique global weak solution, which is bounded for any $t \ge 0$. In the last section, we prove that in the limit case $\big(f_i(u, \nabla u_i) = \sum_{j=1}^d c_{ij} u_j^{\alpha_{ij}} + \overrightarrow{b_i}\nabla(u_i^{m_i})\big)$, the global solvability depends on the spatial structure of Ω, more precisely, we prove that there exist thick domains Ω such that all (nontrivial) positive weak solutions of (1.1)–(1.4) blow up in finite time, while they exist globally and decay uniformly to zero as $t \to \infty$ if Ω is small.

Remark 1. In practice, it is most important to consider a positive initial data but we will assume that it is arbitrary for mathematical considerations. For simplicity when investigating the limit case we may assume without loss of generality that $u_{i0} \ge 0$ in Ω.

2. Statements of main results

The following assumptions will be made throughout the paper, for all $i = 1, 2, \ldots d$:

(H_1) $1 \le m_i < \sigma_i + 1$,

(H_2) $g_i(0) = 0$,

(H_3) g_i and $\overrightarrow{b_i}$ are locally lipschitz in there arguments,

(H_4) there exist positive constants L_i, α_{ij} with $\alpha_{ij} < \sigma_j + 1$ such that

$$\|\overrightarrow{b_i}\| \le L_i,\ |g_i(u)| \le L_i\Big(\sum_{j=1}^d |u_j|^{\alpha_{ij}} + 1\Big),$$

(H_5) $u_{i0} \in L^{\sigma_i+2}(\Omega)$.

Equation (1.1) is degenerate parabolic at the points where u_i vanishes. Therefore the problem (1.1)–(1.4) has, in general, no classical solutions. The weak solution is defined as follows.

Definition 1. A function $(u_1, u_2, \dots u_d)$ is said to be a *weak solution* of problem (1.1)–(1.4) on Q_T if for all $i = 1, 2, \dots d$

(1) $|u_i|^{\sigma_i} u_i \in L^2(Q_T)$,

(2) $\nabla(|u_i|^{\sigma_i} u_i)$ exists in the sense of distributions in Q_T and $\nabla(|u_i|^{\sigma_i} u_i) \in \left(L^2(Q_T)\right)^N$,

(3) $u_i = 0$ on $(0, T) \times \partial\Omega$ in the sense of the traces,

(4) u_i satisfies the identity

$$\begin{aligned}\int_\Omega u_i(x,T)\varphi_i(x,T)\,dx - \int_{Q_T} \varphi_{it} u_i\,dx\,dt + \int_{Q_T} \nabla(|u_i|^{\sigma_i} u_i)\nabla\varphi_i\,dx\,dt \\ = \int_{Q_T} \left(g_i(u)\varphi_i - \overrightarrow{b_i}\nabla\varphi_i|u_i|^{m_i-1}u_i - \operatorname{div}(\overrightarrow{b_i})\varphi_i|u_i|^{m_i-1}u_i\right)dx\,dt \\ + \int_\Omega u_{i0}(x)\varphi_i(0,x)\,dx\end{aligned}$$

for every $\varphi_i \in C^1(\overline{Q_T})$ such that $\varphi_i = 0$ *on* $(0, T) \times \partial\Omega$.

We shall say that u is a global weak solution of problem (1.1)–(1.4) if u is a weak solution on Q_T for all $T > 0$. By blow-up of solutions we mean that the solution is defined in $(0, T)$, $0 < T \le \infty$, and that at time T we have,

$$\lim_{t \nearrow T} \|u(t, .)\|_{L^\infty(\Omega)} = +\infty.$$

With respect to global existence and uniqueness our main result is the following.

Theorem 2.1. *Under the above assumptions, there exists a global weak solution* $u = (u_1, u_2, \dots u_d)$ *of the problem* (1.1)–(1.4), *which has the property that*

$$\|u_i(., t)\|_{L^\infty(\Omega)} \le F(\xi) \quad \textit{for all } t \ge \xi > 0.$$

If moreover $u_{i0} \in L^\infty(\Omega)$ *then* u *is unique in the class of bounded solutions, and has the property that*

$$\|u_i(., t)\|_{L^\infty(\Omega)} \le C \quad \textit{for all } t \ge 0,$$

where $F(\xi)$ *is a positive function depending only* ξ *and* C *is a positive constant depending only on* u_0.

Moreover, if the initial data is positive and the functions g_i *are quasi-positive then* $(u_1, u_2, \dots u_d)$ *is positive.*

The proof is found in sections 5 and 6. Finally, in Section 7, we present the global existence and blow-up results, depending on the range of the parameters in the limit case.

Theorem 2.2. *Let* $f_i(u, \nabla u_i) = c_{i0} + \sum_{j=1}^{d} c_{ij} u_j^{\sigma_j+1} + \overrightarrow{b_i} \nabla(u_i^{\sigma_i+1})$.

(1) *If* $2d \max_{i,j=1,\dots d} c_{ij} + \max_{i=1,\dots d} \|\overrightarrow{b_i}\|(\lambda+1) < 2\lambda$ (λ *is the first eigenvalue of the Laplacian with zero Dirichlet data on* $\partial\Omega$) *then for every positive initial data in* $(L^\infty(\Omega))^d$ *there exists a global weak solution of* (1.1)–(1.4) (*tending to zero in case* $c_{i0} = 0$) *which is unique, positive and globally bounded.*
(2) *If* $\overrightarrow{b_i}$ *is independent of t,* $\overrightarrow{b_i} \in C^\infty(\overline{\Omega})$ *and if* $c_{ii} > \lambda_i$, (λ_i *is the first eigenvalue of* $-\Delta\psi(x) + \overrightarrow{b_i}\nabla\psi(x)$ *with zero Dirichlet data on* $\partial\Omega$) *then any nonnegative* (*non-trivial*) *weak solution of* (1.1)–(1.4) *blows up in finite time.*

3. L^∞-regularity

In this section we give a basic result of L^∞-regularity for weak solutions of (1.1)–(1.4). More precisely, we have the following theorem.

Theorem 3.1. *Let* $(u_1, u_2, \dots u_d)$ *be a weak solution of the problem* (1.1)–(1.4). *Assume that there exists a positive continuous function* F_1 *not depending on* u_0 *such that*

$$\|u_i(t,.)\|_{L^{\sigma_i+1}(\Omega)} \le F_1(\xi) \quad \text{for all } t \in [\xi, T_{\max}),\ i = 1, \dots, d. \tag{3.1}$$

Then there exists a positive continuous function F_∞ *not depending on* u_0 *such that*

$$\|u_i(t,.)\|_{L^\infty(\Omega)} \le F_\infty(\xi) \quad \text{for all } t \in [\xi, T_{\max}),\ i = 1, \dots, d. \tag{3.2}$$

Moreover, if there exists a positive number $C_1(u_0)$ *such that*

$$\|u_i(t,.)\|_{L^{\sigma_i+1}(\Omega)} \le C_1(u_0) \quad \text{for all } t \in [0, T_{\max}),\ i = 1, \dots, d, \tag{3.3}$$

then there exists a positive number $C_\infty(u_0)$ *such that*

$$\|u_i(t,.)\|_{L^\infty(\Omega)} \le C_\infty(u_0) \quad \text{for all } t \in [0, T_{\max}),\ i = 1, \dots, d. \tag{3.4}$$

The proof of the above theorem is obtained by an obvious modification of the techniques of Dung [9]; the following two lemmas serve as the main ingredients.

Lemma 3.2. *Suppose that the nonnegative function y is absolutely continuous and satisfies for almost every t the inequality*

$$y' + \theta y^{\nu} \leq \delta \qquad \text{with } \nu > 1,\ \theta > 0,\ \delta \geq 0. \tag{3.5}$$

Then for all $t > 0$ we have

$$y(t) \leq \left(\frac{\delta}{\theta}\right)^{1/\nu} + \big(\theta(\nu-1)t\big)^{-1/(\nu-1)}. \tag{3.6}$$

In particular, if $\lim_{t\to 0^+} y(t) = y(0)$ *is finite,* (3.6) *becomes*

$$y(t) \leq \max\left\{ y(0), \left(\frac{\delta}{\theta}\right)^{1/\nu} \right\} \qquad \text{for all } t \geq 0. \tag{3.7}$$

The proof can be found in [18, page 167].

Lemma 3.3. *Let $p \in [1,2)$ and $r \in \left[p, 2\frac{N+1}{N}\right)$. Then for any given $\eta > 0$, there exist positive constants $c(\eta)$, q depending only on p and r, such that*

$$\int_{\Omega} |u|^r \leq \eta\left(\int_{\Omega} \|\nabla u\|^2\, dx + \|u\|^2_{L^p(\Omega)} \right) + c(\eta)\|u\|^q_{L^p(\Omega)}.$$

for any $u \in W^{1,2}_p(\Omega)$. Here

$$q = \frac{2r(1-\tau)}{2 - r\tau} \qquad \text{with } \tau := \frac{\frac{1}{p} - \frac{1}{r}}{\frac{1}{p} + \frac{1}{N} - \frac{1}{2}}.$$

In proving local existence for degenerate equations such as (1.1)–(1.4) one standard approach consists in approximating the problem with a sequence of non-degenerate problems which can be solved in a classical sense. In order to do that we consider

- an increasing sequence of positive numbers $(R_\varepsilon)_\varepsilon$ such that $\lim_{\varepsilon\to 0} R_\varepsilon = +\infty$;
- $\psi_\varepsilon \in C_c^\infty(\mathbb{R}^+)$ such that $0 \leq \psi_\varepsilon \leq 1$ and $\psi_\varepsilon(r) = \begin{cases} 1 & \text{if } |r| \leq R_\varepsilon, \\ 0 & \text{if } |r| \geq R_\varepsilon + 1; \end{cases}$
- smooth functions $g_{i\varepsilon}$ such that $g_{i\varepsilon}(r_1, r_2, \dots r_d) = g_i(r_1, r_2, \dots r_d)\psi_\varepsilon(|r_1| + |r_2| + \cdots + |r_d|)$;
- $\phi_\varepsilon(r) := (|r| + \varepsilon)$ for all $r \in \mathbb{R}$;
- a sequence $u_{0\varepsilon} = (u_{10\varepsilon}, u_{20\varepsilon}, \dots u_{d0\varepsilon}) \in \left(C_c^\infty(\Omega)\right)^d$ (which is uniformly bounded in L^∞ if $u_{i0} \in L^\infty$) such that $(u_{i0\varepsilon})_\varepsilon$ tends to u_{i0} in $L^{\sigma_i+2}(\Omega)$.

Consider the following regularizing problems:

$$\partial_t(u_{i\varepsilon}) - (\sigma_i + 1)\,\mathrm{div}\big(\phi_\varepsilon^{\sigma_i}(u_{i\varepsilon})\nabla u_{i\varepsilon}\big) = g_{i\varepsilon}(u_\varepsilon) + \overrightarrow{b_i}\nabla(|u_{i\varepsilon}|^{m_i-1}u_{i\varepsilon}) \quad \text{in } Q_T, \quad (3.8)$$

subject to Dirichlet boundary conditions

$$u_{i\varepsilon} = 0 \quad \text{on } (0,T)\times\partial\Omega, \quad (3.9)$$

and initial conditions

$$u_{i\varepsilon}(0,.) = u_{i0\varepsilon} \quad \text{in } \Omega. \quad (3.10)$$

By [14, Theorem 7.4], there is $T_{\max,\varepsilon} > 0$ such that the problem (3.8)–(3.10) has a unique maximal solution $u_\varepsilon = (u_{1\varepsilon}, u_{2\varepsilon}, \dots u_d) \in \big(W_q^{1,2}(Q_{T_{\max,\varepsilon}})\big)^d$ for all $1 \le q < \infty$.

Moreover, under the additional conditions

(H_6) $u_{i0} \ge 0$, $i = 1, 2, \dots d$,

(H_7) g_i is quasi-positive, that is $g_i(u) \ge 0$ for every $u = (u_1, u_2, \dots u_d)$ such that $u_i = 0$ and $u_j \ge 0$ for $i \ne j$,

we can prove that u_ε is classical and positive, see [12]. In order to prove Theorem 3.1 it suffices to prove the following.

Proposition 3.4. *Suppose there exists a positive continuous function F_1 not depending on ε and u_0 such that*

$$\|u_{i\varepsilon}(t,.)\|_{L^{\sigma_i+1}(\Omega)} \le F_1(\xi) \quad \textit{for all } t \in [\xi, T_{\max}). \quad (3.11)$$

Then there exists a positive continuous function F_∞ not depending on ε and u_0 such that

$$\|u_{i\varepsilon}(t,.)\|_{L^\infty(\Omega)} \le F_\infty(\xi) \quad \textit{for all } t \in [\xi, T_{\max}). \quad (3.12)$$

Alternatively, if there exists a positive finite constant $C_1(u_0)$ not depending on ε such that

$$\|u_{i\varepsilon}(t,.)\|_{L^{\sigma_i+1}(\Omega)} \le C_1(u_0) \quad \textit{for all } t \in [0, T_{\max}), \quad (3.13)$$

then there exists a finite positive constant $C_\infty(u_0)$ not depending on ε such that

$$\|u_{i\varepsilon}(t,.)\|_{L^\infty(\Omega)} \le C_\infty(u_0) \quad \textit{for all } t \in [0, T_{\max}). \quad (3.14)$$

In order to prove this proposition at first we prove the following lemmas.

Lemma 3.5. *Assuming* (3.11)*, there exists a positive continuous function* F_∞ *not depending on* ε *and* u_0 *such that*

$$\|u_{i\varepsilon}(t,.)\|_{L^{\sigma_i+2}(\Omega)} \le F_2(\xi) \quad \text{for all } t \in [\xi, T_{\max}). \tag{3.15}$$

If (3.13) *is satisfied then there exists a finite positive constant* $C_2(u_0)$ *not depending on* ε *such that*

$$\|u_{i\varepsilon}(t,.)\|_{L^{\sigma_i+2}(\Omega)} \le C_2(u_0) \quad \text{for all } t \in [0, T_{\max}). \tag{3.16}$$

Proof. For simplicity, we omit the index ε. By multiplying (3.8) by $|u_i|^{\sigma_i}u_i$, and integrating over Ω, we obtain the following inequality with the help of the Young inequality:

$$\frac{1}{\sigma_i+2}\frac{d}{dt}\int_\Omega |u_i|^{\sigma_i+2}\,dx + \int_\Omega \|\nabla(|u_i|^{\sigma_i}u_i)\|^2\,dx$$
$$\le C(\eta)\sum_{j=1}^d \int_\Omega |u_j|^{\sigma_j+1+\theta}\,dx + \eta\int_\Omega \|\nabla(|u_i|^{\sigma_j}u_i)\|^2\,dx + C(\eta),$$

where $\theta \le \sigma_j + 1$

From Lemma 3.3, if we take into account assumptions on α_{ij} and m_i, we find

$$\frac{d}{dt}\int_\Omega |u_i|^{\sigma_i+2}\,dx + C\int_\Omega \|\nabla(|u_i|^{\sigma_i}u_i)\|^2\,dx$$
$$\le 2\eta\sum_{j=1}^d \int_\Omega \|\nabla(|u_j|^{\sigma_j}u_j)\|^2\,dx + C(\eta)\sum_{j=1}^d \Big(\int_\Omega |u_j|^{\sigma_j+1}\,dx\Big)^q + C(\eta),$$

By adding these inequalities we obtain that, for η sufficiently small,

$$\frac{d}{dt}\sum_{i=1}^d \int_\Omega |u_i|^{\sigma_i+2}\,dx + C\sum_{i=1}^d \int_\Omega \|\nabla(|u_i|^{\sigma_i}u_i)\|^2\,dx \le C\sum_{i=1}^d \Big(\int_\Omega |u_i|^{\sigma_i+1}\,dx\Big)^q + C \tag{3.17}$$

Assuming (3.11), (3.17) can be written in the following form

$$\frac{d}{dt}\sum_{i=1}^d \int_\Omega |u_i|^{\sigma_i+2}\,dx + C\int_\Omega \|\nabla(|u_i|^{\sigma_i}u_i)\|^2\,dx \le C(\xi) \quad \text{for all } t \ge \xi > 0. \tag{3.18}$$

On the other hand, the Hölder and Young inequalities imply

$$\int_\Omega |u_i|^{\sigma_i+2}\,dx \le C\Big(\int_\Omega |u_i|^{2(\sigma_i+1)}\,dx\Big)^{(\sigma_i+2)/2(\sigma_i+1)} \le C\Big(\int_\Omega |u_i|^{2(\sigma_i+1)}\,dx+1\Big)^{(\gamma+1)/2}$$

where $\gamma := \max_{1\le i\le d}\frac{1}{(\sigma_i+1)}$. Then from Lemma 3.3 and Jensen inequality, (3.18) becomes

$$\frac{d}{dt}\int_\Omega \sum_{i=1}^d |u_i|^{\sigma_i+2}\,dx + C\Big(\int_\Omega \sum_{i=1}^d |u_i|^{\sigma_i+2}\,dx\Big)^{2/(\gamma+1)} \le C(\xi) \qquad \text{for all } t\ge\xi>0. \tag{3.19}$$

Alternatively, if (3.13) is satisfied we obtain

$$\frac{d}{dt}\int_\Omega \sum_{i=1}^d |u_i|^{\sigma_i+2}\,dx + C_{13}\Big(\int_\Omega \sum_{i=1}^d |u_i|^{\sigma_i+2}\Big)^{2/(\gamma+1)} \le C(u_0) \qquad \text{for all } t\ge 0. \tag{3.20}$$

Finally, by putting $y(t)=\int_\Omega\sum_{i=1}^d |u_i|^{\sigma_i+2}\,dx$ in (3.19) and (3.20), Lemma 3.2 implies the desired result. □

We now prove inductively that $u_{i\varepsilon}$ is bounded in L^p for each $p\ge\sigma_i+1$.

Lemma 3.6. *Let $p\ge\sigma_i+1$. Assuming* (3.11), *there exists a positive function F_p not depending on u_0 and ε such that*

$$\|u_{i\varepsilon}(t,.)\|_{L^p(\Omega)} \le F_p(\xi) \qquad \textit{for all } t\in[\xi, T_{\max,\varepsilon}). \tag{3.21}$$

If (3.13) *is given, then there exists a positive constant $C_p(u_0)$ not depending on ε such that*

$$\|u_{i\varepsilon}(t,.)\|_{L^p(\Omega)} \le C_p(u_0) \qquad \textit{for all } t\in[0, T_{\max,\varepsilon}). \tag{3.22}$$

Proof. Let $r_k\ge 1$. By multiplying (3.8) by $|u_i|^{r_k(\sigma_i+1)-1}u_i$ and integrating over Ω, we obtain the following with the help of Young inequality

$$\frac{1}{r_k(\sigma_1+1)+1}\frac{d}{dt}\int_\Omega |u_j|^{r_k(\sigma_1+1)+1}\,dx + \frac{4r_k}{(1+r_k)^2}\int_\Omega \|\nabla(|u_j|^{(\sigma_1+1)(r_k+1)/2-1}u_j)\|^2\,dx$$
$$\le C(\eta)\sum_{j=1}^d\int_\Omega |u_j|^{r_k(\sigma_j+1)+\theta}\,dx + \eta\int_\Omega \|\nabla(|u_j|^{(\sigma_1+1)(r_k+1)/2-1}u_j)\|^2\,dx + C(\eta), \tag{3.23}$$

where $\theta \le \sigma_j + 1$. In order to estimate $\int_\Omega |u_j|^{r_k(\sigma_j+1)+\theta}\,dx$ we construct the following sequences:

$$r_k = \lambda^k, \qquad p_{ik} = \frac{2\big(r_{k-1}(\sigma_i+1)+1\big)}{(\sigma_i+1)(1+r_k)} \qquad \text{and} \qquad v_{ik} = \frac{\big((\sigma_i+1)1+r_k\big)}{1+r_k(\sigma_i+1)},$$

where $1 < \lambda < +\min_{i=1,\dots,d}\frac{1}{\sigma_i+1}$. It is obvious that $1 < p_{ik} < 2$ for all $i = 1,\dots,d$. By setting $w_i = |u_i|^{(\sigma_i+1)((r_k+1)/2)-1}u_i$ and applying Lemma 3.3, we can estimate $\int_\Omega |u_j|^{r_k(\sigma_j+1)+\theta}\,dx = \int_\Omega |u_j|^{2(r_k(\sigma_j+1)+\theta)/(r_k+1)(\sigma_j+1)}\,dx$ in term of $\|w_i\|_{L^{p_{ik}}}$ and $\|\nabla w_i\|_{L^2}$. Hence (3.23) becomes

$$\frac{d}{dt}\int_\Omega |w_i|^{2/v_{ik}}\,dx + (1-\eta)\int_\Omega \|\nabla w_i\|^2\,dx \\ \le \eta\sum_{j=1}^d\int_\Omega \|\nabla w_j\|^2\,dx + C(\eta)\sum_{j=1}^d \|w_j\|^{q_k}_{L^{p_{ik}}(\Omega)} + C(\eta).$$

By summing up these inequalities over i we find

$$\frac{d}{dt}\int_\Omega\sum_{i=1}^d |w_i|^{2/v_{ik}}\,dx + (1-2\,d\eta)\int_\Omega\sum_{i=1}^d \|\nabla w_i\|^2\,dx \\ \le 2\,d\eta\int_\Omega\sum_{i=1}^d \|\nabla w_i\|^2\,dx + C(\eta)\sum_{i=1}^d \|w_i\|^{q_k}_{L^{p_{ik}}(\Omega)} + C(\eta). \tag{3.24}$$

We will prove by induction on $k \ge 1$ that

$$\|w_i\|_{L^{p_{ik}}(\Omega)} < F_p(\xi) \qquad \text{for all } t \ge \xi > 0. \tag{3.25}$$

Assuming (3.25) for some k, (3.24) becomes

$$\frac{d}{dt}\int_\Omega\sum_{i=1}^d |w_i|^{2/v_{ik}}\,dx + C\int_\Omega\sum_{i=1}^d \|\nabla w_i\|^2\,dx \le F_p(\xi). \tag{3.26}$$

By combining the Hölder, Sobolev and Young inequalities we get

$$\left(\int_\Omega |w_i|^{2/v_{ik}}\,dx\right)^{v_k} \le C\int_\Omega \|\nabla w_i\|^2\,dx + C, \qquad \text{where } v_k = \min_{i=1,2}(v_{ik}). \tag{3.27}$$

By letting $y_k(t) = \int_\Omega \sum_{1\le i\le 2} |w_i|^{2/v_{ik}}\,dx = \|w_i\|_{L^{p_{ik}}}$ and inserting (3.27) into (3.26), we find

$$\frac{d}{dt}y_k(t) + Cy_k(t)^{v_k} \le C.$$

As a consequence, Lemma 3.2 implies that (3.25) will be satisfied for $k+1$. The lemma now follows by applying Lemma 3.5. □

Next, in order to show that the solution u_ε is uniformly bounded, we make use of the following lemma.

Lemma 3.7. *For any $\lambda \geq 1$ there exist positive constants d_0, d_1, d_2, τ and τ' with τ and τ' not depending on λ such that if* (3.11) *is satisfied then for every $t \geq \xi > 0$ we have*

$$\frac{d}{dt}\int_\Omega \sum_{1\leq i\leq d} |v_i|^{\gamma_i+\lambda}\,dx + d_0\int_\Omega \|\nabla(|v_i|^{(\lambda-1)/2}v_i)\|^2\,dx \leq d_1(\xi)\lambda^\tau \int_\Omega \sum_{1\leq i\leq d} |v_i|^{\gamma_i+\lambda}\,dx + d_2\lambda^\tau.$$

Moreover, if (3.13) *is satisfied then for all $t \geq 0$ we have*

$$\frac{d}{dt}\int_\Omega \sum_{1\leq i\leq d} |v_i|^{\gamma_i+\lambda}\,dx + d_0\int_\Omega \|\nabla(|v_i|^{(\lambda-1)/2}v_i)\|^2\,dx \leq d_1(\|u_0\|)\lambda^\tau \int_\Omega \sum_{1\leq i\leq d} |v_i|^{\gamma_i+\lambda}\,dx + d_2\lambda^\tau,$$

where $v_i = |u_i|^{\sigma_i}u_i$ and $\gamma_i = \frac{1}{\sigma_i+1}$.

Proof. By multiplying (3.8) by $|u_i|^{\lambda(\sigma_i+1)-1}u_i$ and integrating over Ω we can proceed exactly as we did in the proof of the Lemma 3.6, to obtain that

$$\frac{d}{dt}\int_\Omega |v_i|^{\gamma_i+\lambda}\,dx + d_3\int_\Omega \|\nabla(|v_i|^{(\lambda-1)/2}v_i)\|^2\,dx \leq d_4\lambda\int_\Omega \sum_{1\leq i\leq d} |v_i|^{\lambda+\alpha}\,dx + d_5\lambda^2 + C \tag{3.28}$$

where $\alpha \geq 1$. By using the Hölder inequality and the fact that

$$|v_i|^{\lambda+\alpha} = |v_i|^{h(1+\lambda)(\alpha-\gamma_i)/(h(\alpha-\gamma_i)+e)}|v_i|^{(\alpha-\gamma_i)(h(\alpha-1)+e)/(h(\alpha-\gamma_i)+e)}|v_i|^{e(\gamma_i+\lambda)/(h(\alpha-\gamma_i)+e)},$$

where e is a positive number and $h > 0$ is to be chosen below, we get

$$\int_\Omega |v_i|^{\alpha+\lambda}\,dx \leq \left(\int_\Omega |v_i|^{h(1+\lambda)/(h-2)}\,dx\right)^{P_i}\left(\int_\Omega |v_i|^{p^*}\,dx\right)^{R}\left(\int_\Omega |v_i|^{\lambda+\gamma_i}\,dx\right)^{Q_i} \tag{3.29}$$

with

$$P_i = \frac{(h-2)(\alpha-\gamma_i)}{h(\alpha-\gamma_i)+e}, \qquad Q_i = \frac{e}{h(\alpha-\gamma_i)+e}, \qquad R_i = \frac{2(\alpha-\gamma_i)}{h(\alpha-\gamma_i)+e},$$

$$p^* = \frac{1}{2}\big(h(\alpha-1)+e\big).$$

Three cases may occur:

- Consider first the case $N > 2$. In this case it suffices to choose $h = N$. Then we use the compactness of the imbedding $W^{1,2} \hookrightarrow L^{2h/(h-2)}$ to obtain that

$$\int_\Omega |v_i|^{h(1+\lambda)/(h-2)}\,dx \le C(\Omega)\Big[\int_\Omega \|\nabla(|v_i|^{(\lambda-1)/2}v_i)\|^2\,dx + \int_\Omega |v_i|^{1+\lambda}\,dx\Big]^{h/(h-2)}. \tag{3.30}$$

- Next, we consider the case $N = 2$. One can obtain (3.30) by choosing $h > 2$ and using the compactness of the imbedding $W^{1,2} \hookrightarrow L^q$ for all $q \ge 2$.
- Lastly, we consider the case $N = 1$. We choose $h > 2$ and we use the compactness of the imbedding $W^{1,2} \hookrightarrow L^\infty$ to obtain (3.30).

Since p^* is independent of λ, by reporting (3.30) into (3.29) and by using the fact that $\frac{h}{h-2}P_i + Q_i = 1$, Young inequality gives us

$$d_4\lambda\int_\Omega |v_i|^{\alpha+\lambda}\,dx \le \eta\Big[\int_\Omega \|\nabla(|v_i|^{(\lambda-1)/2}v_i)\|^2\,dx + \int_\Omega |v_i|^{1+\lambda}\,dx\Big] + C(\eta)\lambda^\tau\int_\Omega v_i^{\gamma_i+\lambda}\,dx \tag{3.31}$$

where $\tau = \max_{i=1,d}\Big\{\frac{1}{Q_i}\Big\}$ for all $\alpha \ge 1$. In particular, for $\alpha = 1$ we have

$$\int_\Omega |v_i|^{1+\lambda}\,dx \le \eta\int_\Omega \|\nabla(|v_i|^{(\lambda-1)/2}v_i)\|^2\,dx + C(\eta)\lambda^\tau\int_\Omega v_i^{\gamma_i+\lambda}\,dx. \tag{3.32}$$

Assuming (3.11) and by inserting (3.31) and (3.32) into (3.28) we obtain that, for η sufficiently small,

$$\frac{d}{dt}\int_\Omega |v_i|^{\gamma_i+\lambda}\,dx + d_0\int_\Omega \|\nabla(|v_i|^{(\lambda-1)/2}v_i)\|^2\,dx \le d_1(\xi)\lambda^\tau\int_\Omega \sum_{1\le i\le d}|v_i|^{\gamma_i+\lambda}\,dx + d_2(\xi)\lambda^2,$$

for all $t \ge \xi > 0$. By exactly in the same way we can prove that if (3.13) is given then we have

$$\frac{d}{dt}\int_\Omega |v_i|^{\gamma_i+\lambda}\,dx + d_0\int_\Omega \|\nabla(|v_i|^{(\lambda-1)/2}v_i)\|^2\,dx$$
$$\leq d_1(u_0)\lambda^\tau \int_\Omega \sum_{1\leq i\leq d} |v_i|^{\gamma_i+\lambda}\,dx + d_2(u_0)\lambda^2. \tag{3.33}$$

for all $t \geq 0$. This completes the proof of the lemma. □

As a final preliminary step we state the following lemma.

Lemma 3.8. *Let $\lambda_k = 2^k$, $k \in \mathbb{N}$, t and μ be positive constants such that $t - \frac{\mu}{\lambda_k} > 0$. Then there exist positive constants $\varkappa$ and $C_0(\mu)$ such that*

$$y_k(t) \leq U_k(t,\mu), \tag{3.34}$$

where

$$y_k(t) = \int_\Omega \sum_{1\leq i\leq k} |u_i|^{(\sigma_i+1)(\lambda_k+\gamma_i)}\,dx, \qquad k \geq 1,$$

$$U_k(t,\mu) = C_0(\mu)\lambda_k^\varkappa\big(\sup_{s\geq t}\ y_{k-1}(s) + 1\big)^{s_k} \qquad \textit{with } s_k = \frac{\delta+\lambda_{k+1}}{\delta+\lambda_k},$$

where $\delta = \min_{1\leq i\leq 2}\{h-\gamma_i(h-2)\} > 0$, and $h = N$ if $N > 2$, $h = 2$ if $N < 2$.

Proof. Let us construct the following sequences

$$Q_{ik} = \frac{h(\lambda_k+1)-(h-2)(\gamma_i+1)}{h(\lambda_k+1)-(h-2)(\gamma_i+1)},$$
$$P_{ik} = 1 - Q_{ik},$$
$$\overline{P_{ik}} = \frac{h}{h-2}P_{ik},$$
$$s_{ik} = \frac{Q_{ik}}{1-\overline{P_{ik}}} = \frac{h-(h-2)\gamma_i+\lambda_{k+1}}{h-(h-2)\gamma_i+\lambda_k} > 1.$$

The Hölder inequality implies that

$$\int_\Omega |v_i|^{\lambda_k+\gamma_i}\,dx \leq \Big(\int_\Omega |v_i|^{h(1+\lambda_k)/(h-2)}\,dx\Big)^{P_{ik}}\Big(\int_\Omega |v_i|^{\lambda_{k-1}+\gamma_i}\,dx\Big)^{Q_{ik}},$$

from which, with the help of the Sobolev inequality, we obtain

$$\int_\Omega |v_i|^{\lambda_k+\gamma_i}\,dx$$
$$\le C\Big(\int_\Omega \|\nabla(|v_i|^{(\lambda_k-1)/2}v_i\|^2\,dx + \int_\Omega |v_i|^{\lambda_k+\gamma_i}\,dx\Big)^{\overline{P_{ik}}}\Big(\int_\Omega |v_i|^{\lambda_{k-1}+\gamma_i}\,dx\Big)^{Q_{ik}}.$$

The Young inequality asserts then that

$$c\lambda^{\tau_2}\int_\Omega |v_i|^{\lambda_k+\gamma_i}\,dx \le \frac{d_0}{2}\int_\Omega \|\nabla(|v_i|^{(\lambda_k-1)/2}v_i\|^2\,dx + c'\lambda^{\tau_3}\Big(\int_\Omega |v_i|^{\lambda_k+\gamma_i}\,dx\Big)^{s_{ik}} \quad (3.35)$$

The remaining part of the proof follows from the proof of [9, Lemma 4]. □

4. Proof of Proposition 3.4

1. *Suppose that* (3.11) *is given.* Let ξ and ξ' be two positive reals such that $\xi' > \xi > 0$. We put $\mu = \frac{\xi'-\xi}{2}$; $t_0 = \frac{\xi'+\xi}{2} > \xi$; $t_k = t_{k-1} - \mu\lambda_k$. From (3.34) we have

$$1 + \sup_{t\ge t_{k-1}} y_k(t) \le C_0\lambda_k^\sigma\big(1 + \sup_{t\ge t_k} y_{k-1}(t)\big)^{s_k}.$$

By letting $K_\xi = \max_{i=1,\dots d}\sup_{t\ge\xi}(\int_\Omega |v_i|^{\gamma_i+1}\,dx + 1)$ we deduce that

$$\sup_{t\ge t_{k-1}} y_k(t) \le C_0^{A_k}2^{\sigma B_k}K_\xi^{C_k},$$

where

$$A_k = 1 + s_k + s_ks_{k-1} + \dots + s_ks_{k-1}\dots s_1,$$
$$B_k = k + (k-1)s_k + (k-2)s_ks_{k-1} + \dots + s_ks_{k-1}\dots s_1,$$
$$C_k = s_ks_{k-1}\dots s_1 = \frac{\delta+\lambda_{k+1}}{\delta+\lambda_1}.$$

In order to complete the proof it suffices to see that A_k and B_k are of order 2^k as k tends to $+\infty$. We found that

$$\sup_{t\ge t_0} y_k(t) \le \sup_{t\ge t_{k-1}} y_k(t) \le C_0^{A_k}2^{\sigma B_k}K_\xi^{(\delta+\lambda_{k+1})/(\delta+\lambda_1)}. \quad (4.1)$$

By taking the $\frac{1}{\gamma_i+2^k}$ power of both sides of (4.1) and passing to the limit as k tends to $+\infty$, we obtain

$$\sup_{t\ge\xi'}\|v_i(t,.)\|_{L^\infty(\Omega)} \le \lim_{k\to\infty}\sup_{t\ge t_0}\big(y_k(t)\big)^{1/(\gamma_i+\lambda_k)} \le CK_\xi^{2/(\delta+\lambda_1)}.$$

2. *Suppose that* (3.13) *is given.* We need the following lemma due to Alikakos [1].

Lemma 4.1. *Let ω a nonnegative function defined in $(0,\infty)\times\Omega$, satisfying the differential inequality*

$$\frac{\partial}{\partial t}\int_\Omega |\omega|^{\lambda_k+\gamma} \le -\varepsilon_k \int_\Omega |\omega|^{\lambda_k+\gamma} + (a_k+\varepsilon_k)c_k\left[\sup_{t\ge 0}\int_\Omega |\omega|^{\lambda_{k-1}+\gamma}\right]^{p_k}, \qquad k = 1,2,\dots,$$

where a_k, ε_k, c_k are respectively of order $\frac{1}{2^k}$, $2^{\alpha k}$, 2^k as k tends to infinity, α is a positive constant, and $(\lambda_{k-1}+1)p_k \le \lambda_k + 1$. Then there exists a positive constant a such that

$$\sup_{t\ge 0}\|\omega(t,.)\|_{L^\infty} \le a 2^{2(\alpha+2)}K,$$

where $K \ge \max\{1, \sup_{t\ge 0}\|\omega(t,.)\|_{L^{\sigma+1}}, \|\omega(0,.)\|_{L^\infty}\}$.

Now combining (3.33) and (3.35) we obtain that

$$\frac{d}{dt}\int_\Omega |v_i|^{\lambda_k+\gamma} \le \big(-2c + d_1(u_0)\big)\lambda_k^{\tau_2}\int_\Omega |v_i|^{\lambda_k+\gamma} + C\lambda_k^{\tau_3}\left[\sup_{t\ge 0}\int_\Omega |v_i|^{\lambda_{k-1}+\gamma}\right]^{s_{ik}},$$

$k = 1,2,\dots\infty$, which completes the proof, thanks also to Lemma 4.1.

Remark 2. The results of this section can be extended to the following cases.

Case 1.

$$\begin{cases} \partial_t u_i - \Delta(|u_i|^{\sigma_i}u_i) = f_i(t,x,u,\nabla u_i) & \text{in }]0,\infty[\times\Omega \\ \dfrac{\partial}{\partial v}(|u_i|^{\sigma_i}u_i)u_i \le 0 & \text{on }]0,\infty[\times\partial\Omega \\ u(0,.) = u_{i0},\ u_{i0}\in L^\infty(\Omega) & \text{in } \Omega \end{cases}$$

with

- $\sigma_i > 0$,
- $|f_i(t,x,u,\xi)| \le k_1\sum_{1\le j\le d}|u_j|^{\alpha_j} + k_2\|\xi\|^{\delta_i} + k_3$, where
- $k_l \ge 0$; $l = 1,3$; $\alpha_i \in \left[0, \sigma_i + 1 + \frac{\sigma_i+2}{N}\right[$; $\delta_i \in \left[0, \frac{\sigma_i+1}{\sigma_i}\right[$.

Case 2.

$$\begin{cases} \partial_t(u_i) - \Delta(|u_i|^{\sigma_i}u_i) = g_i(t,x,u) + \overrightarrow{b_i}\nabla(|u_i|^{m_i-1}u_i) & \text{in }]0,\infty[\times\Omega, \\ \left.\begin{array}{l} \dfrac{\partial}{\partial v}(u_i|u|^{\sigma_i})u_i \le 0, \text{ or} \\ \sum_{j=1}^N u_i\left[\dfrac{\partial}{\partial x_j}(|u_i|^{\sigma_i}u_i) + b_{ij}|u_i|^{m_i-1}u_iv_j\right] \le 0 \end{array}\right\} & \text{on }]0,\infty[\times\partial\Omega, \\ u_i(0,.) = u_{i0} & \text{in } \Omega, \end{cases}$$

with

- $\sigma_i > 0$,
- there exist $\alpha_j \in \left[0, \sigma_j + 1 + \frac{\sigma_j+2}{N}\right[$ such that for $(t,x) \in \mathbb{R}^+ \times \Omega$ and $u = (u_1, u_2, \dots u_d)$ we have

$$|g_i(t,x,u)| \le k_1 \sum_{1\le j\le d} u_j^{\alpha_j} + k_2,$$

  for some positive constants k_1, k_2,
- $m_i \in \left[0, (\sigma_i + 1)\frac{N+1}{N}\right[$.

5. Global existence

In order to prove the global existence we prove at first the following energy estimates.

Lemma 5.1. *Suppose that the assumptions* (H_1)–(H_5) *are satisfied. Then the solution* u_ε *of* (3.8)–(3.10) *is global* (*that is* $T_{\max,\varepsilon} = \infty$) *and there exists a positive function F not depending on* ε *and* u_0 *such that*

$$\|u_{i\varepsilon}(t,.)\|_{L^\infty} \le F(\xi) \quad \textit{for all } t \ge \xi > 0. \tag{5.1}$$

Moreover, if $u_0 \in \left(L^\infty(\Omega)\right)^d$ *then there exists a positive constant C not depending on* ε *such that*

$$\|u_{i\varepsilon}(t,.)\|_{L^\infty} \le C(\|u_0\|_{L^\infty}) \quad \textit{for all } t \ge 0. \tag{5.2}$$

Proof. By Proposition 3.4, it is enough to show that there is a positive function F_0 such that

$$\|u_{i\varepsilon}(t,.)\|_{L^{\sigma_i+2}(\Omega)} \le F_0(\xi) \quad \text{for all } t \ge \xi > 0,$$

and if $u_0 \in \left(L^\infty(\Omega)\right)^d$ then there is a positive constant C_0 such that

$$\|u_{i\varepsilon}(t,.)\|_{L^{\sigma_i+2}(\Omega)} \le C_0(u_0) \quad \text{for all } t \ge 0.$$

By multiplying (3.8) by $|u_{i\varepsilon}|^{\sigma_i} u_{i\varepsilon}$, integrating over Ω and taking into account that $\alpha_{ij} < \sigma_j + 1$ and $m_i < \sigma_i + 1$, we obtain the following, with the help of Young and Poincaré inequalities: for all $\eta > 0$,

$$\frac{1}{\sigma_i+2}\int_\Omega \partial_t |u_{i\varepsilon}|^{\sigma_i+2}\,dx + \int_\Omega \|\nabla(|u_{i\varepsilon}|^{\sigma_i}u_{i\varepsilon})\|^2\,dx \\ \leq \eta\sum_{j=1}^d \int_\Omega \|\nabla(|u_j|^{\sigma_j}u_j)\|^2\,dx + C(\eta).$$

By adding these inequalities, we find

$$\int_\Omega \sum_{i=1}^d \partial_t |u_{i\varepsilon}|^{\sigma_i+2}\,dx + C(1-d\eta)\int_\Omega \sum_{i=1}^d \|\nabla(|u_{i\varepsilon}|^{\sigma i}u_{i\varepsilon})\|^2\,dx \leq C(\eta). \tag{5.3}$$

By choosing η small enough in the last inequality and using the Poincaré inequality we have

$$\frac{d}{dt}\int_\Omega \sum_{i=1}^d |u_{i\varepsilon}|^{\sigma_i+2}\,dx + C\int_\Omega \sum_{i=1}^d |u_{i\varepsilon}|^{2(\sigma_i+1)}\,dx \leq C. \tag{5.4}$$

By using the Hölder inequality in the second term of the left hand side, we find

$$\sum_{i=1}^d \int_\Omega |u_{i\varepsilon}|^{2(\sigma_i+1)}\,dx \geq C\sum_{i=1}^d \left(\int_\Omega |u_{i\varepsilon}|^{\sigma_i+2}\,dx\right)^{2(\sigma_i+1)/(\sigma_i+2)} \geq C\sum_{i=1}^d \left(\int_\Omega |u_{i\varepsilon}|^{\sigma_i+2}\,dx\right)^{\nu}, \tag{5.5}$$

where $\nu > 1$ depends on σ_i. By inserting (5.5) into (5.4) and writing $y = \sum_{i=1}^d \int_\Omega |u_{i\varepsilon}|^{\sigma_i+2}\,dx$, we obtain the following, by also using Jensen inequality:

$$\frac{d}{dt}y(t) + Cy(t)^\nu \leq C.$$

Thanks also to Lemma 3.2, this completes the proof. □

We now proceed with the proof of global existence. By integrating the differential inequality (5.3) over $[0, T]$ and choosing η sufficiently small, we obtain

$$\int_\Omega |u_{i\varepsilon}|^{\sigma_i+2}(T,x)\,dx + \int_{Q_T} \|\nabla(|u_{i\varepsilon}|^{\sigma i}u_{i\varepsilon})\|^2\,dx\,dt \leq C(T), \qquad i = 1, d.$$

By using the uniform estimate (5.2), multiplying (3.8) by $\phi(u_{i\varepsilon})^{\sigma_i}u_{i\varepsilon}$ and integrating over Q_T, we get

$$\int_0^T \int_\Omega \|\phi(u_{i\varepsilon})^{\sigma_i}\nabla u_{i\varepsilon}\|^2\,dx\,dt \leq C(T), \qquad i = 1, d.$$

By compactness arguments, it follows that there exists a function u_i and a subsequence of $u_{i\varepsilon}$, which we still denote by $u_{i\varepsilon}$, such that

$$(|u_{i\varepsilon}| + \varepsilon)^{\sigma_i} \nabla u_{i\varepsilon} \to |u_i|^{\sigma_i} \nabla u_i \text{ weakly in } L^2(Q_T),$$
$$|u_{i\varepsilon}|^{\sigma_i} u_{i\varepsilon} \to |u_i|^{\sigma_i} u_i \text{ in the strong topology of } L^2(Q_T),$$
$$u_{i\varepsilon}(t,.) \to u_i(t,.) \text{ almost everywhere in } \Omega,$$
$$|u_{i\varepsilon}|^{m_i-1} u_{i\varepsilon} \to |u_i|^{m_i-1} u_i \text{ in the strong topology of } L^2(Q_T),$$
$$g_{i\varepsilon}(u_\varepsilon) \to g_i(u) \text{ almost everywhere in } Q_T.$$

Hence the dominated convergence theorem guarantees that $g_{i\varepsilon}(u_\varepsilon) \to g_i(u)$ in the strong topology of $L^2(Q_T)$. Since u_ε is a smooth solution of (3.8)–(3.10), it clearly satisfies

$$\int_\Omega u_{i\varepsilon}(x,T)\varphi_i(x,T)\,dx - \int_{Q_T} \varphi_{it} u_{i\varepsilon}\,dx\,dt + \int_{Q_T} \nabla(|u_{i\varepsilon}|^{\sigma_i} u_{i\varepsilon}) \nabla\varphi_i\,dx\,dt$$
$$= \int_{Q_T} \left(g_{i\varepsilon}(u_{i\varepsilon}, u_{2\varepsilon})\varphi_i + \overrightarrow{b_i}\nabla\varphi_i |u_{i\varepsilon}|^{m_i-1} u_{i\varepsilon}\right) dx\,dt + \int_\Omega u_{i0\varepsilon}(x)\varphi_i(0,x)\,dx$$

for any test function φ_i. From here, passing to the limit as ε tends to zero we obtain that $u = (u_1, u_2, \dots u_d)$ is indeed a weak solution in the sense of our definition.

Finally, from the fact that, for all $t \geq \xi > 0$, $\|u_{i\varepsilon}(t,.)\|_{L^\infty(\Omega)}$ is uniformly bounded, we can extract a subsequence, still denoted $\left(u_{i\varepsilon}(t,.)\right)_{0<\varepsilon<1}$, such that as ε tends to 0, $\left(u_{i\varepsilon}(t,.)\right)_{0<\varepsilon<1}$ is weakly convergent to $u_i(t,.)$ in $L^p(\Omega)$ for every finite $p \geq 1$. Hence, due to [8], one can extract a subsequence $\left(\omega_{i\varepsilon}(t,.)\right)_{0<\varepsilon<1}$ of convex combinations of elements of $u_{i\varepsilon}(t,.)$ such that $\omega_{i\varepsilon}(t,.) \to u_i(t,.)$ weakly in $L^p(Q_T)$, and almost everywhere in Ω. From the facts just proved it follows that

$$u_i \in L^\infty_{\mathrm{loc}}\left(\xi, \infty; L^\infty(\Omega)\right), \qquad i = 1, 2, \dots, d.$$

Moreover, if $u_0 \in \left(L^\infty(\Omega)\right)^d$ one finds that

$$u_i \in L^\infty_{\mathrm{loc}}\left(0, \infty; L^\infty(\Omega)\right), \qquad i = 1, 2, \dots, d.$$

6. Uniqueness

In this section we consider the question of the uniqueness of a bounded solution. We will always assume that

(H_8) $u_{i0} \in L^\infty(\Omega)$, $i = 1, 2, \dots d$.

Theorem 6.1. *If, in addition to* (H_1)–(H_4), $u_{i0} \in L^\infty(\Omega)$ *then* u *is unique in the class of bounded functions.*

Proof. The proof is a straightforward extension of the one given in [6] in a special situation.

Indeed, suppose on the contrary that there exist two weak solutions $u = (u_1, u_2, \dots, u_d)$ and $\hat{u} = (\hat{u}_1, \hat{u}_2, \dots, \hat{u}_d)$ of problem (1.1)–(1.4) such that $u, \hat{u} \in \left(L^\infty(Q_T)\right)^d$; that is, there exist a positive constant $M(T)$ and a set $J \subset \{1, 2, \dots, d\}$ such that

$$\left(\int_{Q_T} |u_i - \hat{u}_i|^2 \, dx \, dt\right)^{1/2} > M(T) \quad \text{if } i \in J, \text{ and } u_i = \hat{u}_i \text{ if } i \notin J. \tag{6.1}$$

We will reach a contradiction by constructing suitable test functions. In order to do this, let us introduce a function $\Psi_i \in L^\infty(Q_T)$ such that

$$\Psi_i = \begin{cases} \frac{|u_i|^{\sigma_i} u_i - |\hat{u}_i|^{\sigma_i} \hat{u}_i}{u_i - \hat{u}_i} & \text{if } u_i \neq \hat{u}_i, \\ 0 & \text{otherwise.} \end{cases}$$

We consider a sequence of functions $\{\Psi_{i\varepsilon}\}$ such that

i) $\Psi_{i\varepsilon} \in L^\infty(Q_T)$,

ii) $\varepsilon \le \Psi_{i\varepsilon} \le \|\Psi_i\|_{L^\infty(Q_T)} + \varepsilon$,

iii) $\dfrac{\Psi_{i\varepsilon} - \Psi_i}{\sqrt{\Psi_{i\varepsilon}}} \to 0$ in $L^\infty(Q_T)$.

We consider also the adjoint non-degenerate boundary value problem

$$\begin{cases} \partial_t \varphi_{i\varepsilon} + \Psi_{i\varepsilon} \Delta \varphi_{i\varepsilon} = 0 & \text{in } Q_T, \\ \varphi_{i\varepsilon} = 0 & \text{on } (0, T) \times \partial\Omega, \\ \varphi_{i\varepsilon} = \varkappa_i & \text{in } \Omega \times \{t = T\}. \end{cases} \tag{6.2}$$

For any smooth function $\varkappa_i$, with $0 \le \varkappa_i \le 1$, the problem (6.2) has a unique solution $\varphi_{i\varepsilon} \in C^\infty(Q_T)$ satisfying

i) $0 \le \varphi_{i\varepsilon} \le 1$,

ii) $\int_{Q_T} \Psi_{i\varepsilon} (\Delta \varphi_{i\varepsilon})^2 \le C$,

iii) $\sup_{0 \le t \le T} \int_\Omega \|\nabla \varphi_{i\varepsilon}\|^2 \le C$,

where the constant C depends only on $\varkappa_i$. It is obvious that the difference $u_i - \hat{u}_i$ satisfies the following equality:

$$\int_{\Omega}(u_i-\hat{u}_i)\varphi_i(x,T)\,dx+(\sigma_i+1)\int_{Q_T}\nabla[u_i^{\sigma_i}u_i-|\hat{u}_i|^{\sigma_i}\hat{u}_i]\nabla\varphi_i\,dx\,dt$$
$$=\int_{Q_T}(u_i-\hat{u}_i)\varphi_{it}(x,t)\,dx\,dt+\int_{Q_T}\big(g_i(u)-g_i(\hat{u})\big)\varphi_i(x,t)\,dx\,dt$$
$$+\int_{Q_T}\overrightarrow{b_i}\nabla\varphi_i[|u_i|^{m_i-1}u_i-|\hat{u}_i|^{m_i-1}\hat{u}_i] \tag{6.3}$$

for every $\varphi_i\in C^1(\overline{Q_T})$ such that $\varphi_i=0$ on $(0,T)\times\partial\Omega$. By setting $\varphi_i=\varphi_{i\varepsilon}$ and $\varkappa_i=\text{sign}_\varepsilon(u_i-\hat{u}_i)^+$ in (6.3), where sign_ε is a regular approximation of the sign function, we obtain

$$\int_{\Omega}(u_i-\hat{u}_i)^+(x,T)\,dx+\int_{Q_T}\Delta\varphi_{i\varepsilon}(\Psi_{i\varepsilon}-\Psi_i)(u_i-\hat{u}_i)\,dx\,dt$$
$$=\int_{Q_T}\big(g_i(u)-g_i(\hat{u})\big)\varphi_{i\varepsilon}(x,t)+\int_{Q_T}\overrightarrow{b_i}\nabla\varphi_i(|u_i|^{m_i-1}u_i-|\hat{u}_i|^{m_i-1}\hat{u}_i].$$

By using the local Lipschitz continuity of the functions g_i and $|z|^{m_i}z$ and the fact that u_ε is uniformly bounded, and by letting $\varepsilon\to 0$, we obtain the following inequality after the use of Hölder inequality:

$$\int_{\Omega}(u_i-\hat{u}_i)^+(x,T)\,dx\le C\int_{Q_T}\sum_{j=1}^{d}|u_j-\hat{u}_j|\,dx\,dt+C(T)\Big(\int_{Q_T}|u_i-\hat{u}_i|^2\,dx\,dt\Big)^{1/2}. \tag{6.4}$$

Now, if $i\in J$ we have

$$\Big(\int_{Q_T}|u_i-\hat{u}_i|^2\,dx\,dt\Big)^{1/2}\le\frac{\int_{Q_T}|u_i-\hat{u}_i|^2\,dx\,dt}{M(T)}\le C(T)\int_{Q_T}|u_i-\hat{u}_i|\,dx\,dt. \tag{6.5}$$

By combining (6.4), (6.5) and assumption (6.1) we find that

$$\int_{\Omega}(u_i-\hat{u}_i)^+(x,T)\,dx\le\big(C+C(T)\big)\int_{Q_T}\sum_{j=1}^{d}|u_j-\hat{u}_j|\,dx\,dt$$
$$\le\big(C+C(T)\big)\int_{Q_T}\sum_{j\in J}|u_j-\hat{u}_j|\,dx\,dt.$$

By summing up over $j\in J$ we conclude that

$$\int_{\Omega}\sum_{j\in J}(u_j-\hat{u}_j)^+(x,T)\,dx\le d\big(C+C(T)\big)\int_{Q_T}\sum_{j\in J}|u_j-\hat{u}_j|\,dx\,dt. \tag{6.6}$$

In a similar way we can establish that, by letting $\varkappa_i = \operatorname{sgn}_\varepsilon(u_i - \hat{u}_i)^-$, then

$$\int_\Omega \sum_{j\in J}(u_j - \hat{u}_j)^-(x,T)\,dx \le d\big(C + C(T)\big)\int_{Q_T}\sum_{j=1}^{d}|u_j - \hat{u}_j|\,dx\,dt. \tag{6.7}$$

By combining (6.6) and (6.7) we get that

$$\int_\Omega \sum_{j\in J}|u_j - \hat{u}_j|(x,T)\,dx \le 2d\big(C + C(T)\big)\int_{Q_T}\sum_{j\in J}|u_j - \hat{u}_j|\,dx\,dt.$$

We may apply Gronwall's lemma to conclude. □

7. The limit cases

We will show now that in the limit case (namely, $f_i(u,\nabla u_i) = \sum_{j=1}^{d} c_{ij}u_j^{\sigma_j+1} + \overrightarrow{b_i}\nabla(u_i^{\sigma_i+1})$), and depending on the relation between the parameters c_{ij}, λ_i, λ, we get globally bounded weak solutions or blowing up solutions. More precisely, we prove the following.

(1) If Ω is small, in an appropriate sense, all positive weak solutions of (1.1)–(1.4) are global.

(2) If Ω is sufficiently large, all positive weak solutions of (1.1)–(1.4) blow-up (i.e. become unbounded) in finite time.

Hence we deduce that large domains (namely, $\lambda < 1$, which is equivalent to $\lambda_i < 0$) are more unstable than small domains ($\lambda \ge 1$).

Throughout this section we suppose that (H_2), (H_3), (H_6), (H_7) and (H_8) are satisfied.

7.1. Global existence.

Let us consider the problem

$$\begin{cases} \partial_t(u_i) - \Delta(|u_i|^{\sigma_i}u_i) = g_i(u) + \overrightarrow{b_i}\nabla(|u_i|^{m_i-1}u_i) & \text{in }]0,\infty[\times\Omega \\ u_i = 0 & \text{on }]0,\infty[\times\partial\Omega \\ u_i(0,.) = u_{i0} & \text{in } \Omega. \end{cases} \tag{7.1}$$

We suppose that

(H_9) there exist positive constants $c_{ij}, \alpha_{ij}, L_i \ge 0$ such that for all $u_1, u_2 \ge 0$ we have

$$|g_i(u)| = c_{i0} + \sum_{j=1}^{d} c_{ij}u_j^{\alpha_{ij}} \quad \text{and} \quad \|\overrightarrow{b_i}\| \le L_i.$$

Finally, we also suppose that

$$(H_{10})\quad\begin{cases}1.\ \alpha_{ij} < \sigma_j + 1,\ m_i = \sigma_i + 1 \text{ and } \|\overrightarrow{b_i}\| < 2\dfrac{\lambda}{\lambda+1} \text{ for all } i, j = 1, \dots d\\ \text{or}\\ 2.\ \text{there exists } j_0 \in \{1, \dots d\} \text{ such that } \alpha_{ij_0} = \sigma_{j_0} + 1,\ m_i < \sigma_i + 1 \text{ and}\\ c_{ij_0} < \lambda,\ \text{for all } i = 1, \dots d\\ \text{or}\\ 3.\ \alpha_{ij} = \sigma_j + 1,\ m_i < \sigma_i + 1;\ \text{and } d \max\limits_{i,j=1,d} c_{ij} < \lambda \text{ for all } i,j = 1, \dots d\\ \text{or}\\ 4.\ \alpha_{ij} = \sigma_j + 1,\ m_i = \sigma_i + 1;\ \text{and } 2d \max\limits_{i,j=1,d} c_{ij} + \max\limits_{i=1,d} \|\overrightarrow{b_i}\|(\lambda+1) < 2\lambda,\\ \text{for all } i, j = 1, \dots d\end{cases}$$

Theorem 7.1. *Let all the assumptions of this section be fulfilled. Then the problem* (7.1) *has a unique global positive weak solution* $(u_1, u_2, \dots u_d)$ *such that*

$$\|u_i(., t)\|_{L^\infty(\Omega)} \le F(\xi) \quad \textit{for all } t \ge \xi > 0,\ i = 1, 2, \dots, d,$$

and

$$\|u_i(., t)\|_{L^\infty(\Omega)} \le C \quad \textit{for all } t \ge 0,\ i = 1, \dots, d,$$

where $F(\xi)$ *is a positive function not depending on* u_0, *and* C *is a positive constant depending only on* u_0. *Moreover, the semigroup* $S(t)$ *corresponding to the system* (7.1) *possesses a global attractor. Finally, in the fourth case in* (H_{10}), *if we assume that* $c_{i0} = 0$ *for all* $i = 1, \dots d$, *then the solution* u *tends to zero as* t *tends to infinity.*

In proving the existence of a global weak solution, we find *a priori* estimates for smooth solutions of problem (3.8)–(3.10) and proceed as in Section 5. We give the details only in the fourth case of (H_{10}).

Lemma 7.2. *For all* $T > 0$, *there exists a positive function* F, *not depending on* ε, *such that*

$$\|u_{i\varepsilon}(T)\|_{L^\infty(\Omega)}, \|\nabla(u_{i\varepsilon}^{\sigma_i+1})\|^2_{L^2(Q_T)} \le F(T). \tag{7.2}$$

Moreover, in the fourth case of (H_{10}), *if we assume that* $c_{i0} = 0$, *then*

$$\|\nabla(u_{i\varepsilon}^{\sigma_i+1})\|^2_{L^2(Q_T)} \le C, \tag{7.3}$$

with C *is a positive constant independent of* T.

Proof. By multiplying (3.8) by $u_{i\varepsilon}^{\sigma_i+1}$, adding them together, and integrating over Q_T, we obtain the following with the help of the Cauchy–Schwartz inequality:

$$\sum_{i=1}^{d}\frac{1}{\sigma_i+2}\int_{\Omega}u_{i\varepsilon}^{\sigma_i+2}(T)\,dx+\sum_{i=1}^{d}\int_{Q_T}\|\nabla(u_{i\varepsilon}^{\sigma_i+1})\|^2\,dx\,dt$$
$$\leq\sum_{i,j=1}^{d}c_{ij}\int_{Q_T}u_{j\varepsilon}^{2(\sigma_j+1)}\,dx\,dt+\sum_{i=1}^{d}\int_{Q_T}\frac{\|\overrightarrow{b_i}\|}{2}(u_{i\varepsilon}^{2(\sigma_i+1)}+\|\nabla u_{i\varepsilon}^{\sigma_i+1}\|^2)\,dx\,dt$$
$$+\sum_{i=1}^{d}\eta\int_{Q_T}u_{i\varepsilon}^{2(\sigma_i+1)}\,dx\,dt+\sum_{i=1}^{d}\frac{1}{\sigma_i+2}\int_{\Omega}u_{i0\varepsilon}^{\sigma_i+2}\,dx+C(\eta,T)\sum_{i=1}^{d}c_{i0}^2.$$

By letting $M=\max_{i,j=1,\dots d}c_{ij}$ and $b=\max_i=1,\dots d\|\overrightarrow{b_i}\|$ and applying the Poincaré inequality, we get

$$\sum_{i=1}^{d}\frac{1}{\sigma_i+2}\int_{\Omega}u_{i\varepsilon}^{\sigma_i+2}(T)\,dx+\left(\frac{2\lambda-b\lambda-2dM-b}{2\lambda}-\eta\right)\sum_{i=1}^{d}\int_{Q_T}\|\nabla(u_{i\varepsilon}^{\sigma_i+1})\|^2\,dx\,dt$$
$$\leq C(T),$$

and

$$\sum_{i=1}^{d}\frac{1}{\sigma_i+2}\int_{\Omega}u_{i\varepsilon}^{\sigma_i+2}(T)\,dx+\left(\frac{2\lambda-b\lambda-2dM-b}{2\lambda}-\eta\right)\sum_{i=1}^{d}\int_{Q_T}\|\nabla(u_{i\varepsilon}^{\sigma_i+1})\|^2\,dx\,dt$$
$$\leq C,$$

where C is independent of T if $c_{i0}=0$ for all $i=1,\dots d$. Thus, for η small enough we deduce

$$\|u_{i\varepsilon}(T)\|_{L^{\sigma_i+2}(\Omega)},\ \|\nabla(u_{i\varepsilon}^{\sigma_i+1})\|_{L^2(Q_T)}^2\leq C(T),\tag{7.4}$$

and

$$\|u_{i\varepsilon}(T)\|_{L^{\sigma_i+2}(\Omega)},\ \|\nabla(u_{i\varepsilon}^{\sigma_i+1})\|_{L^2(Q_T)}^2\leq C,\tag{7.5}$$

if $c_{i0}=0$, $i=1,\dots d$. Our claim follows then from Theorem 3.1. □

Remark 3. As a conclusion of (7.5) and the Poincaré inequality, we emphasize that if $c_{i0}=0$ then $\|u_{i\varepsilon}^{\sigma_i+1}\|_{L^2(Q_T)}$ is uniformly bounded with respect to T, that is $\|u_{i\varepsilon}^{\sigma_i+1}\|_{L^2(Q_\infty)}$ and then $\|f_{i\varepsilon}(u_\varepsilon,\nabla u_{i\varepsilon})\|_{L^2(Q_\infty)}$ are bounded. Thus we see that $\|f_{i\varepsilon}(u_\varepsilon,\nabla u_{i\varepsilon})\|_{L^2(Q_{t/2,t})}$ tends to zero as $t\to\infty$.

Lemma 7.3. *There is a positive constant C such that for all $t>0$ we have the following inequality:*

$$\|\nabla u_{i\varepsilon}^{\sigma_i+1}(t)\|_{2,\Omega}\leq\frac{2}{t}C+\int_{Q_{t/2,t}}f_{i\varepsilon}^2(u_\varepsilon,\nabla u_{i\varepsilon})\,ds\quad\text{for all } i=1,\dots,d.\tag{7.6}$$

This inequality implies that the solution tends to zero as t tends to ∞, *provided* $c_{i0} = 0$.

Proof. Let $\tau \in \left[\frac{t}{2}, t\right]$, where $t > 0$. By multiplying (3.8) by $(u_{i\varepsilon}^{\sigma_i+1})_t$ and integrating the obtained result over $\Omega \times [\tau, t]$, we obtain

$$I = \left(\frac{2}{\sigma_i + 2}\right)^2 \int_{Q_{t/2,\,t}} \left(\partial_t (u_{i\varepsilon}^{(\sigma_i+1)/2})\right)^2 ds\, dx + \|\nabla u_{i\varepsilon}^{\sigma_i+1}(.,t)\|_{2,\Omega}^2$$
$$\leq \|\nabla u_{i\varepsilon}^{\sigma_i+1}(.,\tau)\|_{2,\Omega}^2 + \int_{Q_{t/2,\,t}} \partial_t (u_{i\varepsilon}^{\sigma_i+1}) f_{i\varepsilon}(u_{1\varepsilon}, u_{2\varepsilon}, \nabla u_{i\varepsilon})\, ds\, dx. \tag{7.7}$$

The Cauchy–Schwartz inequality yields

$$I \leq \left(\frac{2}{\sigma_i + 2}\right)^2 \int_{Q_{t/2,\,t}} \left(\partial_t (u_{i\varepsilon}^{(\sigma_i+1)/2})\right)^2 ds\, dx + \|\nabla u_{i\varepsilon}^{\sigma_i+1}(.,\tau)\|_{2,\Omega}^2$$
$$+ C_1 \int_{Q_{t/2,\,t}} u_{i\varepsilon}^{\sigma_i} f_{i\varepsilon}^2(u_{1\varepsilon}, u_{2\varepsilon}, \nabla u_{i\varepsilon})\, dx\, ds. \tag{7.8}$$

By combining estimates (7.7) and (7.8) we deduce

$$\|\nabla u_{i\varepsilon}^{\sigma_i+1}(.,t)\|_{2,\Omega}^2 \leq \|\nabla u_{i\varepsilon}^{\sigma_i+1}(.,\tau)\|_{2,\Omega}^2 + C_2 \int_{Q_{t/2,\,t}} f_{i\varepsilon}^2(u_{1\varepsilon}, u_{2\varepsilon}, \nabla u_{i\varepsilon})\, dx\, ds. \tag{7.9}$$

By integrating in τ, over $\left[\frac{t}{2}, t\right]$, the previous estimate, we conclude that

$$\frac{t}{2} \|\nabla u_{i\varepsilon}^{\sigma_i+1}(.,t)\|_{2,\Omega}^2$$
$$\leq \int_{Q_{t/2,\,t}} \|\nabla u_{i\varepsilon}^{\sigma_i+1}(.,\tau)\|_{2,\Omega}^2 + C_2 \frac{t}{2} \int_{Q_{t/2,\,t}} f_{i\varepsilon}^2(u_{1\varepsilon}, u_{2\varepsilon}, \nabla u_{i\varepsilon})\, ds\, dx,$$

and this completes the proof. □

7.2. Blow-up results. In the following we assume that

$$\overrightarrow{b_i} \text{ is independent of } t, \qquad \overrightarrow{b_i} \in \left(C^\infty(\overline{\Omega})\right)^N$$

and

$$f_i(u, \nabla u_i) = c_{i0} + \sum_{j=1}^{d} c_{ij} u_j^{\alpha_{ij}} + \overrightarrow{b_i} \nabla (u_i^{m_i}).$$

In this subsection we prove the finite time blow-up results stated in Theorem 2.2. A crucial role is played here by the first eigenvalue of the Dirichlet problem

$$\begin{cases} -\Delta\psi_i(x) + \overrightarrow{b_i}(x)\nabla\psi_i(x) = \lambda_i\psi_i(x) & \text{in } \Omega \\ \psi_i(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

We denote by λ_i the first eigenvalue and by $\psi_i(x)$ the corresponding eigenfunction with the normalization $\psi_i(x) > 0$ in Ω and $\|\psi_i\|_{L^1} = 1$ (see [5]). It is well known that λ_i increases as the size of the domain Ω decreases (see [7]).

Theorem 7.4. *Suppose $c_{ii} > \lambda_i$. Then any positive (nontrivial) weak solution of* (1.1)–(1.4) *blows up in finite time.*

Proof. We multiply the equations defining u_i by ψ_i, add them together and integrate over $(0, t) \times \Omega$, to obtain

$$\sum_{i=1}^{d} \int_\Omega u_i(t)\psi_i\,dx + \lambda_i \sum_{i=1}^{d} \int_{Q_t} u_i^{\sigma_i+1}(s)\psi_i\,dx\,dt$$
$$= \sum_{i,j=1}^{d} c_{ij} \int_{Q_t} u_j^{\sigma_j+1}(s)\psi_i\,dx\,dt + \sum_{i=1}^{d} \left(\int_\Omega u_{i0}\psi_i\,dx + C(t)c_{i0} \right). \tag{7.10}$$

But

$$\sum_{i,j=1}^{d} c_{ij} u_j^{\sigma_j+1}(t)\psi_i \geq M u_i^{\sigma_i+1}(t)\psi_i,$$

where $M = \max_{i=1,\dots d} c_{ii}$. On the other hand, the Hölder inequality yields

$$\int_\Omega u_i^{\sigma_i+1}(t)\psi_i\,dx \geq \left(\int_\Omega u_i(s)\psi_i\,dx \right)^{\sigma_i+1}.$$

By inserting this into (7.10) and denoting $g(s) = \sum_{i=1}^{d} \left(\int_\Omega u_i(s)\psi_i\,dx \right)$, $\sigma = \min_{i=1,\dots d} \sigma_i$, we obtain

$$g(t) \geq (M - \lambda_i) \int_0^t (g(s))^{\sigma+1}\,ds + C.$$

This shows that there exists a finite time T^* such that

$$\lim_{t \nearrow T^*} g(t) = +\infty,$$

hence u blows-up in finite time. □

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