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Regularity and decay properties of weak solutions to Navier–Stokes equations in general domains

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Abstract. In this paper we consider the regularity of weak solutions and find some regular criteria for 3D non-stationary Navier–Stokes equations. Moreover, we establish decay rates for weak solutions in general domains by means of the spectral decomposition method of fractional powers of the Stokes operator.

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1. Introduction and main results

Let Ω be a general domain in $\mathbb{R}^3$, and $0 < T \le \infty$. We consider the initial value problem of the Navier–Stokes equations

$$\begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = f & \text{in } \Omega \times (0,T), \\ \nabla \cdot u = 0, \; u|_{\partial\Omega \times (0,T)} = 0, \; u(x,0) = a, \end{cases} \tag{1.1}$$

where $u = \big(u_1(x,t), u_2(x,t), u_3(x,t)\big)$ and $p = p(x,t)$ denote the unknown velocity vector and the pressure respectively, while $a = a(x)$ is a given initial velocity vector field, and $f = \big(f_1(x,t), f_2(x,t), f_3(x,t)\big)$ is the given external force.

Definition 1.1. u is called a weak solution of (1.1) if $u \in L^\infty\big(0,T; L^2_\sigma(\Omega)\big) \cap L^2_{\mathrm{loc}}\big([0,T); H^1_0(\Omega)\big)$ satisfies

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$$-\int_0^T\int_\Omega u\partial_\tau v\,dx\,d\tau+\int_0^T\int_\Omega \nabla u\cdot\nabla v\,dx\,d\tau+\int_0^T\int_\Omega u\cdot\nabla u\cdot v\,dx\,d\tau$$
$$=\int_\Omega av(0)\,dx+\int_0^T\int_\Omega fv\,dx\,d\tau \qquad \text{for all } v\in C_0^\infty\big([0,T);C_{0,\sigma}^\infty(\Omega)\big),$$

where $a\in L^2_\sigma(\Omega)$ and $f\in L^1\big(0,T;L^2(\Omega)\big)$, $\partial_\tau v=\frac{\partial}{\partial\tau}v(x,\tau)$. Furthermore, we say that u is a strong solution of (1.1) if $u\in L^s\big(0,T;L^q(\Omega)\big)$, $\frac{2}{s}+\frac{3}{q}\le 1$ with $2\le s<\infty$, $3<q\le\infty$. In addition, the weak solution u is said to satisfy the energy inequality if

$$\|u(t)\|_2^2+2\int_0^t\|\nabla u(\tau)\|_2^2\,d\tau\le\|a\|_2^2+2\int_0^t\int_\Omega fu\,dx\,d\tau \tag{1.2}$$

for a.e. $t\in[0,T)$.

The weak solution of (1.1) is so far known to be unique only if it belongs to a certain class of functions which, however, does not cover the whole space $L^\infty\big(0,T;L^2_\sigma(\Omega)\big)\cap L^2_{\mathrm{loc}}\big([0,T);H^1_0(\Omega)\big)$. The results of G. Prodi [15], J. L. Lions and G. Prodi [10], Foias [6], Serrin [17], Kozono and Sohr [9] and others showed the uniqueness in $L^s\big(0,T;L^q(\mathbb{R}^N)\big)$ with $\frac{2}{s}+\frac{N}{q}\le 1$, $N<q\le\infty$ for weak solutions of (1.1) satisfying (1.2). That is, if u is a weak solution of (1.1) satisfying (1.2), and if v is another weak solution of (1.1) in $L^s\big(0,T;L^q(\mathbb{R}^N)\big)$ with s, q as above, then $u=v$ in $\mathbb{R}^N\times[0,T)$. In fact, the uniqueness result also holds for any unbounded domain Ω, see [18], [19]. Recently Escauriaza et al. [5] proved that if $\Omega=\mathbb{R}^3$ and $f=0$, then each suitable weak solution in the class $L^\infty\big(0,T;L^3(\mathbb{R}^3)\big)$ is unique and smooth in $\mathbb{R}^3\times(0,T)$.

It is well known that any weak solution of (1.1) is regular in the Serrin class (see [16], [18], [19]). We try to establish another regular class for the 3D Navier–Stokes system:

$$\begin{cases}\partial_t u-\Delta u+(u\cdot\nabla)u+\nabla p=0 & \text{in } \mathbb{R}^3\times(0,T),\\ \nabla\cdot u=0,\ u(x,0)=a & \text{in } \mathbb{R}^3.\end{cases} \tag{1.3}$$

Recently, Chae and Choe [3] proved regularity by imposing conditions on the gradients of two components of the velocity. Subsequently, Beirão da Veiga [1] also reduced Serrin's condition to two components of the velocity field. Neustupa and Penel [13] verified the regularity for suitable weak solutions of (1.3) if one velocity component is essentially bounded. The regularity with respect to one component of the velocity was also proved by Neustupa et al. in [12]. Here for the first time the authors came up with the inequality $\frac{2}{s}+\frac{3}{q}\le\frac{1}{2}$ in connection with one component of the velocity. Subsequently, He [7] and Zhou [21] also imposed the regularity criterion on one component of the weak solutions of (1.3). In the following we find a new regularity criterion for weak solutions of (1.3), which can be viewed as another form of Serrin's condition.

Theorem 1.2. *Suppose that* $a \in H^1_\sigma(\mathbb{R}^3)$ *and* $0 < T < \infty$. *Assume that* u *is a weak solution of* (1.3) *satisfying the energy inequality* (1.2) *with* $f = 0$. *Let* $u = (u_1, u_2, u_3)$ *satisfy one of the following conditions:*

for some $s \in [2, \infty)$, $q \in (3, \infty]$ *with* $\frac{2}{s} + \frac{3}{q} \le 1$,

$$u_1 \in L^\infty\big(0, T; L^3(\mathbb{R}^3)\big), \qquad u_2, u_3 \in L^s\big(0, T; L^q(\mathbb{R}^3)\big), \tag{1.4}$$

or

$$\begin{aligned} &u_1, u_2 \in L^\infty\big(0, T; L^3(\mathbb{R}^3)\big),\ u_3 \in L^s\big(0, T; L^q(\mathbb{R}^3)\big);\ \textit{moreover} \\ &\|u_1\|_{L^\infty(0,T;L^3(\mathbb{R}^3))} + \|u_2\|_{L^\infty(0,T;L^3(\mathbb{R}^3))} \le \eta_0 \ \textit{for some small} \\ &\textit{number}\ \eta_0 > 0. \end{aligned} \tag{1.5}$$

Then u *is regular on* $\mathbb{R}^3 \times (0, T)$.

Remark. Neustupa and Penel [14] also formulated criterions for regularity (see Theorem 1 in [14]) by means of different assumptions on the first two and on the third component of velocity, where the authors need $q > 6$, however without a smallness condition. Some ideas (taking curl on both sides of the equations of (1.1) for example) in the proofs are similar or even identical, but the corresponding results do not overlap in [14] and the present paper.

In recent years, much attention has been paid to the decay properties of solutions for problem (1.1) in general unbounded domains. Kozono and Ogawa [8] proved that if $a \in D(A^{1/4}) \cap R(A^\mu)$ with $0 \le \mu \le \frac{1}{2}$, there is a strong solution u of (1.1) with $f = 0$ such that if $0 \le \mu < \frac{1}{4}$, then $\|A^\alpha u(t)\|_2 = O(t^{-\mu-\alpha})$ for $0 \le \alpha \le 1$, and if $\frac{1}{4} \le \mu \le \frac{1}{2}$, then $\|A^\alpha u(t)\|_2 = o(t^{-\mu-\alpha})$ for $0 \le \alpha \le 1$. Maremonti [11] also considered the time decay of some strong solution for (1.1) in unbounded domains, and obtained similar results. Crispo and Tartaglione [4] studied the asymptotic stability in the L^2-norm of solutions of (1.1) in three-dimensional unbounded domains with non-compact boundary, that is, they considered the perturbations to the rest state and to the stationary motions. Borchers and Miyakawa [2] considered the L^2-decay for weak solutions of (1.1) in general domains. By using a specific approximate scheme, they first showed the decay of the time average $t^{-1}\int_0^t \|u(s)\|_2\,ds$ for the general weak solution u, which satisfies the energy inequality, and then proved Theorem 1.3 below. However, the authors in [2] believed that it was difficult to obtain the decay properties for weak solutions of (1.1) by applying the spectral decomposition. In this paper, we give an alternative proof of Theorem 1.3 by employing the spectral decomposition method which is simpler and easier than the one in [2].

Theorem 1.3. *Let $\Omega \subseteq \mathbb{R}^3$ be any general C^2-domain, and let $T = \infty$. If $a \in L^2_\sigma(\Omega)$, $f \in L^1(0, \infty; \hat{V}^*)$, and if $\int_0^\infty (s+1)\|f(s)\|_2\, ds < \infty$, then there is a weak solution u of (1.1) satisfying (1.2) such that if $\|e^{-tA}a\|_2 = O(t^{-\alpha})$ for some $\alpha > 0$ as $t \to \infty$, then*

$$\|u(t)\|_2 = \begin{cases} O(t^{-\alpha}) & \text{if } \alpha < \frac{1}{2}, \\ O(t^{\varepsilon - 1/2}) & \text{if } \alpha \geq \frac{1}{2}, \end{cases} \tag{1.6}$$

where $\varepsilon \in \left(0, \frac{1}{2}\right)$, $\hat{V}^*$ is the dual of $V = \hat{H}^1_{0,\sigma}(\Omega)$.

Throughout this paper, we denote the norms of $L^\ell(\Omega)$, $L^s\big(0, T; L^\ell(\Omega)\big)$ $(1 \leq s, \ell < \infty)$ by $\|u\|_\ell = \left(\int_\Omega |u|^\ell\, dx\right)^{1/\ell}$ and $\|u\|_{L^s(0,T;L^\ell(\Omega))} = \left(\int_0^T \|u(t)\|_\ell^s\, dt\right)^{1/s}$ respectively, and positive constants (possibly different from line to line) by C.

2. Regularity criteria for weak solutions of (1.3)

In this section we first state two technical lemmas and then give the proof of Theorem 1.2.

Lemma 2.1. *Suppose that $a \in H^1_\sigma(\mathbb{R}^3)$. Assume that $u = (u_1, u_2, u_3)$ is a strong solution of (1.3) with $\nabla u \in L^\infty\big(0, T; L^2(\mathbb{R}^3)\big)$ and $\Delta u \in L^2\big(\mathbb{R}^3 \times (0, T)\big)$. If $u_3 \in L^s\big(0, T; L^q(\mathbb{R}^3)\big)$ with $\frac{2}{s} + \frac{3}{q} \leq 1$, $2 \leq s \leq \infty$, $3 \leq q \leq \infty$, then for any $0 \leq t < T$, the vorticity $\omega = \operatorname{curl} u = (\omega_1, \omega_2, \omega_3)$ satisfies*

$$\|\omega_3(t)\|_2^2 + \int_0^t \|\nabla\omega_3(\tau)\|_2^2\, d\tau$$
$$\leq \begin{cases} \|\omega_3(0)\|_2^2 + C\|u_3\|^2_{L^s(0,t;L^q(\mathbb{R}^3))}\|\nabla u\|^{4/s}_{L^\infty(0,t;L^2(\mathbb{R}^3))}\|\Delta u\|^{6/q}_{L^2(\mathbb{R}^3\times(0,t))} & \text{if } 3 < q < \infty, \\ \|\omega_3(0)\|_2^2 + C\|u_3\|^2_{L^2(0,t;L^\infty(\mathbb{R}^3))}\|\nabla u\|^2_{L^\infty(0,t;L^2(\mathbb{R}^3))} & \text{if } q = \infty, \\ \|\omega_3(0)\|_2^2 + C\|u_3\|^2_{L^\infty(0,t;L^3(\mathbb{R}^3))}\|\Delta u\|^2_{L^2(\mathbb{R}^3\times(0,t))} & \text{if } q = 3. \end{cases}$$

Proof. We first consider the case when $3 < q < \infty$. After a direct calculation, we find out that $\omega = \mathit{curl}\, u = (\omega_1, \omega_2, \omega_3)$ satisfies

$$\frac{\partial\omega}{\partial t} - \Delta\omega + (u \cdot \nabla)\omega - (\omega \cdot \nabla)u = 0 \qquad \text{in } \mathbb{R}^3 \times (0, T). \tag{2.1}$$

By multiplying both sides of the equation on ω_3 in (2.1) by ω_3, and integrating by parts over $\mathbb{R}^3 \times (0, t)$, we get

$$
\begin{aligned}
&\|\omega_3(t)\|_2^2 + 2\int_0^t \|\nabla\omega_3(\tau)\|_2^2\,d\tau \\
&\quad \le \|\omega_3(0)\|_2^2 + 2\int_0^t\int_{\mathbb{R}^3} |(\omega\cdot\nabla)\omega_3 u_3|\,dx\,d\tau \\
&\quad \le \|\omega_3(0)\|_2^2 + \int_0^t \|\nabla\omega_3(\tau)\|_2^2\,d\tau + \int_0^t\int_{\mathbb{R}^3} |\omega|^2|u_3|^2\,dx\,d\tau \\
&\quad \le \|\omega_3(0)\|_2^2 + \int_0^t \|\nabla\omega_3(\tau)\|_2^2\,d\tau + C\int_0^t\int_{\mathbb{R}^3} |\nabla u|^2|u_3|^2\,dx\,d\tau, \qquad (2.2)
\end{aligned}
$$

where we have used the fact that $|\omega| \le C|\nabla u|$. In the following we will rather use the notation $\frac{1}{s}$ for s^{-1}, for any $s > 0$. Next we observe that $2\left(1-\frac{2}{q}\right)^{-1} \in (2,6)$ for any $q \in (3,\infty)$, and so

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}^3} |\nabla u|^2|u_3|^2\,dx\,d\tau \\
&\quad \le \int_0^t \|u_3\|_q^2 \|\nabla u\|_{2(1-2/q)^{-1}}^2\,d\tau \\
&\quad \le \int_0^t \|u_3\|_q^2 \|\nabla u\|_2^{2(1-3/q)} \|\nabla u\|_6^{6/q}\,d\tau \\
&\quad \le C\|\nabla u\|_{L^\infty(0,t;L^2(\mathbb{R}^3))}^{2(1-3/q)-2(1-2/s-3/q)} \int_0^t \|u_3\|_q^2 \|\nabla u\|_2^{2(1-2/s-3/q)} \|\Delta u\|_2^{6/q}\,d\tau \\
&\quad \le C\|\nabla u\|_{L^\infty(0,t;L^2(\mathbb{R}^3))}^{4/s} \left(\int_0^t \|u_3\|_q^s\,d\tau\right)^{2/s} \left(\int_0^t \|\nabla u\|_2^2\,d\tau\right)^{(1-2/s-3/q)} \left(\int_0^t \|\Delta u\|_2^2\,d\tau\right)^{3/q} \\
&\quad \le C\|a\|_2^{2(1-2/s-3/q)} \|u_3\|_{L^s(0,t;L^q(\mathbb{R}^3))}^2 \|\nabla u\|_{L^\infty(0,t;L^2(\mathbb{R}^3))}^{4/s} \|\Delta u\|_{L^2(\mathbb{R}^3\times(0,t))}^{6/q}. \qquad (2.3)
\end{aligned}
$$

By inserting (2.3) into (2.2) we complete the proof of Lemma 2.1 in the case when $3 < q < \infty$. Following the above arguments, we easily verify the remaining cases when $q = 3$ or $q = \infty$. □

Lemma 2.2. *Suppose that* $a \in H_\sigma^1(\mathbb{R}^3)$. *Assume that* u *is a strong solution of* (1.3) *with* $\nabla u \in L^\infty\big(0,T;L^2(\mathbb{R}^3)\big)$ *and* $\Delta u \in L^2\big(\mathbb{R}^3\times(0,T)\big)$. *If* $u = (u_1,u_2,u_3)$ *satisfies the assumption* (1.4) *or* (1.5), *then*

$$
\sup_{0\le t<T} \|\nabla u(t)\|_2^2 + \int_0^T \|\Delta u(\tau)\|_2^2\,d\tau \le C.
$$

Here $C = C(\|a\|_{H^1(\mathbb{R}^3)},T,q,s)$ *if* (1.4) *holds, and* $C = C(\|a\|_{H^1(\mathbb{R}^3)},T,q,s,\eta_0)$ *if* (1.5) *holds.*

Proof. The equation in (1.3) can be rewritten as follows:

$$\partial_t u - \Delta u + \omega \times u + \frac{1}{2}\nabla|u|^2 + \nabla p = 0 \quad \text{in } \mathbb{R}^3 \times (0,T). \tag{2.4}$$

By multiplying both sides in (2.4) by Δu and integrating by parts, we obtain

$$\begin{aligned}
&\|\nabla u(t)\|_2^2 + 2\int_0^t \|\Delta u(\tau)\|_2^2\,d\tau \\
&\quad = 2\int_0^t\int_{\mathbb{R}^3} (\omega \times u)\cdot \Delta u\,dx\,d\tau + \|\nabla u_0\|_2^2 \\
&\quad = 2\int_0^t\int_{\mathbb{R}^3} \big((\omega_2 u_3 - \omega_3 u_2)\Delta u_1 + (\omega_3 u_1 - \omega_1 u_3)\Delta u_2 \\
&\qquad\qquad + (\omega_1 u_2 - \omega_2 u_1)\Delta u_3\big)\,dx\,d\tau + \|\nabla a\|_2^2.
\end{aligned} \tag{2.5}$$

Case 1. Asssume (1.4) holds. Then we obtain

$$\begin{aligned}
&\left|2\int_0^t\int_{\mathbb{R}^3} \big((\omega_2 u_3 - \omega_3 u_2)\Delta u_1 - \omega_1 u_3 \Delta u_2 + \omega_1 u_2 \Delta u_3\big)\,dx\,d\tau\right| \\
&\quad \le 2\int_0^t\int_{\mathbb{R}^3} |u_3|(|\omega_2|\,|\Delta u_1| + |\omega_1|\,|\Delta u_2|)\,dx\,d\tau \\
&\qquad + 2\int_0^t\int_{\mathbb{R}^3} |u_2|(|\omega_3|\,|\Delta u_1| + |\omega_1|\,|\Delta u_3|)\,dx\,d\tau \\
&\quad \le 2\int_0^t \|u_3\|_q(\|\omega_2\|_{(1/2-1/q)^{-1}}\|\Delta u_1\|_2 + \|\omega_1\|_{(1/2-1/q)^{-1}}\|\Delta u_2\|_2)\,d\tau \\
&\qquad + 2\int_0^t \|u_2\|_q(\|\omega_3\|_{(1/2-1/q)^{-1}}\|\Delta u_1\|_2 + \|\omega_1\|_{(1/2-1/q)^{-1}}\|\Delta u_3\|_2)\,d\tau \\
&\quad \le C\int_0^t (\|u_2\|_q + \|u_3\|_q)\|\nabla u\|_{(1/2-1/q)^{-1}}\|\Delta u\|_2\,d\tau \\
&\quad \le C\int_0^t (\|u_2\|_q + \|u_3\|_q)\|\nabla u\|_2^{1-3/q}\|\nabla u\|_6^{3/q}\|\Delta u\|_2\,d\tau \\
&\quad \le C\int_0^t (\|u_2\|_q + \|u_3\|_q)\|\nabla u\|_2^{1-3/q}\|\Delta u\|_2^{1+3/q}\,d\tau \\
&\quad \le \varepsilon\int_0^t \|\Delta u\|_2^2\,d\tau + C(\varepsilon)\int_0^t (\|u_2\|_q + \|u_3\|_q)^{2(1-3/q)^{-1}}\|\nabla u\|_2^2\,d\tau
\end{aligned}$$

$$
\begin{aligned}
&\le \varepsilon \int_0^t \|\Delta u\|_2^2\, d\tau + C(\varepsilon) \|\nabla u\|_{L^\infty(0,t;L^2(\mathbb{R}^3))}^{2-2(1-3/q)^{-1}(1-2/s-3/q)} \\
&\quad \cdot \int_0^t (\|u_2\|_q + \|u_3\|_q)^{2(1-3/q)^{-1}} \|\nabla u\|_2^{2(1-3/q)^{-1}(1-2/s-3/q)}\, d\tau \\
&\le \varepsilon \int_0^t \|\Delta u\|_2^2\, d\tau + C(\varepsilon) \|\nabla u\|_{L^\infty(0,t;L^2(\mathbb{R}^3))}^{(4/s)(1-3/q)^{-1}} \\
&\quad \times (\|u_2\|_{L^s(0,t;L^q(\mathbb{R}^3))} + \|u_3\|_{L^s(0,t;L^q(\mathbb{R}^3))})^{2(1-3/q)^{-1}} \\
&\quad \cdot \Big(\int_0^t \|\nabla u\|_2^2\, d\tau \Big)^{(1-3/q)^{-1}(1-2/s-3/q)} \\
&\le \varepsilon \int_0^t \|\Delta u\|_2^2\, d\tau + C(\|a\|_2, \varepsilon) \|\nabla u\|_{L^\infty(0,t;L^2(\mathbb{R}^3))}^{(4/s)(1-3/q)^{-1}} \\
&\quad \cdot (\|u_2\|_{L^s(0,t;L^q(\mathbb{R}^3))} + \|u_3\|_{L^s(0,t;L^q(\mathbb{R}^3))})^{2(1-3/q)^{-1}}. \qquad (2.6)
\end{aligned}
$$

By Lemma 2.1, we conclude

$$
\begin{aligned}
&\Big| 2 \int_0^t \int_{\mathbb{R}^3} u_1 (\omega_3 \Delta u_2 - \omega_2 \Delta u_3)\, dx\, d\tau \Big| \\
&\quad \le 2 \int_0^t \int_{\mathbb{R}^3} |u_1| (|\omega_3|\, |\Delta u_2| + |\omega_2|\, |\Delta u_3|)\, dx\, d\tau \\
&\quad \le 2 \int_0^t \|u_1\|_3 (\|\omega_3\|_6 \|\Delta u_2\|_2 + \|\omega_2\|_6 \|\Delta u_3\|_2)\, d\tau \\
&\quad \le C \|u_1\|_{L^\infty(0,T;L^3(\mathbb{R}^3))} \int_0^t (\|\nabla \omega_2\|_2 + \|\nabla \omega_3\|_2) \|\Delta u\|_2\, d\tau \\
&\quad \le C (\|\nabla \omega_2\|_{L^2(\mathbb{R}^3 \times (0,t))} + \|\nabla \omega_3\|_{L^2(\mathbb{R}^3 \times (0,t))}) \|\Delta u\|_{L^2(\mathbb{R}^3 \times (0,t))} \\
&\quad \le C \big(\|\omega_2(0)\|_2 + \|\omega_3(0)\|_2 \big) \|\Delta u\|_{L^2(\mathbb{R}^3 \times (0,t))} \\
&\qquad + C (\|u_2\|_{L^s(0,t;L^q(\mathbb{R}^3))} + \|u_3\|_{L^s(0,t;L^q(\mathbb{R}^3))}) \\
&\qquad \cdot \|\nabla u\|_{L^\infty(0,t;L^2(\mathbb{R}^3))}^{2/s} \|\Delta u\|_{L^2(\mathbb{R}^3 \times (0,t))}^{1+3/q} \\
&\quad \le \varepsilon \int_0^t \|\Delta u\|_2^2\, d\tau + C(\varepsilon) \big(\|\omega_2(0)\|_2^2 + \|\omega_3(0)\|_2^2 \big) \\
&\qquad + C(\varepsilon) (\|u_2\|_{L^s(0,t;L^q(\mathbb{R}^3))} + \|u_3\|_{L^s(0,t;L^q(\mathbb{R}^3))})^{2(1-3/q)^{-1}} \\
&\qquad \cdot \|\nabla u\|_{L^\infty(0,t;L^2(\mathbb{R}^3))}^{(4/s)(1-3/q)^{-1}}. \qquad (2.7)
\end{aligned}
$$

By inserting (2.6) and (2.7) into (2.5) and taking $\varepsilon = \frac{1}{4}$ we obtain, for all $0 \le t < T$ and $\delta > 0$,

$$\|\nabla u(t)\|_2^2 + \int_0^t \|\Delta u(\tau)\|_2^2 \, d\tau \le \begin{cases} C\|\nabla a\|_2^2 + \delta\|\nabla u\|^2_{L^\infty(0,t;L^2(\mathbb{R}^3))} + C(\delta)(\|u_2\|_{L^s(0,t;L^q(\mathbb{R}^3))} \\ \quad + \|u_3\|_{L^s(0,t;L^q(\mathbb{R}^3))})^{2(1-2/s-3/q)^{-1}} \quad \text{if } \frac{2}{s}+\frac{3}{q}<1, \\ C\|\nabla a\|_2^2 + C(\|u_2\|^s_{L^s(0,t;L^q(\mathbb{R}^3))} \\ \quad + \|u_3\|^s_{L^s(0,t;L^q(\mathbb{R}^3))})\|\nabla u\|^2_{L^\infty(0,t;L^2(\mathbb{R}^3))} \quad \text{if } \frac{2}{s}+\frac{3}{q}=1. \end{cases} \tag{2.8}$$

Set

$$g(t',t) = C(\|u_2\|^s_{L^s(t',t;L^q(\mathbb{R}^3))} + \|u_3\|^s_{L^s(t',t;L^q(\mathbb{R}^3))}) \quad \text{with any } 0 < t' < t < T,$$

where C is given in (2.8) with $\frac{2}{s}+\frac{3}{q}=1$.

From the assumption that $u_2, u_3 \in L^s(0,T;L^q(\mathbb{R}^3))$, we infer that $g(t',t)$ is continuous on t', t, and nondecreasing on time $t \in (t',T)$. We assume that $g(0,T) > \frac{1}{2}$, otherwise (2.9) below holds in the case $\frac{2}{s}+\frac{3}{q}=1$. Since $g(0,0)=0$, there exists $t_0 \in (0,T)$ such that $g(0,t_0) = \frac{1}{2}$. So from (2.8) with $\frac{2}{s}+\frac{3}{q}=1$, we get

$$\sup_{0\le t\le t_0} \|\nabla u(t)\|_2^2 + \int_0^{t_0} \|\Delta u(\tau)\|_2^2 \, d\tau \le C\|\nabla a\|_2^2.$$

From t_0 with $u(t_0)$ as the initial value for (1.3), we can find $t_1 \in (t_0,T)$ such that $g(t_0,t_1) = \frac{1}{2}$, and

$$\sup_{t_0\le t\le t_1} \|\nabla u(t)\|_2^2 + \int_{t_0}^{t_1} \|\Delta u(\tau)\|_2^2 \, d\tau \le C\|\nabla u(t_0)\|_2^2 \le C\|\nabla a\|_2^2.$$

By repeating the above process, we obtain an increasing sequence $\{t_k\}_{k=1}^\infty$ satisfying $g(t_{k-1},t_k) = \frac{1}{2}$ (here we always assume $g(t_k,T) > \frac{1}{2}$, otherwise (2.9) below holds in the case $\frac{2}{s}+\frac{3}{q}=1$). Moreover, the following inequality holds for any $k \ge 1$:

$$\sup_{t_{k-1}\le t\le t_k} \|\nabla u(t)\|_2^2 + \int_{t_{k-1}}^{t_k} \|\Delta u(\tau)\|_2^2 \, d\tau \le C_k\|\nabla a\|_2^2.$$

Denote the lengths of intervals (t_{k-1},t_k) by d_k, that is $d_k = t_k - t_{k-1}$. If $d_k \to 0$ as $k \to \infty$ then $g(t_{k-1},t_k) \to 0$ as $k \to \infty$, which is a contradiction with $g(t_{k-1},t_k) = \frac{1}{2}$ for any $k \ge 1$. Therefore, $d_k \nrightarrow 0$ as $k \to \infty$. Since $0 < T < \infty$, after a finite number of steps we obtain that

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$$\sup_{0\le t<T} \|\nabla u(t)\|_2^2 + \int_0^T \|\Delta u(\tau)\|_2^2\, d\tau \le C. \tag{2.9}$$

If $\frac{2}{s} + \frac{3}{q} < 1$, from (2.8), we immediately infer that (2.9) also holds.

Case 2. We now assume that (1.5) holds. Following the proof in Case 1 we know that, for any $\varepsilon > 0$,

$$\begin{aligned} &\left|2\int_0^t \int_{\mathbb{R}^3} u_3(\omega_2 \Delta u_1 - \omega_1 \Delta u_2)\, dx\, d\tau\right| \\ &\quad \le \varepsilon \int_0^t \|\Delta u\|_2^2\, d\tau + C(\|a\|_2, \varepsilon) \|u_3\|_{L^s(0,t;L^q(\mathbb{R}^3))}^{2(1-3/q)^{-1}} \|\nabla u\|_{L^\infty(0,t;L^2(\mathbb{R}^3))}^{(4/s)(1-3/q)^{-1}}, \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} &\left|2\int_0^t \int_{\mathbb{R}^3} \big(u_1(\omega_3 \Delta u_2 - \omega_2 \Delta u_3) + u_2(\omega_1 \Delta u_3 - \omega_3 \Delta u_1)\big)\, dx\, d\tau\right| \\ &\quad \le \int_0^t \int_{\mathbb{R}^3} \big(|u_1|(|\omega_3|\,|\Delta u_2| + |\omega_2|\,|\Delta u_3|) \\ &\qquad + |u_2|(|\omega_1|\,|\Delta u_3| + |\omega_3|\,|\Delta u_1|)\big)\, dx\, d\tau \\ &\quad \le \int_0^t \big(\|u_1\|_3(\|\omega_3\|_6 \|\Delta u_2\|_2 + \|\omega_2\|_6 \|\Delta u_3\|_2) \\ &\qquad + \|u_2\|_3(\|\omega_1\|_6 \|\Delta u_3\|_2 + \|\omega_3\|_6 \|\Delta u_1\|_2)\big)\, d\tau \\ &\quad \le C(\|u_1\|_{L^\infty(0,T;L^3(\mathbb{R}^3))} + \|u_2\|_{L^\infty(0,T;L^3(\mathbb{R}^3))}) \int_0^t \|\Delta u\|_2^2\, d\tau. \end{aligned} \tag{2.11}$$

By inserting (2.10) and (2.11) into (2.5) we obtain, for every $0 \le t < T$,

$$\begin{aligned} &\|\nabla u(t)\|_2^2 + 2\int_0^t \|\Delta u(\tau)\|_2^2\, d\tau \\ &\quad \le \|\nabla a\|_2^2 + \varepsilon \int_0^t \|\Delta u\|_2^2\, d\tau + C(\varepsilon) \|u_3\|_{L^s(0,t;L^q(\mathbb{R}^3))}^{2(1-3/q)^{-1}} \|\nabla u\|_{L^\infty(0,t;L^2(\mathbb{R}^3))}^{(4/s)(1-3/q)^{-1}} \\ &\qquad + C(\|u_1\|_{L^\infty(0,T;L^3(\mathbb{R}^3))} + \|u_2\|_{L^\infty(0,T;L^3(\mathbb{R}^3))}) \int_0^t \|\Delta u\|_2^2\, d\tau. \end{aligned} \tag{2.12}$$

Thanks to assumption (1.5), we can choose $\eta_0 \in \left(0, \frac{1}{2C}\right)$ such that

$$C(\|u_1\|_{L^\infty(0,T;L^3(\mathbb{R}^3))} + \|u_2\|_{L^\infty(0,T;L^3(\mathbb{R}^3))}) \le C\eta_0 \le \frac{1}{2}. \tag{2.13}$$

Then, by taking $\varepsilon = \frac{1}{2}$ in (2.12) and using also (2.13), we conclude that, for $0 \le t < T$ and $\delta > 0$,

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$$\|\nabla u(t)\|_2^2 + \int_0^t \|\Delta u(\tau)\|_2^2 \, d\tau$$
$$\leq \begin{cases} \|\nabla a\|_2^2 + \delta \|\nabla u\|_{L^\infty(0,t;L^2(\mathbb{R}^3))}^2 + C(\delta)\|u_3\|_{L^s(0,t;L^q(\mathbb{R}^3))}^{2(1-2/s-3/q)^{-1}} & \text{if } \frac{2}{s}+\frac{3}{q}<1, \\ \|\nabla a\|_2^2 + C\|u_3\|_{L^s(0,t;L^q(\mathbb{R}^3))}^s \|\nabla u\|_{L^\infty(0,t;L^2(\mathbb{R}^3))}^2 & \text{if } \frac{2}{s}+\frac{3}{q}=1. \end{cases}$$

The next proof is the same to the one in Case 1. So (2.9) also holds in Case 2. From the above arguments in both two cases, we complete the proof of Lemma 2.2. □

Proof of Theorem 1.2. It is well known that for any $a \in H_\sigma^1(\mathbb{R}^3)$ there is a unique strong solution $\tilde{u}$ of (1.3) with $\tilde{u} \in L^\infty(0, T_0; H^1(\mathbb{R}^3)) \cap L^2(0, T_0; H^2(\mathbb{R}^3))$ for some $T_0 \in (0, T)$. Since u is a weak solution of (1.3) satisfying the energy inequality

$$\|u(\eta)\|_2^2 + 2\int_0^\eta \|\nabla u(\tau)\|_2^2 \, d\tau \leq \|a\|_2^2,$$

for any $0 \leq \eta < T_0$, we can conclude (by means of Serrin's uniqueness theorem) that $u = \tilde{u}$ in $\mathbb{R}^3 \times (0, T_0)$. By the *a priori* estimate in Lemma 2.2 and a continuation argument, we can extend the local strong solution u to the whole interval $(0, T)$. □

3. Decay rates for weak solutions of (1.1) in general domains

Before giving the proof of Theorem 1.3, we introduce some notations and useful lemmas, which can be found in [18], [20].

Set

$$\begin{aligned} C_{0,\sigma}^\infty(\Omega) &= \{u \in C_0^\infty(\Omega) : \operatorname{div} u = 0\}, \\ L_\sigma^2(\Omega) &= \text{the closure of } C_{0,\sigma}^\infty(\Omega) \text{ in } L^2(\Omega), \\ H_{0,\sigma}^1(\Omega) &= \text{the closure of } C_{0,\sigma}^\infty(\Omega) \text{ in } H_0^1(\Omega), \\ \hat{H}_{0,\sigma}^1(\Omega) &= \text{the closure of } C_{0,\sigma}^\infty(\Omega) \text{ in } \hat{H}_0^1(\Omega). \end{aligned}$$

Let $\Omega \subseteq \mathbb{R}^N$ $(N \geq 2)$ be any domain, and $A = -P\Delta : D(A) \to L_\sigma^2(\Omega)$ be the Stokes operator for Ω, where $P : L^2(\Omega) \to L_\sigma^2(\Omega)$ is the Helmholtz projection operator. Then A is positive self-adjoint operator with dense domain $D(A) \subseteq L_\sigma^2(\Omega)$, $C_{0,\sigma}^\infty(\Omega) \subseteq D(A) \subseteq H_{0,\sigma}^1(\Omega)$. Moreover, $N(A) = \{v \in D(A) : Av = 0\} = \{0\}$. Since A is a positive self-adjoint operator, there exists a uniquely determined resolution $\{E_\lambda : \lambda \geq 0\}$ of the identity in $L_\sigma^2(\Omega)$ such that A has the spectral representation

$$A = \int_0^\infty \lambda\, dE_\lambda \quad \text{with domain } D(A) = \left\{ v \in L^2_\sigma(\Omega) : \|Av\|_2^2 = \int_0^\infty \lambda^2 d\|E_\lambda v\|_2^2 < \infty \right\}.$$

More generally, for $0 < \alpha < 1$, we define the positive self-adjoint operator

$$A^\alpha = \int_0^\infty \lambda^\alpha\, dE_\lambda \qquad \text{with domain } D(A^\alpha) = \left\{ v \in L^2_\sigma(\Omega) : \int_0^\infty \lambda^{2\alpha} d\|E_\lambda v\|_2^2 < \infty \right\}.$$

Moreover,

$$N(A^\alpha) = \{v \in D(A^\alpha) : A^\alpha v = 0\} = \{0\}.$$

We define the Yosida approximation of the identity I by

$$J_m \triangleq \left(I + \frac{1}{m} A^{1/2} \right)^{-1} = \int_0^\infty (1 + m^{-1}\lambda)^{-1}\, dE_\lambda, \qquad m = 1, 2, \ldots.$$

Then $J_m v \in D(A^{1/2})$ for all $v \in L^2_\sigma(\Omega)$, and the Yosida approximation operator norm is given by

$$\|J_m\|_{L^2_\sigma \to D(A^{1/2})} = \sup_{\lambda \ge 0} (1 + m^{-1}\lambda)^{-1} \le 1.$$

In addition,

$$A^{1/2} J_m = \int_0^\infty \lambda (1 + m^{-1}\lambda)^{-1}\, dE_\lambda$$

is also a bounded operator with the norm

$$\|A^{1/2} J_m\|_{L^2_\sigma \to L^2_\sigma} = \sup_{\lambda \ge 0} \big(\lambda (1 + m^{-1}\lambda)^{-1} \big) \le m.$$

Moreover, $A^{1/2} J_m v = J_m A^{1/2} v$ for all $v \in D(A^{1/2})$, and

$$\lim_{m\to\infty} \|J_m v - v\|_2 = 0 \qquad \text{for all } v \in L^2_\sigma(\Omega),$$
$$\lim_{m\to\infty} \|A^{1/2} J_m v - A^{1/2} v\|_2 = 0 \qquad \text{for all } v \in D(A^{1/2}).$$

Lemma 3.1. *Set* $\beta_0 = \min\{\alpha, \frac{1}{2}, \frac{1}{4}\}$, $\beta_1 = \min\left\{\alpha, \frac{1}{2}, \frac{1}{4} + \frac{\beta_0}{2}\right\}$, *and*

$$\beta_{n+1} = \min\left\{ \alpha, \frac{1}{2}, \frac{1}{4} + \frac{\beta_n}{2} \right\}, \qquad n = 0, 1, \ldots.$$

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Then we have

(i) $\lim_{n\to\infty} \beta_n = \frac{1}{2}$, *and* $0 < \beta_n < \frac{1}{2}$ *for any* $n = 0, 1, \ldots$ *if* $\alpha \geq \frac{1}{2}$,

(ii) $\lim_{n\to\infty} \beta_n = \alpha$, *and* $0 < \beta_n \leq \alpha$ *for any* $n = 0, 1, \ldots$ *if* $0 < \alpha < \frac{1}{2}$.

Proof. We first prove (i). Obviously, $\beta_0 = \min\{\alpha, \frac{1}{2}, \frac{1}{4}\} = \frac{1}{4} < \frac{1}{2}$ if $\alpha \geq \frac{1}{2}$, and

$$\beta_1 = \min\left\{\alpha, \frac{1}{2}, \frac{1}{4} + \frac{1}{2} \times \frac{1}{4}\right\} = \frac{3}{8} < \frac{1}{2}.$$

Assume $\beta_n < \frac{1}{2}$ for any fixed $n \geq 1$. We conclude

$$\frac{1}{4} + \frac{\beta_n}{2} < \frac{1}{4} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2},$$

and then

$$\beta_{n+1} = \min\left\{\alpha, \frac{1}{2}, \frac{1}{4} + \frac{\beta_n}{2}\right\} = \frac{1}{4} + \frac{\beta_n}{2} < \frac{1}{2}.$$

From the above arguments, we conclude that, for any $n = 0, 1, \ldots,$

$$0 < \beta_n < \frac{1}{2} \qquad \text{and} \qquad \beta_{n+1} = \frac{1}{4} + \frac{\beta_n}{2}.$$

Therefore, after an elementary calculation, we find that

$$\beta_{n+1} = \frac{1}{2} + \left(\beta_0 - \frac{1}{2}\right)\left(\frac{1}{2}\right)^{n+1} = \frac{1}{2} - \frac{1}{4}\left(\frac{1}{2}\right)^{n+1}.$$

Hence we immediately get $\lim_{n\to\infty} \beta_n = \frac{1}{2}$.

The proof of (ii) is a little more complicated. We first show that $\{\beta_n\}$ is an increasing sequence. Note that

$$\beta_1 = \min\left\{\alpha, \frac{1}{4} + \frac{\beta_0}{2}\right\} = \begin{cases} \alpha & \text{if } 0 < \alpha < \frac{1}{4}, \\ \min\{\alpha, \frac{3}{8}\} & \text{if } \frac{1}{4} \leq \alpha < \frac{1}{2}, \end{cases} = \begin{cases} \alpha & \text{if } 0 < \alpha < \frac{3}{8}, \\ \frac{3}{8} & \text{if } \frac{3}{8} \leq \alpha < \frac{1}{2}, \end{cases}$$

$$\beta_2 = \min\left\{\alpha, \frac{1}{4} + \frac{\beta_1}{2}\right\} = \begin{cases} \min\{\alpha, \frac{1}{4} + \frac{\alpha}{2}\} & \text{if } 0 < \alpha < \frac{3}{8}, \\ \min\{\alpha, \frac{1}{4} + \frac{1}{2} \cdot \frac{3}{8}\} & \text{if } \frac{3}{8} \leq \alpha < \frac{1}{2}, \end{cases} = \begin{cases} \alpha & \text{if } 0 < \alpha < \frac{7}{16}, \\ \frac{7}{16} & \text{if } \frac{7}{16} \leq \alpha < \frac{1}{2}. \end{cases}$$

Therefore, $\beta_0 \leq \beta_1 \leq \beta_2$. Assume that $\beta_{n-1} \leq \beta_n$ for any fixed n. Then

$$\beta_n = \min\left\{\alpha, \frac{1}{4} + \frac{\beta_{n-1}}{2}\right\} \leq \min\left\{\alpha, \frac{1}{4} + \frac{\beta_n}{2}\right\} = \beta_{n+1},$$

and this shows that $\{\beta_n\}$ is an increasing sequence. Since $\beta_n \le \alpha$ for every $n = 0, 1, 2, \dots$, we infer that $\lim_{n\to\infty} \beta_n = \beta$ for some $\beta \in (0, \alpha]$.

Passing to the limit in the equality $\beta_n = \min\left\{\alpha, \frac{1}{4} + \frac{\beta_{n-1}}{2}\right\}$, we obtain

$$\beta = \min\left\{\alpha, \frac{1}{4} + \frac{\beta}{2}\right\} = \begin{cases} \alpha & \text{if } 0 < \alpha \le \frac{1}{4} + \frac{\beta}{2}, \\ \frac{1}{4} + \frac{\beta}{2} & \text{if } \frac{1}{4} + \frac{\beta}{2} \le \alpha < \frac{1}{2}. \end{cases}$$

Now in case $\beta = \frac{1}{4} + \frac{\beta}{2}$ it follows that $\beta = \frac{1}{2}$, which contradicts the fact that $\frac{1}{2} > \alpha \ge \frac{1}{4} + \frac{\beta}{2} = \frac{1}{2}$. Thus, $\lim_{n\to\infty} \beta_n = \alpha$ if $0 < \alpha < \frac{1}{2}$. □

Lemma 3.2. *Let $\Omega \subseteq \mathbb{R}^3$ be any domain. Then, for every $u \in H^1_{0,\sigma}(\Omega)$, we have that*

$$\left\|E_\lambda P\big((J_m u)\cdot \nabla u\big)\right\|_2 \le C\lambda^{1/2}\|u\|_2^{1/2}\|\nabla u\|_2^{3/2} \qquad m = 1, 2, \dots, \tag{3.1}$$

where C is independent of m and u.

Proof. For any $\phi \in C^\infty_{0,\sigma}(\Omega)$, we have

$$\begin{aligned}
\left|\big(E_\lambda P\big((J_m u)\cdot\nabla u\big),\big)\right| &= \left|\big(E_\lambda P\,\mathrm{div}\big((J_m u)\otimes u\big),\phi\big)\right| \\
&= \left|\big((J_m u)\otimes u, \nabla E_\lambda \phi\big)\right| \\
&\le \|J_m u\|_4\|u\|_4\|\nabla E_\lambda\phi\|_2 \\
&\le C(\|J_m u\|_2\|u\|_2)^{1/4}(\|\nabla J_m u\|_2\|\nabla u\|_2)^{3/4}\|A^{1/2}E_\lambda\phi\|_2 \\
&\le C\|u\|_2^{1/2}(\|A^{1/2}J_m u\|_2\|A^{1/2}u\|_2)^{3/4}\left(\int_0^\infty \mu d_\mu\|E_\mu E_\lambda\phi\|_2^2\right)^{1/2} \\
&\le C\lambda^{1/2}\|u\|_2^{1/2}(\|J_m A^{1/2}u\|_2\|A^{1/2}u\|_2)^{3/4}\left(\int_0^\lambda d_\mu\|E_\mu\phi\|_2^2\right)^{1/2} \\
&\le C\lambda^{1/2}\|u\|_2^{1/2}\|\nabla u\|_2^{3/2}\|\phi\|_2,
\end{aligned}$$

which yields that (3.1). Here we use the fact that

$$\int_0^\infty \mu d_\mu\|E_\mu E_\lambda\phi\|_2^2 = \int_0^\lambda \mu d_\mu\|E_\mu\phi\|_2^2,$$

and that $\|E_\mu\phi\|_2 \le \|\phi\|_2$ for all $\phi \in C^\infty_{0,\sigma}(\Omega)$; the latter follows by observing that

$$\|E_\mu\phi\|_2^2 = (E_\mu\phi, E_\mu\phi) = (\phi, E_\mu E_\mu\phi) = (\phi, E_\mu\phi) \le \|E_\mu\phi\|_2\|\phi\|_2.$$ □

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Proof of Theorem 1.3. It is well known that (see e.g. [18]) that there exists a strong solution $u = u_m(x,t)$ of the problem

$$\begin{cases} \partial_t u - \Delta u + \big((J_m u)\cdot\nabla\big)u + \nabla p = f & \text{in } \Omega\times(0,\infty),\\ \nabla\cdot u = 0,\ u|_{\partial\Omega} = 0,\ u(0) = J_m a. \end{cases} \tag{3.2}$$

Moreover, $u = u_m(x,t)$ satisfies

$$\begin{aligned} &\big(u(t),\phi(t)\big) + \int_s^t \Big(\big(\nabla u(\tau),\nabla\phi(\tau)\big) + \big(J_m u(\tau)\cdot\nabla u(\tau),\phi(\tau)\big)\Big)\,d\tau \\ &\quad = \big(u(s),\phi(s)\big) + \int_s^t \Big(\big(u(\tau),\partial_\tau\phi(\tau)\big) + \big(f(\tau),\phi(\tau)\big)\Big)\,d\tau, \end{aligned} \tag{3.3}$$

for all $0 \le s \le t < \infty$ and any $\phi \in C\big([0,\infty);H^1_{0,\sigma}(\Omega)\big)\cap C^1\big([0,\infty);L^2(\Omega)\big)$. In addition, the energy equality holds, namely

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u(\tau)\|_2^2\,d\tau = \|u(0)\|_2^2 + 2\int_0^t \big(f(\tau),u(\tau)\big)\,d\tau, \tag{3.4}$$

for all $t \ge 0$. From (3.4) we have

$$\frac{d}{dt}\|u(t)\|_2^2 + 2\|\nabla u(t)\|_2^2 = 2\big(f(t),u(t)\big). \tag{3.5}$$

We observe that, for any $\rho > 0$,

$$\begin{aligned} \|\nabla u(t)\|_2^2 &= \|A^{1/2}u(t)\|_2^2 \\ &= \int_0^\infty \lambda d\|E_\lambda u(t)\|_2^2 \\ &\ge \rho\int_\rho^\infty d\|E_\lambda u(t)\|_2^2 \\ &= \rho\Big(\int_0^\infty d\|E_\lambda u(t)\|_2^2 - \int_0^\rho d\|E_\lambda u(t)\|_2^2\Big) \\ &= \rho\big(\|u(t)\|_2^2 - \|E_\rho u(t)\|_2^2\big). \end{aligned} \tag{3.6}$$

By inserting (3.6) into (3.5), we get

$$\frac{d}{dt}\|u(t)\|_2 + \rho\|u(t)\|_2 \le \rho\|E_\rho u(t)\|_2 + \|f(t)\|_2. \tag{3.7}$$

By taking $\phi(\tau) = e^{-(t-\tau)A}E_\rho\psi$ in (3.3), with $\psi \in C^\infty_{0,\sigma}(\Omega)$, we obtain

$$\big(E_\rho u(t), \psi\big) = \big(E_\rho e^{-tA} u(0), \psi\big) - \int_0^t \big(E_\rho e^{-(t-s)A} PJ_m u(s) \cdot \nabla u(s), \psi\big)\, ds$$
$$+ \int_0^t \big(E_\rho e^{-(t-s)A} Pf(s), \psi\big)\, ds. \tag{3.8}$$

From (3.8) and Lemma 3.2, we infer

$$\|E_\rho u(t)\|_2 \le \|e^{-tA} u(0)\|_2 + C\rho^{1/2} \int_0^t \|f(s)\|_{\hat{V}^*}\, ds$$
$$+ C\rho^{1/2} \int_0^t \|u(s)\|_2^{1/2} \|\nabla u(s)\|_2^{3/2}\, ds. \tag{3.9}$$

By combining (3.7) and (3.9) we deduce that

$$\frac{d}{dt}\|u(t)\|_2 + \rho\|u(t)\|_2 \le \|f(t)\|_2 + C\rho\Big\{\|e^{-tA} u(0)\|_2 + \rho^{1/2} \int_0^t \|f(s)\|_{\hat{V}^*}\, ds$$
$$+ \rho^{1/2} \int_0^t \|u(s)\|_2^{1/2} \|\nabla u(s)\|_2^{3/2}\, ds\Big\}. \tag{3.10}$$

Next we observe that

$$\|e^{-tA} u(0)\|_2 = \|e^{-tA} J_m a\|_2 = \|J_m e^{-tA} a\|_2 \le \|e^{-tA} a\|_2 \le C(t+1)^{-\alpha}.$$

So, by setting $\rho = k(t+1)^{-1}$ with some large positive integer k, and by multiplying both sides of (3.10) by $(t+1)^k$, we obtain

$$\frac{d}{dt}\big((t+1)^k \|u(t)\|_2\big) \le C(t+1)^k \{\|f(t)\|_2 + (t+1)^{-1-\alpha} + (t+1)^{-1-1/2}\}$$
$$+ C(t+1)^{k-3/2} \Big(\int_0^t \|u(s)\|_2^2\, ds\Big)^{1/4}$$
$$\times \Big(\int_0^t \|\nabla u(s)\|_2^2\, ds\Big)^{3/4} \tag{3.11}$$

and then

$$\|u(t)\|_2 \le \|a\|_2 (t+1)^{-k} + C(t+1)^{-1} \int_0^\infty (s+1)\|f(s)\|_2\, ds$$
$$+ C(t+1)^{-k} \int_0^t \big((s+1)^{k-1-\alpha} + (s+1)^{k-3/2} + (s+1)^{k-1-1/4}\big)\, ds$$
$$\le C\big((t+1)^{-\alpha} + (t+1)^{-1/2} + (t+1)^{-1/4}\big). \tag{3.12}$$

Let $\beta_0 = \min\{\alpha, \frac{1}{2}, \frac{1}{4}\}$. Then from (3.12) we have

$$\|u(t)\|_2 \le C_0(t+1)^{-\beta_0}. \tag{3.13}$$

By inserting (3.13) into (3.11), we get

$$\frac{d}{dt}\big((t+1)^k\|u(t)\|_2\big) \le C(t+1)^k\{\|f(t)\|_2 + (t+1)^{-1-\alpha} + (t+1)^{-1-1/2}\} + C(t+1)^{k-5/4-\beta_0/2}.$$

Then,

$$\|u(t)\|_2 \le C\big((t+1)^{-\alpha} + (t+1)^{-1/2} + (t+1)^{-1/4-\beta_0/2}\big). \tag{3.14}$$

Let $\beta_1 = \min\left\{\alpha, \frac{1}{2}, \frac{1}{4} + \frac{\beta_0}{2}\right\}$. Then (3.14) can be rewritten as

$$\|u(t)\|_2 \le C_1(t+1)^{-\beta_1}. \tag{3.15}$$

By inserting (3.15) into (3.11) and after a direct calculation, we get

$$\|u(t)\|_2 \le C_2(t+1)^{-\beta_2},$$

where $\beta_2 = \min\left\{\alpha, \frac{1}{2}, \frac{1}{4} + \frac{\beta_1}{2}\right\}$.

By iterating the above arguments, we obtain for any $n = 0, 1, 2, \dots$

$$\|u(t)\|_2 \le C_n(t+1)^{-\beta_n}, \tag{3.16}$$

where

$$\beta_0 = \min\left\{\alpha, \frac{1}{2}, \frac{1}{4}\right\}, \quad \beta_1 = \min\left\{\alpha, \frac{1}{2}, \frac{1}{4} + \frac{\beta_0}{2}\right\}, \quad \beta_{n+1} = \min\left\{\alpha, \frac{1}{2}, \frac{1}{4} + \frac{\beta_n}{2}\right\},$$

and for $n = 0, 1, \dots$

$$C_{n+1} = A_0 + B_0 C_n^{1/2}(1-2\beta_n)^{-1/4} \quad \text{with } A_0, B_0 > 0 \text{ independent of } n. \tag{3.17}$$

Recall that $u = u_m$ in the above arguments satisfies (3.2) and (3.4). By using the convergence properties on J_m we can easily construct a weak solution u of (1.1) satisfying (1.2). Moreover, from Lemma 3.1 we know that if $0 < \alpha < \frac{1}{2}$ then $\lim_{n\to\infty} \beta_n = \alpha$ and $0 < \beta_n < \frac{1}{2}$ for any $n \ge 1$. So, according to (3.17), the limit $\lim_{n\to\infty} C_n$ is finite. By passing (3.16) to the limit as $n \to \infty$ we infer that (1.6) holds for $0 < \alpha < \frac{1}{2}$. If $\alpha \ge \frac{1}{2}$, then $\lim_{n\to\infty} \beta_n = \frac{1}{2}$ and $0 < \beta_n < \frac{1}{2}$ for every $n \ge 1$. So, for any given $\varepsilon \in \left(0, \frac{1}{2}\right)$ there exists a large number $n_\varepsilon > 0$ such that $\beta_{n_\varepsilon} > \frac{1}{2} - \varepsilon$, and from (3.16) we get that

$$\|u(t)\|_2 \le C_{n_\varepsilon}(t+1)^{-\beta_{n_\varepsilon}} \le C_{n_\varepsilon}(t+1)^{-1/2+\varepsilon};$$

that is, (1.6) holds for $\alpha \ge \frac{1}{2}$. This completes the proof of Theorem 1.3. □

Remark. We cannot take $\varepsilon = 0$ in (1.6). Indeed, since the constant C_n in (3.16) satisfies $C_n > B_0 A_0^{1/2}(1-2\beta_{n-1})^{-1/4}$ and by also taking Lemma 3.1 into account, we immediately infer that $C_n \to \infty$ as $n \to \infty$ in case $\alpha \ge \frac{1}{2}$. So, in case $\alpha \ge \frac{1}{2}$ we cannot insure that $\|u(t)\|_2 \le C(t+1)^{-1/2}$ by means of passing (3.16) to the limit.

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