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Holomorphic and pseudo-holomorphic curves on rationally connected varieties

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Abstract. These notes give a short introduction to the study of curves on algebraic varieties. The main emphasis is on families of genus 0 curves. After an elementary proof of the dimension formula for the space of curves, we summarize the basic properties of uniruled and of rationally connected varieties. The last section is devoted to a conjectural characterization of rationally connected varieties using symplectic geometry.

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1. Codimension 1 theory. For details, see, for instance [GH78], Chap. 1.

One of the aims of algebraic geometry is to understand all subvarieties of a given algebraic variety. For simplicity, assume that X is a smooth projective variety over $\mathbb{C}$. We have a very successful—by now classical—theory that describes codimension 1 subvarieties, that is, hypersurfaces or divisors on X.

Each codimension 1 subvariety $Z \subset X$ determines a line bundle (equivalently, a rank 1 locally free sheaf) $L := \mathcal{O}_X(Z)$ and a section $s \in H^0(X, L)$; the constant 1 section of $\mathcal{O}_X(Z)$. (L is determined uniquely, s is determined up to a multiplicative constant.) It is thus sufficient to describe the pairs (L, s). We proceed in three steps.

(1.1) As a topological line bundle, L is determined by its Chern class $c_1(L) \in H^2(X(\mathbb{C}), \mathbb{Z})$. Furthermore, a cohomology class $\alpha \in H^2(X(\mathbb{C}), \mathbb{Z})$ is the Chern class of a holomorphic/algebraic line bundle iff the image of α in $H^2(X(\mathbb{C}), \mathbb{C})$ lies in $H^{1,1}(X(\mathbb{C}), \mathbb{C})$.

(1.2) The set of all holomorphic/algebraic line bundles with a given $c_1(L) \in H^2(X(\mathbb{C}), \mathbb{Z})$ is parametrized by (more precisely, is a principal homogeneous space under) an Abelian variety, called the Picard variety, $\mathrm{Pic}^0(X)$. It is biholomorphic to $H^{0,1}(X(\mathbb{C}), \mathbb{C})/H^1(X(\mathbb{C}), \mathbb{Z})$.

(1.3) The sections of a line bundle L form a vector space $H^0(X,L)$, and nonzero sections up to a multiplicative constant are parametrized by the corresponding projective space $|L| := \mathbb{P}\big(H^0(X,L)^\vee\big)$. In terms of Čech cohomology, computing $H^0(X,L)$ is essentially a linear algebra problem. On the other hand, the Hirzebruch–Riemann–Roch theorem and Serre's vanishing theorem say that, for sufficiently ample L, the dimension of $H^0(X,L)$ can be computed from the Chern classes, thus from topological data.

Summary: The topology of $X(\mathbb{C})$ and Hodge theory determine steps (1.1–2) and linear algebra governs step (1.3).

Ideally, we would like to have a similar description of higher codimension subvarieties, but this goal is very far off.

The aim of these notes is to focus on the next simplest case, the study of 1-dimensional subvarieties, that is, curves in X. Despite Poincaré duality between divisors and curves, the study of curves turns out to be quite a bit harder than the theory of divisors. There are still many deep open questions.

Let us start with the analog of (1.1).

1. Homology classes of curves

Given any curve $C \subset X$, it has a homology class $[C] \in H_2\big(X(\mathbb{C}),\mathbb{Z}\big)$. Our first question is: Which homology classes can be realized by algebraic curves? As before, this class has Hodge type $(1,1)$. We usually think of Hodge theory as living on cohomology groups, so we identify $H_2\big(X(\mathbb{C}),\mathbb{C}\big)$ with $H^{2n-2}\big(X(\mathbb{C}),\mathbb{C}\big)$ (where $n = \dim X$), and think of $[C]$ as an integral cohomology class of type $(n-1,n-1)$.

The hard Lefschetz theorem [GH78], p. 122, implies that if L is an ample divisor then

$$H^{n-1,n-1}\big(X(\mathbb{C}),\mathbb{C}\big) = L^{n-2}\cdot H^{1,1}\big(X(\mathbb{C}),\mathbb{C}\big).$$

In particular, a rational multiple of every curve class can be written as $(L^{n-2}\cdot D)$ for some (not necessarily effective) divisor D on X. This gives the complete answer with rational coefficients. The problem is, however, more subtle over $\mathbb{Z}$.

To illustrate how little is known, let us discuss two conjectures about homology classes of curves.

Consider smooth hypersurfaces $X_e^n \subset \mathbb{P}^{n+1}$ of degree e. By another Lefschetz theorem [GH78], p. 156, the natural map $H_2\big(X(\mathbb{C}),\mathbb{Z}\big) \to H_2(\mathbb{CP}^{n+1},\mathbb{Z})$ is an isomorphism if $n \geq 3$. Thus there is a class $\ell \in H_2\big(X(\mathbb{C}),\mathbb{Z}\big)$ whose image in $H_2(\mathbb{CP}^{n+1},\mathbb{Z})$ is the homology class of a line and $H_2\big(X(\mathbb{C}),\mathbb{Z}\big) = \mathbb{Z}[\ell]$.

The original form of the Hodge conjecture implies that ℓ is the homology class of a ($\mathbb{Z}$-linear combination of) algebraic curves, while the Noether-Lefschetz theorem suggests that all curves on X may be complete intersections with surfaces.

Conjecture 2 ([GH85]). *Let $X^n \subset \mathbb{P}^{n+1}$ be a very general smooth hypersurface and $C \subset X$ an algebraic curve. Assume that* $\deg X$ *is large enough (maybe* $\deg X \geq 2n$ *is sufficient). Then* $[C] \in H_2(X(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}[\ell]$ *is a multiple of* $\deg X \cdot [\ell]$.

A straightforward dimension count shows that every hypersurface $X^n \subset \mathbb{P}^{n+1}$ contains a line if $\deg X \leq 2n - 1$, but the general hypersurface does not contain a line if $\deg X \geq 2n$. (This, however, does not exclude the possibility that it contains a $\mathbb{Z}$-linear combination of algebraic curves whose degree is 1.)

An easy result [Kol92] shows that if $(6, k) = 1$, X is a very general hypersurface of degree $3k^2$ and $C \subset X$ is any curve, then $[C]$ is a multiple of $k \cdot [\ell]$.

By contrast, for every degree there are smooth hypersurfaces that contain a line. Thus topology and Hodge theory together do not give enough information to decide which 2-dimensional homology classes are algebraic.

The opposite may hold for smooth projective Fano varieties (that is, varieties with $-K_X$ ample) or more generally for rationally connected varieties (29). In the Fano case, these conjectures go back to Fano and Iskovskikh; the general case is proposed in [Voi07].

Question 3. Let X be a smooth projective variety.

(1) Is $H_2(X(\mathbb{C}), \mathbb{Z})$ generated by the homology classes of rational curves if X is Fano or, more generally, if X is rationally connected?

(2) Assume that X is Fano and $\eta \in H_2(X(\mathbb{C}), \mathbb{Z})$ such that $m\eta$ is the homology class of an effective algebraic curve for some $m > 0$. Is η the homology class of an effective algebraic curve, all of whose irreducible components are rational?

Both of these are known with $\mathbb{Q}$-coefficients. (3.1) is quite easy [Kol96], IV.3.13, and (3.2) follows from Mori's cone theorem [Mor79].

In dimension 3 the positive answer to (3.1) is a special case of [Voi06]. The original conjecture of Fano and Iskovskikh asks (3.2) in case the second Betti number of $X(\mathbb{C})$ is 1. In dimension 3 this is established as part of the classification of Fano 3-folds.

If X is rationally connected, the technique of "deformation of combs" (cf. [Kol96], Sec. II.7) shows that the subgroup of $H_2(X(\mathbb{C}), \mathbb{Z})$ generated by homology classes of curves (resp. rational curves) is invariant under smooth deformations of X.

Example 4. 1. Let $\pi : X \to \mathbb{P}^n$ be a smooth degree d cover of $\mathbb{P}^n$, given by an affine equation of the form $z^d = f_{dr}(x_1, \ldots, x_n)$ where f_{dr} had degree dr. The branch locus $Z \subset \mathbb{P}^n$ is (the projective closure of) the hypersurface $(f_{dr} = 0)$.

For special choices of f_{dr}, there are lines $L \subset \mathbb{P}^n$ that are d-fold tangent to Z at r-points $P_1, \dots, P_r$. Then $\pi^{-1}(L)$ splits as a union of d irreducible curves, each mapping isomorphically to L. Let $L_i \subset X$ be any of these curves and $P_j \in X$ any of the points. Set $Y := B_{P_j}X$, the blow-up of X at the point P_j with exceptional divisor E. Let $\alpha \in H_2(Y(\mathbb{C}), \mathbb{Z})$ denote the homology class of the birational transform $L_i' \subset Y$ of L_i. The class α is characterized by the properties $\alpha \cdot E = 1$ and $\alpha \cdot \pi_Y^* H = 1$ where H is the hyperplane class on $\mathbb{P}^n$ and $\pi_Y : Y \to \mathbb{P}^n$ the composite of the blow-down map with π. Note that the class α is well defined for any $B_P X$, no matter where the point $P \in X$ lies.

An easy dimension count as in [Kol96], IV.2.12, shows that if $(d-1)r \geq n$ then there is no such line for general $P \in Z$. Moreover, if $r \geq 2$, then we can choose P and Z such that there is a line L_P that is d-fold tangent to Z at P, but every such line has transverse intersection with Z at some other point.

Then $\pi^{-1}(L_P)$ is irreducible and has multiplicity d at P. Hence its birational transform on $B_P X$ represents the homology class $d\alpha$. However, there are no effective curves in the homology classes $\alpha, \dots, (d-1)\alpha$.

If $(d-1)r = n$ then X, and hence $B_P X$, are rationally connected. X is Fano but $B_P X$ is not Fano.

2. By [AM72], there is a smooth, projective, unirational 3-fold X such that $H_2(X(\mathbb{C}), \mathbb{Z})$ contains a 2-torsion element α. One can write $\alpha = [C_1] - [C_2]$ where the $C_i \subset X$ are smooth rational curves.

Using [dJ04], one can show that there is a smooth 4-fold $\pi : Y \to X$ that is analytically locally a $\mathbb{P}^1$-bundle but it has no rational section. Thus every line bundle on Y is of the form $\pi^* L(mK_Y)$ where L is a line bundle on X.

Let $C \cong \mathbb{P}^1$ be a fiber of π, then $(C \cdot c_1(\pi^* L(mK_Y))) = -2m$, hence even. Since $h^2(Y, \mathcal{O}_Y) = 0$, we see that every class in $H^2(X(\mathbb{C}), \mathbb{Z})$ is algebraic, thus $(C \cdot L) \in 2\mathbb{Z}$ for every $L \in H^2(X(\mathbb{C}), \mathbb{Z})$. By Poincaré duality, there is a homology class $\beta \in H_2(X(\mathbb{C}), \mathbb{Z})$ such that $[C] = 2\beta$ but β is not the homology class of an effective algebraic curve. Nonetheless, one can find liftings $C_i' \subset Y$ of the C_i such that $\beta = [C_1'] - [C_2']$. Thus (3.2) does not hold for Y but (3.1) does.

In this example Y is unirational but not Fano. However, I see no reason why similar Fano examples should not exist. Thus the answer to (3.2) may well be negative.

Exercise 5 ([FMSS95]). Let X be a proper toric variety. Then $H_2(X(\mathbb{C}), \mathbb{Z})$ is generated by the homology classes of 1-dimensional orbits.

2. The dimension of the space of curves

Here we consider the analogs of (1.2–3) together for curves, since it is not known how to separate the Abelian part (1.2) from the linear algebra part (1.3). Actually, it seems quite unlikely that considering the two parts separately would make sense for families of curves on varieties of dimension ≥ 3. Our final results are (14) and (15); we build up to them through a series of examples.

Example 6 (Maps of rational curves to $\mathbb{P}^n$). Giving a morphism $f : \mathbb{P}^1 \to \mathbb{P}^n$ whose image has degree d is equivalent to giving $n+1$ homogeneous polynomials $(f_0, \dots, f_n)$ of degree d without common root (up to a multiplicative constant). Thus the space of all such maps is an open subset of a projective space of dimension $(n+1)(d+1) - 1$:

$$\mathrm{Mor}_d(\mathbb{P}^1, \mathbb{P}^n) \subset \mathbb{P}^{(n+1)(d+1)-1}.$$

Example 7 (Maps of rational curves to hypersurfaces). Let $X_e^n \subset \mathbb{P}^{n+1}$ be a hypersurface of degree e given by an equation $G(x_0, \dots, x_{n+1}) = 0$. The image of a morphism $f : \mathbb{P}^1 \to \mathbb{P}^{n+1}$ as in (6) lies in X iff $G(f_0, \dots, f_{n+1}) \equiv 0$. The latter is a homogeneous polynomial of degree de, so its vanishing is equivalent to $de+1$ equations in the coefficients of the f_i. Thus the space of all degree d maps $\mathbb{P}^1 \to X$ is either empty or a closed subset of $\mathrm{Mor}_d(\mathbb{P}^1, \mathbb{P}^{n+1})$ of codimension $\leq de + 1$. In particular

$$\dim_{[f]} \mathrm{Mor}_d(\mathbb{P}^1, X_e^n) \geq (n+2)(d+1) - 1 - (de+1) = (n+2-e)d + n.$$

(There is no reason to assume that $\mathrm{Mor}_d(\mathbb{P}^1, X_e^n)$ is pure dimensional, and $\dim_{[f]}$ denotes the dimension at the point $[f] \in \mathrm{Mor}_d(\mathbb{P}^1, X_e^n)$ corresponding to f. Actually we proved something stronger: the lower bound holds for every irreducible component containing $[f]$.)

With hindsight masquerading as prescience, we can write the right-hand side as

$$\dim_{[f]} \mathrm{Mor}_d(\mathbb{P}^1, X_e^n) \geq c_1(X) \cdot f_*[\mathbb{P}^1] + \dim X. \tag{7.1}$$

Example 8 (Maps of curves to $\mathbb{P}^n$). Now let us replace $\mathbb{P}^1$ by any curve C. To start, one can think of C as a smooth projective curve, but all the arguments work if C is a 1-dimensional, smoothable projective scheme over a field k such that $H^0(C, \mathcal{O}_C) \cong k$.

First, let us fix a line bundle L on C. Giving a morphism $f : C \to \mathbb{P}^n$ such that $f^*\mathcal{O}_{\mathbb{P}^n}(1) \cong L$ is equivalent to giving $n+1$ sections $f_0, \dots, f_n \in H^0(C, L)$ without common root (up to a multiplicative constant). Thus the space of all such maps is an open subset of a projective space of dimension $(n+1)h^0(C, L) - 1$, hence

$$\dim \mathrm{Mor}_L(C, \mathbb{P}^n) = (n+1)h^0(C, L) - 1,$$

where Mor_L denotes those morphisms for which $f^*\mathcal{O}_{\mathbb{P}^n}(1) \cong L$.

Example 9 (Maps of curves to hypersurfaces). Now let $X_e^n \subset \mathbb{P}^{n+1}$ be a hypersurface of degree e given by an equation $G(x_0, \dots, x_{n+1}) = 0$. Then $G(f_0, \dots, f_{n+1})$ is a section of L^e, hence its identical vanishing imposes $h^0(C, L^e)$ conditions. As in (7), the space of all such maps $C \to X$ is either empty or a closed subset of $\mathrm{Mor}_L(C, \mathbb{P}^{n+1})$ of codimension $\leq h^0(C, L^e)$. In particular

$$\dim_{[f]} \mathrm{Mor}_L(C, X_e^n) \geq (n+2)h^0(C, L) - 1 - h^0(C, L^e). \tag{9.1}$$

If $\deg L$ is large enough, then $h^0(C, L) = \deg L + 1 - g$ and we can rewrite the formula as

$$\dim_{[f]} \mathrm{Mor}_L(C, X_e^n) \geq c_1(X_e^n) \cdot f_*[C] + n \cdot \chi(\mathcal{O}_C) - g(C). \tag{9.2}$$

This fits very nicely with (7.1), except for the $-g(C)$ term at the end. Remember now that at the beginning we fixed not just $\deg L$ but L itself. All line bundles of degree d are parametrized by the $g(C)$-dimensional $\mathrm{Jac}_d(C)$, which is (non-canonically) isomorphic to the Jacobian of C (see, for instance, [GH78], Sec. 2.7). Thus, letting L vary should increase the dimension by $g(C)$ and we should get that

$$\dim_{[f]} \mathrm{Mor}(C, X) \geq c_1(X) \cdot \phi_*[C] + \dim X \cdot \chi(\mathcal{O}_C). \tag{9.3}$$

(Note, however, that our results included a caveat that the spaces may be empty. Thus, as we vary L in $\mathrm{Jac}_d(C)$, we could get empty spaces for some L where we expect something positive dimensional. Hence (9.3) is not yet firmly proven.)

Example 10 (Curves on $\mathbb{P}^n$). So far we have been estimating the dimension of the space of morphisms from a fixed curve to $\mathbb{P}^n$ with fixed $f^*\mathcal{O}_{\mathbb{P}^n}(1)$. In order to estimate the dimension of the space of curves on $\mathbb{P}^n$, we need to take three differences into account. (If $h^1(C, \mathcal{O}_C) = 0$ then only the second and third steps are needed.)

First, we need to work out precisely how to let $f^*\mathcal{O}_{\mathbb{P}^n}(1)$ vary in $\mathrm{Jac}_d(C)$.

Second, if $g = 0$ or $g = 1$, then every curve has many automorphisms. Thus a single rational curve $C \subset \mathbb{P}^n$ gives rise to a whole family of birational maps $\mathbb{P}^1 \to C$ parametrized by $\mathrm{Aut}(\mathbb{P}^1) = PGL(2)$. Similarly, every elliptic curve $C \subset \mathbb{P}^n$ gives rise to a 1-dimensional family of maps.

Third, for $g \geq 1$, we can vary the curve C in its moduli space.

To do the first, let $\mathrm{Jac}_d(C)$ denote the family of degree d line bundles on C. If $d \geq 2g - 1$, the universal H^0 gives a vector bundle $\mathscr{H}_d \to \mathrm{Jac}_d(C)$. As before, the degree d maps from C to $\mathbb{P}^n$ are parametrized by an open subset of the projective space bundle

$$\mathbb{P}_{\mathrm{Jac}_d(C)}\big((\mathscr{H}_d^\vee)^{n+1}\big) \to \mathrm{Jac}_d(C).$$

This gives the expected dimension formula

$$\begin{aligned}\dim_{[f]} \operatorname{Mor}_d(C, \mathbb{P}^n) &\geq c_1(\mathbb{P}^n) \cdot f_*[C] + n \cdot \bigl(1 - g(C)\bigr) \\ &= (n+1) \deg C + n \cdot \bigl(1 - g(C)\bigr). \end{aligned} \tag{10.1}$$

For $g \geq 2$, consider a $(3g-3)$-dimensional family $\mathscr{M}_g$ of curves of genus g (in practice, some finite cover of the moduli space) over which there is a universal family of curves $\mathscr{C}_g \to \mathscr{M}_g$. (If C is singular, we use the local deformation space of C as in [Kol96], II.1.11.) Let $\mathscr{J}_d \to \mathscr{M}_g$ denote the universal family of degree d components of the Jacobians. As before, if $d \geq 2g-1$, the universal H^0 gives a vector bundle

$$\mathscr{H}_d \to \mathscr{J}_d \to \mathscr{M}_g,$$

and $\dim \mathscr{H}_d = (3g-3) + g + (d+1-g)$. Thus the degree d maps from genus g curves to $\mathbb{P}^n$ are parametrized by an open subset of the projective space bundle

$$\mathbb{P}_{\mathscr{J}_d}\bigl((\mathscr{H}_d^\vee)^{n+1}\bigr) \to \mathscr{J}_d \to \mathscr{M}_g.$$

Hence, if $C \subset \mathbb{P}^n$ is a smooth curve of degree $\geq 2g(C) - 1$, then

$$\begin{aligned}\dim_C \operatorname{Curves}(\mathbb{P}^n) &\geq c_1(\mathbb{P}^n) \cdot [C] + (n-3) \cdot \bigl(1 - g(C)\bigr) \\ &= (n+1) \deg C + (n-3) \cdot \bigl(1 - g(C)\bigr), \end{aligned} \tag{10.2}$$

where $\operatorname{Curves}(\mathbb{P}^n)$ denotes either the Chow variety or the Hilbert scheme of curves on $\mathbb{P}^n$. (These two spaces agree near smooth or normal subvarieties, [Kol96], Cor. I.6.6.1.) It is easy to check that the formulas also work if $g(C) \leq 1$.

Example 11 (Maps of curves to varieties). Let $X^n \subset \mathbb{P}^N$ be a smooth projective subvariety. In general X needs many defining equations $G_i = 0$. Correspondingly, the image of a morphism $f : C \to \mathbb{P}^N$ lies in X iff $G_i(f_0, \ldots, f_N) \equiv 0$ for every i. This implies that $\operatorname{Mor}_L(C, X)$ is a closed algebraic subvariety of $\operatorname{Mor}_L(C, \mathbb{P}^N)$, but we can not estimate its codimension unless we know the degrees of the equations G_i.

If $X^n \subset \mathbb{P}^N$ is a complete intersection, there is a "natural" choice for the G_i and everything we said before works out. In particular, (9.2–3) still hold. In general, however, we get an estimate that is much worse and depends on the choice of the equations G_i.

One can, however, easily compute from this presentation the tangent spaces of $\operatorname{Mor}_L(C, X)$.

Theorem 12. *Let X be a smooth variety, C a proper curve such that $H^1(C, \mathcal{O}_C) = 0$ and $f : C \to X$ a morphism. Then*

$$T_{[f]}\operatorname{Mor}(C, X) = H^0(C, f^*T_X).$$

Proof. Assume that $X^n \subset \mathbb{P}^N$ is given by the equations $G_i = 0$ and let $f : C \to X$ be given by $(f_0 : \cdots : f_N)$ where the f_i are sections of $f^*\mathcal{O}_X(1)$. Inside $\operatorname{Mor}(C, \mathbb{P}^N)$ the tangent directions are given by the deformations $(f_0 + th_0 : \cdots : f_N + th_N)$ where the h_i are also sections of $f^*\mathcal{O}_X(1)$. The corresponding tangent vector is in the tangent space of $\operatorname{Mor}(C, X)$ iff

$$G_i(f_0 + th_0, \ldots, f_N + th_N) \equiv 0 \bmod (t^2) \qquad \text{for all } i.$$

Using the Taylor expansion, this is equivalent to

$$\sum_j \frac{\partial G_i}{\partial x_j} \cdot h_j = 0 \qquad \forall i.$$

The latter holds if $(h_0, \ldots, h_N)$ maps to a section of $f^*T_X \subset f^*T_{\mathbb{P}^N}$ in the exact sequence

$$0 \to f^*\mathcal{O}_{\mathbb{P}^N} \to f^*\mathcal{O}_{\mathbb{P}^N}(1)^{N+1} \to f^*T_{\mathbb{P}^N} \to 0.$$

Since $f^*\mathcal{O}_{\mathbb{P}^N} = \mathcal{O}_C$, by taking cohomology we get the exact sequence

$$H^0\big(C, f^*\mathcal{O}_{\mathbb{P}^N}(1)\big)^{N+1} \to H^0(C, f^*T_{\mathbb{P}^N}) \to H^1(C, \mathcal{O}_C) = 0, \tag{12.1}$$

thus every section of $H^0(C, f^*T_X)$ is the image of some $(h_0, \ldots, h_N)$. The same argument shows that if $g(C) \neq 0$ then

$$T_{[f]}\operatorname{Mor}_L(C, X) = \ker[H^0(C, f^*T_X) \to H^1(C, \mathcal{O}_C)], \tag{12.2}$$

where the map on the right comes from the sequence (12.1). □

While we usually think of an algebraic variety as sitting inside a larger dimensional projective space, one can also represent an n-dimensional variety as a finite branched cover $\pi : X^n \to \mathbb{P}^n$. This was Riemann's original point of view for algebraic curves. In higher dimensions it can be obtained by repeatedly projecting $X^n \subset \mathbb{P}^N$ from points outside it, until we get a dominant morphism $\pi : X^n \to \mathbb{P}^n$. This will allow us to prove the expected lower bounds for the spaces of maps.

13. Lifting deformations to branched covers. Let $\pi : X \to \mathbb{P}^n$ be a finite surjection with ramification divisor $R \subset X$ and $f : C \to X$ a morphism from a reduced curve to X. Assume the following genericity conditions.

***Assumptions* 13.1.**

i) C is smooth and $\pi \circ f : C \to \mathbb{P}^N$ is unramified at every $p \in f^{-1}(R)$,

ii) near $f(C) \cap R$, R is smooth and the ramification index of π is 2,

iii) R and $f(C)$ intersect transversally.

Equivalently, we can choose local analytic coordinates $(x_1, \dots, x_n)$ near p and $(y_1, \dots, y_n)$ near $\pi(p)$ such that π is given by

$$\pi : (x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1}, x_n^2),$$

and C is parametrized as

$$f : t \mapsto \big(f_1(t), \dots, f_n(t)\big),$$

where $f_i(0) = 0$, $\big(f_1'(0), \dots, f_{n-1}'(0)\big) \neq (0, \dots, 0)$ and $f_n'(0) \neq 0$. Thus $\pi \circ f$ is given by

$$\pi \circ f : t \mapsto \big(f_1(t), \dots, f_{n-1}(t), f_n^2(t)\big) =: \big(g_1(t), \dots, g_n(t)\big). \tag{13.2}$$

Its image is a smooth curve germ (since $\big(g_1'(0), \dots, g_{n-1}'(0)\big) \neq (0, \dots, 0)$) which is simply tangent to the branch locus $(y_n = 0)$ (since $g_n(t) = f_n^2(t)$ vanishes with multiplicity 2 at $t = 0$).

Let us now consider a complex analytic deformation

$$\big(G_1(t,s), \dots, G_n(t,s)\big) \text{ of } \big(g_1(t), \dots, g_n(t)\big),$$

where s varies in a polydisc D^r, the G_i are analytic and $G_i(t,0) = g_i(t)$. When can we lift this local deformation of $\pi \circ f$ to a deformation of f? From (13.2) we see that our only choice is to take $F_i(t,s) = G_i(t,s)$ for $i < n$ and $F_n(t,s) = \sqrt{G_n(t,s)}$. That is, the lifting is possible iff $G_n(t,s)$ is a square.

(There are two possible choices of the square root, but only one of these will agree with $f_n(t,0)$ when $s = 0$. Thus, if a lifting exists, it is unique.)

Lemma 13.3. *There is a hypersurface $H \subset D^r$ such that $\big(G_1(t,s), \dots, G_n(t,s)\big)$ lifts to a deformation $\big(F_1(t,s), \dots, F_n(t,s)\big)$ iff $s \in H$.*

Proof. By assumption, $G_n(t,s)$ contains t^2 with nonzero coefficient. By the Weierstrass preparation theorem (cf. [GH78], p. 8) we can write

$$G_n(t,s) = U(t,s) \cdot \big(t^2 + b(s)t + c(s)\big)$$

where $U(0,0) \neq 0$. Thus $G_n(t,s)$ is a square iff $b(s)^2 - 4c(s) = 0$. Thus $H \subset D^r$ is defined by the equation $b(s)^2 - 4c(s) = 0$. □

Corollary 13.4. *Composing with* π *gives*

$$\pi_* : \mathrm{Mor}(C, X) \to \mathrm{Mor}(C, \mathbb{P}^n) \quad \textit{and} \quad \pi_* : \mathrm{Curves}(X) \to \mathrm{Curves}(\mathbb{P}^n).$$

If C *satisfies the assumptions* (13.1) *then both of these are local embeddings near* $[C]$ *and their image has codimension* $\leq (R \cdot C)$.

Proof. If $\pi : X \to \mathbb{P}^n$ is a local analytic isomorphism between neighborhoods $(x \in U_x)$ and $(\pi(x) \in V_x)$ then everything automatically lifts from V_x to U_x. Thus the only problem is at the points $x \in R \cap C$. For every such point, it is one condition to lift by (13.3). Thus there is a global lifting over the intersection of the hypersurfaces $\{H_x : x \in R \cap C\}$. □

We are now ready to prove the main dimension estimates for the spaces of curves. The formula is very much in the spirit of Riemann's original version of the Riemann–Roch theorem: we estimate $\dim \mathrm{Mor}(C, X)$ from below in terms of intersection numbers involving Chern classes. However, it is not known how to define appropriate analogs of the higher cohomology groups that would make the inequality into an equality, and thus establish a better parallel with the Riemann–Roch theorem.

Theorem 14. *Let* X *be a smooth quasi projective variety,* C *a proper reduced curve and* $f : C \to X$ *a morphism. Then*

$$\begin{aligned} \dim_{[f]} \mathrm{Mor}(C, X) &\geq c_1(X) \cdot f_*[C] + \dim X \cdot \chi(\mathcal{O}_C) \\ &= \deg_C f^* T_X + \dim X \cdot \chi(\mathcal{O}_C) \\ &= H^0(C, f^* T_X) - H^1(C, f^* T_X). \end{aligned}$$

Proof. Choose a general $\pi : X \to \mathbb{P}^n$ such that the assumptions (13.1) hold and the degree of $\pi \circ f$ is high enough. By (8),

$$\dim_{[\pi \circ f]} \mathrm{Mor}(C, \mathbb{P}^n) \geq c_1(\mathbb{P}^n) \cdot (\pi \circ f)_*[C] + \dim X \cdot \chi(\mathcal{O}_C)$$

and by (13.4) this gives that

$$\dim_{[f]} \mathrm{Mor}(C, X) \geq \pi^* c_1(\mathbb{P}^n) \cdot f_*[C] + \dim X \cdot \chi(\mathcal{O}_C) - (R \cdot f_*[C]).$$

By the Hurwitz formula $c_1(X) = \pi^* c_1(\mathbb{P}^n) - R$, giving the inequality in (14).

The first equality in the statement is clear since $c_1(X) = c_1(\det T_X)$ and

$$\deg_C f^* T_X = \deg_C f^* \det T_X = c_1(X) \cdot f_*[C]$$

by the projection formula. The last equality is just Riemann–Roch for curves. □

The same argument also gives the dimension estimate for the space of curves:

Theorem 15. *Let X be a smooth projective variety and $C \subset X$ a smooth curve. Then*

$$\dim_{[C]} \operatorname{Curves}(X) \geq c_1(X) \cdot [C] + (\dim X - 3) \cdot \big(1 - g(C)\big),$$

where $\operatorname{Curves}(X)$ *denotes either the Chow variety or the Hilbert scheme of curves on X.*

It is not hard to modify our arguments to see that (15) also holds if C is a reduced curve with locally smoothable singularities, see [Kol96], II.1.14.

16. A philosophical claim and a challenge. The philosophical claim is that the estimates (14) and (15) are optimal. We will see that in many cases indeed equality holds, thus in this weak sense they are optimal. More substantively, if we take a general almost complex perturbation of the complex structure of X, then (14) and (15) should become equalities at every "interesting" point of the space of pseudo-holomorphic curves. (The formula frequently miscounts the dimension for curves $f : C \to X$ for which $C \to f(C)$ is a multiple cover. In some sense, we can ignore these if we study curves on X. However, if one looks at families of curves on X, such multiple covers naturally arise as limits of embedded curves. For definitions and more details, see Section 5.) One may thus claim that (14) and (15) correctly compute the dimension of the spaces of curves that persist under small almost complex perturbations of X, but they fail to take into account the curves that exist only "accidentally."

For instance, if $X_e^3 \subset \mathbb{P}^4$ is a smooth hypersurface of degree e, then we get that

$$\dim_C \operatorname{Curves}(X_e^3) \geq (5 - e) \deg C,$$

and the philosophical claim is that equality should hold. In particular, for $e = \deg X_e^3 \geq 6$ this predicts that there are no curves at all on X! Equivalently, every curve on a smooth, 3-dimensional hypersurface of degree ≥ 6 is "accidental."

On the other hand, from the point of view of almost complex manifolds, being an algebraic variety is an "accident," and on an algebraic variety there are many curves. One of the biggest challenges of the theory of curves on varieties is to explain how to correct the formulas (14) and (15) for algebraic varieties. After all, our aim is to study algebraic varieties, not almost complex manifolds.

Aside 17. In general, from (9.1), we get the more precise formula

$$\begin{aligned}\dim_{[f]} \operatorname{Mor}_L(C, X) \geq{}& c_1(X) \cdot f_*[C] + \dim X \cdot \chi(\mathcal{O}_C) - g(C)\\ &+ \dim X \cdot h^1(C, L) - h^1(C, L^e).\end{aligned}$$

For instance, if $C \subset X^n \subset \mathbb{P}^{n+1}$ is embedded by (a subsystem of) the canonical system, then

$$\dim_{[f]} \operatorname{Mor}_{K_C}(C, X) \geq c_1(X) \cdot \phi_*[C] + \dim X \cdot \chi(\mathcal{O}_C) - g(C) + \dim X.$$

Thus if a hypersurface contains such a curve, it contains a family of such curves whose dimension is at least ($\dim X$)–larger than one would expect.

3. Free curves and uniruled varieties

As in the Riemann–Roch theorem, the next step is to ask when the inequality in (14) is an equality. The following result describes essentially the only known general case when this holds and the local structure of $\operatorname{Mor}(C, X)$ is fully understood.

Theorem 18. *Let X be a smooth quasi projective variety, C a proper reduced curve and $f : C \to X$ a morphism. If $H^1(C, f^*T_X) = 0$ then* $\operatorname{Mor}(C, X)$ *is smooth at* $[f]$ *of dimension* $\deg_C f^*T_X + \dim X \cdot \chi(\mathcal{O}_C)$.

Proof. Assume first that $H^1(C, \mathcal{O}_C) = 0$. If $H^1(C, f^*T_X) = 0$ then, by (14), $\dim_{[f]} \operatorname{Mor}(C, X) \geq h^0(C, f^*T_X)$. On the other hand, by (12), the tangent space of $\operatorname{Mor}(C, X)$ at $[f]$ is $H^0(C, f^*T_X)$. The dimension of the tangent space is always at least the dimension and equality holds only at smooth points.

The case when $H^1(C, \mathcal{O}_C) \neq 0$ is harder since we have not proved that in (12.2) the map on the right is surjective. For a complete proof, see [Kol96], Sec. I.2. □

The above is a very useful result if there are many curves on X for which the condition $H^1(C, f^*T_X) = 0$ holds. Our next aim is to get a feeling how frequently this happens.

Exercise 19. Let A be an Abelian variety. Then T_A is trivial, hence $h^1(C, f^*T_A) = \dim A \cdot h^1(C, \mathcal{O}_C)$ for every C. Since an Abelian variety does not contain any rational curves, we see that $H^1(C, f^*T_A)$ is never zero.

(More precisely, if $\phi : \mathbb{P}^1 \to A$ is a constant map, then $H^1(\mathbb{P}^1, f^*T_A) = 0$. Here $\operatorname{Mor}(\mathbb{P}^1, A)$ consist only of constant maps, so $\operatorname{Mor}(\mathbb{P}^1, A) \cong A$ which is smooth of dimension $\dim A$. While it may seem silly to think about such cases, it is useful to study maps from reducible curves to a variety that may be constant on some of the irreducible components. In using induction, we frequently need to make sure that our formulas work for constant maps as well.)

Exercise 20. Let $C \subset \mathbb{P}^n$ be a smooth complete intersection of hypersurfaces of degrees $d_1, \ldots, d_{n-1}$. Check that $H^1(C, f^*T_{\mathbb{P}^n}) = 0$ if and only if either $g(C) \leq 1$ or $C \subset \mathbb{P}^n$ is canonically embedded.

Up to permuting the d_i, the first happens only in the cases

$$(1,1,\ldots,1),(2,1,\ldots,1),(3,1,\ldots,1),(2,2,1,\ldots,1).$$

The second possibility holds only in the cases $(4),(3,2),(2,2,2)$.

Exercise 21. Let $f : C \to \mathbb{P}^n$ be a smooth curve such that $\deg C \ge 2g(C) - 1$. Then $H^1(C, f^*T_{\mathbb{P}^n}) = 0$.

Example 22. Let X be a projective homogeneous space (for instance, $\mathbb{P}^n$, a smooth quadric, a Grassmannian, ...). Then T_X is generated by global sections, hence f^*T_X is also generated by global sections for every $f : C \to X$. If $C \cong \mathbb{P}^1$, this implies that $H^1(\mathbb{P}^1, f^*T_X) = 0$.

Thus $\operatorname{Mor}(\mathbb{P}^1, X)$ is everywhere smooth and of the expected dimension.

This might be the only case where $\operatorname{Mor}(\mathbb{P}^1, X)$ is everywhere nice:

Conjecture 23 [CP91]. *Let X be a smooth projective variety. The following are equivalent:*

(1) $H^1(\mathbb{P}^1, f^*T_X) = 0$ *for every* $f : \mathbb{P}^1 \to X$.

(2) *There is a morphism* $p : X \to Y$ *such that*
   (a) $X \to Y$ *is a locally trivial fiber bundle whose fibers are projective homogeneous spaces under a linear algebraic group, and*
   (b) *every map* $\mathbb{P}^1 \to Y$ *is constant.*

(Here we allow the uninteresting special case when every map $\mathbb{P}^1 \to X$ is constant and we take $Y = X$.)

We have basically established above that (23.2) implies (23.1). The converse is known in low dimension; essentially as a consequence of much stronger classification results. The case of complete intersections is proved in [Pan04].

Note also that, at first sight, the blow up $B_p\mathbb{P}^2$ of $\mathbb{P}^2$ at a point seems like a counter example. Indeed, $\operatorname{Aut}(B_p\mathbb{P}^2)$ is transitive away from the exceptional curve $E \subset B_p\mathbb{P}^2$, hence $H^1(\mathbb{P}^1, f^*T_{B_p\mathbb{P}^2}) = 0$, unless $f(\mathbb{P}^1) \subset E$. The normal bundle of E itself is $\mathcal{O}_E(-1)$, thus the tangent bundle restricted to E is $\mathcal{O}_E(2) + \mathcal{O}_E(-1)$ and its H^1 is zero.

However, if $f : \mathbb{P}^1 \to E$ is a degree 2 map, then $f^*T_{B_p\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^1}(4) + \mathcal{O}_{\mathbb{P}^1}(-2)$, and its H^1 is nonzero.

Definition 24. A morphism $f : \mathbb{P}^1 \to X$ is called *free* if f^*T_X is generated by global sections. Equivalently, if $H^1\big(\mathbb{P}^1, f^*T_X(-1)\big) = 0$. Informally, these are the rational curves that can be deformed in every possible direction in X.

Note also that if f^*T_X is generated by global sections over a nonempty open set then it is generated by global sections everywhere.

A morphism $f : \mathbb{P}^1 \to X$ is called *very free* if $H^1(\mathbb{P}^1, f^*T_X(-2)) = 0$. Since every vector bundle on $\mathbb{P}^1$ is a direct sum of line bundles, this is equivalent to saying that

$$f^*T_X = \sum_{i=1}^{\dim X} \mathcal{O}_{\mathbb{P}^1}(a_i) \qquad \text{with } a_i \geq 1.$$

Informally, these are the rational curves that can be deformed in every possible direction in X while keeping a given point fixed.

The following theorem says that, in some sense, every rational curve on a variety is either free or special.

Theorem 25. *Let X be a smooth projective variety. Then there are countably many proper closed subvarieties $V_i \subset X$ such that every $f : \mathbb{P}^1 \to X$ whose image is not contained in any of the V_i is free.*

Proof. There are countably many irreducible components of $\mathrm{Mor}(\mathbb{P}^1, X)$. Let us see how each of them leads to finitely many of the V_is.

To be slightly more general, let $U \subset \mathrm{Mor}(\mathbb{P}^1, X)$ be any irreducible subset that we are interested in. Consider the universal morphism $\mathrm{univ}_U : U \times \mathbb{P}^1 \to X$. If univ_U is not dominant, the closure of $\mathrm{univ}_U(U \times \mathbb{P}^1)$ will be one of the V_is.

Thus assume that univ_U is dominant. Then there is a dense open subset $W \subset \mathbb{P}^1 \times U$ such that $\mathrm{univ}_U : W \to X$ is smooth and its derivative $d(\mathrm{univ}_U) : T_{\mathbb{P}^1 \times U} \to \mathrm{univ}_U^* T_X$ is surjective over W.

Let $\pi_2 : \mathbb{P}^1 \times U \to U$ be the second projection and assume that $u \in \pi_2(W)$. Then

$$\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}^{\dim U} \cong T_{\mathbb{P}^1 \times U}|_{\mathbb{P}^1 \times \{u\}} \to f_u^* T_X$$

is surjective on the open set $W \cap \mathbb{P}^1 \times \{u\}$. Therefore f_u is free.

This takes care of the maps corresponding to points in $\pi_2(W)$. The complement $U \backslash \pi_2(W)$ is a union of lower dimensional subvarieties. We repeat the above argument for each, and so on. Eventually, for any irreducible component $U_i \subset \mathrm{Mor}(\mathbb{P}^1, X)$ we get finitely many proper closed subvarieties $V_{ij} \subset X$ such that for every $[f] \in U_i$, either $f(\mathbb{P}^1)$ is contained in some V_{ij} or f is free.

Since $\mathrm{Mor}(\mathbb{P}^1, X)$ has countably many irreducible components, we may need to exclude countably many subvarieties V_{ij}. (It is very poorly understood when one actually needs countably many exceptions. Such examples are given by $\mathbb{P}^2$ blown up at $m \geq 9$ general points and by general K3 surfaces.) □

This shows that the three conditions in the following definition are equivalent.

Definition 26. A smooth projective variety X over $\mathbb{C}$ is *uniruled* iff the following equivalent conditions hold.

(1) There is a dense open set $X^0 \subset X$ such that for every $x \in X^0$, there is a rational curve through x.

(2) There is a variety Y and a dominant and generically finite map $Y \times \mathbb{P}^1 \dashrightarrow X$.

(3) There is a free morphism $f : \mathbb{P}^1 \to X$.

Thus, if X is not uniruled, then all rational curves on X lie in the union of at most countably many subvarieties $V_i \subsetneq X$. We think of this as having only "few" rational curves on X.

4. Very free curves and rationally connected varieties

This section is mostly a summary of the basic results. For a general overview, see [Kol01]. An introduction with proofs is given in [AK03], while those wishing to go through all the details should consult the original papers or [Kol96].

Being uniruled is not a good structural property, since $Y \times \mathbb{P}^1$ is uniruled for any Y. For over a century it has been an open problem to define a good subclass of uniruled varieties that does not contain any such "mongrel" examples and generalizes to higher dimensions the following result about surfaces

27. Basic trichotomy of surface theory. Let S be a smooth projective surface. Then exactly one of the following holds:

(1) S is not uniruled (and hence has at most countably many rational curves),

(2) S is uniruled but maps to a non-uniruled (equivalently, of genus ≥ 1) curve (and hence all rational curves on S are in the fibers of this map), or

(3) S is birational to $\mathbb{P}^2$ (and hence has many rational curves).

By now it is clear that the key property is to require rational curves not just through any point, but through any pair of points. One can then imagine several variants of this concept. The next result shows that all of these are equivalent. For the proofs, see [KMM92c], [Kol96], [AK03], [San07].

Theorem 28. *Let X be a smooth projective variety over $\mathbb{C}$. The following are equivalent.*

(1) *There is a dense open set $X^0 \subset X$ such that for every $x_1, x_2 \in X^0$, there is a rational curve through x_1 and x_2.*

(2) *For every* $x_1, x_2 \in X$*, there is a rational curve through* x_1 *and* x_2*.*

(3) *For every integer* $m > 0$*, and every* $x_1, \dots, x_m \in X$*, there is a rational curve through* $x_1, \dots, x_m$*.*

(4) *For every* 0*-dimensional subscheme* $Z \subset \mathbb{P}^1$*, every morphism* $f_Z : Z \to X$ *can be extended to a morphism* $f : \mathbb{P}^1 \to X$*.*

(5) *There is a dense open set* $X^0 \subset X$ *such that, for every* $x_1, x_2 \in X^0$*, there is a connected curve with rational components through* x_1 *and* x_2*.*

(6) *There is a very free morphism* $f : \mathbb{P}^1 \to X$*.*

(7) *Let* C *be a smooth curve,* $Z \subset C$ *be a* 0*-dimensional subscheme,* $r > 0$ *an integer and* $W \subset X$ *a subscheme of codimension* ≥ 2*. Then every morphism* $f_Z : Z \to X$ *can be extended to a morphism* $f : C \to X$ *such that*
   (a) $H^1\big(C, f^*T_X(-r)\big) = 0$,
   (b) $f(C \setminus Z)$ *is disjoint from* W*,*
   (c) f *is an embedding on* $C \setminus Z$ *if* $\dim X \geq 3$*,*
   (d) f *is an embedding on* C *if* $\dim X \geq 3$ *and* f_Z *is an embedding.*

Definition 29. Let X be a smooth projective variety over $\mathbb{C}$. We say that X is *rationally connected* if it satisfies the equivalent conditions of (28).

There are two additional (partially conjectural) characterizations that are of interest.

Conjecture 30. *Let* X *be a smooth projective rationally connected variety. Let* C *be a smooth curve,* $D \subset C$ *a Euclidean open set and* $\phi_D : D \to X$ *a holomorphic map. Then there is a sequence of algebraic maps* $f_r : C \to X$ *such that the* $f_r|_D$ *converge to* ϕ_D *in the compact-open topology.*

Moreover, if $Z \subset D$ *is a finite set then we can also assume that* $f_r|_Z = \phi_D|_Z$*.*

31. Loop spaces of rationally connected varieties. [LS07] proves that a variety X is rationally connected iff its loop space ΩX is rationally connected. By a loop space we mean the space of all continuous/differentiable maps form the circle S^1 to X. (The loop spaces have a natural complex Banach manifold structure, see [LS07] for detail.) Aside from technicalities, the key result is the following, which fits very nicely in the sequence of characterizations in (28).

(8) Fix $m > 0$ continuous/differentiable maps $\phi_i : S^1 \to X$ and distinct points $p_i \in \mathbb{P}^1$. Then there is a sequence of continuous/differentiable maps $\Phi_r : S^1 \times \mathbb{CP}^1$ that are algebraic on each $\{s\} \times \mathbb{CP}^1$ such that, as $r \to \infty$, the $\Phi_r|_{S^1 \times \{p_i\}}$ converge to ϕ_i for every i.

Conjecturally, one can even achieve that $\Phi_r|_{S^1\times\{p_i\}} = \phi_i$ for every i, but this is proved only for general ϕ_i.

Being rationally connected is stable under various operations:

Theorem 32 [KMM92c]. *Let X be a smooth projective rationally connected variety. Then:*

(1) *Every smooth projective variety that is birational to X is also rationally connected.*

(2) *Every smooth projective variety that is the image of X by a rational map is also rationally connected.*

(3) *Every smooth projective variety that is a deformation of X is also rationally connected.*

It is also useful to know that many varieties are rationally connected:

Theorem 33 ([Nad91], [KMM92b], [KMM92a], [Cam92]). *Let X be a smooth, projective Fano variety, that is, with $-K_X$ ample. Then X is rationally connected.*

Theorem 34 ([GHS03]). *Let X be a smooth, projective variety and $f : X \to Y$ a morphism. Assume that Y is rationally connected and so are the smooth fibers of f. Then X is also rationally connected.*

We also have the basic trichotomy of algebraic varieties, which is a close analog of (27). The map in (35.2) is called the *MRC-fibration* or *maximal rationally connected fibration* of X.

Theorem 35 ([KMM92c]). *Let X be a smooth projective variety. Then exactly one of the following holds:*

(1) *X is not uniruled* (*and hence has "few" rational curves*),

(2) *there is a map $\pi : X \dashrightarrow Y$ onto a non-uniruled variety with $0 < \dim Y < \dim X$ whose general fibers are rationally connected* (*and hence most rational curves on X are in the fibers of π*), *or*

(3) *X is rationally connected* (*and hence has many rational curves*).

5. Connections with symplectic geometry

For general introductions to the topics in this section, see [FP97], [MS98], [MS04] and for more technical details consult [BF97], [LT98], [LT99].

36. Symplectic structure of varieties. Any smooth projective variety admits a symplectic structure. This can be constructed as follows. On $\mathbb{C}^{n+1}$ consider the Fubini–Study 2-form

$$\omega' := \frac{\sqrt{-1}}{2\pi}\left[\frac{\sum dz_i \wedge d\bar{z}_i}{\sum |z_i|^2} - \frac{(\sum \bar{z}_i\, dz_i) \wedge (\sum z_i\, d\bar{z}_i)}{(\sum |z_i|^2)^2}\right].$$

It is closed, non-degenerate on $\mathbb{C}^{n+1}\setminus\{0\}$ and invariant under scalar multiplication. Thus ω' descends to a symplectic 2-form ω on $\mathbb{CP}^n = (\mathbb{C}^{n+1}\setminus\{0\})/\mathbb{C}^*$. This construction depends on the choice of a basis in $\mathbb{C}^{n+1}$ and one can see that ω is invariant under the unitary group $U(n+1)$ but not under $\mathrm{Aut}(\mathbb{CP}^n) = PGL(n+1)$. Thus it is better to think of $\mathbb{CP}^n$ yielding not just one symplectic manifold $(\mathbb{CP}^n, \omega)$ but rather a whole family of symplectic manifolds parametrized by the connected space $PGL(n+1)/U(n+1)$.

Generalizing this, we say that two symplectic manifolds (M, ω_0) and (M, ω_1) are *symplectic deformation equivalent* if there is a continuous family of symplectic manifolds (M, ω_t) starting with (M, ω_0) and ending with (M, ω_1).

If $X \subset \mathbb{CP}^n$ is any smooth variety, then the restriction $\omega|_X$ makes $X(\mathbb{C})$ into a symplectic manifold. (Note that this $\omega|_X$ has nothing to do with the dualizing sheaf, commonly denoted by ω_X.) The resulting symplectic manifold $\big(X(\mathbb{C}), \omega|_X\big)$ depends on the choice of ω, but the dependence is rather clear. Thus to every smooth projective variety and the choice of a (very) ample cohomology class, the above construction associates a symplectic manifold $\big(X(\mathbb{C}), \omega|_X\big)$ which is unique up to symplectic deformation equivalence.

A powerful way to relate properties of the symplectic manifold $\big(X(\mathbb{C}), \omega|_X\big)$ to the algebraic geometry of X is through the enumerative properties of stable curves.

Definition 37. Let X be a variety. A genus g stable stable curve with n marked points over X is a triple (C, P, f), where

(1) C is a proper connected curve having only nodes,

(2) $P = (p_1, \dots, p_n) \subset C$ is an ordered set of smooth points of C,

(3) $f : C \to X$ is a morphism, and

(4) C has only finitely many automorphisms that fix P and commute with f.

As for any map of a curve to a variety, $f_*[C] \in H_2\big(X(\mathbb{C}), \mathbb{Z}\big)$ is well defined.

For a given $\beta \in H_2\big(X(\mathbb{C}), \mathbb{Z}\big)$ and subvarieties $Z_i \subset X$, let

$$\mathcal{M}_{g,n}(X, \beta, Z_1, \dots, Z_n)$$

denote the set of all genus g stable curves with n marked points such that $f_*[C] = \beta$ and $f(p_i) \in Z_i$ for $i = 1, \dots, n$.

38. Gromov–Witten invariants. It is not hard to define families of stable curves, interpret $\mathscr{M}_{g,n}(X, \beta, Z_1, \dots, Z_n)$ as a moduli functor and define its coarse moduli space $M_{g,n}(X, \beta, Z_1, \dots, Z_n)$. We are mostly interested in the cases when $M_{g,n}(X, \beta, Z_1, \dots, Z_n)$ is a finite set with reduced scheme structure. In these cases functorial definitions add nothing interesting to the picture.

An easy generalization of the dimension estimates in Section 2 shows that

$$\begin{aligned} &\dim M_{g,n}(X, \beta, Z_1, \dots, Z_n) \\ &\qquad \geq c_1(X) \cdot \beta + (\dim X - 3) \cdot \chi(\mathcal{O}_C) - \sum_i (\operatorname{codim}_X Z_i - 1), \end{aligned} \tag{38.1}$$

(There are two ways to think about this formula. First, in the third step of (10) we can use M_g. Then the last term in (38.1) comes from the observation that in a family of curves on X it is $(\operatorname{codim}_X Z - 1)$ conditions to intersect a subvariety $Z \subset X$. Alternatively, we can use $M_{g,n}$ (which adds an extra n to the formula) and note that in a family of pointed curves on X it is $\operatorname{codim}_X Z$ conditions for the marked point to lie on Z. Since there are n of the Z_i, this cancels out the extra n.)

If the right-hand side of (38.1) is 0, we expect that there are only finitely many maps in $\mathscr{M}_{g,n}(X, \beta, Z_1, \dots, Z_n)$ and their number, called a *Gromov–Witten invariant* of X, has enumerative significance.

In fact, one can define a Gromov–Witten invariant

$$\Phi_{g,n}(X, \beta, Z_1, \dots, Z_n) \tag{38.2}$$

whenever the right-hand side of (38.1) is 0, even if $M_{g,n}(X, \beta, Z_1, \dots, Z_n)$ is positive dimensional. One advantage of Gromov–Witten invariants is that they depend on the symplectic structure only. That is, although $M_{g,n}(X, \beta, Z_1, \dots, Z_n)$ visibly depends on the algebraic variety X and the Z_i, the value of the Gromov–Witten invariants depends only on the symplectic deformation class of (M_X, ω), the numbers g, n and the homology classes β, $[Z_i]$:

$$\Phi_{g,n}(X, \beta, Z_1, \dots, Z_n) = \Phi_{g,n}\big((M_X, \omega), \beta, [Z_1], \dots, [Z_n]\big). \tag{38.3}$$

In general, Gromov–Witten invariants are rational numbers which can be negative or zero. (A curve C with automorphisms may count as $1/|\mathrm{Aut}(C)|$ and positive dimensional components may have negative contribution; see (42) for such an example.) There are, however, a few cases when the obvious algebraic count gives the Gromov–Witten invariant.

Example 38.4. Assume that $\mathcal{M}_{g,r}(X, \beta, Z_1, \dots, Z_r) = \emptyset$. Then, for any homology classes $[Z_{r+1}], \dots, [Z_n]$, the Gromov–Witten invariant $\Phi_{g,n}(M_X, \omega, \beta, [Z_1], \dots, [Z_n])$ is zero.

Example 38.5. Assume that for every $(C, f) \in \mathcal{M}_{0,0}(X, \beta)$ we have that $C \cong \mathbb{P}^1$, f is an immersion and $f^*T_X = \mathcal{O}(2) + \mathcal{O}(-1)^{\dim X - 1}$. Then the Gromov–Witten invariant equals the number of such maps:

$$\Phi_{0,0}(X, \beta) = \#\mathcal{M}_{0,0}(X, \beta).$$

Example 38.6. The case when the Z_i are "sufficiently general," can be reduced to (38.5). First we need to assume that, for every i, $f(p_i)$ is a smooth point of Z_i and $p_i = f^{-1}(Z_i)$ scheme theoretically (that is, Z_i and $f(C)$ meet only at $f(p_i)$ and transversally).

If this holds, let $B_Z X \to X$ denote the blow up of $\bigcup Z_i \subset X$, $\tilde{f} : C \to B_Z X$ the lifting of f and $\beta_Z := \tilde{f}_*[C]$. ($B_Z X$ can be a quite singular space. However, by our assumptions, $\bigcup Z_i$ is smooth in a neighborhood of $f(C)$, thus the singularities of $B_Z X$ will not matter to us.)

Under these assumptions, $f \mapsto \tilde{f}$ establishes a bijection

$$\mathcal{M}_{g,n}(X, \beta, Z_1, \dots, Z_n) \leftrightarrow \mathcal{M}_{g,0}(B_Z X, \beta_Z).$$

That is, the blow up removed the subvarieties Z_i and the marked points $P \subset C$ from the picture.

Now assume further that, for every $(C, P, f) \in \mathcal{M}_{0,n}(X, \beta, Z_1, \dots, Z_n)$, the curve C is isomorphic to $\mathbb{P}^1$, f is an embedding and $\tilde{f}^*T_X = \mathcal{O}(2) + \mathcal{O}(-1)^{\dim X - 1}$. Then the Gromov–Witten invariant is just the number of such maps:

$$\Phi_{0,n}(X, \beta, Z_1, \dots, Z_n) = \#\mathcal{M}_{0,n}(X, \beta, Z_1, \dots, Z_n).$$

Example 38.7. More generally, assume that the expected and the true dimensions of $\mathcal{M}_{g,n}(X, \beta, Z_1, \dots, Z_n)$ are both zero and the corresponding maps are birational to their image. Then

$$\Phi_{g,n}(X, \beta, [Z_1], \dots, [Z_n]) \geq \#\mathcal{M}_{g,n}(X, \beta, Z_1, \dots, Z_n).$$

We use these examples to show that being uniruled is a property of the underlying symplectic variety.

Theorem 39 ([Kol98], 4.2.10, [Rua99]). *Let X_1, X_2 be smooth projective varieties and (M_i, ω_i) the corresponding symplectic manifolds. Assume that (M_1, ω_1) is symplectic deformation equivalent to (M_2, ω_2). Then X_1 is uniruled iff X_2 is.*

Proof. Assume that X_1 is uniruled. Fix a very general point $x_1 \in X_1$ such that the corresponding point $x_2 \in X_2$ is also very general. Let H be a very ample divisor on X_1 and $f : \mathbb{P}^1 \to X_1$ a map such that $f(0) = x_1$ and $(H \cdot f_*[\mathbb{P}^1])$ is the smallest possible. Set $\beta := f_*[\mathbb{P}^1]$ and consider $\mathscr{M}_{0,1}(X_1, \beta, x_1)$. Since $(H \cdot f_*[\mathbb{P}^1])$ is the smallest possible, every such curve is irreducible. Since $x_1 \in X_1$ is very general, every such curve is free (25). Set $r := \dim \mathscr{M}_{0,1}(X_1, \beta, x_1)$, let $H_1, \dots, H_{2r} \subset X_1$ be general divisors linearly equivalent to H and consider

$$\mathscr{M}_{0,r+1}(X_1, \beta, x_1, H_1 \cap H_2, \dots, H_{2r-1} \cap H_{2r}),$$

that is, stable rational curves in the homology class β that pass through x_1 and intersect each of the $H_{2j-1} \cap H_{2j}$.

The assumptions of (38.7) are satisfied and therefore

$$\begin{aligned}\Phi_{0,r+1}(X_1, \beta, x_1, [H_1 \cap H_2], \dots, [H_{2r-1} \cap H_{2r}])\\ \geq \# \mathscr{M}_{0,r+1}(X_1, \beta, x_1, H_1 \cap H_2, \dots, H_{2r-1} \cap H_{2r}) > 0.\end{aligned}$$

(By [Kol96], II.3.14.4, all our maps are immersions, thus we could have used the more elementary (38.6).)

Since the Gromov–Witten numbers are symplectic invariants, this implies that

$$\Phi_{0,r+1}(X_2, \beta, x_2, [H_1 \cap H_2], \dots, [H_{2r-1} \cap H_{2r}]) > 0.$$

Thus, by (38.4), $\mathscr{M}_{0,1}(X_2, \beta, x_2)$ is not empty and so X_2 also contains a rational curve through x_2. (Note that $\mathscr{M}_{0,1}(X_2, \beta, x_2)$ may not contain any irreducible curves, but all the irreducible components of the curves in $\mathscr{M}_{0,1}(X_2, \beta, x_2)$ are rational, so we do get an irreducible rational curve through x_2 after all.) Thus X_2 is also uniruled. □

This was only a warm-up for the main question, which would be a significant generalization of (32.3).

Conjecture 40 ([Kol98], 4.2.7). *Let X_1, X_2 be smooth projective varieties and (M_i, ω_i) the corresponding symplectic manifolds. Assume that (M_1, ω_1) is symplectic deformation equivalent to (M_2, ω_2). Then X_1 is rationally connected iff X_2 is.*

This is still wide open, even for 3-folds, despite significant partial results in [Voi08].

41. Instead of discussing the positive results, let us illustrate the problem by a simple example showing why the proof of (39) does not work directly in the rationally connected case.

For $e \geq 0$ consider the minimal ruled surface $\pi : \mathbb{F}_{2e} \to \mathbb{P}^1$. The class of a fiber is denoted by F and the negative section by E_{2e}. Then $H_2(\mathbb{F}_{2e}, \mathbb{Z}) = \mathbb{Z}[E_{2e}] + \mathbb{Z}[F]$, $(E_{2e}^2) = -2e, (E_{2e} \cdot F) = 1$ and $(F^2) = 0$.

Note that $\mathbb{F}_{2e}$ is deformation equivalent to $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$. Under this equivalence, the fibers correspond to each other but E_{2e} corresponds to $E_0 - eF$.

Given 2 general points $p, q \in \mathbb{F}_{2e}$, the smallest degree curve (in any projective embedding) connecting p and q is a reducible curve, consisting of the two fibers F_p (resp. F_q) through p (resp. q) and of E_{2e}. Its homology class is $[E_{2e} + 2F]$ and $E_{2e} + F_p + F_q$ is the unique curve passing through p, q whose homology class is $[E_{2e} + 2F]$.

We may be tempted to believe that having such a curve is a symplectic property. However, this fails, even for smooth algebraic deformations. Indeed, by the above remarks, the homology class $[E_{2e} + 2F]$ becomes $[E_0 + (2 - e)F]$ on $\mathbb{F}_0$, and there is no effective curve on $\mathbb{F}_0$ in the homology class $[E_0 + (2 - e)F]$ if $e > 2$.

We can try next to work with irreducible curves. The smallest homology class that contains an irreducible rational curve through p, q is $[E_{2e} + 2eF]$. The linear system $|E_{2e} + 2eF|$ has dimension $2e + 1$ and its general member is a smooth rational curve. Those curves that pass through p, q form a linear subsystem of dimension $2e - 1$. At first sight it seems that we can repeat the arguments of (39).

There is, however, a hitch. We need to consider not the space of curves in $\mathbb{F}_{2e}$ but the space of maps of curves to $\mathbb{F}_{2e}$. For irreducible curves, these two spaces are essentially the same, but in general problems arise with multiple covers. In our case, we can have maps whose set-theoretic image is $F_p + F_q + E_{2e}$, but give a degree $2e - 1$ cover over F_p. These have a moduli space of dimension $2(2e - 1) - 2 = 4e - 4$. Thus, for $e \geq 2$, this has greater dimension than the "main component" which is birational to the linear system $|E_{2e} + 2eF|(-p - q)$.

As we see in (42), larger dimensional components may give a negative contribution to a Gromov–Witten invariant which may cancel the positive contribution given by the irreducible curves.

Actually, in this case, we end up with a correct argument if we follow the method of (39).

By fixing $2e - 1$ other general points $r_1, \ldots, r_{2e-1}$, we see that there is a unique curve in $|E_{2e} + 2eF|$ passing through all the points $p, q, r_1, \ldots, r_{2e-1}$. This curve is smooth and irreducible and gives 1 for the value of the Gromov–Witten invariant. The larger dimensional components do not contribute anything. This, however, seems more luck than a general principle.

The following example, based on a suggestion of R. Pandharipande, illustrates that negative contributions from too large components can cancel out a nice curve, even for algebraic deformations.

Example 42. In $\mathbb{P}^3$ consider the family of quadrics

$$Q_t := (x_0^2 - x_1 x_2 - t^2 x_3^2 = 0).$$

For $t \neq 0$ we get a smooth quadric, isomorphic to $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$. Let $|L_t|$ denote one of the two families of lines. For $t = 0$ we get a singular quadric. We can resolve the singularity by blowing up $(x_0 - tx_3 = x_1 = 0)$. We get a family of smooth surfaces. For $t \neq 0$ we still have $\mathbb{F}_0$, but for $t = 0$ we get $\mathbb{F}_2$, the blow-up of Q_0 at the origin. Let $E_2 \subset \mathbb{F}_2$ be the exceptional curve and $|L_0|$ the birational transforms of the family of lines on Q_0. Note that as $t \to 0$, the limit of the family of lines $|L_t|$ is the family $E_2 + |L_0|$ of reducible curves.

Let $|H|$ be the pull-back of the family of hyperplane sections of Q_0 to $\mathbb{F}_2$. Its singular members form the family $E_0 + 2|L_0|$.

Over the pair of lines $(st = 0) \subset \mathbb{C}^2_{s,t}$ consider a family of curves and smooth surfaces as follows.

(1) Over the t-axis, we have the family Q_t degenerating to $\mathbb{F}_2$ and curves $2L_t$ degenerating to $2E_2 + 2L_0$.

(2) Over the s-axis, we have the trivial family of $\mathbb{F}_2$ with curves $E_2 + H_s$ degenerating to $2E_2 + 2L_0$.

Set $X := \mathbb{F}_2 \times \mathbb{P}^1$, let β be the class of $\{(\text{point})\} \times \mathbb{P}^1$ and set $Z_i := (E_2 + H^i) \times \{p_i\}$ where $p_i \in \mathbb{P}^1$ are two distinct points and $H^i \in |H|$ are two smooth hyperplane sections intersecting at 2 distinct points $q_1, q_2 \in \mathbb{F}_2$.

Consider $\mathcal{M}_{0,2}(X, \beta, Z_1, Z_2)$. It consist of two isolated points corresponding to the curves $\{q_i\} \times \mathbb{P}^1 \hookrightarrow \mathbb{F}_2 \times \mathbb{P}^1$ (each contributing 1 to the Gromov–Witten invariant) and a 1-dimensional component of curves of the form $\{(\text{point})\} \times \mathbb{P}^1$ that are contained in $E_2 \times \mathbb{P}^1$. The expected dimension is 0.

If we move over to $\mathbb{F}_0 \times \mathbb{P}^1$, the Z_i can be represented by curves of the form $L_i + L_i'$ where the lines L_i, L_i' are in the same family of lines. Thus we can choose Z_1 and Z_2 to be disjoint, showing that $\mathcal{M}_{0,2}(X, \beta, Z_1, Z_2) = \emptyset$.

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