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An oscillation criterion for linear difference equations with general delay argument

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Abstract. This paper presents a new sufficient condition for the oscillation of all solutions of linear difference equations with general delay argument. The significance of this condition is demonstrated by comparing with known oscillation conditions. An example illustrating the results is also given.

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1. Introduction

An important question in the qualitative theory of difference equations is that of deriving sufficient conditions for the oscillation of the solutions of delay difference equations. The oscillation theory of difference equations has been extensively developed. See [1]–[22], [24]–[30], [32]–[34] and the references cited therein. This paper is devoted to the oscillation of the solutions to linear difference equations with a general delay argument.

Consider the delay difference equation

$$\Delta x(n) + p(n)\, x\big(\tau(n)\big) = 0, \tag{1.1}$$

where $\big(p(n)\big)_{n\geq 0}$ is a sequence of nonnegative real numbers and $\big(\tau(n)\big)_{n\geq 0}$ is a sequence of integers such that

$$\tau(n) \leq n-1 \quad \text{for all } n \geq 0 \qquad \text{and} \qquad \lim_{n\to\infty} \tau(n) = \infty.$$

Here Δ stands for the usual forward difference operator defined by

$$\Delta h(n) = h(n+1) - h(n), \qquad n \geq m,$$

for any sequence of real numbers $\big(h(n)\big)_{n \geq m}$.

Define

$$k = -\min_{n \geq 0} \tau(n).$$

(Clearly, k is a positive integer.)

By a *solution* of the delay difference equation (1.1), we mean a sequence of real numbers $\big(x(n)\big)_{n \geq -k}$ which satisfies (1.1) for all $n \geq 0$. It is clear that, for each choice of real numbers $c_{-k}, c_{-k+1}, \ldots, c_{-1}, c_0$, there exists a unique solution $\big(x(n)\big)_{n \geq -k}$ of (1.1) which satisfies the initial conditions $x(-k) = c_{-k}, x(-k+1) = c_{-k+1}, \ldots, x(-1) = c_{-1}, x(0) = c_0$.

As usual, a solution $\big(x(n)\big)_{n \geq -k}$ of the delay difference equation (1.1) is called *oscillatory* if the terms $x(n)$ of the sequence are neither eventually positive nor eventually negative, and otherwise the solution is said to be *nonoscillatory*.

In the special case where the delay $\big(n - \tau(n)\big)_{n \geq 0}$ is a constant, the delay difference equation (1.1) becomes

$$\Delta x(n) + p(n)x(n-k) = 0, \tag{1.2}$$

where k is a positive integer.

Strong interest in the delay difference equation (1.1) is motivated by the fact that it represents a discrete analogue of the (first order) delay differential equation (see, for example, [23] and the references cited therein)

$$x'(t) + p(t)x\big(\tau(t)\big) = 0, \tag{1.3}$$

where p is a nonnegative continuous real-valued function on the interval $[0, \infty)$, and τ is a continuous real-valued function on $[0, \infty)$ such that

$$\tau(t) \leq t \quad \text{for all } t \geq 0 \quad \text{and} \quad \lim_{t \to \infty} \tau(t) = \infty.$$

In particular, the delay difference equation (1.2) represents a discrete analogue of the (first order) delay differential equation

$$x'(t) + p(t)x(t - T) = 0, \tag{1.4}$$

where T is a positive real constant.

Since 1989, many researchers have studied systematically the oscillation of the solutions of the delay difference equation (1.1) (and, especially, of the equation (1.2)) and a large number of oscillation criteria have been obtained, which should be looked as discrete analogues of corresponding criteria for the oscillation of the solutions of the delay differential equation (1.3) (and, especially, of the equation (1.4)). See [2]–[10], [12]–[22], [24]–[30], [32]–[34] and the references cited therein. It is the purpose of the present work to establish a new sufficient condition for the oscillation of the solutions of the delay difference equation (1.1); the oscillation criterion obtained should be looked upon as a discrete analogue of an oscillation result due to Yu, Wang, Zhang and Qian [31] for the delay differential equation (1.3).

In 1989, Erbe and Zhang [10] proved that each one of the conditions

$$\liminf_{n\to\infty} p(n) > \frac{k^k}{(k+1)^{k+1}} \tag{1.5}$$

and

$$\limsup_{n\to\infty} \sum_{j=n-k}^{n} p(j) > 1 \tag{1.6}$$

is sufficient for all solutions of (1.2) to be oscillatory. In the same year, 1989, Ladas, Philos and Sficas [15] established that all solutions of (1.2) are oscillatory if

$$\liminf_{n\to\infty} \left[\frac{1}{k}\sum_{j=n-k}^{n-1} p(j)\right] > \frac{k^k}{(k+1)^{k+1}}. \tag{1.7}$$

(Clearly, condition (1.7) improves (1.5)). A substantial improvement of this oscillation criterion has been presented, in 2004, by Philos, Purnaras and Stavroulakis [22].

We now turn to the general case of the delay difference equation (1.1). The oscillation condition (1.6) can be extended to the equation (1.1). More precisely, if the sequence $\big(\tau(n)\big)_{n\geq 0}$ is assumed to be increasing, then from Chatzarakis, Koplatadze and Stavroulakis [2] it follows that all solutions of (1.1) are oscillatory if

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} p(j) > 1. \tag{1.8}$$

In 1991, Philos [18] extended the oscillation criterion (1.7) to the general case of the equation (1.1), by establishing that, if the sequence $\big(\tau(n)\big)_{n\geq 0}$ is increasing, then the condition

$$\liminf_{n\to\infty} \left[\frac{1}{n-\tau(n)}\sum_{j=\tau(n)}^{n-1} p(j)\right] > \limsup_{n\to\infty} \frac{\big(n-\tau(n)\big)^{n-\tau(n)}}{\big(n-\tau(n)+1\big)^{n-\tau(n)+1}} \tag{1.9}$$

suffices for the oscillation of all solutions of (1.1). This oscillation result has recently improved substantially by Philos and Purnaras [21] (the results in [21] extend the ones given in [22] concerning the special case of equation (1.2)).

In 2008, Chatzarakis, Koplatadze and Stavroulakis [3] proved that if

$$\limsup_{n\to\infty}\sum_{j=\tau(n)}^{n-1} p(j) < \infty$$

and

$$\liminf_{n\to\infty}\sum_{j=\tau(n)}^{n-1} p(j) > \frac{1}{e}, \tag{1.10}$$

then all solutions of (1.1) are oscillatory. It should be emphasized that, in this oscillation criterion, no assumption on the increasing character of the sequence $\left(\tau(n)\right)_{n\geq 0}$ is imposed. (In some particular cases, related conditions can be found in Zhang and Tian [33], [34].)

It is interesting to establish sufficient conditions for the oscillation of all solutions of the delay difference equation (1.1), in the case where neither (1.8) nor (1.9) or (1.10) is fulfilled. This question has been investigated by several authors in the special case of equation (1.2), when neither (1.6) nor (1.7) is satisfied. See, for example, Chatzarakis and Stavroulakis [5] and the references cited therein. In the case of equation (1.1) with a general delay argument, this question was investigated for the first time by Chatzarakis, Koplatadze and Stavroulakis [2]. In the special case that the sequence $\left(\tau(n)\right)_{n\geq 0}$ is increasing, the conditions established in [2] can be formulated as follows. Set

$$\alpha = \liminf_{n\to\infty}\sum_{j=\tau(n)}^{n-1} p(j). \tag{1.11}$$

Then

(I) if $0 < \alpha \leq 1$ and

$$\limsup_{n\to\infty}\sum_{j=\tau(n)}^{n} p(j) > 1 - (1 - \sqrt{1-\alpha})^2 \tag{1.12}$$

or

(II) if $0 < \alpha < 1$, $p(n) \geq 1 - \sqrt{1-\alpha}$ for all large n, and

$$\limsup_{n\to\infty}\sum_{j=\tau(n)}^{n} p(j) > 1 - \alpha\frac{1-\sqrt{1-\alpha}}{\sqrt{1-\alpha}}, \tag{1.13}$$

then all solutions of the delay difference equation (1.1) are oscillatory.

Later these results have been improved by Chatzarakis, Philos and Stavroulakis in [4]. In the special case where the sequence $\big(\tau(n)\big)_{n\geq 0}$ is increasing, the results in [4] can be formulated as follows:

Lemma 1.1 ([4]). *Assume that the sequence* $\big(\tau(n)\big)_{n\geq 0}$ *is increasing, and set* α *as in* (1.11). *Let* $\big(x(n)\big)_{n\geq -k}$ *be a nonoscillatory solution of the delay difference equation* (1.1). *Then we have:*

(i) *If* $0 < \alpha \leq \frac{1}{2}$, *then*

$$\liminf_{n\to\infty} \frac{x(n+1)}{x\big(\tau(n)\big)} \geq \frac{1}{2}(1 - \alpha - \sqrt{1 - 2\alpha}). \tag{1.14}$$

(ii) *If* $0 < \alpha \leq 6 - 4\sqrt{2}$ *and, in addition,*

$$p(n) \geq \frac{\alpha}{2} \qquad \text{for all large } n, \tag{1.15}$$

then

$$\liminf_{n\to\infty} \frac{x(n+1)}{x\big(\tau(n)\big)} \geq \frac{1}{4}(2 - 3\alpha - \sqrt{4 - 12\alpha + \alpha^2}). \tag{1.16}$$

Theorem 1.1 ([4]). *Assume that the sequence* $\big(\tau(n)\big)_{n\geq 0}$ *is increasing, and define* α *by* (1.11). *Then we have:*

(I) *If* $0 < \alpha \leq \frac{1}{2}$, *then the condition*

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} p(j) > 1 - \frac{1}{2}(1 - \alpha - \sqrt{1 - 2\alpha}) \tag{1.17}$$

is sufficient for all solutions of the delay difference equation (1.1) *to be oscillatory.*

(II) *If* $0 < \alpha \leq 6 - 4\sqrt{2}$ *and, in addition,* (1.15) *holds, then the condition*

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} p(j) > 1 - \frac{1}{4}(2 - 3\alpha - \sqrt{4 - 12\alpha + \alpha^2}) \tag{1.18}$$

is sufficient for all solutions of (1.1) *to be oscillatory.*

Provided that $0 < \alpha \leq 6 - 4\sqrt{2}$ (clearly, $6 - 4\sqrt{2} < \frac{1}{2}$), we have

$$\frac{1}{4}(2 - 3\alpha - \sqrt{4 - 12\alpha + \alpha^2}) > \frac{1}{2}(1 - \alpha - \sqrt{1 - 2\alpha}).$$

Therefore, when $0 < \alpha \leq 6 - 4\sqrt{2}$ and (1.15) holds, inequality (1.16) improves (1.14), and condition (1.18) is weaker than (1.17).

In the present paper, the authors study further the equation (1.1) and essentially improve the upper bound of the ratio $x(\tau(n))/x(n+1)$ for possible nonoscillatory solutions $(x(n))_{n \geq -k}$ of (1.1), when neither (1.8) nor (1.9) or (1.10) is satisfied, and derive a new sufficient oscillation condition. This condition essentially inproves the known conditions (1.12), (1.13) and (1.17). An example illustrating the results is given.

2. Statement of the results and comments

Our main result is Theorem 2.1 stated below. The proof of this theorem is essentially based on the following lemma.

Lemma 2.1. *Assume that the sequence $(\tau(n))_{n \geq 0}$ is increasing. Moreover, assume that $0 < \alpha \leq -1 + \sqrt{2}$, where α is defined by* (1.11). *Then every nonoscillatory solution $(x(n))_{n \geq -k}$ of the delay difference equation* (1.1) *satisfies*

$$\liminf_{n \to \infty} \frac{x(n+1)}{x(\tau(n))} \geq \frac{1}{2}(1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}). \tag{2.1}$$

Theorem 2.1. *Let the assumptions of Lemma* 2.1 *hold. Then the condition*

$$\limsup_{n \to \infty} \sum_{j=\tau(n)}^{n} p(j) > 1 - \frac{1}{2}(1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}) \tag{2.2}$$

is sufficient for all solutions of the delay difference equation (1.1) *to be oscillatory.*

Remark 2.1. Observe that, when $0 < \alpha \leq -1 + \sqrt{2}$, it is easy to see that

$$\frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2} > \alpha \frac{1 - \sqrt{1 - \alpha}}{\sqrt{1 - \alpha}} > \frac{1 - \alpha - \sqrt{1 - 2\alpha}}{2} > (1 - \sqrt{1 - \alpha})^2$$

and therefore the condition (2.2) is weaker than the conditions (1.12), (1.17) and (1.13).

Remark 2.2. We note that, in the special case of the delay difference equation (1.2), Lemma 2.1 and Theorem 2.1 have been presented by Chen and Yu [6]. We also notice that Lemma 2.1 and Theorem 2.1 should be looked upon as discrete analogues of corresponding results due to Yu, Wang, Zhang and Qian [31] concerning the solutions of the delay differential equation (1.3) (and, especially, of the delay differential equation (1.4)).

Now we define

$$\sigma(n) = \max_{0 \le s \le n} \tau(s) \qquad \text{for } n \ge 0. \tag{2.3}$$

Clearly, the sequence of integers $(\sigma(n))_{n \ge 0}$ is increasing. Moreover, as it has been shown in [2], it holds that

$$\liminf_{n \to \infty} \sum_{j=\sigma(n)}^{n-1} p(j) = \liminf_{n \to \infty} \sum_{j=\tau(n)}^{n-1} p(j). \tag{2.4}$$

Following Chatzarakis, Koplatadze and Stavroulakis [2], one can use (2.4) and apply Lemma 2.1 in [2] (cf. Philos [18] and Kordonis and Philos [12]) to establish the following generalization of Theorem 2.1.

Theorem 2.1′. *Let the sequence* $(\sigma(n))_{n \ge 0}$ *be defined by* (2.3). *Moreover, assume that* $0 < \alpha \le -1 + \sqrt{2}$, *where* α *is defined by* (1.11). *Then the condition*

$$\limsup_{n \to \infty} \sum_{j=\sigma(n)}^{n} p(j) > 1 - \frac{1}{2}\left(1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}\right)$$

is sufficient for all solutions of the delay difference equation (1.1) *to be oscillatory.*

In Theorem 2.1′ it is not assumed that the sequence $(\tau(n))_{n \ge 0}$ is increasing. Note that, if $(\tau(n))_{n \ge 0}$ is increasing, then the sequence $(\sigma(n))_{n \ge 0}$ coincides with $(\tau(n))_{n \ge 0}$.

It must be noted that an analogous generalization of Theorem 1.1 has been presented in [4]. More precisely, the assumption that the sequence $(\tau(n))_{n \ge 0}$ is increasing is removed, but conditions (1.17) and (1.18) are replaced by the conditions

$$\limsup_{n \to \infty} \sum_{j=\sigma(n)}^{n} p(j) > 1 - \frac{1}{2}\left(1 - \alpha - \sqrt{1 - 2\alpha}\right)$$

and

$$\limsup_{n \to \infty} \sum_{j=\sigma(n)}^{n} p(j) > 1 - \frac{1}{4}\left(2 - 3\alpha - \sqrt{4 - 12\alpha + \alpha^2}\right),$$

respectively, where $(\sigma(n))_{n \ge 0}$ is defined by (2.3).

3. Proofs of Lemma 2.1 and Theorem 2.1

Proof of Lemma 2.1. Define

$$q(t) = p(n) \quad \text{for } n \le t < n+1 \ (n = 0, 1, \dots).$$

Clearly, q is a nonnegative real-valued function on the interval $[0, \infty)$, which is continuous on each one of the intervals $(n, n+1)$ $(n = 0, 1, \dots)$. Note that $q(n) = p(n)$ for every integer $n \ge 0$. Furthermore, consider the real-valued function σ defined on the interval $[0, \infty)$ by

$$\sigma(t) = \tau(n) \quad \text{for } n \le t < n+1 \ (n = 0, 1, \dots).$$

It is obvious that for each $n = 0, 1, \dots$ the function σ is continuous on $(n, n+1)$. We notice that $\sigma(n) = \tau(n)$ for all integers $n \ge 0$. We can immediately see that

$$\sigma(t) < t \quad \text{for all } t \ge 0 \qquad \text{and} \qquad \lim_{t\to\infty} \sigma(t) = \infty.$$

Also, as the sequence $\big(\tau(n)\big)_{n\ge 0}$ is assumed to be increasing, we observe that the function σ is increasing on $[0, \infty)$.

Let $\big(x(n)\big)_{n\ge -k}$ be a solution of the delay difference equation (1.1). We define

$$y(t) = x(n) + \big(\Delta x(n)\big)(t-n) \quad \text{for } n \le t < n+1 \ (n = -k, -k+1, \dots).$$

It is clear that

$$y(n) = x(n) \quad \text{for all integers } n \ge -k.$$

Moreover, it is easy to verify that the real-valued function y is continuous on the interval $[-k, \infty)$. Also, we see that y is continuously differentiable on each one of the intervals $(n, n+1)$ $(n = -k, -k+1, \dots)$ with

$$y'(t) = \Delta x(n) \quad \text{for } n < t < n+1 \ (n = -k, -k+1, \dots).$$

Furthermore, as $\big(x(n)\big)_{n\ge -k}$ satisfies (1.1) for all integers $n \ge 0$, we can easily conclude that the function y satisfies

$$y'(t) + q(t)y\big(\sigma(t)\big) = 0 \quad \text{for } n < t < n+1 \ (n = 0, 1, \dots). \tag{3.1}$$

Next assume that the solution $\big(x(n)\big)_{n\ge -k}$ of (1.1) is nonoscillatory. Then it is either eventually positive or eventually negative. As $\big(-x(n)\big)_{n\ge -k}$ is also a solution of (1.1), we may (and do) restrict ourselves only to the case where $x(n) > 0$ for all large n. Let $\rho \ge -k$ be an integer such that $x(n) > 0$ for all $n \ge \rho$, and consider an integer $r \ge 0$ so that $\tau(n) \ge \rho$ for $n \ge r$ (clearly, $r > \rho$). Then it follows immediately from (1.1) that $\Delta x(n) \le 0$ for every $n \ge r$, which means that the sequence $(x(n))_{n\ge r}$ is decreasing. Furthermore, it is not difficult to conclude that the function y is positive on the interval $[\rho, \infty)$ and that y is decreasing on $[r, \infty)$.

Consider an arbitrary real number ε with $0 < \varepsilon < \alpha$. Then we can choose an integer $n_0 > r$ such that $\tau(n) \ge r$ for $n \ge n_0$, and

$$\sum_{j=\tau(n)}^{n-1} p(j) > \alpha - \varepsilon \qquad \text{for every } n \ge n_0.$$

For any point $t \ge n_0$, there exists an integer $n \ge n_0$ such that $n \le t < n+1$, and consequently

$$\int_{\sigma(t)}^{t} q(s)\,ds = \int_{\tau(n)}^{t} q(s)\,ds \ge \int_{\tau(n)}^{n} q(s)\,ds = \sum_{j=\tau(n)}^{n-1} p(j) > \alpha - \varepsilon.$$

So we have

$$\int_{\sigma(t)}^{t} q(s)\,ds > \alpha - \varepsilon \qquad \text{for all } t \ge n_0. \tag{3.2}$$

Furthermore, we will establish the following claim.

Claim. *For each point $t \ge n_0$, there exists a $t^* > t$ such that $\sigma(t^*) < t$ and*

$$\int_{t}^{t^*} q(s)\,ds = \alpha - \varepsilon. \tag{3.3}$$

To prove this claim, let us consider an arbitrary point $t \ge n_0$. Set

$$f(v) = \int_{t}^{v} q(s)\,ds \qquad \text{for } v \ge t.$$

We see that $f(t) = 0$. Moreover, it is not difficult to show that (3.2) guarantees that

$$\int_{0}^{\infty} q(s)\,ds = \infty$$

and, in particular,

$$\int_{t}^{\infty} q(s)\,ds = \infty,$$

i.e., $\lim_{v\to\infty} f(v) = \infty$. Thus, as the function f is continuous on the interval $[t, \infty)$, there always exists a point $t^* > t$ so that $f(t^*) = \alpha - \varepsilon$, i.e., such that (3.3) is satisfied. By using (3.2) (for the point t^*) as well as (3.3), we obtain

$$\int_{\sigma(t^*)}^{t} q(s)\,ds = \int_{\sigma(t^*)}^{t^*} q(s)\,ds - \int_{t}^{t^*} q(s)\,ds > (\alpha - \varepsilon) - (\alpha - \varepsilon) = 0$$

and consequently we necessarily have $\sigma(t^*) < t$. Our claim has been proved.

Now we choose an integer $N > n_0$ such that $\tau(n) \geq n_0$ for every $n \geq N$.

Let us consider an arbitrary point $t \geq N$. By our claim, there exists a $t^* > t$ such that $\sigma(t^*) < t$, and (3.3) holds. From (3.1) it follows that

$$y(t) = y(t^*) + \int_{t}^{t^*} q(s)y\big(\sigma(s)\big)\,ds. \tag{3.4}$$

Let s be any point with $t \leq s \leq t^*$. As the function σ is increasing on $[0, \infty)$, we have $n_0 \leq \sigma(t) \leq \sigma(s) \leq \sigma(t^*) < t$, and $r \leq \sigma(u) \leq \sigma(t)$ for every u with $\sigma(s) \leq u \leq t$. Thus, by taking into account the fact that the function y is decreasing on $[r, \infty)$, from (3.1) we obtain

$$\begin{aligned} y\big(\sigma(s)\big) &= y(t) + \int_{\sigma(s)}^{t} q(u)y\big(\sigma(u)\big)\,du \\ &\geq y(t) + \left[\int_{\sigma(s)}^{t} q(u)\,du\right] y\big(\sigma(t)\big) \\ &= y(t) + \left[\int_{\sigma(s)}^{s} q(u)\,du - \int_{t}^{s} q(u)\,du\right] y\big(\sigma(t)\big). \end{aligned}$$

So, by applying (3.2) (for the point s), we get

$$y\big(\sigma(s)\big) > y(t) + \left[(\alpha - \varepsilon) - \int_{t}^{s} q(u)\,du\right] y\big(\sigma(t)\big).$$

As this inequality holds true for all s with $t \leq s \leq t^*$, it follows from (3.4) that

$$\begin{aligned} y(t) \geq\ & y(t^*) + \int_{t}^{t^*} q(s)\left\{ y(t) + \left[(\alpha - \varepsilon) - \int_{t}^{s} q(u)\,du\right] y\big(\sigma(t)\big)\right\} ds \\ =\ & y(t^*) + \left[\int_{t}^{t^*} q(s)\,ds\right] y(t) + \left\{(\alpha - \varepsilon)\int_{t}^{t^*} q(s)\,ds \right. \\ & \left. - \int_{t}^{t^*} q(s)\left[\int_{t}^{s} q(u)\,du\right] ds\right\} y\big(\sigma(t)\big) \end{aligned}$$

and consequently, in view of (3.3),

$$y(t) \geq y(t^*) + (\alpha - \varepsilon) y(t) + \left\{ (\alpha - \varepsilon)^2 - \int_t^{t^*} q(s) \left[\int_t^s q(u)\, du \right] ds \right\} y(\sigma(t)). \quad (3.5)$$

Noting the known formula

$$\int_t^{t^*} q(s) \left[\int_t^s q(u)\, du \right] ds = \int_t^{t^*} q(u) \left[\int_u^{t^*} q(s)\, ds \right] du$$

or

$$\int_t^{t^*} q(s) \left[\int_t^s q(u)\, du \right] ds = \int_t^{t^*} q(s) \left[\int_s^{t^*} q(u)\, du \right] ds,$$

we have

$$\begin{aligned} \int_t^{t^*} q(s) \left[\int_t^s q(u)\, du \right] ds &= \frac{1}{2} \left\{ \int_t^{t^*} q(s) \left[\int_t^s q(u)\, du \right] ds + \int_t^{t^*} q(s) \left[\int_s^{t^*} q(u)\, du \right] ds \right\} \\ &= \frac{1}{2} \int_t^{t^*} q(s) \left[\int_t^s q(u)\, du + \int_s^{t^*} q(u)\, du \right] ds \\ &= \frac{1}{2} \int_t^{t^*} q(s) \left[\int_t^{t^*} q(u)\, du \right] ds = \frac{1}{2} \left[\int_t^{t^*} q(s)\, ds \right]^2 \end{aligned}$$

and therefore, by (3.3),

$$\int_t^{t^*} q(s) \left[\int_t^s q(u)\, du \right] ds = \frac{1}{2} (\alpha - \varepsilon)^2.$$

Hence, (3.5) is written as

$$y(t) \geq y(t^*) + (\alpha - \varepsilon) y(t) + \frac{1}{2} (\alpha - \varepsilon)^2 y(\sigma(t)). \quad (3.6)$$

This gives

$$y(t) > (\alpha - \varepsilon) y(t) + \frac{1}{2} (\alpha - \varepsilon)^2 y(\sigma(t)),$$

i.e.,

$$y(t) > \frac{(\alpha - \varepsilon)^2}{2[1 - (\alpha - \varepsilon)]} y(\sigma(t)).$$

(Note that $0 < a - \varepsilon < a \leq -1 + \sqrt{2} < 1$.) We have thus proved that

$$y(t) > \lambda_1 y\big(\sigma(t)\big) \qquad \text{for all } t \geq N, \tag{3.7}$$

where

$$\lambda_1 = \frac{(\alpha - \varepsilon)^2}{2[1 - (\alpha - \varepsilon)]}.$$

Now, in view of (3.7) (for the point t^*), we have

$$y(t^*) > \lambda_1 y\big(\sigma(t^*)\big).$$

But since $\sigma(t^*) < t$ and the function y is decreasing on $[r, \infty)$, we also have

$$y\big(\sigma(t^*)\big) \geq y(t).$$

Combining the last two inequalities, we obtain

$$y(t^*) > \lambda_1 y(t)$$

and hence (3.6) yields

$$y(t) > \lambda_1 y(t) + (\alpha - \varepsilon) y(t) + \frac{1}{2}(\alpha - \varepsilon)^2 y\big(\sigma(t)\big)$$

or

$$[1 - (\alpha - \varepsilon) - \lambda_1] y(t) > \frac{1}{2}(\alpha - \varepsilon)^2 y\big(\sigma(t)\big).$$

This implies, in particular, that

$$1 - (\alpha - \varepsilon) - \lambda_1 > 0.$$

Consequently,

$$y(t) > \frac{(\alpha - \varepsilon)^2}{2[1 - (\alpha - \varepsilon) - \lambda_1]} y\big(\sigma(t)\big).$$

Thus, it has been shown that

$$y(t) > \lambda_2 y\big(\sigma(t)\big) \qquad \text{for all } t \geq N,$$

where

$$\lambda_2 = \frac{(\alpha - \varepsilon)^2}{2[1 - (\alpha - \varepsilon) - \lambda_1]}.$$

Following the above procedure, we can inductively construct a sequence of positive real numbers $(\lambda_\nu)_{\nu \geq 1}$ with

$$1 - (\alpha - \varepsilon) - \lambda_\nu > 0 \qquad (\nu = 1, 2, \ldots)$$

and

$$\lambda_{\nu+1} = \frac{(\alpha - \varepsilon)^2}{2[1 - (\alpha - \varepsilon) - \lambda_\nu]} \qquad (\nu = 1, 2, \ldots)$$

such that

$$y(t) > \lambda_\nu y\big(\sigma(t)\big) \qquad \text{for all } t \geq N \ (\nu = 1, 2, \ldots). \tag{3.8}$$

As $\lambda_1 > 0$, we obtain

$$\lambda_2 = \frac{(\alpha - \varepsilon)^2}{2[1 - (\alpha - \varepsilon) - \lambda_1]} > \frac{(\alpha - \varepsilon)^2}{2[1 - (\alpha - \varepsilon)]} = \lambda_1,$$

i.e., $\lambda_2 > \lambda_1$. By an easy induction, one can immediately see that the sequence $(\lambda_\nu)_{\nu \geq 1}$ is strictly increasing. Furthermore, by taking into account the fact that the function y is decreasing on $[r, \infty)$ and using (3.8) (for $t = N$), we get

$$y(N) > \lambda_\nu y\big(\sigma(N)\big) \geq \lambda_\nu y(N) \qquad (\nu = 1, 2, \ldots).$$

Therefore, for each integer $\nu \geq 1$, we have $\lambda_\nu < 1$. This ensures that the sequence $(\lambda_\nu)_{\nu \geq 1}$ is bounded. Since $(\lambda_\nu)_{\nu \geq 1}$ is a strictly increasing and bounded sequence of positive real numbers, it follows that $\lim_{\nu \to \infty} \lambda_\nu$ exists as a positive real number. Set

$$\Lambda = \lim_{\nu \to \infty} \lambda_\nu.$$

Then (3.8) gives

$$y(t) \geq \Lambda y\big(\sigma(t)\big) \qquad \text{for all } t \geq N. \tag{3.9}$$

Because of the definition of $(\lambda_\nu)_{\nu \geq 1}$, it holds

$$\Lambda = \frac{(\alpha - \varepsilon)^2}{2[1 - (\alpha - \varepsilon) - \Lambda]},$$

i.e.,

$$\Lambda^2 - [1-(\alpha-\varepsilon)]\Lambda + \frac{1}{2}(\alpha-\varepsilon)^2 = 0.$$

Hence, either

$$\Lambda = \frac{1}{2}[1-(\alpha-\varepsilon) - \sqrt{1-2(\alpha-\varepsilon)-(\alpha-\varepsilon)^2}]$$

or

$$\Lambda = \frac{1}{2}[1-(\alpha-\varepsilon) + \sqrt{1-2(\alpha-\varepsilon)-(\alpha-\varepsilon)^2}].$$

In both cases, we have

$$\Lambda \geq \frac{1}{2}[1-(\alpha-\varepsilon) - \sqrt{1-2(\alpha-\varepsilon)-(\alpha-\varepsilon)^2}]$$

and consequently (3.9) yields

$$y(t) \geq \frac{1}{2}[1-(\alpha-\varepsilon) - \sqrt{1-2(\alpha-\varepsilon)-(\alpha-\varepsilon)^2}]y(\sigma(t)) \qquad \text{for all } t \geq N. \quad (3.10)$$

Let n be an arbitrary integer with $n \geq N$. Then, by (3.10),

$$y(t) \geq \frac{1}{2}[1-(\alpha-\varepsilon) - \sqrt{1-2(\alpha-\varepsilon)-(\alpha-\varepsilon)^2}]y(\sigma(t)) \qquad \text{for } n \leq t < n+1.$$

But, $y(\sigma(t)) = y(\tau(n)) = x(\tau(n))$ for $n \leq t < n+1$. So,

$$y(t) \geq \frac{1}{2}[1-(\alpha-\varepsilon) - \sqrt{1-2(\alpha-\varepsilon)-(\alpha-\varepsilon)^2}]x(\tau(n)) \qquad \text{for } n \leq t < n+1$$

and therefore

$$\lim_{t\to(n+1)-0} y(t) \geq \frac{1}{2}[1-(\alpha-\varepsilon) - \sqrt{1-2(\alpha-\varepsilon)-(\alpha-\varepsilon)^2}]x(\tau(n)).$$

Note that $\lim_{t\to(n+1)-0} y(t) = y(n+1) = x(n+1)$. We have thus proved that

$$x(n+1) \geq \frac{1}{2}[1-(\alpha-\varepsilon) - \sqrt{1-2(\alpha-\varepsilon)-(\alpha-\varepsilon)^2}]x(\tau(n)) \qquad \text{for all } n \geq N. \quad (3.11)$$

Finally, we see that (3.11) is written as

$$\frac{x(n+1)}{x(\tau(n))} \geq \frac{1}{2}[1-(\alpha-\varepsilon)-\sqrt{1-2(\alpha-\varepsilon)-(\alpha-\varepsilon)^2}] \quad \text{for every } n \geq N$$

and consequently

$$\liminf_{n\to\infty} \frac{x(n+1)}{x(\tau(n))} \geq \frac{1}{2}[1-(\alpha-\varepsilon)-\sqrt{1-2(\alpha-\varepsilon)-(\alpha-\varepsilon)^2}].$$

The last inequality holds true for all real numbers ε with $0 < \varepsilon < \alpha$. Hence, we can obtain (2.1).

The proof of our lemma is now complete. □

Proof of Theorem 2.1. Assume, for the sake of contradiction, that there exists a nonoscillatory solution $(x(n))_{n\geq -k}$ of the delay difference equation (1.1). Since $(-x(n))_{n\geq -k}$ is also a solution of (1.1), we can confine our discussion only to the case where the solution $(x(n))_{n\geq -k}$ is eventually positive. Consider an integer $\rho \geq -k$ so that $x(n) > 0$ for every $n \geq \rho$, and let $r \geq 0$ be an integer such that $\tau(n) \geq \rho$ for $n \geq r$ (clearly, $r > \rho$). Then from (1.1) we immediately obtain $\Delta x(n) \leq 0$ for all $n \geq r$, and consequently the sequence $(x(n))_{n\geq r}$ is decreasing.

Now, we choose an integer $n_0 > r$ such that $\tau(n) \geq r$ for $n \geq n_0$. Furthermore, we consider an integer $N > n_0$ so that $\tau(n) \geq n_0$ for $n \geq N$. Then, as the sequence $(\tau(n))_{n\geq 0}$ is increasing and the sequence $(x(n))_{n\geq r}$ is decreasing, it follows from (1.1) that, for every $n \geq N$,

$$x(\tau(n)) - x(n+1) = \sum_{j=\tau(n)}^{n} p(j)x(\tau(j)) \geq \Big[\sum_{j=\tau(n)}^{n} p(j)\Big]x(\tau(n)).$$

This gives

$$\sum_{j=\tau(n)}^{n} p(j) \leq 1 - \frac{x(n+1)}{x(\tau(n))} \qquad \text{for all } n \geq N.$$

Hence,

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} p(j) \leq 1 - \liminf_{n\to\infty} \frac{x(n+1)}{x(\tau(n))}.$$

But, in view of Lemma 2.1, inequality (2.1) holds. So, we obtain

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} p(j) \le 1 - \frac{1}{2}(1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}),$$

which contradicts condition (2.2).

The proof of the theorem is complete. □

4. Example

We illustrate the significance of our results by the following example in which the delay difference equation (1.1) is considered with a variable delay argument.

Example 4.1. Let α be a real number with $0 < \alpha \le 1/e$, and define

$$A_1 = 1 - (1 - \sqrt{1 - \alpha})^2, \qquad A_2 = 1 - \frac{1}{2}(1 - \alpha - \sqrt{1 - 2\alpha}),$$
$$A_3 = 1 - \alpha \frac{1 - \sqrt{1 - \alpha}}{\sqrt{1 - \alpha}}$$

and

$$A_4 = 1 - \frac{1}{2}(1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}).$$

Note that (cf. Remark 2.1) $A_1 > A_2 > A_3 > A_4$. Next, we consider a positive real number d such that $A_4 - \alpha < d < A_3 - \alpha$ (we notice that $A_4 > \alpha$). So, we have $A_1 > A_2 > A_3 > \alpha + d > A_4$. Furthermore, let β be a real number with $0 < \beta < 1$, and set $c = \frac{\alpha}{-\ln \beta}$ and $r = 2 + \left[\frac{1}{\beta}\right]$ ($\left[\frac{1}{\beta}\right]$ denotes the greatest integer less than or equal to $\frac{1}{\beta}$).

Consider now the delay difference equation (1.1) with

$$p(n) = \begin{cases} \frac{c}{n} & \text{if } n \in \{1, 2, \ldots\} \backslash \{r, r^2, \ldots\}, \\ d & \text{if } n \in \{0, r, r^2, \ldots\}, \end{cases}$$

and

$$\tau(0) = -1 \qquad \text{and} \qquad \tau(n) = [\beta n] \qquad (n = 1, 2, \ldots).$$

Here $(p(n))_{n \ge 0}$ is a sequence of positive real numbers, and $(\tau(n))_{n \ge 0}$ is a sequence of integers such that $\tau(n) \le n - 1$ for all $n \ge 0$, and $\lim_{n\to\infty} \tau(n) = \infty$. Moreover, we note that the sequence $(\tau(n))_{n \ge 0}$ is increasing.

We will first show that

$$\lim_{n\to\infty}\sum_{j=[\beta n]}^{n-1}\frac{c}{j}=\alpha. \tag{4.1}$$

To this end, we obtain, for n sufficiently large,

$$\sum_{j=[\beta n]}^{n-1}\frac{c}{j}\geq c\sum_{j=[\beta n]}^{n-1}\int_j^{j+1}\frac{ds}{s}=c\int_{[\beta n]}^{n}\frac{ds}{s}=c\ln\frac{n}{[\beta n]}$$

and

$$\sum_{j=[\beta n]}^{n-1}\frac{c}{j}\leq c\sum_{j=[\beta n]}^{n-1}\int_{j-1}^{j}\frac{ds}{s}=c\int_{[\beta n]-1}^{n-1}\frac{ds}{s}=c\ln\frac{n-1}{[\beta n]-1}.$$

But, it is easy to see that

$$\lim_{n\to\infty}\left(c\ln\frac{n}{[\beta n]}\right)=\lim_{n\to\infty}\left(c\ln\frac{n-1}{[\beta n]-1}\right)=c\ln\frac{1}{\beta}=\alpha.$$

From the above it is clear that (4.1) holds true. In particular, it follows from (4.1) that

$$\lim_{n\to\infty}\sum_{j=[\beta r^n]}^{r^n-1}\frac{c}{j}=\alpha. \tag{4.2}$$

Observe that

$$r^{n-1}<[\beta r^n]<r^n-1 \qquad \text{for large } n. \tag{4.3}$$

Indeed, for any integer $n\geq 0$, we have $[\beta r^n]\leq \beta r^n$ and, since $\frac{\beta r^n}{r^n-1}\to\beta<1$, as $n\to\infty$, it holds that $[\beta r^n]<r^n-1$, for all large n. On the other hand, for $n\geq 0$, we obtain $[\beta r^n]-r^{n-1}>(\beta r^n-1)-r^{n-1}=(\beta r-1)r^{n-1}-1$. But $\beta r-1 =\beta\big(2+\big[\frac{1}{\beta}\big]\big)-1>\beta\big(1+\frac{1}{\beta}\big)-1=\beta>0$ and so $\lim_{n\to\infty}\big((\beta r-1)r^{n-1}-1\big)=\infty$, which quarantees that $\lim_{n\to\infty}([\beta r^n]-r^{n-1})=\infty$. Hence, $[\beta r^n]-r^{n-1}>0$, for all large n. Therefore, (4.3) has been proved.

Now, in view of (4.3), we get

$$\sum_{j=[\beta r^n]}^{r^n-1}p(j)=\sum_{j=[\beta r^n]}^{r^n-1}\frac{c}{j} \qquad \text{for all large } n$$

and consequently, because of (4.2),

$$\lim_{n\to\infty}\sum_{j=[\beta r^n]}^{r^n-1}p(j)=\alpha. \tag{4.4}$$

Furthermore, since $d \geq \frac{c}{j}$ for all large j, we obtain

$$\sum_{j=[\beta n]}^{n-1} p(j) \geq \sum_{j=[\beta n]}^{n-1} \frac{c}{j} \qquad \text{for all large } n,$$

which, by virtue of (4.1), gives

$$\liminf_{n\to\infty} \sum_{j=[\beta n]}^{n-1} p(j) \geq \alpha. \tag{4.5}$$

From (4.4) and (4.5) it follows that

$$\liminf_{n\to\infty} \sum_{j=[\beta n]}^{n-1} p(j) = \alpha. \tag{4.6}$$

Next we shall prove that

$$\limsup_{n\to\infty} \sum_{j=[\beta n]}^{n} p(j) = \alpha + d. \tag{4.7}$$

Observe that

$$\sum_{j=[\beta r^n]}^{r^n} p(j) = \sum_{j=[\beta r^n]}^{r^n-1} p(j) + d \qquad \text{for all large } n,$$

and so, because of (4.4),

$$\lim_{n\to\infty} \sum_{j=[\beta r^n]}^{r^n} p(j) = \alpha + d. \tag{4.8}$$

Furthermore, we see that

$$\lim_{n\to\infty}\left(\frac{\ln n}{\ln r} - \frac{\ln[\beta n]}{\ln r}\right) = \lim_{n\to\infty}\left(\frac{\ln \frac{n}{[\beta n]}}{\ln r}\right) = \frac{\ln \frac{1}{\beta}}{\ln r} < 1,$$

which implies that

$$\frac{\ln n}{\ln r} - \frac{\ln[\beta n]}{\ln r} < 1 \qquad \text{for sufficiently large } n.$$

Hence, for each large n, there exists at most one integer n^* so that

$$\frac{\ln[\beta n]}{\ln r} \le n^* \le \frac{\ln n}{\ln r} \qquad \text{or} \qquad \ln[\beta n] \le n^* \ln r \le \ln n,$$

i.e., such that

$$[\beta n] \le r^{n^*} \le n.$$

By taking into account this fact, we obtain

$$\sum_{j=[\beta n]}^{n} p(j) \le \sum_{j=[\beta n]}^{n} \frac{c}{j} + d = \sum_{j=[\beta n]}^{n-1} \frac{c}{j} + \frac{c}{n} + d$$

for all large n. Thus, by using (4.1), we derive

$$\limsup_{n\to\infty} \sum_{j=[\beta n]}^{n} p(j) \le \alpha + d. \tag{4.9}$$

From (4.8) and (4.9) we conclude that (4.7) is always valid.

Here, we observe that (4.6) coincides with (1.11). Moreover, since

$$A_4 < \alpha + d < A_3 < A_2 < A_1,$$

it follows from (4.7) that condition (2.2) of Theorem 2.1 is satisfied and therefore all solutions of (1.1) are oscillatory. Observe, however, that none of the conditions (1.13), (1.17) and (1.12) is satisfied. In addition, we immediately see that conditions (1.8), (1.9) and (1.10) are also not satisfied.

References

[1] R. P. Agarwal, M. Bohner, S. R. Grace, and D. O'Regan, *Discrete oscillation theory*. Hindawi Publishing Corporation, New York 2005. Zbl 1084.39001 MR 2179948

[2] G. E. Chatzarakis, R. Koplatadze, and I. P. Stavroulakis, Oscillation criteria of first order linear difference equations with delay argument. *Nonlinear Anal.* **68** (2008), 994–1005. Zbl 1144.39003 MR 2382315

[3] G. E. Chatzarakis, R. Koplatadze, and I. P. Stavroulakis, Optimal oscillation criteria for first order difference equations with delay argument. *Pacific J. Math.* **235** (2008), 15–33. Zbl 1153.39010 MR 2379767

[4] G. E. Chatzarakis, Ch. G. Philos, and I. P. Stavroulakis, On the oscillation of the solutions to linear difference equations with variable delay. *Electron. J. Differential Equations* **2008** (2008), no. 50. Zbl 1165.39004 MR 2392954

[5] G. E. Chatzarakis and I. P. Stavroulakis, Oscillations of first order linear delay difference equations. *Aust. J. Math. Anal. Appl.* **3** (2006), no. 1 Art. 14. Zbl 1096.39003 MR 2223018

[6] M.-P. Chen and J. S. Yu, Oscillations of delay difference equations with variable coefficients. In *Proceedings of the First International Conference on Difference Equations*, Gordon and Breach, London 1994, 105–114. Zbl 0860.39022 MR 1678657

[7] S. S. Cheng and G. Zhang, "Virus" in several discrete oscillation theorems. *Appl. Math. Lett.* **13** (2000), 9–13. Zbl 1013.39007 MR 1755736

[8] Y. Domshlak, Sturmian comparison method in oscillation study for discrete difference equations. I, II. *Differential Integral Equations* **7** (1994), 571–582, 583–592. Zbl 0790.39003 Zbl 0790.39004 MR 1255906

[9] Y. Domshlak, What should be a discrete version of the Chanturia-Koplatadze lemma? *Funct. Differ. Equ*. **6** (1999), 299–304. Zbl 1033.39019 MR 1809196

[10] L. H. Erbe and B. G. Zhang, Oscillation of discrete analogues of delay equations. *Differential Integral Equations* **2** (1989), 300–309. Zbl 0723.39004 MR 983682

[11] I. Győri and G. Ladas, *Oscillation theory of delay differential equations*. Oxford Math. Monogr., The Clarendon Press, New York 1991. Zbl 0780.34048 MR 1168471

[12] I.-G. E. Kordonis and Ch. G. Philos, Oscillation and nonoscillation in linear delay or advanced difference equations. *Math. Comput. Modelling* **27** (1998), 11–21. Zbl 0992.39010 MR 1621805

[13] G. Ladas, Oscillations of equations with piecewise constant mixed arguments. In *Differential equations and applications* (Columbus, OH, 1988), vol. II, Ohio University Press, Athens, 64–69. Zbl 0711.34083 MR 1026200

[14] G. Ladas, Explicit conditions for the oscillation of difference equations. *J. Math. Anal. Appl.* **153** (1990), 276–287. Zbl 0718.39002 MR 1080131

[15] G. Ladas, Ch. G. Philos, and Y. G. Sficas, Sharp conditions for the oscillation of delay difference equations. *J. Appl. Math. Simulation* **2** (1989), 101–111. Zbl 0685.39004 MR 1010549

[16] Z. Luo and J. Shen, New results for oscillation of delay difference equations. *Comput. Math. Appl.* **41** (2001), 553–561. Zbl 0996.39007 MR 1822581

[17] Z. Luo and J. Shen, New oscillation criteria for delay difference equations. *J. Math. Anal. Appl.* **264** (2001), 85–95. Zbl 1038.39008 MR 1868329

[18] Ch. G. Philos, On oscillations of some difference equations. *Funkcial. Ekvac.* **34** (1991), 157–172. Zbl 0734.39004 MR 1116887

[19] Ch. G. Philos, Oscillations in a nonautonomous delay logistic difference equation. *Proc. Edinburgh Math. Soc.* (2) **35** (1992), 121–131. Zbl 0737.39003 MR 1150958

[20] Ch. G. Philos and I. K. Purnaras, Oscillations in linear difference equations with variable coefficients. *J. Appl. Math. Stochastic Anal.* **4** (1991), 241–258. Zbl 0747.39003 MR 1115156

[21] Ch. G. Philos and I. K. Purnaras, Sufficient conditions for the oscillation of linear difference equations with variable delay. *J. Difference Equ. Appl.* **14** (2008), 629–644. Zbl 1147.39004 MR 2417013

[22] Ch. G. Philos, I. K. Purnaras, and I. P. Stavroulakis, Sufficient conditions for the oscillation of delay difference equations. *J. Difference Equ. Appl.* **10** (2004), 419–435. Zbl 1052.39011 MR 2047221

[23] Y. G. Sficas and I. P. Stavroulakis, Oscillation criteria for first-order delay equations. *Bull. London Math. Soc.* **35** (2003), 239–246. Zbl 1035.34075 MR 1952402

[24] J. H. Shen and I. P. Stavroulakis, Oscillation criteria for delay difference equations. *Electron. J. Differential Equations* **2001** (2001), no. 10. Zbl 0964.39009 MR 1811783

[25] I. P. Stavroulakis, Oscillations of delay difference equations. *Comput. Math. Appl.* **29** (1995), 83–88. Zbl 0832.39002 MR 1321260

[26] I. P. Stavroulakis, Oscillation criteria for first order delay difference equations. *Mediterr. J. Math.* **1** (2004), 231–240. Zbl 1072.39010 MR 2089091

[27] X. H. Tang and J. S. Yu, Oscillation of delay difference equation. *Comput. Math. Appl.* **37** (1999), 11–20. Zbl 0937.39012 MR 1688201

[28] X. H. Tang and J. S. Yu, A further result on the oscillation of delay difference equations. *Comput. Math. Appl.* **38** (1999), 229–237. Zbl 0976.39005 MR 1729819

[29] X. H. Tang and J. S. Yu, Oscillations of delay difference equations. *Hokkaido Math. J.* **29** (2000), 213–228. Zbl 0958.39015 MR 1745511

[30] X. H. Tang and R. Y. Zhang, New oscillation criteria for delay difference equations. *Comput. Math. Appl.* **42** (2001), 1319–1330. Zbl 1002.39022 MR 1861531

[31] J. S. Yu, Z. C. Wang, B. G. Zhang, and X. Z. Qian, Oscillations of differential equations with deviating arguments. *Panamer. Math. J.* **2** (1992), 59–78. Zbl 0845.34082 MR 1160129

[32] J. S. Yu, B. G. Zhang, and Z. C. Wang, Oscillation of delay difference equations. *Appl. Anal.* **53** (1994), 117–124. Zbl 0729.39003 MR 1379188

[33] B. G. Zhang and C. J. Tian, Oscillation criteria for difference equations with unbounded delay. *Comput. Math. Appl.* **35** (1998), 19–26. Zbl 0907.39016 MR 1604779

[34] B. G. Zhang and C. J. Tian, Nonexistence and existence of positive solutions for difference equations with unbounded delay. *Comput. Math. Appl.* **36** (1998), 1–8. Zbl 0932.39007 MR 1631051

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