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Variants of the Diophantine equation $n! + 1 = y^2$

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Abstract. In this note we study variants of the Brocard–Ramanujan Diophantine equation $n! + 1 = y^2$. For example, Berend and Harmse [1] proved that the equation $n! = y^r(y+1)$ has only finitely many positive integer solutions (n, y) when $r \geq 4$ is a fixed integer. Here we find all the integer solutions of this equation when $r = 2, 3$ under the additional assumption that $y + 1$ is square-free or cube-free, respectively.

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1. Introduction

The question *"Pour quelles valeurs du nombre entier x l'expression*

$$1 \cdot 2 \cdot 3 \cdot 4 \ldots x + 1$$

est-elle un carré parfait?" was asked by Brocard in [3] and [4]. In 1913, Ramanujan (see [16] and [17]) posed the same problem as follows: *"The number* $n! + 1$ *is a square for* $n = 4, 5, 7$. *Find other values."* The Diophantine equation

$$n! + 1 = y^2 \tag{1}$$

is now referred to as the *Brocard–Ramanujan* Diophantine equation. Finding all the integer solutions (n, y) of the Diophantine equation (1) is still an open problem (see D25 in [10]). Let us give a quick history of this problem. In his contribution to the problem, Gerardin [8] assumed that the equation (1) has no solutions in the range $7 < n < 25$. Using Gerardin's ideas, Gupta [9] proved that the equation (1) has no solutions other than the known ones for $n \leq 63$. In 1993, Overholt [15] showed that the weak form of Szpiro's conjecture implies that the equation (1) has only finitely many solutions. We recall that the weak form of Szpiro's conjecture is a special case of the *abc*-conjecture and asserts that there exists a constant C such that if a and b are coprime positive integers then

$$a + b < \operatorname{rad}\big(ab(a+b)\big)^C,$$

where for a positive integer N we write $\operatorname{rad}(N)$ for the product of all distinct primes dividing N. In 2000, Berndt and Galway [2] used a computational method and extended Gupta's calculations to $n \le 10^9$. No new solution was found.

Many variants of the Brocard–Ramanujan Diophantine equation have also been studied. In 1935, Erdős and Obláth [7] showed that the Diophantine equation

$$y^d \pm 1 = n!$$

has no positive integer solutions (y, d, n) with $y > 1$ and $d \ge 3$. In 1996, Dabrowski [5] studied the Diophantine equation

$$n! + A = y^2 \tag{2}$$

in positive integers n and y when A is a fixed nonzero integer. He proved that if A is not a square, then the equation (2) has only a finite number of positive integer solutions (n, y). He also showed that the weak form of Szpiro's conjecture implies that the equation (2) has only finitely many solutions if A is a square. Using the same method, Dufour and Kihel [6] proved that the weak form of Hall's conjecture, which too is a special case of the *abc*-conjecture, implies that the equation (2) has only finitely many positive integer solutions (n, y). The weak form of Hall's conjecture mentioned above asserts that for every $\varepsilon > 0$ there exists a positive constant C_ε depending on ε only such that if x and y are positive integers with $x^3 \ne y^2$, then

$$\max\{|x^3|, |y^2|\} < C_\varepsilon |x^3 - y^2|^{6+\varepsilon}.$$

They also proved that if the integer A is not a qth power of an integer, then the Diophantine equation

$$n! + A = y^q,$$

has only finitely many positive integer solutions (n, y). In [14], Luca proved that the *abc*-conjecture implies that for any $P \in \mathbb{Z}[X]$ of degree at least 2 the Diophantine equation

$$P(y) = n!$$

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has only finitely many integer solutions (y, n) with $n \geq 0$. In 2004, Kihel and Nwabueze [13] used p-adic linear forms in logarithms to show that if $P \in \mathbb{Z}[X]$ is any nonzero polynomial, then any positive integer solution (x, y, n, q) with $q > 1$ to the Diophantine equation

$$P(n)n! + x^q = y^q \qquad \text{with } y \equiv 1 \pmod 2 \tag{3}$$

satisfies

$$\frac{2^n}{n+1} < y^{4000(\log 2)^2(\log q)^2}.$$

Moreover, it was checked computationally that the Diophantine equation

$$1 + \prod_{\substack{k \nmid n \\ 1 \leq k \leq n}} k = y^2, \tag{4}$$

has only the positive integer solutions $(n, y) = (4, 2)$ and $(5, 5)$ in the range $n \leq 10^5$. Recently, Kihel and Luca [12] studied variants of the Brocard–Ramanujan Diophantine equation (1) in which $n!$ is replaced by the product of the positive integers $k \leq n$ not dividing n (appearing in (4), for instance). For example, with the method from [12] one can easily show that the Diophantine equation (3) has only finitely many integer solutions (x, y, n, q) with x and y coprime and $q \geq 3$.

In the two notes [19] and [20], the third author has studied simultaneous Diophantine equations of the form $x! + A = y^2$. Moreover, he proved that the set

$$S = \{|n! - y^2| \mid (n, y) \in \mathbb{N}\}$$

is of asymptotic density zero. He remarked that the Diophantine equation $n! + 505 = y^2$ has the three solutions $(n, y) = (4, 23), (5, 25), (6, 35)$ and asked if there are any others. The fact that there are only finitely many such solutions follows already from Dabrowski's 1996 result [5] mentioned earlier with $A = 505$.

The aim of this paper is to extend the work done by the third author. So, in Section 2 we answer the question asked in [19] and [20] by showing that there are no other solutions (n, y) to the Diophantine equation $n! + 505 = y^2$ except for the three given above. We also prove some other results related to the Diophantine equation $n! + A = y^2$. In Section 3, we study the Diophantine equation $n! = y^r(y+1)$ for $r = 2, 3$. We prove that the only positive integer solutions (y, n) with $y + 1$ is cube-free are $(y, n) = (1, 2), (2, 4)$ if $r = 3$, and the unique positive integer solution is $(y, n) = (1, 2)$ if $r = 2$. The more general equation

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$n! = y^r(y+1)$ was studied by Berend and Harmse in [1]. They proved that the above Diophantine equation has only finitely many positive integer solutions (n, y) when $r \geq 4$ if a fixed integer and left open the case $1 \leq r \leq 3$. Thus, we give a partial answer to this problem for $r = 3$. In the last section, we prove that the set

$$S = \left\{ \left| \prod_{\substack{k \nmid n \\ 1 \leq k \leq n}} k - y^2 \right| \mid (n, y) \in \mathbb{N} \right\}$$

is of asymptotic density zero.

2. On the equation $n! + A = y^2$

Let A be a fixed nonzero integer.

Theorem 2.1. *Assume that p is a prime such that $p \parallel A$. If (n, y) are positive integers such that*

$$n! + A = y^2, \tag{5}$$

then $n < 2p$.

Proof. Indeed, if $n \geq 2p$, then $p^2 \mid n!$. Since $p \mid A$, we get that $p \mid n! + A$, therefore $p \mid y^2$. Hence, $p \mid y$. In particular, p^2 divides both $n!$ and y^2, thus also $A = y^2 - n!$, which is a contradiction. □

Remark 2.2. The only positive integer solutions (n, y) of the Diophantine equation

$$n! + 505 = y^2 \tag{6}$$

are $(n, y) = (4, 23), (5, 25), (6, 35)$. Indeed, to see this observe that $5 \parallel 505$, so by Theorem 2.1 we must have $n < 10$. A quick computation finishes the job.

Next, we look at the Diophantine equation $n! + A = y^2$ when $A = 3k + 2$ or $A = 4k + 3$. The result is the following.

Theorem 2.3. *Let $k \geq 0$ be a fixed integer. All positive integer solutions (n, y) of the Diophantine equation*

$$n! + 3k + 2 = y^2 \tag{7}$$

have $n < 3$. *All positive integer solutions* (n, y) *of the Diophantine equation*

$$n! + 4k + 3 = y^2 \tag{8}$$

have $n < 4$. *Moreover, each of the Diophantine equations* (7) (8) *has at most one solution.*

Proof. If $n \geq 3$ in equation (7) or $n \geq 4$ in equation (8), we then get that $2 \equiv y^2 \pmod 3$ and $3 \equiv y^2 \pmod 4$, respectively, which is impossible. For the last part, assume say that both $n = 1$ and $n = 2$ yield positive integer solutions y to (7). Then $3k + 3 = y_1^2$ and $3k + 4 = y_2^2$, giving $y_2^2 - y_1^2 = 1$, which does not have positive integer solutions (y_1, y_2). A similar argument applies to show that equation (8) has at most one positive integer solution (n, y). □

The next remark generalizes Theorem 2.2 in [20].

Remark 2.4. All positive integer solutions (n_1, n_2, A, y) of the simultaneous equations

$$n_1! + A + 1 = y^2 \tag{9}$$

and

$$n_2! + A = y^2 \tag{10}$$

have $(n_1, n_2) = (2, 1)$. In fact, subtracting the two equations above we get $n_1! - n_2! = 1$. It is easy to see that the only positive integer solution of this last equation is $(n_1, n_2) = (2, 1)$.

3. On the Diophantine equation $n! = y^r(y + 1)$

In this section we consider the Diophantine equation

$$n! = y^r(y + 1) \qquad \text{with } r = 2, 3. \tag{11}$$

We have the following result.

Theorem 3.1. *The only positive integer solutions* (y, n) *of the Diophantine equation* (11) *with* $r = 3$ *and* $y + 1$ *cube-free are* $(y, n) = (1, 2), (2, 4)$.

Proof. Since $y + 1$ is cube-free, we get that

$$y + 1 \leq \Big(\prod_{p \leq n} p\Big)^2.$$

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Thus,

$$n! = y^3(y+1) < (y+1)^4 < \Big(\prod_{p \le n} p\Big)^8.$$

From the elementary inequality $n! \ge (n/e)^n$, we get that

$$n \log(n/e) \le \log n! < 8 \sum_{p \le n} \log p.$$

Theorem 6 of [18] shows that

$$\sum_{p \le n} \log p < 1.001102n$$

holds for all $n \ge 1$. Thus, we get that $n < e^{1+8\cdot 1.001102}$ or $n \le 8174$.

Next we ran a computation showing that in fact $n < 80$. Here is how we checked it. For each $n \in [80, 8200]$, we checked that there is some prime p among the first 9 (namely, $p \le 23$), such that the exponent of p in $n!$ is not a multiple of 3. In particular, if $n! = y^3(y+1)$ for some n in our range, then $p \mid y+1$. Since $n \ge 80 > 3p$, it follows that the exponent of p in $n!$ is $\ge \lfloor n/p \rfloor \ge 3$, so if $y+1$ is cube-free, then also $p \mid y$, which is a contradiction. Interestingly enough, $n = 8230$ has the property that the exponent of p in $n!$ is a multiple of 3 for all primes $p \le 23$. Finally, one checks with Mathematica that $(n, y) = (2, 1), (4, 2)$ are the only positive integer solutions to the Diophantine equation $n! = y^3(y+1)$ in the range $n \le 80$. □

We use a similar method to obtain the following result.

Theorem 3.2. *The only positive integer solution* (y, n) *of the Diophantine equation* (11) *with* $r = 2$ *and* $y+1$ *square-free is* $(y, n) = (1, 2)$.

Proof. Using exactly the method used in the proof of Theorem 3.1, we get that

$$(n/e)^n \le n! < (y+1)^3 \le \Big(\prod_{p \le n} p\Big)^3 < e^{3\times 1.001102n}.$$

Thus, $n < e^{1+3\cdot 1.001102}$, yielding $n \le 54$. A very quick computation for solving the cubic Diophantine equation (11) with $r = 2$ and $1 \le n \le 54$, gives the unique solution $(y, n) = (1, 2)$. □

4. On a thin set of integers

In this section we look at the set

$$S = \Big\{ \Big| \prod_{\substack{k \nmid n \\ 1 \le k \le n}} k - y^2 \Big| \mid (n, y) \in \mathbb{N} \Big\}. \tag{12}$$

We put $S(T) = S \cap [1, T]$. We prove the following result.

Theorem 4.1. *The estimate* $\#S(T) = T^{1/2+o(1)}$ *holds as* $T \to \infty$.

Proof. Throughout this proof, we use the Landau symbolds O and o and the Vinogradov symbols $\gg$ and $\ll$ with their usual meanings.

Let T be a large positive integer. For the lower bound on $\#S(T)$, we take pairs (n, y) where $n = 1$ and $y \in \{1, \dots, \lfloor \sqrt{T} \rfloor\}$.

From now on, we deal with the upper bound. Let $m \in S(T)$. We may assume that m is not a square since there are $\lfloor \sqrt{T} \rfloor$ perfect squares $\le T$. For a positive integer n we put

$$M_n := \prod_{\substack{k \nmid n \\ 1 \le k \le n}} k = \frac{n!}{n^{\tau(n)/2}},$$

where $\tau(n)$ stands for the number of divisors of n.

Lemma 4.2. *There exists* n_0 *such that if* $n > n_0$ *and* $p < n^{7/8}$ *is a prime, then the exponent of* p *in the prime factorization of* M_n *is* $> n^{1/8}/2$.

Proof. The exponent of p in $n!$ is $\ge \lfloor n/p \rfloor > n/p - 1 > n^{1/8} - 1$. The exponent of p in $n^{\tau(n)/2}$ is $\ll \log(n^{\tau(n)/2}) \ll \tau(n) \log n = n^{o(1)}$ as $n \to \infty$. Thus, the exponent of p in M_n is $\ge n^{1/8} - n^{o(1)}$ as $n \to \infty$, which implies the desired lower bound for large n. □

Now write $m \in S(T)$ as $m = |M_n - y^2|$. Thus, $y^2 - M_n = \eta m$, where $\eta \in \{\pm 1\}$. In what follows, we show that $n < T^{1/2}$ if $T > T_0$, where T_0 is a sufficiently large positive real number. Assume that this is not so. Let D be the part of $\gcd(M_n, m)$ build up of primes $p \le T^{7/16}$. For each prime $p \le T^{7/16}$, let α_p, β_p and γ_p be the exact exponents at which p appears in the prime factorizations of M_n, m and y^2, respectively (some of these exponents are zero if p does not divide D). For such values of p, it follows that $p < n^{7/8}$, so by the above Lemma 4.2 we have that $\alpha_p \gg n^{1/8} \gg T^{1/16}$. Clearly, since $m \le T$, we have that $\beta_p \ll \log T$. Thus, if T is sufficiently large, we then have that $\alpha_p - \beta_p \ge 3$ for all primes

$p \le T^{7/16}$. Writing now $M_n = DM'_n$, $m = Dm_1$, $y^2 = Dz_1$, we get that

$$z_1 - M'_n = \eta m_1.$$

Furthermore, every prime $p \le T^{7/16}$ divides M'_n. This shows that if $p \mid D$, then $\beta_p = \gamma_p$. Since γ_p is even, we get that D is a perfect square, so $z_1 = y^2/D = y_1^2$ is a perfect square also. We thus get that

$$y_1^2 - M'_n = \eta m_1.$$

Furthermore, M'_n is a multiple of 8 (since $\alpha_2 - \beta_2 \ge 3$) and y_1 is odd, so $\eta m_1 \equiv 1 \pmod 8$. If $p \le T^{7/16}$ is any odd prime, then reducing the above relation modulo p we get $\eta m_1 \equiv y^2 \pmod p$, so $(\eta m_1 \mid p) = 1$. Here and in what follows, for integers a and $b > 1$ and odd, we use $(a \mid b)$ for the Jacobi symbol. Note that $m_1 > 1$ since m is not a perfect square. If $\eta = 1$, then $m_1 \equiv 1 \pmod 8$, and the relation $(m_1 \mid p) = 1$ implies by the Quadratic Reciprocity Law that $(p \mid m_1) = 1$. The same is true for $p = 2$ since $m_1 \equiv 1 \pmod 8$. Hence, $(k \mid m_1) = 1$ for all $k \le T^{7/16}$ when $\eta = 1$. The same conclusion holds when $\eta = -1$. Indeed, then $m_1 \equiv 7 \pmod 8$, so by the Quadratic Reciprocity Law,

$$\begin{aligned} 1 &= (-m_1 \mid p) = (-1)^{(p-1)/2}(m_1 \mid p) = (-1)^{(p-1)/2}(-1)^{((p-1)/2)\cdot((m_1-1)/2)}(p \mid m_1) \\ &= (-1)^{p-1}(p \mid m_1) = (p \mid m_1). \end{aligned}$$

The fact that $(2 \mid m_1) = 1$ follows because $m_1 \equiv 7 \pmod 8$. Hence, in this case too we have that $(k \mid m_1) = 1$ holds for all $k \le T^{7/16}$. By the Burgess bound for character sums (see [11, Theorem 12.5]), we get that

$$\lfloor T^{7/16} \rfloor = \sum_{k \le T^{7/16}} (k \mid m_1) \ll T^{7/32+o(1)} m_1^{3/16} \le T^{7/32+3/16+o(1)} = T^{13/32+o(1)}$$

as $T \to \infty$, which is of course a contradiction for large values of T. This contradiction shows that indeed $n < T^{1/2}$ if T is large.

Let $z(T) := 10 \log T / \log\log T$. Assume that $n > z(T)$. Then, by the Stirling formula,

$$M_n = \frac{n!}{n^{\tau(n)/2}} = \exp\big(n \log n + O\big(\tau(n) \log n\big)\big) = \exp\big(\big(1 + o(1)\big) n \log n\big) \ge T^{10+o(1)},$$

showing that for large values of T we have that $m = O(M_n^{1/9})$. Thus, the equation $y^2 - M_n = \eta m$ gives

$$y^2 = M_n + O(M_n^{1/9}),$$

so

$$y = M_n^{1/2}\big(1 + O(M_n^{-8/9})\big)^{1/2} = M_n^{1/2} + O(M_n^{-7/18}) = M_n^{1/2} + o(1)$$

as $T \to \infty$, implying that if n is fixed, then y must be the closest integer to $M_n^{1/2}$. Thus, if $n > z(T)$, then y is uniquely determined in terms of n. This shows that the number of possibilities for m is at most the number of possibilities for n; hence, $O(T^{1/2})$.

Next fix n with $n \le z(T)$. Let $y_1, \dots, y_k$ be all the possibilities for y such that $\eta = 1$. Then the numbers

$$y_1^2 - M_n, y_2^2 - M_n, \dots, y_k^2 - M_n \text{ are all in } S(T). \tag{13}$$

Assume that $y_1 < \cdots < y_k$. Then the difference between the first and last of the above numbers in (13) is $y_k^2 - y_1^2 > (y_k - y_1)^2 \ge (k-1)^2$ and it is obviously $\le T$. Hence, $k \ll T^{1/2}$. In the same way, one proves that if $n \le z(T)$ is fixed, then there are only $O(T^{1/2})$ possibilities for y such that $m \in S(T)$ and $\eta = -1$. Summing now up over all the possibilities for n, we get that there are $O\big(T^{1/2}z(T)\big) = T^{1/2+o(1)}$ as $T \to \infty$ such possiblities for our m. This completes the proof of the theorem. □

Remark 4.3. By partial summation, Theorem 4.1 implies that the series

$$\sum_{m \in S} \frac{1}{m^{1/2-\varepsilon}} < \infty \qquad \text{for all } \varepsilon > 0.$$

An identical proof works to show that, as $T \to \infty$, there are at most $T^{1/2+o(1)}$ positive integers $m \le T$ of the form $|n! - y^2|$ for some natural numbers n and y. Note that there are also at least $T^{1/2+o(1)}$ such integers m as well (just take $n = 1$ and $y \in \{1, \dots, \lfloor T^{1/2} \rfloor\}$), so up to the exponent of $o(1)$ our result is best possible. In [20], the third author has proved that the set

$$S' = \{|x! - y^2| \mid (x, y) \in \mathbb{N}\}. \tag{14}$$

is of asymptotic density zero, although the upper bound on the counting function of S' obtained there is much weaker than the present one.

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