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Series of reciprocal products with factors from linear recurrence sequences

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Abstract. In this article we study the values of power series $\sum z^n / \prod_{i=0}^{j} W_{(n+im)k}$ at certain points of their domain of meromorphy from the arithmetical point of view. (W_n) is a sequence of non-zero integers satisfying a recurrence $W_{n+1} = pW_n + qW_{n-1}$ with non-zero integers p, q such that the discriminant $\Delta = p^2 + 4q$ is positive but not a square. The main interest is to characterize the situations, where these values lie in the real quadratic number field $\mathbb{Q}(\sqrt{\Delta})$ or even in $\mathbb{Q}$, but we also include some transcendence problems.

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1. Introduction and main result

Melham [6] considered, for given positive real p, the binary linear recurrence

$$W_n = pW_{n-1} + W_{n-2} \qquad (n = 2, 3, \ldots), \tag{1}$$

which is uniquely determined by the pair (W_0, W_1) of initial values. For (W_0, W_1) being $(0, 1)$ or $(2, p)$, the corresponding sequences are denoted by (U_n) and (V_n), respectively, and these can be explicitly given in the form

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad V_n = \alpha^n + \beta^n \tag{2}$$

for any $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$. Here α, β are the two real roots of the companion polynomial $X^2 - pX - 1$ of (1). In the case $p = 1$, (U_n) and (V_n) are the Fibonacci and Lucas sequence, respectively (which we denote, as usual, by (F_n) and (L_n)), and in the case $p = 2$ the Pell and Pell–Lucas sequence, respectively.

The main objective of Melham's note [6] is to express the series

$$\sum_{n=1}^{\infty} \frac{1}{W_{nk} W_{(n+m)k}} \tag{3}$$

in the two cases $W_n = U_n$ and $W_n = V_n$ in closed form by values of the Lambert series $\sum_{n=1}^{\infty} z^n/(1-z^n)$ at the points β^{2k}, β^{4k}, β^{8k} in the first case, where β^{4k} can be omitted in the second one. Here $k, m \in \mathbb{N} := \mathbb{N}_0 \backslash \{0\}$ are odd and β satisfies $0 > \beta > -1$. The particular case $m = 1$ of this problem has been treated earlier by André-Jeannin [1].

The main aim of the present paper is to investigate arithmetically series of type (3), in fact, far-reaching generalizations of them, without recourse to their representation by the above-mentioned Lambert series. Plainly, for that purpose we need arithmetical hypotheses on the parameter p in (1). More generally, we shall consider linear recurrences of the form

$$W_n = pW_{n-1} + qW_{n-2} \qquad (n = 2, 3, \ldots), \tag{4}$$

with $p, q, W_0, W_1 \in \mathbb{Z}$ and $pqW_1 \neq 0$ implying, in particular, $W_n \in \mathbb{Z}$ for any $n \in \mathbb{N}_0$. Then the companion polynomial $X^2 - pX - q$ of (4) has two non-zero roots, which are distinct (and denoted by α, β as above) if the expression $\Delta := p^2 + 4q$ is non-zero. Moreover, both of them are real and have distinct absolute values if we suppose that even $\Delta > 0$. Without loss of generality, we will always assume that $|\alpha| > |\beta|$. Requiring additionally that Δ is not a square, both roots are irrational and generate the same real quadratic number field $\mathbb{Q}(\sqrt{\Delta})$.

Moreover, it is easily seen that every W_n can be written in the form

$$W_n = g\alpha^n + h\beta^n \qquad (n \in \mathbb{N}_0), \tag{5}$$

where g and h can be expressed by α, β and the initial values W_0, W_1 in the form

$$g = \frac{W_1 - W_0\beta}{\alpha - \beta}, \qquad h = \frac{W_0\alpha - W_1}{\alpha - \beta} \tag{6}$$

implying that $g, h \in \mathbb{Q}(\sqrt{\Delta}) \backslash \{0\}$. From (5) it is clear that at most one W_n, $n \in \mathbb{N}_0$, can vanish.

We now state our main result, from which we will deduce later, by suitable specialization, arithmetic data on Melham's series (3). But first we recall that the denominator of an algebraic number δ, denoted by $\mathrm{den}(\delta)$, is the smallest $d \in \mathbb{N}$ such that $d \cdot \delta$ is an algebraic integer.

Theorem 1.1. *Suppose that* $(W_n) \in \mathbb{Z}^{\mathbb{N}_0}$, *with* $W_n \neq 0$ *for any* $n \in \mathbb{N}$, *satisfies the linear recurrence* (4) *with* $p \in \mathbb{Z}$, $q \in \{1, -1\}$ *and* $\Delta = p^2 + 4q > 0$ *not a square. Assume that the roots* α, β *of the companion polynomial of* (4) *satisfy* $|\alpha| > |\beta|$, *and that the* $g, h \in K^{\times}$ *in the representation* (5) *of the* W_n *are such that* $\frac{g}{h}$ *is a unit in the ring of integers* O_K *of* $K := \mathbb{Q}(\sqrt{\Delta})$. *Finally, let* $j, k, m \in \mathbb{N}$ *and* $\ell \in \mathbb{N}_0$.

Then, for the meromorphic function $\mathscr{W}_j(z; k, \ell, m)$, *defined in* $|z| < |\alpha|^{jk}$ *by the power series*

$$\sum_{n=1}^{\infty} \frac{z^n}{\prod_{i=0}^{j-1} W_{(n+im)k+\ell}}, \tag{7}$$

the following alternative holds.

(i) *For every* $\gamma \in K^{\times}$ *with* $\gamma \neq (-q)^{\rho k} \alpha^{(j-2\rho)k}$ *for all* $\rho \in \mathbb{Z}$, $\rho < j$, *and with* $\operatorname{den}(\gamma^{-1}\alpha^{(j+2\tau-2)k}) < \left|\frac{\alpha}{\beta}\right|^{k/4}$ *for some* $\tau \in \mathbb{N}_0$ *one has* $\mathscr{W}_j(\gamma; k, \ell, m) \notin K$. *This is true if* m *is odd; for even* m, *however, one has to suppose additionally that* $\gamma \neq -(-q)^{\rho k} \alpha^{(j-2\rho)k}$ *for* $\rho = 1, \ldots, j-1$.

(ii) *For* $\gamma = (-q)^{\rho k} \alpha^{(j-2\rho)k}$ *with some* $\rho \in \{1, \ldots, j-1\}$ *if* m *is odd, and for* $\gamma = \pm(-q)^{\rho k} \alpha^{(j-2\rho)k}$ *with some* $\rho \in \{1, \ldots, j-1\}$ *if* m *is even, the number* $\mathscr{W}_j(\gamma; k, \ell, m)$ *lies in* K *and can be explicitly determined.*

Remark 1.2. If $j = 1$ the power series (7) does not depend on m, hence we write from now on $\mathscr{W}(z; k, \ell)$ instead of $\mathscr{W}_1(z; k, \ell, m)$. Note that, in this case, the alternative (ii) cannot occur and, moreover, the additional hypothesis on γ for even m can be dropped from (i).

Remark 1.3. The hypotheses on p, q and Δ in our theorem imply $|p| \geq 1$ if $q = 1$, and $|p| \geq 3$ if $q = -1$. The assumption $q \in \{1, -1\}$ guarantees that not only α, β are in O_K but also α^{-1}, β^{-1}. This will be needed later in the proof. The hypothesis on the quotient $\frac{g}{h}$ can be checked in each case via (6). It should be pointed out that, *from now on*, we define U_n, V_n as in (2), no matter if $q = 1$ (as at the very beginning) or $q = -1$. Then, in the 'U-case' we have to take $g = -h = \frac{1}{\alpha - \beta}$ in (5), and $g = h = 1$ in the 'V-case'. Hence, in these two standard cases, the condition on $\frac{g}{h}$ is satisfied.

Remark 1.4. By $\alpha, \alpha^{-1} \in O_K$, every expression of the form $\gamma^{-1}\alpha^{(j+2\tau-2)k}$ with $\gamma = \pm\alpha^{\omega}$, $\omega \in \mathbb{Z}$, has denominator 1. Hence, for those γ's, the denominator condition needed only in case (i) is satisfied.

In the next section, we will present several applications of Theorem 1.1 with $\gamma = \pm 1$. It should be noted that the consideration of the points $\gamma = \pm\alpha^{\omega}$, $\omega \in \mathbb{Z}$, would be equally possible but technically more unpleasant. On the other hand, by

(7), $\mathscr{W}_2(1;k,0,m)$ is exactly Melham's series (3), for which, under the conditions of our theorem, the following characterization will be deduced in Corollary 2.3:

$$\mathscr{W}_2(1;k,\ell,m) \notin K \Leftrightarrow q = 1 \text{ and } km \text{ odd.}$$

Notice that in Theorem 2 of [6] the series $\mathscr{W}_2(1;k,0,m)$ was considered in just this case $q = 1$, km odd for the particular sequences (W_n) being (U_n) or (V_n). In these two cases, we will obtain in Corollaries 2.4 and 2.5 below more detailed information by looking a bit more precisely to the proof of Theorem 1.1.

2. Some corollaries and transcendence problems

These will concern the two cases $\gamma = \pm 1$ for odd j, for j divisible by 4, and for $j = 2$.

Corollary 2.1. *Assume the hypotheses of Theorem* 1.1 *and let j be odd. Then the values of both series $\mathscr{W}_j(1;k,\ell,m)$ and $\mathscr{W}_j(-1;k,\ell,m)$ are not in $K = \mathbb{Q}(\sqrt{\Delta})$.*

Corollary 2.2. *Assume the hypotheses of Theorem* 1.1 *and let j be a multiple of* 4. *Then $\mathscr{W}_j(1;k,\ell,m) \in K$ holds always, but $\mathscr{W}_j(-1;k,\ell,m) \in K$ holds if and only if m is even.*

Corollary 2.3. *Suppose that the conditions of Theorem* 1.1 *are fulfilled. Then the following two equivalences hold:*

a) $\mathscr{W}_2(1;k,\ell,m) \notin K \Leftrightarrow q = 1$ *and km is odd,*

b) $\mathscr{W}_2(-1;k,\ell,m) \notin K \Leftrightarrow (-q)^k = 1$ *and m is odd.*

If either $q = -1$ or $q = 1$ and km is even, then $\mathscr{W}_2(1;k,\ell,m)$ is in K, and the same holds for $\mathscr{W}_2(-1;k,\ell,m)$ if m is even or $(-q)^k = -1$. According to part (ii) in our theorem, these sums can be explicitly determined. Our next corollary precisely describes in both situations the conditions for $\mathscr{W}_2(\pm 1;k,\ell,m)$ to lie in $K \backslash \mathbb{Q}$ or in $\mathbb{Q}$, at least in the two standard cases (U_n) and (V_n) of (W_n), where we self-evidently write $\mathscr{U}$ and $\mathscr{V}$ instead of $\mathscr{W}$.

Corollary 2.4. *Suppose that the conditions of Theorem* 1.1 *are satisfied. Then the following is true.*

a) *The values of the sums $\mathscr{U}_2(1;k,\ell,m)$, $\mathscr{V}_2(1;k,\ell,m)$ lie in $K \backslash \mathbb{Q}$ if either $q = -1$ or $q = 1$ and k is even, but they lie in $\mathbb{Q}$ if $q = 1$, k is odd and m is even.*

b) *The values of the sums $\mathscr{U}_2(-1;k,\ell,m)$, $\mathscr{V}_2(-1;k,\ell,m)$ lie in $K \backslash \mathbb{Q}$ if $(-q)^k = -1$, but they lie in $\mathbb{Q}$ if $(-q)^k = 1$ and m is even.*

This corollary is purely qualitative. In the subsequent one, we explicitly determine the sums $\mathscr{W}_2(\pm 1; k, \ell, m)$ in the U- and V-case under the conditions stated in Corollary 2.4 being necessary and sufficient that their values belong to $K\backslash\mathbb{Q}$ or to $\mathbb{Q}$.

Corollary 2.5. *Assume the conditions of Theorem* 1.1.

a) *If either $q = -1$ or $q = 1$ and k is even, then*

$$\mathscr{U}_2(1; k, \ell, m) = \frac{(-q)^\ell}{2U_{mk}}\Big(\sum_{n=1}^{m}\frac{V_{nk+\ell}}{U_{nk+\ell}} - m\sqrt{\Delta}\Big) =: D(U),$$

$$\mathscr{V}_2(1; k, \ell, m) = \frac{(-q)^\ell}{2U_{mk}}\Big(\frac{m}{\sqrt{\Delta}} - \sum_{n=1}^{m}\frac{U_{nk+\ell}}{V_{nk+\ell}}\Big) =: D(V),$$

and if $q = 1$, k is odd and m is even, then

$$\mathscr{U}_2(1; k, \ell, m) = \frac{U_k}{U_{mk}}\sum_{i=1}^{m/2}\frac{1}{U_{(2i-1)k+\ell}U_{2ik+\ell}} =: S(U),$$

$$\mathscr{V}_2(1; k, \ell, m) = \frac{U_k}{U_{mk}}\sum_{i=1}^{m/2}\frac{1}{V_{(2i-1)k+\ell}V_{2ik+\ell}} =: S(V).$$

b) *If $(-q)^k = -1$, then $\mathscr{U}_2(-1; k, \ell, m) = D(U)$ and $\mathscr{V}_2(-1; k, \ell, m) = D(V)$. If $(-q)^k = 1$ and m is even, then $\mathscr{U}_2(-1; k, \ell, m) = -S(U)$ and $\mathscr{V}_2(-1; k, \ell, m) = -S(V)$.*

In particular, if $k = 1$, $\ell = 0$ and m is even and U, V are the usual Fibonacci and Lucas numbers, respectively, we deduce from the third formula in a) that

$$\sum_{n=1}^{\infty}\frac{1}{F_nF_{n+m}} = \frac{1}{F_m}\sum_{i=1}^{m/2}\frac{1}{F_{2i-1}F_{2i}}, \qquad \sum_{n=1}^{\infty}\frac{1}{L_nL_{n+m}} = \frac{1}{F_m}\sum_{i=1}^{m/2}\frac{1}{L_{2i-1}L_{2i}}.$$

Both formulae are due to Brousseau [2]. Moreover, we find from the first formula in b) and from $L_n = 2F_{n-1} + F_n$ that

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{F_nF_{n+m}} = \frac{1}{2F_m}\Big(\sum_{n=1}^{m}\frac{L_n}{F_n} - m\sqrt{5}\Big) = \frac{1}{F_m}\Big(m\frac{1-\sqrt{5}}{2} + \sum_{n=1}^{m}\frac{F_{n-1}}{F_n}\Big), \qquad (8)$$

a formula due to Rabinowitz [8]. Using instead the second formula in b) and $5F_n = 2L_{n-1} + L_n$ we obtain the following Lucas analogue:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{L_n L_{n+m}} = -\frac{1}{5F_m}\Big(m\frac{1-\sqrt{5}}{2} + \sum_{n=1}^{m} \frac{L_{n-1}}{L_n}\Big).$$

Notice that the two last formulae hold for arbitrary positive integers m.

Our last corollary will concern the case $j = 4$. Under the conditions of Theorem 1.1, Corollary 2.2 tells us precisely when $\mathscr{W}_4(\pm 1; k, \ell, m)$ lies in K. Having Corollaries 2.4 and 2.5 in mind, we may ask if it is possible to describe here also the exact conditions for these values to lie in $K \backslash \mathbb{Q}$ or in $\mathbb{Q}$. For reasons to be explained in Remark 4.2 below, we are not able to solve this problem in full generality, even not in the general U- or V-case (as we did right before for $j = 2$) but just in the particular case $p = q = 1$ of Fibonacci and Lucas sequences, where we will write $\mathscr{F}$ and $\mathscr{L}$ instead of $\mathscr{W}$, $\mathscr{U}$, $\mathscr{V}$.

Corollary 2.6. *For $m \in \mathbb{N}$ the following assertions hold.*

a) *Both $\mathscr{F}_4(1; 1, 0, m)$ and $\mathscr{L}_4(1; 1, 0, m)$ lie in $\mathbb{Q}(\sqrt{5}) \backslash \mathbb{Q}$,*

b) *both $\mathscr{F}_4(-1; 1, 0, m)$ and $\mathscr{L}_4(-1; 1, 0, m)$ are rational if m is even, but do not lie in $\mathbb{Q}(\sqrt{5})$ if m is odd.*

As a matter of fact, we shall prove in Section 4 that

$$\mathscr{F}_4(j; 1, 0, m) = a_j(m)\mathscr{F}_2(-j; 1, 0, m) + b_j(m) \qquad \text{for } j \in \{1, -1\} \tag{9}$$

(and two similar formulae for the $\mathscr{L}$'s) hold with certain explicit rational $a_j(m)$, $b_j(m)$ satisfying $a_j(m) \neq 0$ for any $m \in \mathbb{N}$. Using (9), b) from Corollary 2.4 implies a), and a) from Corollary 2.4 implies the part of even m in b), whereas $\mathscr{F}_4(-1; 1, 0, m) \notin \mathbb{Q}(\sqrt{5})$ for odd m is already known from Corollary 2.2 or can be seen from a) in Corollary 2.3.

To deal also with a numerical example, we compute from (20) below $a_1(2) = -\frac{5}{24}$, $b_1(2) = -\frac{113}{2880}$ in (9), and using (8) for $m = 2$, i.e., $\mathscr{F}_2(-1; 1, 0, 2) = 2 - \sqrt{5}$, we find

$$\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2} F_{n+4} F_{n+6}} = \frac{5\sqrt{5}}{24} - \frac{1313}{2880}.$$

Some transcendence problems. We conclude this section by a few questions on transcendence and algebraic independence. For example, one could adopt the principle that every function value appearing in our theorem or in its corollaries, that is not in our quadratic field K, should be transcendental. Another good open problem, proposed by Ribenboim [9], p. 60, is the algebraic independence of the three series $\mathscr{F}_j(1; 1, 0, 1)$, $j = 1, 2, 3$.

The only known transcendence results in our topic stem from the fact, proved by Duverney, the Nishiokas and Shiokawa [5] (see also [4]), that the series

$$\sum_{n=1}^{\infty}\frac{1}{U_{2n-1}},\qquad \sum_{n=1}^{\infty}\frac{1}{V_{2n}},\qquad \sum_{n=1}^{\infty}\frac{1}{V_n}$$

are transcendental, the first two if $q=1$, the third if $q=-1$. These assertions were deduced from Nesterenko's [7] famous transcendence theorems on modular functions. In our notation, the above three series are $\mathscr{U}(1;2,1)$, $\mathscr{V}(1;2,0)$, $\mathscr{V}(1;1,0)$ (compare Remark 1.2). Using formula (15) below with $\gamma=\alpha^{(j-1)k}$ one concludes the transcendence of

$$\mathscr{U}_j(\alpha^{2(j-1)};2,1,m),\qquad \mathscr{V}_j(\alpha^{2(j-1)};2,0,m),\qquad \mathscr{V}_j(\alpha^{j-1};1,0,m)$$

for any $j,m\in\mathbb{N}$, the first two if $q=1$, the third if $q=-1$.

3. Proof of Theorem 1.1

We first investigate the power series (7) analytically. For $j=1$ it reduces to

$$\sum_{n=1}^{\infty}\frac{z^n}{W_{nk+\ell}},\tag{10}$$

on which it was shown in Lemma 3 of [3]: The series (10), convergent in $|z|<|\alpha|^k$, has a meromorphic continuation to the whole complex plane. This continuation $\mathscr{W}(z;k,\ell)$ has its poles exactly at the points $\alpha^k\left(\frac{\alpha}{\beta}\right)^{ik}$, $i\in\mathbb{N}_0$, and they are all simple.

Denoting, for arbitrary $j\in\mathbb{N}$, the power series (7), convergent in $|z|<|\alpha|^{jk}$, by $\mathscr{W}_j(z;k,\ell,m)$ as in our theorem, we next clarify the connection of $\mathscr{W}_j$ and $\mathscr{W}_{j+1}$. For that purpose we start from the identity

$$W_{(n+jm)k+\ell}-\beta^{jmk}W_{nk+\ell}=g(\alpha^{jmk}-\beta^{jmk})\alpha^{nk+\ell}$$

following from (5). Multiplying this identity by $z^n/(\prod_{i=0}^{j}W_{(n+im)k+\ell})$ and summing over all $n\in\mathbb{N}$ we obtain after an easy calculation that

$$\left(1-\left(\frac{\beta^{jk}}{z}\right)^m\right)\mathscr{W}_j(z;k,\ell,m)+\left(\frac{\beta^{jk}}{z}\right)^m\sum_{n=1}^{m}\frac{z^n}{\prod_{i=0}^{j-1}W_{(n+im)k+\ell}}$$
$$=g\alpha^{\ell}(\alpha^{jmk}-\beta^{jmk})\mathscr{W}_{j+1}(\alpha^k z;k,\ell,m).\tag{11}$$

By the induction assumption, $\mathcal{W}_j(z;k,\ell,m)$ is holomorphic in $|z|<|\alpha|^{jk}$, hence has no pole at all points $\beta^{jk}e^{2\pi i\mu/m}$ $(\mu=0,\dots,m-1)$. Thus, we conclude from (11): $p\in\mathbb{C}^\times$ is a pole of $\mathcal{W}_{j+1}$ if and only if $p\alpha^{-k}$ is a pole of $\mathcal{W}_j$, i.e., if and only if $p=\alpha^{(j+1)k}\left(\frac{\alpha}{\beta}\right)^{ik}$ for suitable $i\in\mathbb{N}_0$. With

$$\begin{aligned} A_j(z) &:= \frac{z^m-\beta^{jkm}}{C_jz^m},\\ B_j(z) &:= \frac{\beta^{jkm}}{C_jz^m}\sum_{n=1}^{m}\frac{z^n}{\prod_{i=0}^{j-1}W_{(n+im)k+\ell}},\\ C_j &:= g\alpha^\ell(\alpha^{jmk}-\beta^{jmk}), \end{aligned} \tag{12}$$

equation (11) can be equivalently written as

$$\mathcal{W}_{j+1}(z)=A_j\left(\frac{z}{\alpha^k}\right)\mathcal{W}_j\left(\frac{z}{\alpha^k}\right)+B_j\left(\frac{z}{\alpha^k}\right) \tag{13}$$

if we suppress for the moment the parameters ℓ, m and (partly) k. From this we obtain by iteration

$$\mathcal{W}_j(z)=\mathcal{W}\left(\frac{z}{\alpha^{(j-1)k}}\right)\prod_{\rho=1}^{j-1}A_\rho\left(\frac{z}{\alpha^{(j-\rho)k}}\right)+\sum_{\rho=1}^{j-1}B_\rho\left(\frac{z}{\alpha^{(j-\rho)k}}\right)\prod_{\iota=\rho+1}^{j-1}A_\iota\left(\frac{z}{\alpha^{(j-\iota)k}}\right) \tag{14}$$

for $j=1,2,\dots$, with the usual convention to interpret empty products or sums as 1 or 0, respectively. Thus, (14) is trivial for $j=1$. If (14) is true for some $j\in\mathbb{N}$, then (13) leads to

$$\begin{aligned} \mathcal{W}_{j+1}(z) = {} & \mathcal{W}\left(\frac{z}{\alpha^{jk}}\right)\prod_{\rho=1}^{j}A_\rho\left(\frac{z}{\alpha^{(j-\rho+1)k}}\right)\\ & +\sum_{\rho=1}^{j-1}B_\rho\left(\frac{z}{\alpha^{(j-\rho+1)k}}\right)\prod_{\iota=\rho+1}^{j}A_\iota\left(\frac{z}{\alpha^{(j-\iota+1)k}}\right)+B_j\left(\frac{z}{\alpha^k}\right), \end{aligned}$$

whence formula (14) holds for $j+1$ instead of j. Clearly, (14) describes the connection of the function $\mathcal{W}_j(z;k,\ell,m)$ with $\mathcal{W}(z;k,\ell)$.

Having all analytic tools for the proof of our theorem, we next quote Theorem 2 from [3] as our main arithmetic tool.

Lemma 3.1. *Let K be an algebraic number field and O_K its ring of integers. Assume that $\alpha,\beta\in K^\times$ have the following three properties: $\frac{\alpha}{\beta}$ is a unit in O_K; the inclusion $K\subset\mathbb{Q}\left(\left(\frac{\alpha}{\beta}\right)^s\right)$ holds for every $s\in\mathbb{N}$; the inequalities $\left|\frac{\alpha}{\beta}\right|>1$ and $\left|\left(\frac{\alpha}{\beta}\right)^{\{\sigma\}}\right|<1$*

hold for any $\sigma \in \mathrm{Aut}(\overline{\mathbb{Q}}|\mathbb{Q})\setminus\{\mathrm{id}\}$. *Assume further* $g, h \in K^\times$ *such that* $\frac{g}{h}$ *is a unit in* O_K, *and suppose that the* W_n *from* (5) *are non-zero for any* $n \in \mathbb{N}$. *For fixed* $k \in \mathbb{N}$, $\ell \in \mathbb{N}_0$, *let* $\mathscr{W}(z; k, \ell)$ *denote the meromorphic function defined in* $|z| < |\alpha|^k$ *by the power series* (10).

Then $\mathscr{W}(\gamma; k, \ell) \notin K$ *holds for every* $\gamma \in K^\times \setminus \alpha^k \left(\frac{\alpha}{\beta}\right)^{k\mathbb{N}_0}$ *satisfying the condition* $\mathrm{den}\left(\gamma^{-1}\beta^k\left(\frac{\alpha}{\beta}\right)^{\tau k}\right)^{2[K:\mathbb{Q}]} < \left|\frac{\alpha}{\beta}\right|^k$ *for some* $\tau \in \mathbb{N}_0$.

Assuming now all hypotheses of Theorem 1.1, we obviously intend to apply the preceding lemma with $K = \mathbb{Q}(\sqrt{\Delta})$. From $\alpha\beta = -q \in \{1, -1\}$ we conclude that $\frac{\alpha}{\beta} = \pm\alpha^2$ and $\frac{\beta}{\alpha} = \pm\beta^2$ are both in O_K, whence $\frac{\alpha}{\beta}$ is a unit. Since this one lies outside the unit circle, by an assumption of our theorem, its only conjugate lies in the unit disk. From

$$\begin{aligned}\left(\frac{\alpha}{\beta}\right)^s &= (-q\alpha^2)^s \\ &= \left(\frac{-q}{4}\right)^s (p \pm \sqrt{\Delta})^{2s} \\ &= \left(\frac{-q}{4}\right)^s \left(\sum_{j=0}^{s} \binom{2s}{2j} p^{2(s-j)}\Delta^j \pm \sqrt{\Delta}\sum_{j=0}^{s-1}\binom{2s}{2j+1} p^{2(s-j)-1}\Delta^j\right)\end{aligned}$$

it is evident that $\left(\frac{\alpha}{\beta}\right)^s$ lies in $\mathbb{Q}(\sqrt{\Delta})\setminus\mathbb{Q}$ for every $s \in \mathbb{N}$. Hence all three assumptions on α, β in Lemma 3.1 are verified. Thus, Lemma 3.1 implies that $\mathscr{W}(\gamma; k, \ell) \notin K$ for any $\gamma \in K^\times$ with $\gamma \neq (-q)^{\nu k}\alpha^{(2\nu+1)k}$ for $\nu = 0, 1, \ldots$ and $\mathrm{den}(\gamma^{-1}\alpha^{(2\tau-1)k}) < \left|\frac{\alpha}{\beta}\right|^{k/4} = |\alpha|^{k/2}$ for suitable $\tau \in \mathbb{N}_0$, which proves our theorem in the case $j = 1$.

Suppose that $j \geq 2$. We use formula (14), which we rewrite in the more detailed form evaluated at the point $z = \gamma$:

$$\mathscr{W}_j(\gamma; k, \ell, m) = \mathscr{W}\left(\frac{\gamma}{\alpha^{(j-1)k}}\right)\prod_{\rho=1}^{j-1} A_\rho\left(\frac{\gamma}{\alpha^{(j-\rho)k}}\right) + D_j(\gamma; k, \ell, m). \tag{15}$$

Here the rational function D_j could be written down explicitly using (14) and the definitions in (12). The explicit form of A_ρ in (12) shows that

$$A_\rho\left(\frac{\gamma}{\alpha^{(j-\rho)k}}\right) = \frac{\gamma^m - \alpha^{(j-\rho)mk}\beta^{\rho mk}}{C_\rho \gamma^m}.$$

This vanishes if and only if $\left(\frac{\gamma}{\alpha^{(j-\rho)k}\beta^{\rho k}}\right)^m = 1$ for some $\rho \in \{1, \ldots, j-1\}$. Since $K \subset \mathbb{R}$ we have only to find the real numbers γ satisfying the last equation. By $\beta = -\frac{q}{\alpha}$, it says

$$\left(\frac{\gamma}{(-q)^{\rho k}\alpha^{(j-2\rho)k}}\right)^m = 1 \qquad \text{for some } \rho \in \{1, \dots, j-1\}.$$

For odd m, this is equivalent to $\gamma = (-q)^{\rho k}\alpha^{(j-2\rho)k}$ for some $\rho \in \{1, \dots, j-1\}$, and for even m with $\gamma = \pm(-q)^{\rho k}\alpha^{(j-2\rho)k}$ for such a ρ. Precisely, if one of the cases characterized here occurs, then $\mathscr{W}_j(\gamma; k, \ell, m) = D_j(\gamma; k, \ell, m) \in K$ holds. In each other case we have $\prod_{\rho=1}^{j-1} A_\rho\left(\frac{\gamma}{\alpha^{(j-\rho)k}}\right) \in K^\times$ and then, by (15), we deduce the claim of Theorem 1.1 from the basis step $j = 1$.

4. Proof of the Corollaries 2.1–2.6

Proof of Corollary 2.1. As we know from Remark 1.4, the denominator condition in (i) of Theorem 1.1 is satisfied for $\gamma = \pm 1$. On the other hand, $j - 2\rho \neq 0$ holds for any $\rho \in \mathbb{Z}$ since j is odd, whence $|\pm(-q)^{\rho k}\alpha^{(j-2\rho)k}| \neq 1$ for any such ρ, and case (i) applies to both $\mathscr{W}_j(\pm 1; k, \ell, m)$. □

Proof of Corollary 2.2. Suppose $4 \mid j$. If $\gamma = 1$ we apply (ii) of our theorem with $\rho = \frac{j}{2}$, which is even. Let $\gamma = -1$. If m is even, then again (ii) proves the claim, but if m is odd, the inequality $(-q)^{\rho k}\alpha^{(j-2\rho)k} \neq -1$ holds for every $\rho \in \mathbb{Z}$ and then (i) proves the assertion in this case. □

Proof of Corollary 2.3. Suppose that $j = 2$.

a) For the equivalence of this case we use Theorem 1.1 with $\gamma = 1$. If either $q = -1$ or $q = 1$, m odd, k even, or $q = 1$, m even, then the assumptions in (ii) are fulfilled (taking $\rho = 1$), whence $\mathscr{W}_2(1; k, \ell, m) \in K$. But if $q = 1$ and k, m odd, we can apply case (i) to obtain $\mathscr{W}_2(1; k, \ell, m) \notin K$.

b) To prove the equivalence here, we put $\gamma = -1$. If m is even, we can fulfill the condition $\gamma = \pm(-q)^{\rho k}\alpha^{(j-2\rho)k}$ using $\rho = 1$ as before and a suitable choice of the sign. But if m is odd and $(-q)^k = -1$, the expression $(-q)^{\rho k}\alpha^{(j-2\rho)k}$ becomes -1 for $\rho = 1$. Hence (ii) applies again, and $\mathscr{W}_2(-1; k, \ell, m)$ lies in both situations in K. Supposing m odd, $(-q)^k = 1$, the first half of (i) shows that $\mathscr{W}_2(-1; k, \ell, m) \notin K$. □

Proof of Corollary 2.5. a) Taking again $j = 2$, $\gamma = 1$, the last considerations of Section 3 show that $A_1(\alpha^{-k}) = 0$ if either $q = -1$ or $q = 1$ and mk even, whence, by (14), (12) and $\alpha\beta = -q$,

$$\mathscr{W}_2(1; k, \ell, m) = B_1\left(\frac{1}{\alpha^k}\right) = \frac{1}{g(\alpha^{mk} - \beta^{mk})}\sum_{n=1}^{m}\frac{1}{\alpha^{nk+\ell}W_{nk+\ell}}. \tag{16}$$

To evaluate the finite sum appearing here in the two cases (U_n) and (V_n) of (W_n), we start from $\alpha^n - \beta^n = \sqrt{\Delta} U_n$, $\alpha^n + \beta^n = V_n$ (compare (2)) implying that $2\beta^n = V_n - \sqrt{\Delta} U_n$. Hence we obtain from (16)

$$\mathscr{U}_2(1; k, \ell, m) = \frac{1}{U_{mk}} \sum_{n=1}^{m} \frac{(-q)^{nk+\ell} \beta^{nk+\ell}}{U_{nk+\ell}} = \frac{(-q)^{\ell}}{2U_{mk}} \sum_{n=1}^{m} (-q)^{nk} \left(\frac{V_{nk+\ell}}{U_{nk+\ell}} - \sqrt{\Delta} \right). \quad (17)$$

The sum $\sum_{n=1}^{m} (-q)^{nk}$ equals m if either $q = -1$ or $q = 1$, k even, proving the first formula in a). If $q = 1$, k odd and m even, the sum $\sum_{n=1}^{m} (-q)^{nk}$ vanishes and (17) yields that

$$\mathscr{U}_2(1; k, \ell, m) = \frac{(-1)^{\ell}}{2U_{mk}} \sum_{n=1}^{m} (-1)^n \frac{V_{nk+\ell}}{U_{nk+\ell}}. \quad (18)$$

To evaluate this sum still further, we use the formula

$$U_N V_{N+k} - U_{N+k} V_N = -2(-q)^N U_k \qquad \text{for any } N, k \in \mathbb{N}_0, \quad (19)$$

which can be easily checked via (2) and $\alpha\beta = -q$. Applying (19) we obtain from formula (18) that

$$\mathscr{U}_2(1; k, \ell, m) = \frac{(-1)^{\ell}}{2U_{mk}} \sum_{i=1}^{m/2} \frac{-2(-1)^{(2i-1)k+\ell} U_k}{U_{(2i-1)k+\ell} U_{2ik+\ell}},$$

proving the third assertion in a).

We similarly obtain from (16) that

$$\mathscr{V}_2(1; k, \ell, m) = \frac{(-q)^{\ell}}{2\sqrt{\Delta} U_{mk}} \sum_{n=1}^{m} (-q)^{nk} \left(1 - \sqrt{\Delta} \frac{U_{nk+\ell}}{V_{nk+\ell}} \right),$$

leading to the second and fourth formula in a) depending on the case q, k, m.

b) We take $j = 2$, $\gamma = -1$ and suppose that either m is even or m is odd and $(-q)^k = -1$. After an easy calculation we obtain that

$$\mathscr{W}_2(-1; k, \ell, m) = B_1\left(-\frac{1}{\alpha^k} \right) = \frac{1}{g(\alpha^{mk} - \beta^{mk})} \sum_{n=1}^{m} \frac{(-1)^n}{\alpha^{nk+\ell} W_{nk+\ell}}.$$

Proceeding as in a) we now have to use that $\sum_{n=1}^{m} (-(-q)^k)^n$ equals m if $(-q)^k = -1$ but vanishes if $(-q)^k = 1$ and m is even. □

Proof of Corollary 2.6. As we saw at the end of Section 2, it is enough to prove (9), and these formulae are immediate consequences of the subsequent lemma. □

Lemma 4.1. *The following equations hold for any $m \in \mathbb{N}$:*

$$F_{2m}F_{3m}z^{2m}\mathscr{F}_4(-z;1,0,m)$$
$$= (z^{2m} - L_{2m}z^m + 1)\mathscr{F}_2(z;1,0,m) - \sum_{n=1}^{2m}\frac{z^n}{F_nF_{n+m}} + L_{2m}\sum_{n=1}^{m}\frac{z^{m+n}}{F_nF_{n+m}},$$
$$-5F_{2m}F_{3m}z^{2m}\mathscr{L}_4(-z;1,0,m)$$
$$= (z^{2m} - L_{2m}z^m + 1)\mathscr{L}_2(z;1,0,m) - \sum_{n=1}^{2m}\frac{z^n}{L_nL_{n+m}} + L_{2m}\sum_{n=1}^{m}\frac{z^{m+n}}{L_nL_{n+m}}.$$

Applying the first formula with $z = -1$, we get equation (9) for $j = 1$ with

$$a_1(m) = \frac{2 - (-1)^m L_{2m}}{F_{2m}F_{3m}},$$
$$b_1(m) = \frac{1}{F_{2m}F_{3m}}\Big(L_{2m}\sum_{n=1}^{m}\frac{(-1)^{m+n}}{F_nF_{n+m}} - \sum_{n=1}^{2m}\frac{(-1)^n}{F_nF_{n+m}}\Big). \tag{20}$$

Note here that $L_{2m} \neq \pm 2$, hence $a_1(m) \neq 0$ for every $m \in \mathbb{N}$.

Proof of Lemma 4.1. We first show that

$$F_{n+2m}F_{n+3m} + F_nF_{n+m} - L_{2m}F_nF_{n+3m} = (-1)^n F_{2m}F_{3m}, \tag{21}$$
$$L_{n+2m}L_{n+3m} + L_nL_{n+m} - L_{2m}L_nL_{n+3m} = -5(-1)^n F_{2m}F_{3m} \tag{22}$$

for any $m, n \in \mathbb{N}_0$. For this we quote the formulae

$$F_{i+j} = F_{i-1}F_j + F_iF_{j+1}, \qquad L_{i+j} = L_{i-1}F_j + L_iF_{j+1}, \tag{23}$$
$$F_{n+1}^2 - F_nF_{n+1} - F_n^2 = (-1)^n, \qquad L_{n+1}^2 - L_nL_{n+1} - L_n^2 = -5(-1)^n, \tag{24}$$

valid for any $i, j, n \in \mathbb{N}_0$ with the conventions $F_{-1} := 1$, $L_{-1} := -1$. They can be easily verified by induction using the recurrence relation (4) of the F's and L's, respectively. Denoting now the left-hand side of (21) (and (22)) by $\Phi_{m,n}^F$ and $\Phi_{m,n}^L$, respectively, the first formula in (23) leads to

$$\Phi_{m,n}^F = (F_{2m-1}F_n + F_{2m}F_{n+1})(F_{3m-1}F_n + F_{3m}F_{n+1}) + F_n(F_{m-1}F_n + F_mF_{n+1})$$
$$- L_{2m}F_n(F_{3m-1}F_n + F_{3m}F_{n+1})$$
$$= F_{2m}F_{3m}F_{n+1}^2 + A_mF_nF_{n+1} + B_mF_n^2.$$

Here we put

$$A_m := (F_{2m-1}F_{3m} + F_{2m}F_{3m+1} + F_m - L_{2m}F_{3m}) - F_{2m}F_{3m}.$$

Using the first formula in (23) again and

$$F_{4m+i} + F_i = L_{2m}F_{2m+i} \tag{25}$$

for $i = m$, we obtain that $A_m = -F_{2m}F_{3m}$. On the other hand, we define

$$B_m := (F_{2m-1}F_{3m-1} + F_{2m}F_{3m} + F_{m-1} - L_{2m}F_{3m-1}) - F_{2m}F_{3m},$$

whence $B_m = -F_{2m}F_{3m}$ using the first formula in (23) and (25) for $i = m - 1$. Thus,

$$\Phi^F_{m,n} = F_{2m}F_{3m}(F^2_{n+1} - F_nF_{n+1} - F^2_n)$$

yielding (21), by the first formula in (24).

The proof of (22) is similar. Of course, instead of the first formula in (23) we have to apply the second one, and the final result is

$$\Phi^L_{m,n} = F_{2m}F_{3m}L^2_{n+1} + A_mL_nL_{n+1} + B_mL^2_n = F_{2m}F_{3m}(F^2_{n+1} - F_nF_{n+1} - F^2_n),$$

hence (22), by the second formula in (24).

To prove now Lemma 4.1, we multiply (21) and (22) by $(-z)^n/(F_nF_{n+m}F_{n+2m}F_{n+3m})$ and $(-z)^n/(L_nL_{n+m}L_{n+2m}L_{n+3m})$, respectively. Summing over all $n \in \mathbb{N}$ we obtain both formulae, as claimed. □

Remark 4.2. Let us try to decide if, e.g., $\mathscr{W}_4(1; k, \ell, m)$ lies in $K \backslash \mathbb{Q}$ or in $\mathbb{Q}$ using the method of proof of Corollary 2.5 for $\mathscr{W}_2(1; k, \ell, m)$. We then apply (14) to $j = 2$, $z = 1$ and see from (12) that $A_2(\alpha^{-2k}) = \left(1 - (\alpha\beta)^{2mk}\right)/C_2 = 0$. Thus, again formula (14) leads to

$$\mathscr{W}_4(1; k, \ell, m) = B_2\left(\frac{1}{\alpha^{2k}}\right)A_3\left(\frac{1}{\alpha^k}\right) + B_3\left(\frac{1}{\alpha^k}\right).$$

But the relative complexity, in particular, of the B's appearing here (compare (12)) prevents us from reaching our goal along these lines, even in the U- or V-case.

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