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The number of continuous curves in digital geometry

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Abstract. As a model for continuous curves in digital geometry, we study the Khalimsky-continuous functions defined on the integers and with values in the set of integers or the set of natural numbers. We determine the number of such functions on a given interval. It turns out that these numbers are related to the Delannoy and Schröder arrays, and a relation between these numbers is established.

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1. Introduction

After the advent of computers, Euclidean geometry is no longer suitable for image processing: all fundamental concepts such as lines, curves and surfaces have to be redefined. A new kind of geometry, taking into account the discrete nature of the pixels building up the images, has to be created—digital geometry is being built up to solve this problem. However, there is in general no unique solution to the problems we face.

Just like the set of all functions $\mathbb{R} \to \mathbb{R}$ is not a good model for curves in Euclidean geometry, the set of all functions $\mathbb{Z} \to \mathbb{Z}$ is not a good model for the curves we want to study in digital geometry. We need some kind of restriction, analogous to continuity or smoothness in the real case.

A suitable model are the continuous functions $\mathbb{Z} \to \mathbb{Z}$, provided that we can define a reasonable topology on the set $\mathbb{Z}$ of integers. In this paper we shall do so, choosing the Khalimsky topology, which makes the digital space $\mathbb{Z}^n$ connected; see Khalimsky et al. [5].

We shall define the Khalimsky topology on $\mathbb{Z}^n$ in Section 1.1 in a simple way, by just defining open subsets of $\mathbb{Z}$ and then going to higher dimensions using the product topology. After equipping the discrete space $\mathbb{Z}^n$ with a topology, we are able to speak about a continuous function. We will review the definition of Khalimsky-continuous function in Section 1.2. For more information in these subjects see Kiselman [6] and Melin [12], [10].

Some combinatorial work has already been done in digital geometry. We can mention here the pioneering work on the number of discrete segments of slope $0 \le \alpha \le 1$ and length L which was done by Berenstein and Lavine [2]. Work on the number of digital straight line segments was done by Bédaride et al. [1] and they went on to determine the number of digital segments of given length and height 2. More information about digital straight line segments can be found in Kiselman [6], Klette and Rosenfeld [8], [9], Melin [11], [10], and Samieinia [13].

Another combinatorial theme in digital geometry is digital curves. One of the articles on this topic is the one by Huxley and Zunić [4], who studied the number of different digital discs consisting of N points and showed an upper bound for it. In earlier papers, Samieinia [13], [14], we have studied the Khalimsky-continuous functions from a combinatorial point of view. We went on to show that these functions, when they have two points in their codomain, yield a new example of the classical Fibonacci sequence. For the case of three or four points in their codomain, some new sequences were presented.

In this paper we shall first determine the number of Khalimsky-continuous functions with codomain $\mathbb{Z}$ and show that it has the same recursion relation as the Pell numbers, but with different initial values. This enumeration gives also an example of Delannoy numbers. Then we shall determine the number of Khalimsky-continuous functions with codomain $\mathbb{N}$. In this case we obtain a sequence by summing up of two consecutive numbers of other sequences. We note as a byproduct some relations between the Schröder numbers; see Corollary 3.6. It turns out that there is a relation between the Delannoy and Schröder numbers, studied in Section 3. We review the definition of Delannoy and Schröder numbers as well as some of their properties in Section 1.3.

1.1. The Khalimsky topology. We present the Khalimsky topology using a topological basis. For every even integer m, the set $\{m-1, m, m+1\}$ is open, and for every odd integer n, the singleton set $\{n\}$ is open. A basis is given by

$$\{\{2n+1\}, \{2n-1, 2n, 2n+1\} \mid n \in \mathbb{Z}\}.$$

It follows that even points are closed. A digital interval $[a,b]_{\mathbb{Z}} = [a,b] \cap \mathbb{Z}$ with the subspace topology is called a *Khalimsky interval*. On the digital plane $\mathbb{Z}^2$, the Khalimsky topology is given by the product topology. A point with both coordinates odd is open. If both coordinates are even, the point is closed. These types of points are called *pure*. Points with one even and one odd coordinate are neither

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open nor closed; these are called *mixed*. By the Khalimsky topology we can see easily that the mixed point $m = (m_1, m_2)$ is connected to its four neighbors,

$$(m_1 \pm 1, m_2) \quad \text{and} \quad (m_1, m_2 \pm 1),$$

whereas the pure point $p = (p_1, p_2)$ is connected to all 8-neighbors,

$$(p_1 \pm 1, p_2),\ (p_1, p_2 \pm 1),\ (p_1 + 1, p_2 \pm 1) \text{ and } (p_1 - 1, p_2 \pm 1).$$

More information on the Khalimsky plane and the Khalimsky topology can find in the Lecture notes by Kiselman [6]. Erik Melin has worked and developed it in the various directions [10].

1.2. Khalimsky-continuous function. When we equip $\mathbb{Z}$ with the Khalimsky topology, we may speak of continuous functions $\mathbb{Z} \to \mathbb{Z}$, i.e., functions for which the inverse image of open sets are open. It is easily proved that a continuous function f is Lipschitz with constant 1. This is however not sufficient for continuity. It is not hard to prove that $f : \mathbb{Z} \to \mathbb{Z}$ is continuous if and only if (i) f is Lipschitz with constant 1 and (ii) for every x, $x \not\equiv f(x) \pmod 2$ implies $f(x \pm 1) = f(x)$. For more information see [12].

We observe that the following functions are continuous:

(1) $\mathbb{Z} \ni x \mapsto a \in \mathbb{Z}$, where a is constant;

(2) $\mathbb{Z} \ni x \ni x \mapsto \pm x + c \in \mathbb{Z}$, where c is an even constant;

(3) $\max(f, g)$ and $\min(f, g)$ if f and g are continuous.

Actually every continuous function on a bounded Khalimsky interval can be obtained by a finite succession of the rules (1), (2), (3); see Kiselman [6].

1.3. Delannoy and Schröder numbers. The Delannoy numbers $d_{i,j}$ were introduced by Henri Delannoy [3]. They satisfy

$$d_{i,j} = d_{i-1,j} + d_{i,j-1} + d_{i-1,j-1},$$

with conditions $d_{0,0} = 1$ and $d_{i,j} = 0$ for $i < 0$ or $j < 0$. The numbers $(d_{i,i})_{i \geq 0} = (1, 3, 13, 63, 321, 1683, 8989, 48639, \ldots)$ (the sequence number A001850 in Sloane [16]) are known as the central Delannoy numbers. In Section 2 we shall show that the number of Khalimsky-continuous functions with codomain $\mathbb{Z}$ gives an example of Delannoy numbers. If we consider instead such functions with codomain $\mathbb{N}$, then we get an example of other numbers, which are called the Schröder numbers, named for Ernst Schröder. He found these numbers while enumerating unrestricted bracketings of words. We define these numbers by the array $r_{i,j}$ such that $r_{i,j} = r_{i-1,j} + r_{i,j-1} + r_{i-1,j-1}$ with conditions $r_{0,0} = 1$ and $r_{i,j} = 0$ if $j < 0$ or $i < j$. The numbers $(r_{i,i})_{i \geq 0} = 1, 2, 6, 22, 90, \ldots$ are known as the large Schröder numbers.

Sulanke [17] presented the relation between the central Delannoy numbers and the Schröder numbers as $d_{n,n} = \sum_i r_{i,2n-i}$. It means that the central Delannoy number $d_{n,n}$ is the sum of the $(2n+1)$-st diagonal of the Schröder numbers. We will see also another relation between these two numbers in Section 3, which is $d_{i,j} = r_{i,j} + r_{i+1,j-1} + \cdots + r_{n,0}$ for $i \geq j$.

Other work that deals with the relation between the Delannoy and the Schröder numbers was done by Joachim Schröder [15]. He introduced generalized Schröder numbers $\mathrm{Schr}(i, j, l)$ as the number of lattice paths from $(0,0)$ to (i, j) with unit steps $(1,0)$, $(0,1)$ and $(1,1)$, which never go below the line $y = lx$.

We shall see in Sections 2 and 3 how these two kinds of numbers appear in enumerating of digital continuous curves.

2. Continuous curves with codomain $\mathbb{Z}$

There are connections between many mathematical problems and the Delannoy numbers. Sulanke (2003) listed 29 different contexts where the central Delannoy numbers appear. A classical example is the number of paths from $(0,0)$ to (n,n) using the steps $(0,1)$, $(1,0)$, and $(1,1)$. The 30th example was mentioned in Kiselman [7]. We present this example in detail in Theorem 2.2. To prove the statements of this section we need to use some of the properties of the Khalimsky topology which we shall state in the following lemma.

Lemma 2.1. *Suppose that f_n^s for $|s| \leq n$ is the number of Khalimsky-continuous functions $f : [0,n]_{\mathbb{Z}} \to \mathbb{Z}$ such that $f(0) = 0$ and $f(n) = s$. Then*

$$f_{2k}^s = \begin{cases} f_{2k-1}^{s-1} + f_{2k-1}^s + f_{2k-1}^{s+1}, & |s| = 2t \text{ for } t = 0, \ldots, k-1, \\ f_{2k-1}^s, & |s| = 2t-1 \text{ for } t = 1, \ldots, k, \\ f_{2k-1}^{2k-1}, & s = 2k, \\ f_{2k-1}^{-2k+1}, & s = -2k, \end{cases} \tag{1}$$

and

$$f_{2k+1}^s = \begin{cases} f_{2k}^{s-1} + f_{2k}^s + f_{2k}^{s+1}, & |s| = 2t-1 \text{ for } t = 1, \ldots, k, \\ f_{2k}^s, & |s| = 2t \text{ for } t = 0, \ldots, k, \\ f_{2k}^{2k}, & s = 2k+1, \\ f_{2k}^{-2k}, & s = -(2k+1). \end{cases} \tag{2}$$

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Proof. Let $f : [0, n]_{\mathbb{Z}} \to \mathbb{Z}$ be a continuous function with $f(2k) = s$, $|s| \neq 2k$ and even. Using the rules (1) and (2) in Section 1.2; $f(2k-1)$ can be one of the values s, $s+1$ or $s-1$. Thus we get the first line of equation (1). The other relations can be achieved similarly. □

Theorem 2.2. *Let f_n^s, $|s| \leq n$, be the number of Khalimsky-continuous functions $f : [0, n]_{\mathbb{Z}} \to \mathbb{Z}$ such that $f(0) = 0$ and $f(n) = s$, and $d_{i,j}$ be the Delannoy numbers. Then we have that $f_n^s = d_{i,j}$ for $i = \frac{1}{2}(n+s)$ and $j = \frac{1}{2}(n-s)$, where $n + s \in 2\mathbb{Z}$ and $f_n^s = f_{n-1}^s$ for $n + s$ odd.*

Proof. We shall use induction to prove the result. It is easy to see that $f_0^0 = 1 = d_{0,0}$, $f_1^1 = 1 = d_{1,0}$, $f_1^{-1} = 1 = d_{0,1}$ and $f_2^0 = 3 = d_{1,1}$. Suppose that the formula is true for $n < 2k$. We shall show that the result is true for $n = 2k$. We consider s such that $2k + s \in 2\mathbb{Z}$; hence s is an even number. For $|s| \neq 2k$, the equation (1) implies that

$$f_{2k}^s = f_{2k-1}^{s-1} + f_{2k-2}^s + f_{2k-1}^{s+1}. \tag{3}$$

By the statement we have

$$f_{2k-1}^{s-1} + f_{2k-2}^s + f_{2k-1}^{s+1} = d_{i-1,j} + d_{i-1,j-1} + d_{i,j-1}, \tag{4}$$

where

$$\frac{2k+s}{2} = i \quad \text{and} \quad \frac{2k-s}{2} = j. \tag{5}$$

Thus by (3), (4) and (5), we get the result. Suppose now $|s| = 2k$. Without loss of generality we may assume that s is positive. Using equation (1) and the statement we get $f_{2k}^s = f_{2k-1}^{s-1} = d_{2k-1,0}$. We can see easily that $d_{2k-1,0} = d_{2k,0}$. Hence we have the result in this case. The proof for $n = 2k + 1$ can be done in the same way. For odd $n + s$, equations (1) and (2) give the result. □

Theorem 2.3. *Let f_n be the number of Khalimsky-continuous functions $f : [0, n]_{\mathbb{Z}} \to \mathbb{Z}$ such that $f(0) = 0$. Then*

$$f_n = 2f_{n-1} + f_{n-2} \quad \text{for } n \geq 2. \tag{6}$$

Proof. Let f_n^s be the number of Khalimsky-continuous function $f : [0, n]_{\mathbb{Z}} \to \mathbb{Z}$ such that $f(0) = 0$ and $f(n) = s$. We have $f_n = \sum_{s=-n}^{n} f_n^s$, but with the

Khalimsky topology we can conclude that we have symmetry for f_n^s, that is, $f_n^s = f_n^{-s}$ for $s = 1, \dots, n$. Therefore we can consider another formulation for f_n, i.e.,

$$f_n = f_n^0 + 2\sum_{s=1}^{n} f_n^s. \tag{7}$$

Moreover, using equation (1), we see that

$$f_{2k}^s = \begin{cases} f_{2k-1}^{s-1} + f_{2k-1}^s + f_{2k-1}^{s+1}, & s = 2t \text{ for } t = 1, \dots, k-1, \\ f_{2k-1}^s, & s = 2t-1 \text{ for } t = 1, \dots, k, \\ f_{2k-1}^{2k-1}, & s = 2k, \\ f_{2k-1}^0 + 2f_{2k-1}^1, & s = 0. \end{cases} \tag{8}$$

We shall show the formula for $n = 2k+1$,

$$f_{2k+1} = f_{2k+1}^0 + 2\sum_{s=1}^{2k+1} f_{2k+1}^s = f_{2k+1}^0 + 2f_{2k+1}^{2k+1} + 2\sum_{t=1}^{k} f_{2k+1}^{2t} + 2\sum_{t=1}^{k} f_{2k+1}^{2t-1}. \tag{9}$$

Equation (9) comes from (7) and the simple separation of odd and even indices. Plugging equation (2) into (9) gives us

$$f_{2k+1} = f_{2k}^0 + 2f_{2k}^{2k} + 2\sum_{t=1}^{k} f_{2k}^{2t} + 2\sum_{t=1}^{k} (f_{2k}^{2t-2} + f_{2k}^{2t-1} + f_{2k}^{2t}),$$

and then with a simple calculation,

$$f_{2k+1} = f_{2k}^0 + 2f_{2k}^{2k} + 2\sum_{t=1}^{k} f_{2k}^{2t} + 2\sum_{t=1}^{k} f_{2k}^{2t-2} + 2\sum_{t=1}^{k} f_{2k}^{2t-1} + 2\sum_{t=1}^{k} f_{2k}^{2t}. \tag{10}$$

We have

$$2\sum_{t=1}^{k} f_{2k}^{2t-2} = 2f_{2k}^0 + 2\sum_{t=2}^{k} f_{2k}^{2t-2} = 2f_{2k}^0 + 2\sum_{t=1}^{k-1} f_{2k}^{2t}. \tag{11}$$

Therefore, by putting (11) in (10) and using (7);

$$f_{2k+1} = 2f_{2k} + 2f_{2k}^{2k} + f_{2k}^0 + 2\sum_{t=1}^{k-1} f_{2k}^{2t} - 2\sum_{t=1}^{k} f_{2k}^{2t-1}. \tag{12}$$

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Inserting (8) into (12) gives us

$$f_{2k+1} = 2f_{2k} + 2f_{2k-1}^{2k-1} + f_{2k-1}^{0} + 2f_{2k-1}^{1} + 2\sum_{t=1}^{k-1} f_{2k-1}^{2t-1} + 2\sum_{t=1}^{k-1} f_{2k-1}^{2t} + 2\sum_{t=1}^{k-1} f_{2k-1}^{2t+1} - 2\sum_{t=1}^{k} f_{2k-1}^{2t-1}. \tag{13}$$

By a simple calculation we have the two following equations,

$$2f_{2k-1}^{2k-1} + 2\sum_{t=1}^{k-1} f_{2k-1}^{2t-1} - 2\sum_{t=1}^{k} f_{2k-1}^{2t-1} = 0, \tag{14}$$

and

$$2f_{2k-1}^{1} + 2\sum_{t=1}^{k-1} f_{2k-1}^{2t+1} = 2\sum_{t=1}^{k} f_{2k-1}^{2t-1}. \tag{15}$$

Finally, by putting (14) and (15) into (13) and by using (7), we obtain the desired formula. □

The sequence in Theorem 2.3 is a well-known sequence, and appears as sequence number A078057 in Sloane's Encyclopedia. It is given by the explicit formula $f_n = \frac{1}{2}[(1+\sqrt{2})^n + (1-\sqrt{2})^n]$. Actually f_n has the same recursion formula as the Pell numbers P_n, but with different initial values. The sequence (P_n) is defined as

$$P_n = \begin{cases} 0, & n = 0, \\ 1, & n = 1, \\ 2P_{n-1} + P_{n-2}, & n \geq 2. \end{cases}$$

The reader can find more information about this sequence in item (A000129) of the encyclopedia.

From Theorem 2.3 we can easily conclude that f_n tends to the Silver Ratio $1+\sqrt{2}$ as n tends to infinity.

Corollary 2.4. *Let f_n be the number of Khalimsky-continuous functions $f : [0,n]_{\mathbb{Z}} \to \mathbb{Z}$ such that $f(0) = 0$. Then $f_n/f_{n-1} \to 1+\sqrt{2}$ as $n \to \infty$.*

The following table shows the values of f_n^s and f_n for $0 \leq n \leq 9$.

9										1
8									1	1
7								1	1	17
6							1	1	15	15
5						1	1	13	13	113
4					1	1	11	11	85	85
3				1	1	9	9	61	61	377
2			1	1	7	7	41	41	231	231
1		1	1	5	5	25	25	129	129	681
0	1	1	3	3	13	13	63	63	321	321
−1		1	1	5	5	25	25	129	129	681
−2			1	1	7	7	41	41	231	231
−3				1	1	9	9	61	61	377
−4					1	1	11	11	85	85
−5						1	1	13	13	113
−6							1	1	15	15
−7								1	1	17
−8									1	1
−9										1
f_n	1	3	7	17	41	99	239	577	1393	3363

3. Continuous curves with codomain $\mathbb{N}$

In this section we consider g_n^s as the number of Khalimsky-continuous functions $g : [0, n] \to \mathbb{N}$ where $g(0) = 0$, $g(n) = s$ and $\mathbb{N} \ni s \leq n$. We put then some properties of this function in Lemma 3.1. These properties can lead us to Theorem 3.2.

Lemma 3.1. *Suppose that g_n^s is the number of Khalimsky-continuous functions $g : [0, n] \to \mathbb{N}$ for $g(0) = 0$, $g(n) = s$ and $s \in \mathbb{N}$, $s \leq n$. Then we have that*

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$$g_{2k}^{s} = \begin{cases} g_{2k-1}^{s-1} + g_{2k-1}^{s} + g_{2k-1}^{s+1}, & s = 2t \text{ for } t = 1, \dots, k-1, \\ g_{2k-1}^{0} + g_{2k-1}^{1}, & s = 0, \\ g_{2k-1}^{2k-1}, & s = 2k, \\ g_{2k-1}^{s}, & s = 2t-1 \text{ for } t = 1, \dots, k, \end{cases} \tag{16}$$

and

$$g_{2k+1}^{s} = \begin{cases} g_{2k}^{s-1} + g_{2k}^{s} + g_{2k}^{s+1}, & s = 2t-1 \text{ for } t = 1, \dots, k, \\ g_{2k}^{s}, & s = 2t \text{ for } t = 0, \dots, k, \\ g_{2k}^{2k}, & s = 2k+1. \end{cases} \tag{17}$$

Proof. Same as Lemma 2.1, the proof can be done by using the rules (1) and (2) stated in Section 1.2. □

In the next theorem we shall see how Schröder numbers appear in the numeration of Khalimsky-continuous functions with codomain $\mathbb{N}$.

Theorem 3.2. *Let* $g_n^s = \operatorname{card}\{g : [0, n] \to \mathbb{N} \mid g(0) = 0, g(n) = s\}$ *for* $s \in \mathbb{N}$ *and* $s \le n$, *and* $r_{i,j}$ *be the Schröder numbers. Then we have* $g_n^s = r_{i,j}$ *for* $i = \frac{1}{2}(n+s)$ *and* $j = \frac{1}{2}(n-s)$, *where* $n + s \in 2\mathbb{N}$.

Proof. We shall use induction. The result for $n = 1, 2$ can be obtained easily, i.e., $g_0^0 = r_{0,0}$ and $g_1^1 = r_{1,0}$. Suppose that the formula is true for $t < 2k$. We shall show the result for $t = 2k$. We consider s such that $2k + s \in 2\mathbb{N}$, hence s is an even number. Using equation (16) and the statement, we see that for $s \neq 0$,

$$g_{2k}^{s} = r_{i-1,j} + r_{i,j-1} + r_{i-1,j-1}, \qquad \text{where } i = \frac{2k+s}{2},\ j = \frac{2k-s}{2}. \tag{18}$$

Thus for the case $s \neq 0$ we are done.

Suppose now $s = 0$. Using again (16) and the statement imply that

$$g_{2k}^{0} = g_{2k-2}^{0} + g_{2k-1}^{1} = r_{i-1,j-1} + r_{i,j-1}, \qquad \text{where } i = \frac{2k}{2},\ j = \frac{2k}{2}. \tag{19}$$

Since $i - 1 < j$, we have $r_{i-1,j} = 0$. Thus by adding it to the equation (19) we get the result in this case. The proof for $t = 2k + 1$ can be done similarly. □

In the following theorem we see that the Delannoy numbers appear also in the enumerating of continuous curves with codomain $\mathbb{N}$. Then by Theorems 3.3 and 3.2, we conclude a relation between the Delannoy and Schröder arrays.

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Theorem 3.3. *Let g_n^s be the number of Khalimsky-continuous functions $g : [0,n] \to \mathbb{N}$ such that $g(0) = 0$ and $g(n) = s$ for $s \in \mathbb{N}$ and $s \le n$. Let $p_{t,n} = \sum_{i=0}^{(n-t)/2} g_n^{t+2i}$, where $0 \le t \le n$ and $n + t \in 2\mathbb{N}$. Then $p_{t,n} = d_{i,j}$, where $i = \frac{n+t}{2}$, $j = \frac{n-t}{2}$ and $d_{i,j}$ is a Delannoy number.*

Proof. We use induction to prove. For $t = n = 0$ and $t = n = 1$ the result is clear. First we consider that $n = 2k$ and the formula in the statement is true for $n < 2k$. By the statement we have

$$p_{t,2k} = g_{2k}^t + g_{2k}^{t+2} + \cdots + g_{2k}^{2k-2} + g_{2k}^{2k}.$$

Let $t \neq 0$. Equation (16) implies that

$$\begin{aligned} llp_{t,2k} = {} & g_{2k-1}^{t-1} + g_{2k-1}^{t+1} + g_{2k-2}^{t} + g_{2k-1}^{t+1} + g_{2k-1}^{t+3} + g_{2k-2}^{t+2} + \cdots \\ & \cdots + g_{2k-1}^{2k-3} + g_{2k-1}^{2k-1} + g_{2k-2}^{2k-2} + g_{2k-1}^{2k-1}. \end{aligned} \tag{20}$$

The first column in the right-hand side of equation (20) is equal to $p_{t-1,2k-1}$. The second and third columns are equal to $p_{t+1,2k-1}$ and $p_{t,2k-2}$, respectively. Thus

$$p_{t,2k} = p_{t-1,2k-1} + p_{t+1,2k-1} + p_{t,2k-2}. \tag{21}$$

By the statement and (21) we have

$$p_{t,2k} = d_{i-1,j} + d_{i,j-1} + d_{i-1,j-1} \qquad \text{for } i = \frac{n+t}{2} \text{ and } j = \frac{n-t}{2}.$$

This is the result for $n = 2k$ and when $t \neq 0$. We can proceed similarly for the other cases. □

Corollary 3.4. *Let $r_{i,j}$ and $d_{i,j}$ be the Schröder numbers and Delannoy numbers, respectively. Then $d_{i,j} = \sum_{l=0}^{j} r_{i+l,j-l}$ for $i \ge j$.*

Proof. Theorem 3.3 leads us to the following equation for $0 \le t \le n$ and $n + t \in 2\mathbb{N}$;

$$d_{i,j} = g_n^t + g_n^{t+2} + \cdots + g_n^n, \qquad \text{where } i = \frac{n+t}{2} \text{ and } j = \frac{n-t}{2}. \tag{22}$$

By Theorem 3.2 and equation (22) we have

$$d_{i,j} = r_{i,j} + r_{i+1,j-1} + \cdots + r_{n,0},$$

which is equal to $\sum_{l=0}^{j} r_{i+l,j-l}$. □

The following table shows the values of $p_{t,n}$ and consequently we can see the relation between these numbers and the Delannoy numbers.

$t \setminus n$	0	1	2	3	4	5	6	7	8	9
0	1		3		13		63		321	
1		1		5		25		129		681
2			1		7		41		231	
3				1		9		61		377
4					1		11		85	
5						1		13		113
6							1		15	
7								1		17
8									1	
9										1

Proposition 3.5. *Let g_n^s be the number of Khalimsky-continuous functions $g : [0,n]_{\mathbb{Z}} \to \mathbb{N}$ such that $g(0) = 0$ and $g(n) = s$ for $s \in \mathbb{N}$ and $s \le n$. Then*

$$g_{2k}^0 = 2\sum_{i=1}^{k} g_{2k-i-1}^{i-1},$$

$$g_{2k+1}^1 = 2\sum_{i=1}^{k}(g_{2k-i-1}^{i-1} + g_{2k-i}^{i}),$$

$$g_{2k+s}^s = 2\sum_{i=1}^{k}\sum_{j=0}^{s} g_{2k-i+j-1}^{i+j-1}, \qquad 2k+s \in 2\mathbb{N}.$$

Proof. The proof consists of an induction. First we prove the result for g_{2k}^0. Using (16) and (17) implies that

$$\begin{aligned} g_{2k}^0 &= g_{2k-2}^0 + g_{2k-1}^1 \\ &= g_{2k-2}^0 + (g_{2k-2}^0 + g_{2k-3}^1 + g_{2k-2}^2) \\ &= 2g_{2k-2}^0 + g_{2k-3}^1 + g_{2k-2}^2, \end{aligned} \tag{23}$$

and also

$$g_{2k-2}^2 = g_{2k-3}^1 + g_{2k-4}^2 + g_{2k-3}^3. \tag{24}$$

We insert (24) into (23) to get

$$g_{2k}^0 = 2g_{2k-2}^0 + 2g_{2k-3}^1 + g_{2k-4}^2 + g_{2k-3}^3.$$

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We can continue in the same way to get

$$g_{2k}^{0} = 2g_{2k-2}^{0} + \cdots + 2g_{k+1}^{k-3} + 2g_{k}^{k-2} + 2g_{k-1}^{k-1} = 2\sum_{i=1}^{k} g_{2k-i-1}^{i-1}. \tag{25}$$

We now prove the statement for s while we assume that the formula is correct for the natural numbers less than s.

Equations (16) and (17), and the induction assumption imply that

$$\begin{aligned} g_{2k+s}^{s} &= g_{2k+s-1}^{s-1} + g_{2k+s-2}^{s} + g_{2k+s-1}^{s+1} \\ &= 2\sum_{i=1}^{k}\sum_{j=0}^{s-1} g_{2k-i+j-1}^{i+j-1} + g_{2k+s-2}^{s} + g_{2k+s-1}^{s+1}. \end{aligned} \tag{26}$$

Using again the equations (16) and (17) we get

$$g_{2k+s-1}^{s+1} = g_{2k+s-2}^{s} + g_{2k+s-3}^{s+1} + g_{2k+s-2}^{s+2}. \tag{27}$$

If we insert (27) into (26), we get

$$g_{2k+s}^{s} = 2\sum_{i=1}^{k}\sum_{j=0}^{s-1} g_{2k-i+j-1}^{i+j-1} + 2g_{2k+s-2}^{s} + g_{2k+s-3}^{s+1} + g_{2k+s-2}^{s+2}.$$

If we go on until we have g_{k+s+1}^{k+s-1}, then we get

$$g_{2k+s}^{s} = 2\sum_{i=1}^{k}\sum_{j=0}^{s-1} g_{2k-i+j-1}^{i+j-1} + 2g_{2k+s-2}^{s} + \cdots + 2g_{k+s+1}^{k+s-3} + g_{k+s}^{k+s-2} + g_{k+s+1}^{k+s-1}. \tag{28}$$

By equations (16) and (17),

$$g_{k+s+1}^{k+s-1} = g_{k+s}^{k+s-2} + g_{k+s-1}^{k+s-1} + g_{k+s}^{k+s}. \tag{29}$$

We need just to observe the equations (28) and (29) to get the equation

$$g_{2k+s}^{s} = 2\sum_{i=1}^{k}\sum_{j=0}^{s-1} g_{2k-i+j-1}^{i+j-1} + 2g_{2k+s-2}^{s} + \cdots + 2g_{k+s}^{k+s-2} + g_{k+s-1}^{k+s-1} + g_{k+s}^{k+s}.$$

The definition of g_n^s implies that $g_{k+s-1}^{k+s-1} = g_{k+s}^{k+s}$. Therefore

$$g_{2k+s}^{s} = 2\sum_{i=1}^{k}\sum_{j=0}^{s} g_{2k-i+j-1}^{i+j-1}.$$ □

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As a consequence of Proposition 3.5 and Theorem 3.2 we obtain the following relations between the Schröder numbers:

Corollary 3.6. *Let* $r_{i,j}$ *be the Schröder numbers. Then*

$$r_{k,k} = 2\sum_{i=1}^{k} r_{k-1,k-i},$$

$$r_{k+1,k} = 2\sum_{i=1}^{k} (r_{k-1,k-i} + r_{k,k-i}),$$

$$r_{k+s,k} = 2\sum_{i=1}^{k}\sum_{j=0}^{s} r_{k+j-1,k-i}.$$

The results in Corollaries 3.4 and 3.6 can of course also be obtained by simple induction. Here they were obtained as a byproduct of our study of digital curves.

In the following theorem we will see that the number of Khalimsky-continuous functions with codomain $\mathbb{N}$ can be obtained by summing up of two consecutive numbers of other sequences.

Theorem 3.7. *Let* g_n *be the number of Khalimsky-continuous functions* $g : [0,n] \to \mathbb{N}$ *such that* $g(0) = 0$. *Let* $p_n = p_{0,n}$ *for* n *even and* $p_n = p_{1,n}$ *for* n *odd, where* $p_{t,n}$ *are the numbers defined in Theorem* 3.3. *Then*

$$g_n = p_n + p_{n-1}. \tag{30}$$

Proof. Let $g_n^s = \operatorname{card}\{g : [0,n] \to \mathbb{N}; g(0) = 0 \text{ and } g(s) = n\}$. Therefore it is clear that

$$g_n = \sum_{s=0}^{n} g_n^s. \tag{31}$$

Suppose that $n = 2k$. By (31) we have

$$g_{2k} = \sum_{t=0}^{k} g_{2k}^{2t} + \sum_{t=1}^{k} g_{2k}^{2t-1}.$$

By the definition of the sequence p_n and equation (16),

$$g_{2k} = p_{2k} + p_{2k-1}.$$

The proof for $n = 2k + 1$ can be obtained in the same way. □

In the following table we can see the values of g_n^s, g_n and p_n.

$s \setminus n$	0	1	2	3	4	5	6	7	8	9	10
0	1	1	2	2	6	6	22	22	90	90	394
1	0	1	1	4	4	16	16	68	68	304	304
2	0	0	1	1	6	6	30	30	146	146	714
3	0	0	0	1	1	8	8	48	48	264	264
4	0	0	0	0	1	1	10	10	70	70	430
5	0	0	0	0	0	1	1	12	12	96	96
6	0	0	0	0	0	0	1	1	14	14	126
7	0	0	0	0	0	0	0	1	1	16	16
8	0	0	0	0	0	0	0	0	1	1	18
9	0	0	0	0	0	0	0	0	0	1	1
10	0	0	0	0	0	0	0	0	0	0	1
11	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0
g_n	1	2	4	8	18	38	88	192	450	1002	2364
p_n	1	1	3	5	13	25	63	129	321	681	1683

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