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Kähler-Sasaki geometry of toric symplectic cones in action-angle coordinates

Miguel Abreu*

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Abstract. In the same way that a contact manifold determines and is determined by a symplectic cone, a Sasaki manifold determines and is determined by a suitable Kähler cone. Kähler-Sasaki geometry is the geometry of these cones.

This paper presents a symplectic action-angle coordinates approach to toric Kähler geometry and how it was recently generalized, by Burns–Guillemin–Lerman and Martelli–Sparks–Yau, to toric Kähler–Sasaki geometry. It also describes, as an application, how this approach can be used to relate a recent new family of Sasaki–Einstein metrics constructed by Gauntlett–Martelli–Sparks–Waldram in 2004, to an old family of extremal Kähler metrics constructed by Calabi in 1982.

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1. Introduction

This paper presents a particular symplectic approach to understand the work of Boyer–Galicki [9], Lerman [21], Gauntlett–Martelli–Sparks–Waldram [17, 18], Burns–Guillemin–Lerman [11] and Martelli–Sparks–Yau [26], regarding the following general geometric set-up:

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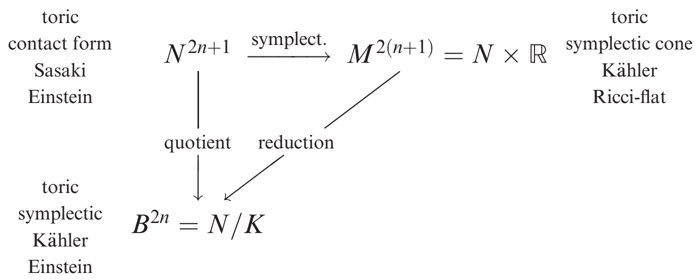

The basic example is given by $B = \mathbb{CP}^n$ with the Fubini-Study metric, $N = S^{2n+1}$ with the round metric and $M = \mathbb{R}^{2(n+1)} \backslash \{0\}$ with the flat Euclidean metric.

Let us start with a few comments on the top row of this diagram. A *contact* manifold determines, via *symplectization*, and is determined, via $\mathbb{R}$-quotient, by a *symplectic cone*. Hence, contact geometry can be thought of as the $\mathbb{R}$-invariant or $\mathbb{R}$-equivariant geometry of symplectic cones. Similarly, a *Sasaki* manifold determines and is determined by a suitable *Kähler cone*. Hence, Sasaki geometry can be thought of as the $\mathbb{R}$-invariant or $\mathbb{R}$-equivariant geometry of these cones and that is what we mean by *Kähler–Sasaki geometry*.

Recall that the symplectization M of a (co-oriented) contact manifold N is diffeomorphic to $N \times \mathbb{R}$, but not in a canonical way. The choice of a contact form on N gives rise to a choice of such a splitting diffeomorphism. Since any Sasaki manifold comes equipped with a contact form, any Kähler–Sasaki cone comes equipped with a splitting diffeomorphism.

In our symplectic approach, a suitable Kähler cone is a symplectic cone equipped with what we will call a *Sasaki complex structure*, i.e., a suitable compatible complex structure. Such a cone will be called a *Kähler–Sasaki cone* and the corresponding Kähler metric will be called a *Kähler–Sasaki metric*.

When a Kähler–Sasaki metric is *Ricci-flat*, the associated Sasaki metric is *Einstein* with positive scalar curvature. There is a lot of interest on *Sasaki–Einstein metrics* due to their possible relation with superconformal field theory via the conjectural AdS/CFT correspondence. For example, the above mentioned work of Gauntlett–Martelli–Sparks–Waldram, a group of mathematical physicists, is motivated by this.

Regarding the left column of the above diagram, recall that a choice of a *contact form* on a contact manifold N gives rise to a *Reeb vector field* K. Denote also by K the contact $\mathbb{R}$-action given by its flow. The *quotient* $B := N/K$, when suitably defined, is a symplectic singular space. When N^{2n+1} is Sasaki (resp. Sasaki–Einstein with scalar curvature $= n(2n+1)$), the Reeb vector field K generates an isometric flow and the quotient B^{2n} is Kähler (resp. *Kähler–Einstein* with scalar curvature $= 2n(n+1)$).

As Boyer–Galicki point out in the Preface of their recent book [10], Sasaki geometry of N is then naturally "sandwiched" between two Kähler geometries:

(i) the Kähler geometry of the associated symplectic cone M;

(ii) the Kähler geometry of the base symplectic quotient B.

As it turns out, there is a direct symplectic/Kähler way to go from (i) to (ii): *symplectic/Kähler reduction*. That is why the symplectic approach of this paper will mostly forget N and use only the diagonal part of the above diagram, i.e., M, B and the reduction arrow between the two.

The word *toric* implies that M and B admit a combinatorial characterization via the images of the moment maps for the corresponding torus actions:

(i) a *polyhedral cone* $C \subset \mathbb{R}^{n+1}$ for the toric symplectic cone $M^{2(n+1)}$;

(ii) a *convex polytope* $P \subset \mathbb{R}^n$ for the toric symplectic space B^{2n}.

The *symplectic reduction* relation between M and B corresponds to C being a cone over P.

The word toric also implies that, in suitable symplectic *action-angle coordinates*, the relevant compatible complex structures on M and B can be described via *symplectic potentials*, i.e., appropriate real functions on C and P. It follows from a theorem of Calderbank–David–Gauduchon [13] that the *Kähler reduction* relation between M and B gives rise to a direct explicit relation between the corresponding symplectic potentials on C and P. As an application, we can use this to show that a particular family of Kähler–Einstein spaces, contained in a more general family of local $U(n)$-invariant extremal Kähler metrics constructed by Calabi in 1982 [12], gives rise to Ricci-flat Kähler–Sasaki metrics on certain toric symplectic cones.

More precisely, let n, m and k be integers such that

$$n \geq 2, \qquad k \geq 1 \qquad \text{and} \qquad 0 \leq m < kn.$$

Consider the cone $C(k,m) \subset \mathbb{R}^{n+1}$ with $n+2$ facets defined by the following normals:

$$\begin{aligned} v_i &= (\vec{e}_i, 1), \qquad i = 1, \dots, n-1, \\ v_n &= \big((m+1)\vec{e}_n - \vec{d}, 1\big), \\ v_- &= (k\vec{e}_n, 1), \\ v_+ &= (-\vec{e}_n, 1), \end{aligned}$$

where $\vec{e}_i \in \mathbb{R}^n$, $i = 1, \dots, n$, are the canonical basis vectors and $\vec{d} = \sum_{i=1}^{n} \vec{e}_i \in \mathbb{R}^n$. Each of these cones $C(k,m) \subset \mathbb{R}^{n+1}$ is good in the sense of Definition 3.9, hence defines a toric symplectic cone $M^{2(n+1)}_{k,m}$. Because their defining normals lie on a fixed hyperplane in $\mathbb{R}^{n+1}$, the first Chern class of all these symplectic cones is zero.

Theorem 1.1. *When*

$$\frac{(k-1)n}{2} < m < kn \tag{1}$$

the toric symplectic cone $M^{2(n+1)}_{k,m}$ *has a Ricci-flat Kähler–Sasaki metric. The corresponding reduced toric Kähler–Einstein space belongs to Calabi's family.*

Let $N^{2n+1}_{k,m}$ denote the corresponding toric Sasaki–Einstein manifold. Using a result of Lerman [24], one can easily check that $N^{2n+1}_{k,m}$ is simply connected iff

$$\gcd(m+n, k+1) = 1. \tag{2}$$

When $n = 2$ one can determine an explicit relation between $N^5_{k,m}$ and the simply connected toric Sasaki–Einstein 5-manifolds $Y^{p,q}$, $0 < q < p$, $\gcd(q,p) = 1$, constructed by Gauntlett–Martelli–Sparks–Waldram [17]. In fact, as we will see, the associated 3-dimensional moment cones are $\mathrm{SL}(3,\mathbb{Z})$ equivalent iff $k = p-1$ and $m = p+q-2$. Note that in this case

$$\frac{(k-1)n}{2} < m < kn \iff 0 < q < p$$

and

$$\gcd(m+n, k+1) = 1 \iff \gcd(q,p) = 1.$$

Since

$$Y^{p,q} \cong S^2 \times S^3 \text{ for all } 0 < q < p \text{ such that } \gcd(q,p) = 1,$$

we conclude that

$$N^5_{k,m} \cong S^2 \times S^3 \text{ for all } k,m \in \mathbb{N} \text{ satisfying (1) and (2) (with } n = 2).$$

Gauntlett–Martelli–Sparks–Waldram construct in [18] higher dimensional generalizations of the manifolds $Y^{p,q}$. They do not describe their exact diffeomorphism type and they do not write down the associated moment cones. The later should be $\mathrm{SL}(n+1,\mathbb{Z})$ equivalent to the cones $C(k,m) \subset \mathbb{R}^{n+1}$, with $k,m \in \mathbb{N}$ satisfying (1) and (2), while the former should be diffeomorphic to the corresponding $N^{2n+1}_{k,m} \subset M^{2(n+1)}_{k,m}$. The cones $C(k,m) \subset \mathbb{R}^{n+1}$ can be used to determine the diffeomorphism type of these manifolds. The following theorem is a particular example of that.

Theorem 1.2. *Given $n \geq 2$ and $m \in \mathbb{N}$, consider the (toric) complex manifold of real dimension $2n$ given by*

$$H_m^{2n} := \mathbb{P}(\mathcal{O}(-m) \oplus \mathbb{C}) \to \mathbb{CP}^{n-1}.$$

When $k = 1$ and $0 < m < n$, the toric symplectic cone $M_{1,m}^{2(n+1)}$ is diffeomorphic to the total space of the anti-canonical line bundle of H_m^{2n} minus its zero section, while the toric contact manifold $N_{1,m}^{2n+1}$ is diffeomorphic to the total space of the corresponding circle bundle.

Remark 1.3. Theorems 1.1 and 1.2 give rise to two natural sub-actions of the torus action on the toric contact manifold $N_{1,m}^{2n+1}$:

(i) the $\mathbb{R}$-action given by the flow of the Reeb vector field K, determined by the contact form associated with the Sasaki–Einstein metric given by Theorem 1.1;

(ii) the S^1-action coming from the identification between $N_{1,m}^{2n+1}$ and an S^1-bundle over H_m^{2n}.

Although in other more *regular* examples, like the basic one given by an odd-dimensional round sphere, the analogues of these two actions coincide, they cannot coincide in the present situation. If that were the case, we would have that H_m^{2n} could be identified with $N_{1,m}^{2n+1}/K$ and would then admit a Kähler–Einstein metric. That is well known to be false. In fact, the complex manifolds H_m^{2n} are used by Calabi [12] as examples that do not admit any Kähler–Einstein metric but do admit explicit extremal Kähler metrics.

As we will see, the quotient $N_{1,m}^{2n+1}/K$ can be identified via its moment polytope as a toric symplectic quasifold, in the sense of Prato [27].

The paper is organized as follows. In Section 2 we give some background on symplectic toric orbifolds and recall the definition and properties of symplectic potentials for toric compatible complex structures. Section 3 is devoted to symplectic cones, their relation with co-oriented contact manifolds and the classification of toric symplectic cones via their moment polyhedral cones. The definition and basic properties of (toric) Kähler–Sasaki cones is the subject of Section 4, which includes a brief description of their relation with (toric) Sasaki manifolds. Cone action-angle coordinates and symplectic potentials are introduced in Section 5, where we also discuss the behaviour of symplectic potentials and toric Kähler–Sasaki metrics under symplectic reduction. Section 6 contains the proofs of Theorems 1.1 and 1.2.

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2. Toric Kähler orbifolds

In this section, after some preliminary background on symplectic toric orbifolds, we recall the definition and some properties of symplectic potentials for compatible toric complex structures in action-angle coordinates, including a formula for the scalar curvature of the corresponding toric Kähler metric. For details see [3, 2].

2.1. Preliminaries on toric symplectic orbifolds.

Definition 2.1. A *toric symplectic orbifold* is a connected $2n$-dimensional symplectic orbifold (B, ω) equipped with an effective Hamiltonian action $\tau : \mathbb{T}^n \to \mathrm{Diff}(B, \omega)$ of the standard (real) n-torus $\mathbb{T}^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n$. The corresponding *moment map*, well defined up to addition by a constant, will be denoted by $\mu : B \to \mathfrak{t}^* \cong \mathbb{R}^n$.

When B is a compact smooth manifold, the Atiyah-Guillemin-Sternberg convexity theorem states that the image $P = \mu(B) \subset \mathbb{R}^n$ of the moment map μ is the convex hull of the image of the points in B fixed by $\mathbb{T}^n$, i.e., a convex polytope in $\mathbb{R}^n$. A theorem of Delzant [14] then says that the convex polytope $P \subset \mathbb{R}^n$ completely determines the toric symplectic manifold, up to equivariant symplectomorphisms.

In [25] Lerman and Tolman generalize these two theorems to orbifolds. While the convexity theorem generalizes word for word, one needs more information than just the convex polytope P to generalize Delzant's classification theorem.

Definition 2.2. A convex polytope P in $\mathbb{R}^n$ is called *simple* and *rational* if:

(1) there are n edges meeting at each vertex p;

(2) the edges meeting at the vertex p are rational, i.e., each edge is of the form $p + tv_i$, $0 \le t \le \infty$, where $v_i \in \mathbb{Z}^n$;

(3) the $v_1, \dots, v_n$ in (2) can be chosen to be a $\mathbb{Q}$-basis of the lattice $\mathbb{Z}^n$.

A *facet* is a face of P of codimension one. Following Lerman–Tolman, we will say that a *labeled polytope* is a rational simple convex polytope $P \subset \mathbb{R}^n$, plus a positive integer (*label*) attached to each of its facets.

Two labeled polytopes are *isomorphic* if one can be mapped to the other by a translation, and the corresponding facets have the same integer labels.

Remark 2.3. In Delzant's classification theorem for compact symplectic toric manifolds, there are no labels (or equivalently, all labels are equal to 1) and the polytopes that arise are slightly more restrictive: the "$\mathbb{Q}$" in (3) is replaced by "$\mathbb{Z}$". These are called *Delzant polytopes*.

Remark 2.4. Each facet F of a rational simple convex polytope $P \subset \mathbb{R}^n$ determines a unique lattice vector $v_F \in \mathbb{Z}^n \subset \mathbb{R}^n$: the primitive inward pointing normal lattice vector. A convenient way of thinking about a positive integer label $m_F \in \mathbb{N}$ associated to F is by dropping the primitive requirement from this lattice vector: consider $m_F v_F$ instead of v_F.

In other words, a labeled polytope can be defined as a rational simple polytope $P \subset \mathbb{R}^n$ with an inward pointing normal lattice vector associated to each of its facets. When dealing with the effect of affine transformations on labeled polytopes it will also be useful to allow more general inward pointing normal vectors (see the end of this section).

Theorem 2.5 (Lerman–Tolman). *Let (B, ω, τ) be a compact toric symplectic orbifold, with moment map $\mu : B \to \mathbb{R}^n$. Then $P \equiv \mu(B)$ is a rational simple convex polytope. For every facet F of P, there exists a positive integer m_F, the label of F, such that the structure group of every $p \in \mu^{-1}(\breve{F})$ is $\mathbb{Z}/m_F\mathbb{Z}$ (here $\breve{F}$ is the relative interior of F).*

Two compact toric symplectic orbifolds are equivariant symplectomorphic (with respect to a fixed torus acting on both) if and only if their associated labeled polytopes are isomorphic. Moreover, every labeled polytope arises from some compact toric symplectic orbifold.

Recall that a *Kähler orbifold* can be defined as a symplectic orbifold (B, ω) equipped with a *compatible* complex structure $J \in \mathscr{I}(B, \omega)$, i.e., a complex structure on B such that the bilinear form

$$g_J(\cdot\,,\cdot) := \omega(\cdot, J\cdot)$$

defines a *Riemannian metric*. The proof of Theorem 2.5, in both manifold and orbifold cases, gives an explicit construction of a canonical model for each toric symplectic orbifold, i.e., it associates to each labeled polytope P an explicit toric symplectic orbifold (B_P, ω_P, τ_P) with moment map $\mu_P : B_P \to P$. Moreover, this explicit construction consists of a certain symplectic reduction of the standard $\mathbb{C}^d$, for $d =$ number of facets of P, to which one can apply the Kähler reduction theorem of Guillemin and Sternberg [20]. Hence, the standard complex structure on $\mathbb{C}^d$ induces a canonical $\mathbb{T}^n$-invariant complex structure J_P on B_P, compatible with ω_P. In other words, each toric symplectic orbifold is Kähler and to each labeled polytope $P \subset \mathbb{R}^n$ one can associate a canonical toric Kähler orbifold $(B_P, \omega_P, J_P, \tau_P)$ with moment map $\mu_P : B_P \to P$.

2.2. Symplectic potentials for toric compatible complex structures. Toric compatible complex structures, and corresponding Kähler metrics, can be described using the following symplectic set up.

Let $\breve{P}$ denote the interior of P, and consider $\breve{B}_P \subset B_P$ defined by $\breve{B}_P = \mu_P^{-1}(\breve{P})$. One can easily check that $\breve{B}_P$ is a smooth open dense subset of B_P, consisting of all the points where the $\mathbb{T}^n$-action is free. It can be described as

$$\breve{B}_P \cong \breve{P} \times \mathbb{T}^n = \{(x, y) \,|\, x \in \breve{P} \subset \mathbb{R}^n, \; y \in \mathbb{R}^n/2\pi\mathbb{Z}^n\},$$

where (x, y) are symplectic or *action-angle* coordinates for ω_P, i.e.,

$$\omega_P = dx \wedge dy = \sum_{j=1}^{n} dx_j \wedge dy_j.$$

If J is any ω_P-compatible toric complex structure on B_P, the symplectic (x, y)-coordinates on $\breve{B}_P$ can be chosen so that the matrix that represents J in these coordinates has the form

$$\begin{bmatrix} 0 & \vdots & -S^{-1} \\ \cdots & \cdots & \cdots \\ S & \vdots & 0 \end{bmatrix}$$

where $S = S(x) = [s_{jk}(x)]_{j,k=1}^{n,n}$ is a symmetric and positive-definite real matrix. A simple computation shows that the vanishing of the Nijenhuis tensor, i.e., the integrability condition for the complex structure J, is equivalent to S being the Hessian of a smooth function $s \in C^\infty(\breve{P})$, i.e.,

$$S = \mathrm{Hess}_x(s), \qquad s_{jk}(x) = \frac{\partial^2 s}{\partial x_j \partial x_k}(x), \qquad 1 \le j, k \le n.$$

Holomorphic coordinates for J are given in this case by

$$z(x, y) = u(x, y) + iv(x, y) = \frac{\partial s}{\partial x}(x) + iy.$$

We will call s the *symplectic potential* of the compatible toric complex structure J. Note that the Kähler metric $g_J(\cdot\,,\cdot) = \omega_P(\cdot\,, J\cdot)$ is given in these (x, y)-coordinates by the matrix

$$\begin{bmatrix} S & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & S^{-1} \end{bmatrix}. \tag{3}$$

Remark 2.6. A beautiful proof of this local normal form for toric compatible complex structures is given by Donaldson in [16] (see also [4]). It illustrates a small part of his formal general framework for the action of the symplectomorphism group of a symplectic manifold on its space of compatible complex structures (cf. [15]).

We will now characterize the symplectic potentials that correspond to toric compatible complex structures on a toric symplectic orbifold (B_P, ω_P, τ_P). Every convex rational simple polytope $P \subset \mathbb{R}^n$ can be described by a set of inequalities of the form

$$\langle x, v_r \rangle + \rho_r \geq 0, \qquad r = 1, \dots, d,$$

where d is the number of facets of P, each v_r is a primitive element of the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ (the inward-pointing normal to the r-th facet of P), and each ρ_r is a real number. Following Remark 2.4, the labels $m_r \in \mathbb{N}$ attached to the facets can be incorporated in the description of P by considering the affine functions $\ell_r : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\ell_r(x) = \langle x, m_r v_r \rangle + \lambda_r,$$

where $\lambda_r = m_r \rho_r$ and $r = 1, \dots, d$. Then x belongs to the r-th facet of P iff $\ell_r(x) = 0$, and $x \in \breve{P}$ iff $\ell_r(x) > 0$ for all $r = 1, \dots, d$.

The following two theorems are proved in [2]. The first is a straightforward generalization to toric orbifolds of a result of Guillemin [19].

Theorem 2.7. *Let (B_P, ω_P, τ_P) be the symplectic toric orbifold associated to a labeled polytope $P \subset \mathbb{R}^n$. Then, in suitable action-angle (x, y)-coordinates on $\breve{B}_P \cong \breve{P} \times \mathbb{T}^n$, the symplectic potential $s_P \in C^\infty(\breve{P})$ of the canonical compatible toric complex structure J_P is given by*

$$s_P(x) = \frac{1}{2} \sum_{r=1}^{d} \ell_r(x) \log \ell_r(x).$$

The second theorem provides the symplectic version of the $\partial\bar{\partial}$-lemma in this toric orbifold context.

Theorem 2.8. *Let J be any compatible toric complex structure on the symplectic toric orbifold (B_P, ω_P, τ_P). Then, in suitable action-angle (x, y)-coordinates on $\breve{B}_P \cong \breve{P} \times \mathbb{T}^n$, J is given by a symplectic potential $s \in C^\infty(\breve{P})$ of the form*

$$s(x) = s_P(x) + h(x),$$

where s_P is given by Theorem 2.7, h is smooth on the whole P, and the matrix $S = \mathrm{Hess}(s)$ is positive definite on $\breve{P}$ and has determinant of the form

$$\mathrm{Det}(S) = \left(\delta \prod_{r=1}^{d} \ell_r\right)^{-1},$$

with δ being a smooth and strictly positive function on the whole P.

Conversely, any such potential s determines a complex structure on $\breve{B}_P \cong \breve{P} \times \mathbb{T}^n$, that extends uniquely to a well defined compatible toric complex structure J on the toric symplectic orbifold (B_P, ω_P, τ_P).

2.3. Scalar curvature. We now recall from [1] a particular formula for the scalar curvature in action-angle (x, y)-coordinates. A Kähler metric of the form (3) has scalar curvature Sc given by

$$Sc = -\sum_{j,k} \frac{\partial}{\partial x_j}\left(s^{jk} \frac{\partial \log \mathrm{Det}(S)}{\partial x_k}\right),$$

which after some algebraic manipulations becomes the more compact

$$Sc = -\sum_{j,k} \frac{\partial^2 s^{jk}}{\partial x_j \partial x_k}, \tag{4}$$

where the s^{jk}, $1 \le j, k \le n$, are the entries of the inverse of the matrix $S = \mathrm{Hess}_x(s)$, $s \equiv$ symplectic potential (Donaldson gives in [16] an appropriate interpretation of this formula, by viewing the scalar curvature as the moment map for the action of the symplectomorphism group on the space of compatible complex structures).

2.4. Symplectic potentials and affine transformations. The labeled polytope $P \subset \mathbb{R}^n$ of a symplectic toric orbifold is only well defined up to translations, since the moment map is only well defined up to addition of constants. Moreover, the twisting of the action by an automorphism of the torus $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$ corresponds to an $\mathrm{SL}(n,\mathbb{Z})$ transformation of the polytope. Since these operations have no effect on a toric Kähler metric, symplectic potentials should have a natural transformation property under these affine maps. While the effect of translations is trivial to analyse, the effect of $\mathrm{SL}(n,\mathbb{Z})$ transformations is more interesting. In fact: *symplectic potentials transform quite naturally under any* $\mathrm{GL}(n,\mathbb{R})$ *linear transformation*.

Let $T \in \mathrm{GL}(n,\mathbb{R})$ and consider the linear symplectic change of action-angle coordinates

$$x := T^{-1}x' \quad \text{and} \quad y := T^t y'.$$

Then

$$P' = \bigcap_{a=1}^{d} \{x' \in \mathbb{R}^n \mid \ell'_a(x') := \langle x', v'_a\rangle + \lambda'_a \geq 0\}$$

becomes

$$P := T^{-1}(P') = \bigcap_{a=1}^{d} \{x \in \mathbb{R}^n \mid \ell_a(x) := \langle x, v_a\rangle + \lambda_a \geq 0\}$$

with

$$v_a = T^t v'_a \quad \text{and} \quad \lambda_a = \lambda'_a,$$

and symplectic potentials transform by

$$s = s' \circ T \quad (\text{in particular, } s_P = s_{P'} \circ T).$$

The corresponding Hessians are related by

$$S = T^t(S' \circ T)T$$

and

$$Sc = Sc' \circ T.$$

For the purposes of this paper, the point of this discussion is the following. Let $P \subset \mathbb{R}^n$ be a labeled polytope and $P' = T(P) \subset \mathbb{R}^n$ for some arbitrary $T \in \mathrm{GL}(n,\mathbb{R})$. Supose that

$$s' : \breve{P}' \to \mathbb{R}$$

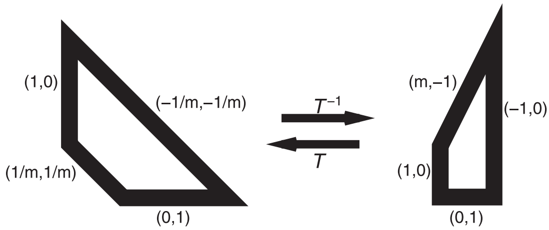

Figure 1. Hirzebruch surfaces.

is of the form specified in Theorem 2.8 (with $s_{P'} = s_P \circ T^{-1}$). Then

$$s := s' \circ T : \breve{P} \to \mathbb{R}$$

also has the form specified in Theorem 2.8 and, consequently, is the symplectic potential of a well defined toric compatible complex structure on the toric symplectic orbifold (B_P, ω_P). Moreover, since $Sc = Sc' \circ T$, we have that

$$Sc' = \text{constant} \iff Sc = \text{constant}.$$

Example 2.9. Figure 1 illustrates two equivalent descriptions of a toric symplectic rational ruled 4-manifold or, equivalently, of a Hirzebruch surface

$$H_m^2 := \mathbb{P}\big(\mathcal{O}(-m) \oplus \mathbb{C}\big) \to \mathbb{CP}^1, \qquad m \in \mathbb{N}.$$

The linear map $T \in \mathrm{GL}(2, \mathbb{R})$ relating the two is given by

$$T = \begin{bmatrix} m & -1 \\ 0 & 1 \end{bmatrix}$$

The inward pointing normal that should be considered for each facet is specified. The right polytope is a standard Delzant polytope for the Hirzebruch surface H_m^2. The left polytope is very useful for the constructions of Section 6 and was implicitly used by Calabi in [12].

3. Toric symplectic cones

In this section, after defining symplectic cones and briefly reviewing their direct relation with co-oriented contact manifolds, we consider toric symplectic cones and their classification via good moment cones.

Definition 3.1. A *symplectic cone* is a triple (M, ω, X), where (M, ω) is a connected symplectic manifold, i.e., $\omega \in \Omega^2(M)$ is a closed and non-degenerate 2-form, and $X \in \mathscr{X}(M)$ is a vector field generating a proper $\mathbb{R}$-action $\rho_t : M \to M$, $t \in \mathbb{R}$, such that $\rho_t^*(\omega) = e^{2t}\omega$. Note that the *Liouville vector field* X satisfies $\mathscr{L}_X\omega = 2\omega$, or equivalently

$$\omega = \frac{1}{2} d\big(\imath(X)\omega\big).$$

A *compact* symplectic cone is a symplectic cone (M, ω, X) for which the quotient $M/\mathbb{R}$ is compact.

Definition 3.2. A *co-orientable contact manifold* is a pair (N, ξ), where N is a connected odd dimensional manifold and $\xi \subset TN$ is a maximally non-integrable hyperplane distribution globally defined by some *contact form* $\alpha \in \Omega^1(N)$, i.e.,

$$\xi = \ker \alpha \quad \text{and} \quad d\alpha|_\xi \text{ is non-degenerate.}$$

A *co-oriented* contact manifold is a triple $(N, \xi, [\alpha])$, where (N, ξ) is a co-orientable contact manifold and $[\alpha]$ is the conformal class of some contact form α, i.e.,

$$[\alpha] = \{e^h\alpha \,|\, h \in C^\infty(N)\}.$$

Given a co-oriented contact manifold $(N, \xi, [\alpha])$, with contact form α, let

$$M := N \times \mathbb{R}, \qquad \omega := d(e^t\alpha) \qquad \text{and} \qquad X := 2\frac{\partial}{\partial t},$$

where t is the $\mathbb{R}$ coordinate. Then (M, ω, X) is a symplectic cone, usually called the *symplectization* of $(N, \xi, [\alpha])$.

Conversely, given a symplectic cone (M, ω, X) let

$$N := M/\mathbb{R}, \qquad \xi := \pi_*\big(\ker\big(\imath(X)\omega\big)\big) \qquad \text{and} \qquad \alpha := s^*\big(\imath(X)\omega\big),$$

where $\pi : M \to N$ is the natural principal $\mathbb{R}$-bundle quotient projection and $s : N \to M$ is any global section (note that such global sections always exist, since any principal $\mathbb{R}$-bundle is trivial). Then $(N, \xi, [\alpha])$ is a co-oriented contact manifold whose symplectization is the symplectic cone (M, ω, X).

In fact, we have that

$$\text{co-oriented contact manifolds} \overset{1:1}{\longleftrightarrow} \text{symplectic cones}$$

(see Chapter 2 of [22] for details). Under this bijection, compact toric contact manifolds, Sasaki manifolds and Sasaki–Einstein metrics correspond respectively to the toric symplectic cones, Kähler–Sasaki cones and Ricci-flat Kähler–Sasaki metrics that are the subject of this paper.

Example 3.3. The most basic example of a symplectic cone is $\mathbb{R}^{2(n+1)}\backslash\{0\}$ with linear coordinates

$$(u_1, \dots, u_{n+1}, v_1, \dots, v_{n+1}),$$

symplectic form

$$\omega_{\mathrm{st}} = du \wedge dv := \sum_{j=1}^{n+1} du_j \wedge dv_j$$

and Liouville vector field

$$X_{\mathrm{st}} = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} := \sum_{j=1}^{n+1}\left(u_j\frac{\partial}{\partial u_j} + v_j\frac{\partial}{\partial v_j}\right).$$

The associated co-oriented contact manifold is isomorphic to $(S^{2n+1}, \xi_{\mathrm{st}})$, where $S^{2n+1} \subset \mathbb{C}^{n+1}$ is the unit sphere and ξ_{st} is the hyperplane distribution of complex tangencies, i.e.,

$$\xi_{\mathrm{st}} = TS^{2n+1} \cap iTS^{2n+1}.$$

Example 3.4. Let Q be a manifold and denote by M the cotangent bundle of Q with the zero section deleted: $M := T^*Q\backslash 0$. We have that M is a symplectic cone since the proper $\mathbb{R}$-action $\rho_t : M \to M$, given by $\rho_t(q, p) = (q, e^{2t}p)$, expands the canonical symplectic form exponentially. The associated co-oriented contact manifold is the co-sphere bundle S^*Q.

Example 3.5. Let (B, ω) be a symplectic manifold such that the cohomology class $\frac{1}{2\pi}[\omega] \in H^2(B, \mathbb{R})$ is integral, i.e., in the image of the natural map $H^2(B, \mathbb{Z}) \to H^2(B, \mathbb{R})$. Suppose that $H^2(B, \mathbb{Z})$ has no torsion, so that the above natural map is injective and we can consider $H^2(B, \mathbb{Z}) \subset H^2(B, \mathbb{R})$. Denote by $\pi : N \to B$ the principle circle bundle with first Chern class

$$c_1(N) = \frac{1}{2\pi}[\omega].$$

A theorem of Boothby and Wang [8] asserts that there is a connection 1-form α on N with $d\alpha = \pi^*\omega$ and, consequently, α is a contact form. We will call $\big(N, \xi := \ker(\alpha)\big)$ the *Boothby–Wang* manifold of (B, ω). The associated symplectic cone is the total space of the corresponding line bundle $L \to B$ with the zero section deleted.

When $B = \mathbb{CP}^n$, with its standard Fubini–Study symplectic form, we recover Example 3.3, i.e., $(N, \xi) \cong (S^{2n+1}, \xi_{\mathrm{st}})$.

Definition 3.6. A *toric symplectic cone* is a symplectic cone (M, ω, X) of dimension $2(n+1)$ equipped with an effective X-preserving symplectic $\mathbb{T}^{n+1}$-action, with moment map $\mu : M \to \mathfrak{t}^* \cong \mathbb{R}^{n+1}$ such that $\mu(\rho_t(m)) = e^{2t}\rho_t(m)$ for all $m \in M$, $t \in \mathbb{R}$. Its *moment cone* is defined to be the set

$$C := \mu(M) \cup \{0\} \subset \mathbb{R}^{n+1}.$$

Remark 3.7. On a symplectic cone (M, ω, X), any X-preserving symplectic group action is Hamiltonian.

Example 3.8. Consider the usual identification $\mathbb{R}^{2(n+1)} \cong \mathbb{C}^{n+1}$ given by

$$z_j = u_j + iv_j, \qquad j = 1, \ldots, n+1,$$

and the standard $\mathbb{T}^{n+1}$-action defined by

$$(y_1, \ldots, y_{n+1}) \cdot (z_1, \ldots, z_{n+1}) = (e^{-iy_1} z_1, \ldots, e^{-iy_{n+1}} z_{n+1}).$$

The symplectic cone $(\mathbb{R}^{2(n+1)} \backslash \{0\}, \omega_{\mathrm{st}}, X_{\mathrm{st}})$ of Example 3.3 equipped with this $\mathbb{T}^{n+1}$-action is a toric symplectic cone. The moment map $\mu_{\mathrm{st}} : \mathbb{R}^{2(n+1)} \backslash \{0\} \to \mathbb{R}^{n+1}$ is given by

$$\mu_{\mathrm{st}}(u_1, \ldots, u_{n+1}, v_1, \ldots, v_{n+1}) = \frac{1}{2}(u_1^2 + v_1^2, \ldots, u_{n+1}^2 + v_{n+1}^2),$$

and the moment cone is $C = (\mathbb{R}_0^+)^{n+1} \subset \mathbb{R}^{n+1}$.

In [21] Lerman completed the classification of compact toric symplectic cones, initiated by Banyaga and Molino [6], [7], [5] and continued by Boyer and Galicki [9]. The ones that are relevant for toric Kähler–Sasaki geometry are characterized by having good moment cones.

Definition 3.9 (Lerman). A cone $C \subset \mathbb{R}^{n+1}$ is *good* if there exists a minimal set of primitive vectors $v_1, \ldots, v_d \in \mathbb{Z}^{n+1}$, with $d \geq n+1$, such that

(i) $C = \bigcap_{a=1}^{d} \{x \in \mathbb{R}^{n+1} \mid \ell_a(x) := \langle x, v_a \rangle \geq 0\}$.

(ii) any codimension-k face F of C, $1 \leq k \leq n$, is the intersection of exactly k facets whose set of normals can be completed to an integral base of $\mathbb{Z}^{n+1}$.

Theorem 3.10 (Banyaga–Molino, Boyer–Galicki, Lerman). *For each good cone $C \subset \mathbb{R}^{n+1}$ there exists a unique compact toric symplectic cone $(M_C, \omega_C, X_C, \mu_C)$ with moment cone C.*

Remark 3.11. The compact toric symplectic cones characterized by this theorem will be called *good* toric symplectic cones. Like for compact toric symplectic manifolds, the existence part of the theorem follows from an explicit symplectic reduction construction starting from a symplectic vector space (see [21]).

Example 3.12. Let $P \subset \mathbb{R}^n$ be an *integral Delzant polytope*, i.e., a Delzant polytope with integral vertices or, equivalently, the moment polytope of a compact toric symplectic manifold (B_P, ω_P, μ_P) such that $\frac{1}{2\pi}[\omega] \in H^2(B_P, \mathbb{Z})$. Then its *standard cone*

$$C := \{z(x,1) \in \mathbb{R}^n \times \mathbb{R} \mid x \in P, z \geq 0\} \subset \mathbb{R}^{n+1} \tag{5}$$

is a good cone. Moreover

(i) the toric symplectic manifold (B_P, ω_P, μ_P) is the $S^1 \cong \{\mathbf{1}\} \times S^1 \subset \mathbb{T}^{n+1}$ *symplectic reduction* of the toric symplectic cone $(M_C, \omega_C, X_C, \mu_C)$ (at level one).

(ii) $\left(N_C := \mu_C^{-1}(\mathbb{R}^n \times \{1\}), \alpha_C := \left(\imath(X_C)\omega_C\right)|_{N_C}\right)$ is the *Boothby–Wang* manifold of (B_P, ω_P). The restricted $\mathbb{T}^{n+1}$-action makes it a *toric contact manifold.*

(iii) (M_C, ω_C, X_C) is the *symplectization* of (N_C, α_C).

See Lemma 3.7 in [23] for a proof of these facts.

If $P \subset \mathbb{R}^n$ is the standard simplex, i.e., $B_P = \mathbb{CP}^n$, then its standard cone $C \subset \mathbb{R}^{n+1}$ is the moment cone of $(M_C = \mathbb{C}^{n+1} \backslash \{0\}, \omega_{\mathrm{st}}, X_{\mathrm{st}})$ equipped with the $\mathbb{T}^{n+1}$-action given by

$$\begin{aligned}&(y_1, \ldots, y_n, y_{n+1}) \cdot (z_1, \ldots, z_n, z_{n+1}) \\ &\quad = (e^{-i(y_1+y_{n+1})} z_1, \ldots, e^{-i(y_n+y_{n+1})} z_n, e^{-iy_{n+1}} z_{n+1}).\end{aligned}$$

The moment map $\mu_C : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{R}^{n+1}$ is given by

$$\mu_C(z) = \frac{1}{2}(|z_1|^2, \ldots, |z_n|^2, |z_1|^2 + \cdots + |z_n|^2 + |z_{n+1}|^2)$$

and

$$N_C := \mu_C^{-1}(\mathbb{R}^n \times \{1\}) = \{z \in \mathbb{C}^{n+1} \mid \|z\|^2 = 2\} \cong S^{2n+1}.$$

Remark 3.13. Up to a possible twist of the action by an automorphism of the torus $\mathbb{T}^{n+1}$, any good toric symplectic cone can be obtained via an orbifold version of the Boothby–Wang construction of Example 3.5, where the base is a toric symplectic orbifold. In fact, up to an $\mathrm{SL}(n+1,\mathbb{Z})$ transformation, any good moment cone can be written as the standard cone, given by (5), of a labeled polytope.

4. Toric Kähler–Sasaki cones

In this section we define (toric) Kähler–Sasaki cones, present their basic properties and briefly describe their relation with (toric) Sasaki manifolds.

Definition 4.1. A *Kähler–Sasaki cone* is a symplectic cone (M,ω,X) equipped with a compatible complex structure $J \in \mathscr{I}(M,\omega)$ such that the *Reeb vector field* $K := JX$ is Kähler, i.e.,

$$\mathscr{L}_K\omega = 0 \quad \text{and} \quad \mathscr{L}_K J = 0.$$

Note that K is then also a *Killing* vector field for the Riemannian metric g_J.

Any such J will be called a *Sasaki* complex structure on the symplectic cone (M,ω,X) and the associated metric g_J will be called a *Kähler–Sasaki* metric. The space of all Sasaki complex structures will be denoted by $\mathscr{I}_S(M,\omega,X)$.

Given a Kähler–Sasaki cone (M,ω,X,J), define a smooth positive function $r : M \to \mathbb{R}^+$ by

$$r := \|X\| = \|JX\| = \|K\|,$$

where $\|\cdot\|$ denotes the norm associated with the metric g_J. One easily checks that

(i) K is the *Hamiltonian* vector field of $-r^2/2$;

(ii) X is the *gradient* vector field of $r^2/2$.

Define $\alpha \in \Omega^1(M)$ by

$$\alpha := \iota(X)\omega/r^2.$$

We then have that

$$\omega = d(r^2\alpha)/2, \quad \alpha(K) \equiv 1 \quad \text{and} \quad \mathscr{L}_X\alpha = 0.$$

If we now define

$$N := \{r = 1\} \subset M \quad \text{and} \quad \xi := \ker\alpha|_N,$$

we have that

$$(N,\xi,\alpha|_N, g_J|_N) \text{ is a } \textit{Sasaki manifold}$$

(see [10] for the definition of a Sasaki manifold). In fact, one can easily check from the definitions that

$$\text{Sasaki manifolds} \overset{1:1}{\longleftrightarrow} \text{Kähler–Sasaki cones.}$$

Given a Kähler–Sasaki cone (M, ω, X, J), let

$$B := M/\!\!/K = N/K$$

be the symplectic reduction of (M, ω) by the action of $K = JX$ and denote by $\pi : N \to B$ the quotient projection. When B is smooth, we have that $\pi^*(TB) \cong \xi$ and $J|_\xi$ induces an almost complex structure on B which, by the already mentioned Kähler reduction theorem of Guillemin and Sternberg [20], is integrable. Hence,

$$(B, d\alpha|_\xi, J|_\xi) \text{ is a Kähler manifold.}$$

The smoothness of B is related with the regularity of the Kähler–Sasaki cone.

Definition 4.2. A Kähler–Sasaki cone (M, ω, X, J), with Reeb vector field $K = JX$, is said to be

(i) *regular* if K generates a *free* S^1-action,

(ii) *quasi-regular* if K generates a *locally free* S^1-action,

(iii) *irregular* if K generates an *effective* $\mathbb{R}$-action.

Hence, B is

(i) a *smooth* Kähler manifold if the Kähler–Sasaki cone is *regular*,

(ii) a Kähler *orbifold* if the Kähler–Sasaki cone is *quasi-regular*,

(iii) only a Kähler *quasifold*, in the sense of Prato [27], if the Kähler–Sasaki cone is *irregular*.

Remark 4.3. Note that the Sasaki manifold determined, as above, by a Kähler–Sasaki cone is always smooth.

Definition 4.4. A *toric Kähler–Sasaki cone* is a good toric symplectic cone (M, ω, X, μ) equipped with a *toric* Sasaki complex structure $J \in \mathscr{I}_S^{\mathbb{T}}(M, \omega)$, i.e., a Sasaki complex structure invariant under the torus action. The associated metric g_J will be called a toric Kähler–Sasaki metric.

Remark 4.5. (i) It follows from Theorem 3.10 and Remark 3.11 that any good toric symplectic cone has toric Sasaki complex structures. These will be described in the next section.

(ii) On a toric Kähler–Sasaki cone (M, ω, X, μ, J), the Kähler action generated by the Reeb vector field $K = JX$ corresponds to the action generated by a fixed vector in the Lie algebra of the torus (see Lemma 5.3 below).

(iii) The Kähler reduction $B := M /\!\!/ K$ of a toric Kähler–Sasaki cone is a toric Kähler space: manifold (regular case), orbifold (quasi-regular case) or quasifold (irregular case).

Example 4.6. The toric symplectic cone $(\mathbb{R}^{2(n+1)} \backslash \{0\}, \omega_{\mathrm{st}}, X_{\mathrm{st}}, \mu_{\mathrm{st}})$ of Example 3.8, equipped with the standard linear complex structure $J_0 : \mathbb{R}^{2(n+1)} \to \mathbb{R}^{2(n+1)}$ given by

$$J_0 = \begin{bmatrix} 0 & \vdots & -I \\ \cdots & \cdots & \cdots \\ I & \vdots & 0 \end{bmatrix}$$

is a toric Kähler–Sasaki cone.

5. Cone action-angle coordinates and symplectic potentials

As described in Section 2, the space $\mathscr{I}^{\mathbb{T}}$ of toric compatible complex structures on a compact toric symplectic orbifold can be effectively parametrized, using global action-angle coordinates, by symplectic potentials, i.e., certain smooth real valued functions on the corresponding labeled polytope. In this section we present the analogue of this fact for the space $\mathscr{I}_S^{\mathbb{T}}$ of toric Sasaki complex structures on a good toric symplectic cone, due to Burns–Guillemin–Lerman [11] and Martelli–Sparks–Yau [26]. We will also discuss how symplectic potentials and toric Kähler–Sasaki metrics behave under symplectic reduction.

Let $C \subset \mathbb{R}^{n+1}$ be a good cone and (M, ω, X, μ) the corresponding good toric symplectic cone (we omit the subscript C to simplify the notation). Let $\breve{C}$ denote the interior of C, and consider $\breve{M} \subset M$ defined by $\breve{M} = \mu^{-1}(\breve{C})$. One can easily check that $\breve{M}$ is a smooth open dense subset of M, consisting of all the points where the $\mathbb{T}^n$-action is free. One can use the explicit model for (M, ω, X, μ), given by the symplectic reduction construction mentioned in Remark 3.11, to show that $\breve{M}$ can be described as

$$\breve{M} \cong \breve{C} \times \mathbb{T}^n = \{(x, y) \mid x \in \breve{C}, y \in \mathbb{T}^{n+1} \equiv \mathbb{R}^{n+1}/2\pi\mathbb{Z}^{n+1}\},$$

where in these (x, y) coordinates we have

$$\omega|_{\breve{M}} = dx \wedge dy, \qquad \mu(x, y) = x \qquad \text{and} \qquad X|_{\breve{M}} = 2x\frac{\partial}{\partial x} = 2\sum_{i=1}^{n+1} x_i \frac{\partial}{\partial x_i}.$$

Definition 5.1. Any such set of coordinates will be called *cone action-angle* coordinates.

If J is any ω-compatible toric complex structure on M such that $\mathscr{L}_X J = 0$, i.e., for which the Liouville vector field X is holomorphic, the cone action-angle (x, y)-coordinates on $\breve{M}$ can be chosen so that the matrix that represents J in these coordinates has the form

$$\begin{bmatrix} 0 & \vdots & -S^{-1} \\ \cdots & \cdots & \cdots \\ S & \vdots & 0 \end{bmatrix}, \tag{6}$$

where $S = S(x) = [s_{ij}(x)]_{i,j=1}^{n+1,n+1}$ is a symmetric and positive-definite real matrix. The integrability condition for the complex structure J is again equivalent to S being the Hessian of a smooth real function $s \in C^\infty(\breve{C})$, i.e.,

$$S = \mathrm{Hess}_x(s), \qquad s_{ij}(x) = \frac{\partial^2 s}{\partial x_i \partial x_j}(x), \qquad 1 \leq i, j \leq n+1, \tag{7}$$

and holomorphic coordinates for J are again given by

$$z(x, y) = u(x, y) + iv(x, y) = \frac{\partial s}{\partial x}(x) + iy.$$

The condition $\mathscr{L}_X J = 0$ is equivalent to

$$S(e^{2t}x) = e^{-2t} S(x) \qquad \text{for all } t \in \mathbb{R},\ x \in \breve{C}, \tag{8}$$

i.e., equivalent to S being homogeneous of degree -1 in x.

Remark 5.2. A proof of these facts can be given by combining Donaldson's method of proof in the polytope case (cf. Remark 2.6) with the Sasaki condition on the complex structure J.

The Reeb vector field $K := JX$ of such a toric complex structure (cf. Definition 4.1) is given by

$$K = \sum_{i=1}^{n+1} b_i \frac{\partial}{\partial y_i} \qquad \text{with } b_i = 2\sum_{j=1}^{n+1} s_{ij} x_j.$$

Lemma 5.3 (Martelli–Sparks–Yau). *If* $S(x) = [s_{ij}(x)]$ *is homogeneous of degree* -1*, then the corresponding Reeb vector field* $K = (\mathbf{0}, K_s)$*, with* $K_s := (b_1, \ldots, b_{n+1})$, *is a* constant *vector. In other words, the action generated by K corresponds to the action generated by a fixed vector in the Lie algebra of the torus. In particular, K is Kähler and*

$$\textit{regularity of the toric Kähler–Sasaki cone} \iff \textit{rationality of } K_s \in \mathbb{R}^{n+1}.$$

The norm of the Reeb vector field is given by

$$\|K\|^2 = \|(\mathbf{0}, K_s)\|^2 = b_i s^{ij} b_j = b_i s^{ij}(2s_{jk}x_k) = 2b_i x_i = 2\langle x, K_s\rangle.$$

Hence

$$\|K\| > 0 \iff \langle x, K_s\rangle > 0 \quad \text{and} \quad \|K\| = 1 \iff \langle x, K_s\rangle = 1/2.$$

Definition 5.4 (Martelli–Sparks–Yau). The *characteristic hyperplane* H_K and *polytope* P_K of a toric Kähler–Sasaki cone (M, ω, X, μ, J), with moment cone $C \subset \mathbb{R}^{n+1}$, are defined as

$$H_K := \{x \in \mathbb{R}^{n+1} \mid \langle x, K_s\rangle = 1/2\} \quad \text{and} \quad P_K := H_K \cap C.$$

Remark 5.5. Note that $N := \mu^{-1}(H_K)$ is a toric Sasaki manifold and P_K is the moment polytope of $B = M /\!\!/ K$. Moreover, we see that K gives rise to compatible splitting identifications $M = N \times \mathbb{R}$ and $C = P_K \times \mathbb{R}$.

As we have just seen, any toric Sasaki complex structure $J \in \mathscr{I}_S^{\mathbb{T}}(\breve{M}, \omega, X)$ can be written in suitable cone action-angle coordinates (x, y) on $\breve{M} \cong \breve{C} \times \mathbb{T}^{n+1}$ in the form (6), with S satisfying (7) and (8).

Definition 5.6. The corresponding smooth real function $s \in C^\infty(\breve{C})$ will be called the *symplectic potential* of the toric Sasaki complex structure

Example 5.7. Consider the toric Kähler–Sasaki cone of Example 4.6. In cone action-angle coordinates (x, y) on

$$\breve{C} \times \mathbb{T}^{n+1} = (\mathbb{R}^+)^{n+1} \times \mathbb{T}^{n+1},$$

the symplectic potential

$$s : \breve{C} = (\mathbb{R}^+)^{n+1} \to \mathbb{R}$$

of the toric Sasaki complex structure J_0 is given by

$$s(x) = \frac{1}{2}\sum_{a=1}^{n+1} x_a \log x_a.$$

We will now characterize the space of smooth real functions $s \in C^\infty(\breve{C})$ that are the symplectic potential of some toric Sasaki complex structure $J \in \mathcal{I}_S^{\mathbb{T}}(M, \omega, X)$.

The Kähler reduction theorem of Guillemin and Sternberg can also be applied to the symplectic reduction construction mentioned in Remark 3.11. Hence, given a good cone $C \subset \mathbb{R}^{n+1}$, defined by

$$C = \bigcap_{a=1}^{d} \{x \in \mathbb{R}^{n+1} \mid \ell_a(x) := \langle x, v_a \rangle \geq 0\}$$

as in Definition 3.9, the explicit model for the corresponding good toric symplectic cone (M, ω, X, μ) has a canonical toric Sasaki complex structure $J_C \in \mathcal{I}_S^{\mathbb{T}}(M, \omega, X)$. Its symplectic potential is given by the following particular case of a theorem proved by Burns–Guillemin–Lerman in [11].

Theorem 5.8. *In appropriate action-angle coordinates* (x, y), *the* canonical symplectic potential $s_C : \breve{C} \to \mathbb{R}$ *for* $J_C|_{\breve{C}}$ *is given by*

$$s_C(x) = \frac{1}{2} \sum_{a=1}^{d} \ell_a(x) \log \ell_a(x).$$

One checks easily that $\mathrm{Hess}_x(s_C)$ is homogeneous of degree -1. The corresponding Reeb vector field $K = (\mathbf{0}, K_C)$ is given by

$$K_C = \sum_{a=1}^{d} v_a. \tag{9}$$

Example 5.9. The symplectic potential presented in Example 5.7 is the canonical symplectic potential of the corresponding good cone $C = (\mathbb{R}_0^+)^{n+1} \subset \mathbb{R}^{n+1}$ and

$$K_C = (1, \dots, 1) \in \mathbb{R}^{n+1}.$$

Example 5.10. The standard cone over the standard simplex, considered in Example 3.12, is given by

$$C = \bigcap_{a=1}^{n+1} \{x \in \mathbb{R}^{n+1} \mid \ell_a(x) := \langle x, v_a \rangle \geq 0\},$$

where

$$v_a = e_a, \quad a = 1, \dots, n, \quad \text{and} \quad v_{n+1} = (-1, \dots, -1, 1).$$

Hence, defining

$$r = \sum_{a=1}^{n} x_a,$$

we have that

$$s_C(x) = \frac{1}{2}\Big(\sum_{a=1}^{n} x_a \log x_a + (x_{n+1} - r)\log(x_{n+1} - r)\Big)$$

and

$$K_C = \sum_{a=1}^{n+1} v_a = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}.$$

Remark 5.11. Examples 5.9 and 5.10 are isomorphic to each other under a $\mathrm{SL}(n+1, \mathbb{Z})$ transformation.

Let $s, s' : \breve{C} \to \mathbb{R}$ be two symplectic potentials defined on the interior of a cone $C \subset \mathbb{R}^{n+1}$. Then

$$K_s = K_{s'} \iff (s - s') + \text{const. is homogeneous of degree 1.}$$

Given $b \in \mathbb{R}^{n+1}$, define

$$s_b(x) := \frac{1}{2}(\langle x, b\rangle \log\langle x, b\rangle - \langle x, K_C\rangle \log\langle x, K_C\rangle), \tag{10}$$

with K_C given by (9). Then $s := s_C + s_b$ is such that $K_s = b$. If C is good, this symplectic potential s defines a *smooth Sasaki* complex structure on the corresponding good toric symplectic cone (M, ω, X, μ) iff

$$\langle x, b\rangle > 0 \qquad \text{for all } x \in C\backslash\{0\}, \text{ i.e., } b \in \breve{C}^*,$$

where $C^* \subset \mathbb{R}^{n+1}$ is the *dual cone*

$$C^* := \{x \in \mathbb{R}^{n+1} \mid \langle v, x\rangle \geq 0 \text{ for all } v \in C\}.$$

This dual cone can be equivalently defined as

$$C^* = \bigcap_{\alpha} \{x \in \mathbb{R}^{n+1} \mid \langle \eta_\alpha, x\rangle \geq 0\},$$

where $\eta_\alpha \in \mathbb{Z}^{n+1}$ are the primitive generating edges of C.

Theorem 5.12 (Martelli–Sparks–Yau [26]). *Any toric Sasaki complex structure $J \in \mathscr{I}_S^{\mathbb{T}}$ on a good toric symplectic cone (M, ω, X, μ), associated to a good moment cone $C \in \mathbb{R}^{n+1}$, is given by a symplectic potential $s : \breve{C} \to \mathbb{R}$ of the form*

$$s = s_C + s_b + h,$$

where s_C is the canonical potential, s_b is given by (10) *with $b \in \breve{C}^*$, and $h : C \to \mathbb{R}$ is homogeneous of degree* 1 *and smooth on $C \backslash \{0\}$.*

5.1. Symplectic reduction of symplectic potentials

Proposition 5.13 (Calderbank–David–Gauduchon [13]). *Symplectic potentials* restrict *naturally under toric symplectic reduction.*

More precisely, suppose (M_P, ω_P, μ_P) is a toric symplectic reduction of (M_C, ω_C, μ_C). Then there is an affine inclusion $P \subset C$ and

$$\text{any } \tilde{J} \in \mathscr{I}^{\mathbb{T}}(M_C, \omega_C) \text{ induces a reduced } J \in \mathscr{I}^{\mathbb{T}}(M_P, \omega_P).$$

This proposition says that if

$$\tilde{s} : \breve{C} \to \mathbb{R} \text{ is a symplectic potential for } \tilde{J}$$

then

$$s := \tilde{s}|_{\breve{P}} : \breve{P} \to \mathbb{R} \text{ is a symplectic potential for } J.$$

This property can be used to prove Theorems 2.7 and 5.8. It is also particularly relevant for the following class of symplectic potentials.

Definition 5.14. Let $P \subset \mathbb{R}^n$ be a convex polytope and $C \subset \mathbb{R}^{n+1}$ its standard cone given by (5). Given a symplectic potential $s : \breve{P} \to \mathbb{R}$, define its *Boothby–Wang* symplectic potential $\tilde{s} : \breve{C} \to \mathbb{R}$ by

$$\tilde{s}(x, z) := zs(x/z) + \frac{1}{2} z \log z \qquad \text{for all } x \in \breve{P},\ z \in \mathbb{R}^+. \tag{11}$$

Note that

$$K_{\tilde{s}} = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}.$$

Example 5.15. In general,

$$\widetilde{s_P} \neq s_C.$$

If $P = \bigcap_{a=1}^{d} \{x \in \mathbb{R}^n \mid \ell_a(x) := \langle x, v_a \rangle + \lambda_a \geq 0\}$, consider

$$s(x) = s_P(x) - \frac{1}{2}\ell_\infty(x) \log \ell_\infty(x),$$

where $\ell_\infty(x) := \sum_a \ell_a(x) = \langle x, v_\infty \rangle + \lambda_\infty$. Then

$$\tilde{s}(x, z) = s_C(x, z) + s_b(x, z)$$

where s_b is given by (10) with $b = (0, \dots, 0, 1)$.

5.2. Toric Kähler–Sasaki–Einstein metrics

Proposition 5.16. *Let $P \subset \mathbb{R}^n$ be a convex polytope and $C \subset \mathbb{R}^{n+1}$ its standard cone defined by* (5). *Given a symplectic potential $s : \breve{P} \to \mathbb{R}$, let $\tilde{s} : \breve{C} \to \mathbb{R}$ be its Boothby–Wang symplectic potential given by* (11). *Then*

$$\widetilde{Sc}(x, z) = \frac{Sc(x/z) - 2n(n+1)}{z}.$$

In particular,

$$\widetilde{Sc} \equiv 0 \iff Sc \equiv 2n(n+1)$$

and, when this happens, the corresponding toric Sasaki metric has constant positive scalar curvature $= n(2n+1)$. Moreover, s defines a toric Kähler–Einstein metric with $Sc \equiv 2n(n+1)$ if and only if $\tilde{s}$ defines a toric Ricci-flat Kähler metric. If this happens, the corresponding toric Sasaki metric is Einstein.

Proof. The relation between Sc and $\widetilde{Sc}$ follows by direct application of formula (4) for the scalar curvature to the symplectic potentials s and $\tilde{s}$.

The last statement follows from the above symplectic reduction property of symplectic potentials and a well known fact in Sasaki geometry (see [10]): on a Sasaki manifold of dimension $2n+1$ the following are equivalent:

(i) the Sasaki metric is Einstein with scalar curvature equal to $n(2n+1)$;

(ii) the transversal Kähler metric is Einstein with scalar curvature equal to $2n(n+1)$;

(iii) the cone Kähler metric is Ricci-flat. □

6. New Sasaki–Einstein from old Kähler–Einstein

In 1982 Calabi [12] constructed, in local complex coordinates, a general 4-parameter family of $U(n)$-invariant extremal Kähler metrics, which he used to put an extremal Kähler metric on

$$H_m^{2n} = \mathbb{P}\big(\mathcal{O}(-m) \oplus \mathbb{C}\big) \to \mathbb{CP}^{n-1},$$

for all $n, m \in \mathbb{N}$ and any possible Kähler cohomology class. In particular, when $n = 2$, on all Hirzebruch surfaces.

When written in action-angle coordinates, using symplectic potentials, Calabi's family can be seen to contain many other interesting cohomogeneity one special Kähler metrics. Besides the ones discussed in [4] and some of the Bochner-Kähler orbifold examples presented in [2], it also contains a 1-parameter family of Kähler–Einstein metrics that are directly related to the Sasaki–Einstein metrics constructed by Gauntlett–Martelli–Sparks–Waldram [17], [18] in 2004.

Consider symplectic potentials $s_A : \breve{P}_A \subset (\mathbb{R}^+)^n \to \mathbb{R}$ of the form

$$s_A(x) = \frac{1}{2}\left(\sum_{i=1}^{n}\left(x_i + \frac{1}{n+1}\right)\log\left(x_i + \frac{1}{n+1}\right) + h_A(r)\right),$$

where

$$r = x_1 + \cdots + x_n,$$

the polytope $\breve{P}_A$ will be determined below and

$$h_A''(r) = -\frac{1}{r + \frac{n}{n+1}} + \frac{\left(r + \frac{n}{n+1}\right)^{n-1}}{p_A(r)},$$

with

$$p_A(r) := \left(r + \frac{n}{n+1}\right)^n \left(\frac{1}{n+1} - r\right) - A \quad \text{and} \quad 0 < A < \frac{n^n}{(n+1)^{n+1}}. \tag{12}$$

One can check (see [4]) that this family of symplectic potentials defines a 1-parameter family of local Kähler–Einstein metrics with $Sc = 2n(n+1)$.

Let $-a$ and b denote the first negative and positive zeros of p_A. Then

$$p_A(r) = (r+a)(b-r)q_A(r), \tag{13}$$

where q_A is a polynomial of degree $n-1$,

$$0 < a < \frac{n}{n+1}, \qquad 0 < b < \frac{1}{n+1} \qquad \text{and}$$
$$\left(\frac{n}{n+1} - a\right)^n \left(\frac{1}{n+1} + a\right) = A = \left(\frac{n}{n+1} + b\right)^n \left(\frac{1}{n+1} - b\right).$$

From (12) and (13) we get that

$$\begin{aligned} p_A'(r) &= -(n+1)r\left(r + \frac{n}{n+1}\right)^{n-1} \\ &= (b-r)q_A(r) - (r+a)q_A(r) + (r+a)(b-r)q_A'(r), \end{aligned}$$

which for $r = -a$ and $r = b$ implies that

$$\begin{aligned} q_A(-a) &= \frac{(n+1)a}{a+b}\left(\frac{n}{n+1} - a\right)^{n-1}, \\ q_A(b) &= \frac{(n+1)b}{a+b}\left(b + \frac{n}{n+1}\right)^{n-1}. \end{aligned}$$

This means in particular that

$$\begin{aligned} \frac{\left(r + \frac{n}{n+1}\right)^{n-1}}{p_A(r)} &= \frac{\left(r + \frac{n}{n+1}\right)^{n-1}}{(r+a)(b-r)q_A(r)} \\ &= \frac{\frac{1}{(n+1)a}}{r+a} + \frac{\frac{1}{(n+1)b}}{b-r} + \frac{\cdots}{q_A(r)}. \end{aligned}$$

Hence, the symplectic potential s_A defines a Kähler–Einstein metric with $Sc = 2n(n+1)$ on the toric quasifold determined by the polytope $P_A \subset \mathbb{R}^n$ defined by the following inequalities:

$$x_i + \frac{1}{n+1} \geq 0, \qquad i = 1, \ldots, n,$$
$$\frac{1}{(n+1)a}(r+a) \geq 0 \quad \text{and} \quad \frac{1}{(n+1)b}(b-r) \geq 0.$$

Since P_A is never $\mathrm{GL}(n, \mathbb{R})$ equivalent to a Delzant polytope, these Kähler–Einstein quasifolds do not give rise to any interesting Kähler–Einstein smooth manifolds. However, they do give rise to interesting Sasaki–Einstein smooth manifolds. In fact, for suitable values of the parameter A, the polytope P_A determines via (5) a standard cone $C_A \subset \mathbb{R}^{n+1}$ that is $\mathrm{GL}(n+1,\mathbb{R})$ equivalent to one of the good cones $C(k,m) \subset \mathbb{R}^{n+1}$ defined in the Introduction. The Boothby–Wang symplectic potential

$$\tilde{s}_A : C_A \subset \mathbb{R}^{n+1} \to \mathbb{R},$$

determined by s_A via (11), will then define a Ricci-flat Kähler metric on the toric "quasicone" determined by C_A and, for these appropriate values of A, also a Ricci-flat Kähler metric on the smooth toric symplectic cone determined by the appropriate $C(k,m)$ and a Sasaki–Einstein metric on the corresponding smooth toric contact manifold, thus proving Theorem 1.1.

The facets of C_A are defined by the set of defining normals

$$\begin{aligned} v_i' &= \left(\vec{e}_i, \frac{1}{n+1}\right), \qquad i = 1, \dots, n, \\ v_a' &= \left(\frac{\vec{d}}{(n+1)a}, \frac{1}{n+1}\right), \\ v_b' &= \left(-\frac{\vec{d}}{(n+1)b}, \frac{1}{n+1}\right), \end{aligned}$$

where $\vec{e}_i \in \mathbb{R}^n$, $i = 1, \dots, n$, are the canonical basis vectors and $\vec{d} = \sum_{i=1}^n \vec{e}_i \in \mathbb{R}^n$. To suitably express the condition implying that the cone C_A is $\mathrm{GL}(n+1,\mathbb{R})$ equivalent to one of the good cones $C(k,m) \subset \mathbb{R}^{n+1}$, it is convenient to introduce the auxiliar real parameter

$$\lambda_A := \frac{b}{a} \cdot \frac{n - (n+1)a}{n + (n+1)b}.$$

Note that λ_A assumes all values in the open interval $(0,1)$, since A varies in the open interval $\left(0, \frac{n^n}{(n+1)^{n+1}}\right)$.

Proposition 6.1. *If $\lambda_A \in (0,1)$ can be written in the form*

$$\lambda_A = \frac{kn - m}{n + m}, \tag{14}$$

with $k, m \in \mathbb{N}$ satisfying

$$\frac{(k-1)n}{2} < m < kn, \tag{15}$$

then C_A *is* $\mathrm{GL}(n+1,\mathbb{R})$ *equivalent to the cone* $C(k,m) \subset \mathbb{R}^{n+1}$ *defined by the following normals:*

$$\begin{aligned} v_i &= (\vec{e}_i, 1), \qquad i = 1, \dots, n-1, \\ v_n &= \big((m+1)\vec{e}_n - \vec{d}, 1\big), \\ v_- &= (k\vec{e}_n, 1), \\ v_+ &= (-\vec{e}_n, 1). \end{aligned} \tag{16}$$

Proof. Consider $T \in \mathrm{GL}(n+1,\mathbb{R})$ defined by

$$\begin{aligned} T^t(\vec{e}_i, 0) &= (\vec{e}_i - \gamma\vec{e}_n, 0), \qquad i = 1, \dots, n-1, \\ T^t(\vec{e}_n, 0) &= \big((m+1-\gamma)\vec{e}_n - \vec{d}, 0\big), \\ T^t(\vec{0}, 1) &= \big((n+1)\gamma\vec{e}_n, n+1\big), \end{aligned}$$

for some $\gamma \in \mathbb{R}$. Then

$$\begin{aligned} T^t(v_i') &= v_i, \qquad i = 1, \dots, n, \\ T^t(v_a') &= v_- \text{ iff } \gamma = \frac{k(n+1)a - m}{(n+1)a - n}, \\ T^t(v_b') &= v_+ \text{ iff } \gamma = \frac{m-(n+1)b}{n+(n+1)b}. \end{aligned}$$

This implies that C_A is $\mathrm{GL}(n+1,\mathbb{R})$ equivalent to $C(k,m)$, provided that

$$\frac{k(n+1)a - m}{(n+1)a - n} = \frac{m-(n+1)b}{n+(n+1)b},$$

which is equivalent to (14). □

Remark 6.2. Note that, in the action-angle coordinates associated with the cone $C(k,m)$, the Reeb vector field of the Ricci-flat Kähler–Sasaki metric is

$$K = \big(\mathbf{0}, T^t(\vec{0},1)\big) \qquad \text{with} \qquad T^t(\vec{0},1) = \big((n+1)\gamma\vec{e}_n, n+1\big).$$

Since

$$\gamma = \frac{k(n+1)a - m}{(n+1)a - n} = \frac{m-(n+1)b}{n+(n+1)b},$$

the (ir)regularity of K is determined by the (ir)rationality of the admissible values of a or, equivalently, b.

For $k = 1$ we have $0 < m < n$ and each cone $C(1, m) \subset \mathbb{R}^{n+1}$ is the standard cone over the integral Delzant polytope $P(m) \subset \mathbb{R}^n$ defined by the affine functions

$$\begin{aligned}
\ell_i(x) &= \langle x, \vec{e}_i \rangle + 1, \qquad i = 1, \ldots, n-1, \\
\ell_n(x) &= \langle x, (m+1)\vec{e}_n - \vec{d} \rangle + 1, \\
\ell_-(x) &= \langle x, \vec{e}_n \rangle + 1, \\
\ell_+(x) &= \langle x, -\vec{e}_n \rangle + 1.
\end{aligned}$$

If $n = 2$ then $m = 1$ and $P(1) \subset \mathbb{R}^2$ is well known to be a polytope for the first Hirzebruch surface

$$H_1^4 = \mathbb{P}(\mathcal{O}(-1) \oplus \mathbb{C}) \to \mathbb{CP}^1.$$

In fact, one easily checks that $P(m) \subset \mathbb{R}^n$, $0 < m < n$, defines a smooth compact toric symplectic manifold (H_m^{2n}, ω), where

$$H_m^{2n} = \mathbb{P}(\mathcal{O}(-m) \oplus \mathbb{C}) \to \mathbb{CP}^{n-1} \quad \text{and} \quad [\omega] = 2\pi c_1(H_m^{2n}).$$

Hence the Sasaki–Einstein manifold $N_{1,m}^{2n+1}$ is diffeomorphic to the corresponding Boothby–Wang manifold, cf. Example 3.5, which is the circle bundle of the anti-canonical line bundle of H_m^{2n}. This proves Theorem 1.2.

Remark 6.3. In general, i.e., when $1 < k \in \mathbb{N}$, the cones $C(k, m) \subset \mathbb{R}^{n+1}$ are standard cones over labeled polytopes $P(k, m) \subset \mathbb{R}^n$ and the corresponding manifolds $N_{k,m}^{2n+1}$ are given by an orbifold version of the Boothby–Wang construction.

We will now check that, when $n = 2$, the cones $C(k, m) \subset \mathbb{R}^3$, with $k, m \in \mathbb{N}$ satisfying (15) and the condition of simply connectedness

$$\gcd(m + n, k + 1) = 1,$$

are $SL(3, \mathbb{Z})$ equivalent to the cones $C_{p,q} \subset \mathbb{R}^3$ associated to the Sasaki–Einstein 5-manifolds $Y^{p,q}$, $0 < q < p$, $\gcd(q, p) = 1$, constructed by Gauntlett–Martelli–Sparks–Waldram [17]. The defining normals of the cones $C(k, m) \subset \mathbb{R}^3$ are v_1, v_2, v_- and v_+ defined by (16) with $n = 2$. According to [26], the cones $C_{p,q} \subset \mathbb{R}^3$ have defining normals given by

$$\mu_1 = (1, p - q - 1, p - q), \qquad \mu_2 = (1, 1, 0),$$
$$\mu_- = (1, 0, 0) \quad \text{and} \quad \mu_+ = (1, p, p).$$

Consider the linear map $T_{k,m} \in \mathrm{SL}(3,\mathbb{Z})$ defined by the matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ k-m-1 & -1 & k \\ k-m & -1 & k \end{bmatrix}.$$

When $k = p-1$ and $m = p+q-2$ we have that

$$T_{k,m}(v_1) = \mu_1, \quad T_{k,m}(v_2) = \mu_2, \quad T_{k,m}(v_-) = \mu_- \quad \text{and} \quad T_{k,m}(v_+) = \mu_+,$$

i.e., $T_{k,m} \in \mathrm{SL}(3,\mathbb{Z})$ provides the required equivalence.

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Miguel Abreu, Centro de Análise Matemática, Geometria e Sistemas Dinâmicos, Departamento de Matemática, Instituto Superior Técnico, Av. Rovisco Pais, 1049-001 Lisboa, Portugal

E-mail: mabreu@math.ist.utl.pt