

Symplectic categories

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Abstract. Quantization problems suggest that the category of symplectic manifolds and symplectomorphisms be augmented by the inclusion of canonical relations as morphisms. These relations compose well when a transversality condition is satisfied, but the failure of the most general compositions to be smooth manifolds means that the canonical relations do not comprise the morphisms of a category.

We discuss several existing and potential remedies to the nontransversality problem. Some of these involve restriction to classes of lagrangian submanifolds for which the transversality property automatically holds. Others involve allowing lagrangian “objects” more general than submanifolds.

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1. Introduction

This paper is based on two lectures given at the Geometry Summer School at the Instituto Superior Tecnico in Lisbon, in July of 2009. We describe several ongoing efforts to build categories whose objects are symplectic manifolds and whose morphisms are canonical relations. The lectures also included a discussion of categories whose hom-objects are symplectic manifolds, with the composition of morphisms a canonical relation, but we do not include that topic in this paper.

The material presented here is based on the work of many people, including the author’s work in progress with Alberto Cattaneo, Benoit Dherin, Shamgar Gurevich, and Ronny Hadani.

1.1. Canonical relations as morphisms in a category. A *canonical relation* between symplectic manifolds M and N is, by definition, a lagrangian submanifold

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of $M \times \bar{N}$, where \bar{N} is N with its symplectic structure multiplied by -1 . For example, the graph of a symplectomorphism is a canonical relation, as is any product of lagrangian submanifolds in M and N .

In his work on Fourier integral operators, Hörmander [13], following Maslov [16], observed that, under a transversality assumption, the set-theoretic composition of two canonical relations is again a canonical relation, and that this composition is a “classical limit” of the composition of certain operators.

Shortly thereafter, Sniatycki and Tulczyjew [22] defined *symplectic relations* as isotropic¹ submanifolds of products and showed that this class of relations was closed under “clean” composition (see Section 2 below). They also observed that the natural relation between a symplectic manifold and the quotient of a submanifold by the kernel of the pulled-back symplectic form is a symplectic relation.

Following in part some (unpublished) ideas of the author, Guillemin and Sternberg [10] observed that the linear canonical relations (i.e., lagrangian subspaces of products of symplectic vector spaces) could be considered as the morphisms of a category, and they constructed a partial quantization of this category (in which lagrangian subspaces are enhanced by half-densities. The automorphism groups in this category are the linear symplectic groups, and the restriction of the Guillemin-Sternberg quantization to each such group is a metaplectic representation. On the other hand, the quantization of certain compositions of canonical relations leads to ill-defined operations at the quantum level, such as the evaluation of a delta “function” at its singular point, or the multiplication of delta functions.

The quantization of the linear symplectic category was part of a larger project of quantizing canonical relations (enhanced with extra structure, such as half-densities) in a functorial way, and this program was set out more formally by the present author in [28] and [29]. It was advocated there that canonical relations should be considered as the morphisms of a “category”, and that quantization should be a functor from there to a category of linear spaces and linear maps, consistent with some additional structures. The word “category” appears in quotation marks above because the composition of canonical relations can fail to be a canonical relation, as will be explained in detail below, so we do not have a category. Briefly, there are two problems.

- The composition of two canonical relations may fail even to be a manifold.
- In the linear symplectic category, where each space of morphisms has a natural topology as a lagrangian grassmannian manifold, the composition operation is discontinuous.

These two problems are both related to the possible failure of a transversality condition. It is in general hard to remedy this by imposing conditions on *individ-*

¹The calculus of coisotropic relations does not seem to have been introduced until much later, in [31].

ual canonical relations which are significantly weaker than local invertibility, though we do present in Section 5 below such a condition which yields a category whose objects are *germs* of symplectic manifolds around lagrangian submanifolds. Otherwise, we must do something more drastic, using objects more general than canonical relations as the morphisms in our category, or modifying the notion of category itself.

We begin this paper with a general discussion of the category in which the morphisms are arbitrary relations between sets. We then take a first look at the composition of linear and nonlinear canonical relations and at reduction by coisotropic submanifolds of symplectic manifolds. Next, we present a simple idea of Wehrheim and Woodward [24], who embed the canonical relations in a true category **SYMP** in what is in some sense the optimal way. Then we describe the category of microfolds [3], whose objects are germs of symplectic manifolds around lagrangian submanifolds. At this point, we are already outside the setting in which the morphisms are maps between sets, but we go even further in the next section by describing the construction by Wehrheim and Woodward [24] of a 2-category of which **SYMP** could be thought of as the “coarse moduli space”. After that, we review and extend an idea of Sabot [19], who deals with the discontinuities of the composition of linear canonical relations by forming the closure of the graph of the composition operation. The result is a multiple-valued operation whose graph is an algebraic subvariety of a product of Grassmann manifolds. Finally, we attempt to unify and extend the examples above by using the language of simplicial spaces, in which categories and groupoids appear as objects satisfying special “Kan conditions” (see, for example, [32]). For each of the remedies above, we briefly discuss the quantization of the resulting structure.

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2. Relations and their composition

We begin with the category **REL** whose objects are sets and for which the morphism space $\mathbf{REL}(X, Y)$ is simply the set of all subsets of $X \times Y$. We consider each such relation f as a morphism *to* X *from* Y . Linear [affine] subspaces of vector [affine] spaces form a subcategory of **REL**.

The natural exchange mappings $X \times Y \rightarrow Y \times X$ define a contravariant *transposition* functor $f \mapsto f^t$ from **REL** to itself.

For any relation to X from Y , X is the *target* and Y the *source*. The image $f(Y)$ of f under projection to X is the *range* of f , and the image $f^t(X) \subseteq Y$ is the *domain* of f . f is *surjective* if its range equals its target, and *cosurjective* if its domain equals its source (i.e., if it is “defined everywhere”). f is *injective* if, for any $x \in X$, there is at most one $(x, y) \in f$, and *coinjective* if there is at most one $(x, y) \in f$ for any $y \in Y$ (i.e. if it is “single valued”). Thus, the cosurjective and coinjective relations in $X \times Y$ are the graphs of mappings to X from Y . For any $y \in Y$, $f(y)$ denotes the image of f on $\{y\}$, i.e., the subset $\{x \in X \mid (x, y) \in f\}$ of X .

The composition $f \circ g$ of $f \in \mathbf{REL}(X, Y)$ with $g \in \mathbf{REL}(Y, Z)$ is

$$\{(x, z) \mid \exists y \in Y \text{ such that } (x, y) \in f \text{ and } (y, z) \in g\}.$$

It is useful to think of this as the result of a sequence of three operations: first, form the product $f \times g \subset X \times Y \times Y \times Z$; second, intersect it with $X \times \Delta_Y \times Z$, where Δ_Y is the diagonal in $Y \times Y$, to obtain the fibre product $f \times_Y g$; third, project this intersection into $X \times Z$.

When X and Y are manifolds and f is a (locally closed) submanifold of $X \times Y$, f is a *smooth relation*. When $f \in \mathbf{REL}(X, Y)$ and $g \in \mathbf{REL}(Y, Z)$ are smooth, the pair (f, g) is *transversal* if $f \times g$ is transversal to $X \times \Delta_Y \times Z$, so that their intersection $f \times_Y g$ is again a manifold. A transversal pair is *strongly transversal*, and we will write $f \pitchfork g$, if the projection map from $f \times_Y g$ to $X \times Z$ is an embedding onto a locally closed submanifold, in which case the image $f \circ g$ is again a smooth relation. When a pair is not strongly transversal, its composition may fail to be a submanifold, so the smooth relations do not form a subcategory of \mathbf{REL} .

A pair (f, g) of linear or affine relations is transversal if and only if the domain of f is transversal to the range of g as subspaces of Y , in which case the pair is necessarily strongly transversal. In particular, $f \pitchfork g$ whenever f is cosurjective or g is surjective. Transversality of smooth (f, g) is detected by the same criterion, applied fibrewise to the tangent relations Tf and Tg . If f is the graph of a smooth mapping to X from Y , then $f \pitchfork g$ is a transversal pair for any smooth relation g to Y from Z . Similarly, if g is the transpose of the graph of a smooth mapping from Y to Z , then $f \pitchfork g$ for every smooth relation f to X from Y . In particular, the category of smooth manifolds and smooth mappings is a subcategory of \mathbf{REL} .

There is a condition weaker than transversality which, together with an embedding condition, still insures that the composition of two smooth relations is again smooth. The pair (f, g) is *clean* if the fibre product $f \times_Y g$ is a submanifold of $X \times Y \times Y \times Z$, and if the natural inclusion of $T(f \times_Y g)$ in the fibre product tangent bundle $Tf \times_{TY} Tg$ is an equality (equivalently, if $T(f \circ g) = Tf \circ Tg$), and if the differential of the projection from $f \times_Y g$ to $X \times Z$ has constant rank. If this

projection is a submersion onto a locally closed submanifold of $X \times Z$, then $f \circ g$ is again a smooth relation, and the pair (f, g) is *immaculate*. For example, any composable pair of linear or affine relations is immaculate.

When X , Y , and Z are symplectic manifolds, $C = X \times \Delta_Y \times Z$ is a coisotropic submanifold of $X \times \bar{Y} \times Y \times \bar{Z}$, and the leaves of the characteristic foliation are connected components of the fibres of the projection from C to $X \times \bar{Z}$. It follows that, for canonical relations, the constant rank condition in the definition of a clean pair follows from the other conditions. For any transversal pair, the projection from $f \times_Y g$ to $X \times \bar{Z}$ is an immersion.

Finally, we note that any (lagrangian) submanifold L of a (symplectic) manifold X may be thought of as a smooth (canonical) relation to X from a point or to a point from X . Although points are neither initial nor terminal objects in our categories of relations, they still play a special role. For instance, the composition of L to a point from X with L' to X from a point is nonempty if and only if L and L' have nonempty intersection. Upon quantization, a point usually becomes the scalars \mathbb{C} , and the intersection of two lagrangian submanifolds represents geometrically the inner product of quantum states to which they correspond.

3. Canonical relations and coisotropic reduction

There are several connections between canonical relations and the reduction of symplectic manifolds by coisotropic submanifolds. In this section, we also give examples of the bad behavior of composition of canonical relations.

3.1. The linear case. We begin with the linear case, though much of what we write here applies immediately to the case of manifolds. (See the following section for further details.)

If C is a coisotropic subspace of a symplectic vector space X , the quotient X_C of C by the kernel C^\perp of the induced bilinear form carries a natural symplectic structure and is called the *reduced space*. It is connected to X by the canonical relation

$$r_c = \{(x, y) \in X_C \times X \mid y \in C \text{ and } x = [y]\},$$

where $[y]$ is the equivalence class of y modulo C^\perp . The composition of r_c with a lagrangian subspace L in X (a linear canonical relation to X from a point) gives the reduced lagrangian subspace $L_C = (L \cap C)/(L \cap C^\perp)$ in X_C . The composition is transversal, hence strongly transversal, exactly when L is transversal to C . We will refer to “transversal reduction” in this situation. Denoting by $\text{Lag}(X)$ the grassmannian of lagrangian subspaces of a symplectic vector space X , we will denote by R_C the operation $\text{Lag}(X_C) \leftarrow \text{Lag}(X)$ given by composition with r_c .

For any linear canonical relation $f \in \mathbf{REL}(X, Y)$, the range and domain are coisotropic subspaces of X and Y respectively, and f induces an isomorphism $X_{f(Y)} \xleftarrow{f_{\text{red}}} Y_{f^i(X)}$ between the reduced spaces, giving a natural factorization

$$f = r_{f(Y)}^t \circ f_{\text{red}} \circ r_{f^i(X)}$$

of any linear canonical relation as the (transversal!) composition of a transposed reduction, a symplectomorphism, and a reduction. In particular, any surjective linear canonical relation is essentially a reduction, and any cosurjective one is essentially a transposed reduction.

The composition of linear canonical relations is itself an instance of reduction. As we have already mentioned in the previous section, $X \times \Delta_Y \times Z$ is a coisotropic subspace of $X \times \bar{Y} \times Y \times \bar{Z}$, and

$$(X \times \Delta_Y \times Z)^\perp = \{0_X\} \times \Delta_Y \times \{0_Y\},$$

so $(X \times \bar{Y} \times Y \times \bar{Z})_{X \times \Delta_Y \times Z}$ is naturally isomorphic to $X \times \bar{Z}$. Under this isomorphism, the composed relation is the reduction of the product $f \times g$, and the composition is transversal if and only if the reduction is.

The following example shows that composition of linear canonical relations is not a continuous map between lagrangian grassmannians. If L_1 and L_2 are lagrangian subspaces of X , then $(L_1 \times L_2) \circ (L_2 \times L_1) = L_1 \times L_1$. If L_1 and L_2 are transversal, then $L_1 \times L_2$ is the limit as $a \rightarrow 0$ of the graphs $\Gamma_a = \{(T_a x, x)\}$ of the symplectomorphisms T_a defined by $T_a x = a^{-1}x$ for $x \in L_1$ and $T_a x = ax$ for $x \in L_2$. Similarly, $L_2 \times L_1$ is the limit of $\Gamma_{a^{-1}}$. The compositions $\Gamma_a \circ \Gamma_{a^{-1}}$ are all equal to the diagonal Γ_1 and hence so is their limit, but the composition of the limits is $L_1 \times L_1$.

3.2. The nonlinear case. We turn now to the case of manifolds. Any coisotropic submanifold C of a symplectic manifold X carries a *characteristic distribution* $TC^\perp \subseteq TC$ which is, by definition, the kernel of the pullback to C of the symplectic form on X . TC^\perp consists of the values of hamiltonian vector fields whose hamiltonians vanish on C and is always an integrable distribution, hence tangent to a foliation which we will denote by \mathcal{C}^\perp . If \mathcal{C}^\perp is *simple* in the sense that its leaves are the fibres of a submersion, then the leaf space $X_C = C/\mathcal{C}^\perp$ is again a symplectic manifold, and the reduction operation

$$r_C = \{(x, y) \in X_C \times X \mid x = [y], y \in C\},$$

where $[y]$ is the leaf of \mathcal{C}^\perp containing y , is a canonical relation to X_C from X .

Let π_C be the projection from C to X_C . For any lagrangian $L \subset X$ (a canonical relation to X from a point), the composition $r_C \circ L = r_C(L)$ is just the reduc-

tion $L_C = \pi_C(L \cap C)$. The composition is transversal when L is transversal to C , in which case the restriction of π_C to $L \cap C$ is an immersion onto L_C , but it is not necessarily injective. Furthermore, if L is not transversal to C , then $L \cap C$ may not even be a manifold. There, a reduction operation R_C to lagrangian submanifolds of X_C from the lagrangian submanifolds of X is not just discontinuous as in the linear case, but not even defined everywhere.

3.3. Examples. Here are two examples of the bad behavior of composition, in each case the result of a reduction operation on lagrangian submanifolds. Both examples are based on the construction by Hörmander [13] of lagrangian submanifolds of cotangent bundles in terms of phase functions, as interpreted by Guillemin and Sternberg [9] as an instance of coisotropic reduction.

In these examples, X is $T^*\mathbb{R}^2$, with canonical coordinates (q_1, q_2, p_1, p_2) , and C is the coisotropic submanifold defined by $p_2 = 0$. The leaves of the characteristic foliation are then the lines in the q_2 -direction, and the reduced manifold X_C may be identified with $T^*\mathbb{R}$ with coordinates (q_1, p_1) .

Let L_S be the image of the differential of a function $S(q_1, q_2)$, i.e., the lagrangian surface in X defined by $p_i = \partial S / \partial q_i$. To determine $R_C(L_S)$, we first see where $p_2 = 0$ on L , which amounts to finding the critical points of S along the fibres of the projection $(q_1, q_2) = 0$. For each such critical point, we get the point $(q_1, \partial S / \partial q_1)$ on L_C .

Now let S be $q_2^3 + q_1^2 q_2$, so that L_S is defined by $p_1 = 2q_1 q_2$ and $p_2 = 3q_2^2 + q_1^2$. Then $L_S \cap C$ is just a single point, and so is its image $(L_S)_C$ in the plane X_C . This is certainly not a lagrangian submanifold.

For the second example, we let ε be a parameter and let S_ε be the function $\varepsilon q_1 q_2$. The lagrangian submanifolds L_{S_ε} obviously depend smoothly on ε . When $\varepsilon \neq 0$, we have $q_1 = 0$ for all fibre-critical points of S , and L_{S_ε} is the line $q_1 = 0$. But when $\varepsilon = 0$, all points of \mathbb{R}^2 are fibre-critical, and L_{S_0} is the line $p_1 = 0$. Thus, the reduction operation R_C is not continuous when applied to linear canonical relations to $T^*\mathbb{R}^2$ from a point.

4. The Wehrheim–Woodward category

Wehrheim and Woodward [24] begin with the following construction to circumvent the problem of bad compositions. (Our terminology and notation throughout this section differ somewhat from theirs.) The result is in some sense the minimal way to produce a category whose morphisms include all of the canonical relations.

Definition 4.1. The Wehrheim–Woodward category **SYMP** is the category whose objects are symplectic manifolds and whose morphisms are generated by

the canonical relations, subject to the relation that the composition of f and g in **SYMP** is equal to the composition in **REL** when $f \pitchfork g$.

More explicitly, as in [24], we may begin with the category whose objects are symplectic manifolds and whose morphisms are sequences (f_1, \dots, f_r) of canonical relations which are composable in **REL**. We also include an empty sequence for each object, which functions as an identity morphism. Composition is given by concatenation of sequences. Set-theoretic composition of relations defines a functor from this category to **REL**.

Now introduce the smallest equivalence relation which is closed under composition from both sides, for which (f, g) is equivalent to fg when (f, g) is a strongly transversal pair, and for which each empty sequence is equivalent to the graph of the identity map on the corresponding object. The equivalence classes are the morphisms in **SYMP**, and the composition functor above descends to give a functor from **SYMP** to **REL**. It follows that distinct canonical relations, considered sequences with a single entry, give distinct morphisms in **REL**. The identity morphisms are the (equivalence classes of the) diagonals $\Delta_Y \subset Y \times \bar{Y}$. A morphism in **SYMP** is called a *generalized lagrangian correspondence* in [24], and a generalized lagrangian correspondence to X from a point is a *generalized lagrangian submanifold*.

SYMP is characterized by the universal property that any map Φ to any category \mathcal{C} from symplectic manifolds and canonical relations such that $\Phi(f \circ g) = \Phi(f)\Phi(g)$ whenever $f \pitchfork g$ factors uniquely through **SYMP**. It is thus tempting to think of **SYMP** as a universal quantization category. On the other hand, quantization of canonical relations by operators on function spaces requires enhancement of the morphisms by some extra structure, such as half-densities or half-forms. Thus, it is natural to try to extend **SYMP** and its variants by building larger categories with forgetful functors to **SYMP**. It will also be important to extend to such categories the basic operations on canonical relations, such as transpose and cartesian products.

The construction of **SYMP** is merely the beginning of what Wehrheim and Woodward do in [24]. Imposing topological conditions (involving Chern classes, Maslov classes, etc.) on the symplectic manifolds and canonical relations they define a certain subcategory² **SYMP'** of **SYMP** whose objects are “admissible” symplectic manifolds and whose morphisms are generated by “admissible” canonical relations. For each admissible manifold X , they construct a Donaldson-Fukaya category $\text{Don}^\sharp(X)$ whose objects are generalized lagrangian submanifolds in X and whose morphism spaces are Floer cohomology groups. Composition of morphisms involves counting pseudoholomorphic curves. For each admissible relation f to X from Y , they construct a functor $\text{Don}^\sharp(f)$ to $\text{Don}^\sharp(X)$ from

²Actually, there are two categories, one for exact symplectic manifolds and one for monotone ones.

$\text{Don}^\sharp(Y)$ such that, when (f, g) is a strongly transversal pair, the functors $\text{Don}^\sharp(f \circ g)$ and $\text{Don}^\sharp(f) \circ \text{Don}^\sharp(g)$ are naturally equivalent. Don^\sharp then extends to a functor from SYMP' to the category whose objects are categories and whose morphisms are natural equivalence classes of functors. Since the target category of this functor has an additive structure, we may view Don^\sharp as a kind of “quantization” of SYMP' .

We will continue our discussion of SYMP' and its quantization in Section 6.

5. Cotangent lifts and symplectic micromorphisms

This section summarizes some results from [3].

There is a natural but quite limited collection of symplectic manifolds and canonical relations which form a subcategory of SYMP . It is the image of a contra-variant functor T^* from the category MAN of smooth manifolds and smooth maps. Namely, for every smooth manifold A , T^*A is its cotangent bundle with the canonical symplectic structure, and for every smooth map³ $\phi : A \rightarrow B$, $T^*\phi$ is its *cotangent lift*, defined as the canonical relation $\{((x, (T\phi)^*(\eta)), (\phi(x), \eta)) \mid x \in A \text{ and } \eta \in T_{\phi(x)}^*B\}$ to T^*A from T^*B . As a manifold, $T^*\phi$ may be identified with the pulled back vector bundle $\phi^*(T^*B)$ over A . It is also the image of the conormal bundle to the graph of ϕ under the symplectomorphism from $T^*(A \times B)$ to $T^*A \times \overline{T^*B}$ given by reversing the sign of cotangent vectors to B . (A slightly different version of this map is called the Schwartz transform in [1].) It is easy to check that T^* embeds MAN as a subcategory $T^*\text{MAN}$ of SYMP (in particular, that composition of cotangent lifts is always strongly transversal).

It turns out that we can isolate a property of cotangent lifts which makes their compositions strongly transversal, and then we can look for more general situations where this property is satisfied. Remembering that the pair (T^*A, Z_A) , with Z_A the zero section, is the local model for any pair consisting of a symplectic manifold and its lagrangian submanifold, we make the following definition.

Definition 5.1. Let (X, A) and (Y, B) be pairs consisting of a symplectic manifold and a lagrangian submanifold. A canonical relation f to X from Y is *liftlike* with respect to A and B if there is a smooth map $\phi : A \rightarrow B$ such that

$$\begin{aligned} f(b) &= \phi^{-1}(b) && \text{for all } b \in B, \\ Tf(v) &= (T\phi)^{-1}(v) && \text{for all } v \in T_bB, \end{aligned}$$

where Tf is the tangent bundle of f considered as a submanifold of $TX \times TY$, hence a relation to TX from TY .

³Note that the arrow here goes from left to right.

Every cotangent lift is liftlike with respect to the zero sections, and the composition of liftlike relations is always transversal near the graph of the underlying map between lagrangian submanifolds, but in order to get a category we must localize around those submanifolds. (The use of the prefix “micro” below is meant to correspond to its use by Milnor [18] in the term “microbundle”).

Definition 5.2. A *manifold pair* consists of a manifold M and a closed submanifold $A \subseteq M$. Two manifold pairs (M, A) and (N, B) will be considered equivalent if $A = B$ and if there is a manifold pair (U, A) such that U is an open subset in both M and N simultaneously. A *microfold* is an equivalence class $[M, A]$ of manifold pairs (M, A) . The (well defined) submanifold A is the *core* of $[M, A]$.

Note that we require equality of neighborhoods and not merely diffeomorphism for two manifold pairs to be equivalent.

Most of the standard constructions on manifolds carry over to microfolds. In particular, a *submicrofold* of a microfold $[M, A]$ is a microfold $[N, B]$ such that $N \subseteq M$ and $B \subseteq A$, and the product $[M, A] \times [N, B]$ is $[M \times N, A \times B]$. A relation between two microfolds is just a submicrofold of their product. It is (the graph of) a *map* $[M, A] \leftarrow [N, B]$ if it has a representative which is a map. This makes the microfolds into a category **MIC**. There is a natural forgetful core functor $[M, A] \mapsto A$ and a cross section thereof $A \mapsto [A, A]$ (with the obvious actions on morphisms).

Definition 5.3. A *symplectic microfold* is a microfold $[M, A]$ together with a germ around A of symplectic structure on M for which A is lagrangian. $[T^*A, A]$ with the canonical symplectic structure on T^*A is the *cotangent microbundle* of A . A *symplectic micromorphism* to $[M, A]$ from $[N, B]$ is a lagrangian submicrofold of $[M, A] \times [\bar{N}, B]$ having a representative which is liftlike with respect to A and B . The associated map $\phi : A \rightarrow B$ is the *core map* of the micromorphism.

By design, any composition of symplectic micromorphisms is strongly transversal. Furthermore, these morphisms are closed under composition, so the symplectic microfolds and micromorphisms form a category **MICSYMP**. A symplectic micromorphism is a map if and only if it is invertible; these maps are just the symplectomorphisms in the micro world. A basic result in [27] is (without the microfold terminology) that every symplectic microfold is symplectomorphic to the cotangent microbundle of its core. Thus, the restriction of **MICSYMP** to the cotangent microbundles is a full subcategory containing the image of the functor $T^* : \mathbf{MAN} \rightarrow \mathbf{MICSYMP}$, but with many more morphisms. There is also a forgetful functor **CORE** : **MICSYMP** \rightarrow **MAN**.

The category **MICSYMP**, or rather its extension by a category of enhanced micromorphisms, carrying half-densities, should be quantized by a functor to a

category of semiclassical Fourier integral operators. What we have so far [4] is a category **FOUR** whose objects are manifolds and whose morphisms are certain operators between smooth half-densities on these manifolds which are formal series in a parameter \hbar . There is a “wavefront” functor from **FOUR** to **MICSYMP** (in fact, to the subcategory whose objects are cotangent microbundles) for which the inverse image of the identity morphism over the cotangent microbundle of a manifold A is an algebra of semiclassical pseudodifferential operators on A . This functor lifts to a principal symbol functor which attaches a half density to the wavefront of any operator. What is missing is a total symbol calculus which can make a symbol functor injective. Also missing is a construction of operators from general symplectic micromorphisms which are not acting on cotangent bundles. Even for cotangent bundles, the total symbol construction and its inverse appear to depend on extra structure, such as connections or local coordinates. The latter allows one to represent symplectic micromorphisms by generating families which serve as phase functions for the explicit construction of the kernels of operators as oscillatory integrals. The general problem is reminiscent of that of passing from local deformation quantizations of Poisson manifolds [15] to global ones, as in [5]. Perhaps some of the methods of the latter paper will be helpful.

Finally, we note that monoidal objects in the category **MICSYMP** are essentially local symplectic groupoids in the sense of [30] and correspond to Poisson manifolds. The construction of such objects in the formal and analytic categories was carried out in [2] and [6] using the “tree-level” part of the Kontsevich star product. A good quantization theory for **MICSYMP** should produce algebras from these monoidal objects.

6. The Wehrheim–Woodward 2-category

In the remaining sections, rather than restricting the nature of our canonical relations, we extend the notion of what a category should be. We begin by returning to the linear case.

The second version of Wehrheim and Woodward’s quantization follows the “groupoid philosophy” that, given an equivalence relation on a set S , one should always try to replace it by the finer structure of a category whose objects are the elements of S and whose isomorphism classes are the equivalence classes. **SYMP'** thus becomes a 2-category. In fact, recalling that any morphism f in **SYMP'**(X, Y) is a generalized lagrangian submanifold in $X \times \bar{Y}$, we may define the 2-morphism space **SYMP'**(f, g) to be the Floer cohomology group which comprises the morphism space to f from g in $\text{Don}^\sharp(X \times \bar{Y})$.

Similarly, for the category of categories which is the target of Don^\sharp , rather than simply identifying natural equivalence classes of functors, it is appropriate

to introduce the 2-category structure in which the natural equivalences become 2-morphisms. Wehrheim and Woodward now assign to each Floer cohomology class in $\mathbf{SYMP}'(\mathbf{f}, \mathbf{g})$ a natural transformation to $\text{Don}^{\sharp}(\mathbf{f})$ from $\text{Don}^{\sharp}(\mathbf{g})$ in such a way that they obtain a 2-functor from the 2-category \mathbf{SYMP}' of (admissible) symplectic manifolds, generalized canonical relations, and Floer cohomology classes, to the 2-category of categories, functors, and natural transformations.

The paragraphs above are merely a schematic description of a *tour de force* of symplectic topology using the authors' theory (see [25] and [26]) of “quilted pseudoholomorphic curves”. These are, roughly speaking, piecewise pseudoholomorphic curves satisfying “seam” conditions along smooth (real) curves separating the smooth pieces of the domain. But each piece of the curve maps to a different manifold, with the seams constrained by canonical relations.

7. Composition of linear canonical relations as a rational map

As we observed the end of Section 3.1, the composition of linear canonical relations is not continuous. In a paper about spectral analysis on fractal graphs, Sabot [19] introduced, in the special case of the composition of linear symplectic reductions with lagrangian subspaces, a construction which “fills in” the discontinuities of the composition operation on linear canonical relations. Composition now becomes multiple valued, just as a discontinuous step function on the line becomes multiple valued if the gaps in its graph are filled in with vertical line segments.

What follows below comes from Sabot's construction applied to general compositions, using the fact (see Section 3) that composition is also a special case of reduction. (We save the details for a future article.) Although we mostly have the case of vector spaces over \mathbb{R} in mind, Sabot works over \mathbb{C} . In view of the possible more general applications to finite fields suggested by [11] and [12], we will carry out as much as possible of the construction over an arbitrary field k .

For symplectic vector spaces X and Y over k , let S_{XY} denote the grassmannian (a manifold when k is \mathbb{R} or \mathbb{C}) of all lagrangian subspaces of $X \times Y$. For three spaces, X , Y , and Z (not necessarily distinct), composition of linear canonical relations is a mapping $M_{XYZ} : S_{XY} \times S_{YZ} \rightarrow S_{XZ}$. The restriction of this mapping to the transversal pairs is continuous, with graph

$$T_{XYZ} \stackrel{\text{def}}{=} \{(f_{XY}, f_{YZ}, f_{XZ}) \mid f_{XY} \pitchfork f_{YZ} \text{ and } f_{XY} \circ f_{YZ} = f_{XZ}\} \subset S_{XY} \times S_{YZ} \times S_{XZ}.$$

T_{XYZ} is dense in the graph of M_{XYZ} , but its closure \bar{T}_{XYZ} in $S_{XY} \times S_{YZ} \times S_{XZ}$ contains more. Sabot's description of \bar{T}_{XYZ} in the case of symplectic reduction extends to general compositions as follows.

For a composable pair (f_{XY}, f_{YZ}) , we measure their failure to be transversal by the *deficiency* $d(f_{XY}, f_{YZ})$, defined as the codimension of $(f_{XY} \times f_{YZ}) \oplus (X \times \Delta_Y \times Z)$ in $X \times \bar{Y} \times Y \times \bar{Z}$, which is also the dimension of the intersection $(f_{XY} \times f_{YZ}) \cap (\{0_X\} \times \Delta_Y \times \{0_Z\})$.

Theorem 7.1. *When k is \mathbb{R} or \mathbb{C} , \bar{T}_{XYZ} is an algebraic variety consisting of all triples (f_{XY}, f_{YZ}, f_{XZ}) for which the codimension of $f_{XZ} \cap (f_{XY} \circ f_{YZ})$ in $f_{XY} \circ f_{YZ}$ is at most $d(f_{XY}, f_{YZ})$. T_{XYZ} is the set of its regular points.*

We may think of \bar{T}_{XYZ} as the graph of a “continuous multiple valued function” \bar{M}_{XYZ} whose value on (f_{XY}, f_{YZ}) is a subvariety $f_{XY} \bullet f_{YZ}$ of S_{XZ} containing the usual composition $f_{XY} \circ f_{YZ}$. It is perhaps worth noting that the subvariety $f_{XY} \bullet f_{YZ}$ is a higher Maslov cycle in the sense of [7]. When k is an arbitrary field, we may take Theorem 7.1 as a *definition* of \bar{T}_{XYZ} and this multiple valued composition operation

Functions like \bar{M}_{XYZ} given by relations which are the closure of graphs of smooth maps are known as “rational maps” [8]. The next step (work in progress) is to make the operation \bullet into the composition operation in a category **LINSYMP** whose objects are symplectic vector spaces, but which is “enriched over” a category **RAT** whose objects are algebraic varieties and whose morphisms are rational maps. In other words, the morphism spaces in **LINSYMP** will be the grassmannians of canonical relations, but the composition operations will be rational maps.

Remark 7.2. Although the morphisms in **RAT** itself are relations, the composition operation there is not that of **REL**, but rather the operation which assigns to rational maps f and g the closure of their set theoretic product. To have the composition defined at all, one needs to assume that the rational maps are “dominant” in the sense of having dense range. This raises the further complication that certain parts of the structure, in particular the inclusion of the units, cannot be dominant. It begins to appear that making **RAT** into a category suitable for defining internal categories involves issues similar the ones we have been dealing with for symplectic categories. The only way out may be the simplicial approach described in Section 8.

As noted in the introduction, quantization of linear symplectomorphisms is not without its problems. Although Guillemin and Sternberg [10] show how to quantize symplectic vector spaces and to associate operators to canonical relations carrying half-densities, these operators may be unbounded and cannot always be composed. In fact, the undefined compositions of operators correspond precisely to nontransversal compositions of canonical relations. A simple example is the operator on “functions” on \mathbb{R} which assigns to each u the product of $u(0)$ with a

delta function at 0. This operator, which is a quantization of $L \times L$, where L is the fibre over 0 in $T^*\mathbb{R}$, cannot be composed with itself.

The discussion above suggests that we should try to modify the target category for the usual quantization, either by introducing a composition of operators which is multiple valued, or by looking at a category whose morphism spaces, while identified with spaces of linear operators, themselves admit “rational maps” which may not be defined (or may be multiply-defined) on individual operators.

Remark 7.3. The “completion” of the composition of canonical relations to something larger also occurs in the microlocal theory of sheaves [14].

Remark 7.4. Segal [21] suggests another approach to building and quantizing a well-behaved category of linear canonical relations. For any symplectic vector space (X, ω) (which could be a product $Y \times \bar{Z}$), the positive lagrangian subspaces L of the complexification $V_{\mathbb{C}}$ (with symplectic structure extend from V by complex bilinearity) are those for which $i\omega(v, \bar{v}) > 0$ for all nonzero $v \in V$. Positive canonical relations are always transversally composable, and their composition is smooth. They do not quite form a category, since the identities are missing, but they may be added “by hand”. Quantization of this category (with the objects enhanced by metaplectic structure and the morphisms enhanced by half-forms) may be done without any obstructions. Taking the real limit remains a problem, though.

Remark 7.5. It would be interesting to see how the linear theory extends to lagrangian affine subspaces of symplectic affine spaces. The hermitian line bundles of geometric quantization may play a more important role here.

8. The simplicial picture

The structure of a category \mathcal{C} can be encoded in that of a simplicial object $N(\mathcal{C})$ called its *nerve*. We recall (see for example [20], [32], or any book on homological algebra) that a simplicial object is a collection of objects S^k ($k = 0, 1, 2, \dots$) in some base category (such as sets, topological spaces, or manifolds) together with, for each k , $k + 1$ face morphisms $S^k \rightarrow S^{k-1}$ and $k + 1$ degeneracy morphisms $S^k \rightarrow S^{k+1}$ satisfying the composition laws of the generators of the category of order-preserving mappings among the sets $\Sigma^n = \{0, 1, \dots, n\}$. The elements of S^0 are sometimes called *vertices* and those of S^1 *edges*. In $N(\mathcal{C})$, the vertices are the objects of \mathcal{C} and the edges are the morphisms, with the faces of a morphism being its target and source, and the degeneracy operator taking each object to its identity morphism. It is convenient to write $\mathcal{C}^{[k]}$ for $N^k(\mathcal{C})$. The composition of morphisms is encoded in $\mathcal{C}^{[2]}$, whose elements are the composable pairs (f, g) , with the face operators taking each such pair to f , fg , and g , while the degeneracy

operators take f to pairs with an identity morphism appended to one side or the other. The rest of the structure is determined by this part, with $\mathcal{C}^{[k]}$ being the composable k -tuples, face operators given by the composition of pairs of adjacent entries or elimination of the entry on one end or the other, and degeneracy operators by the insertion of identities. The associative and identity axioms are equivalent to the compatibility conditions.

A simplicial topological (possibly discrete) space S has a *geometric realization* $|S|$ which is obtained from the disjoint union of $S^k \times \Delta_k$ for all k , where Δ_k is the usual k -simplex, by gluing them together using rules derived from the face and degeneracy operators. The usual cohomology of $|S|$ is called the cohomology of the simplicial space. For instance, if S is the nerve $N(G)$ of a group G , $|N(G)|$ is a model for the classifying space BG , and its cohomology is the group cohomology of G .

Not every simplicial object comes from a category or groupoid. The ones that do are characterized by so-called “horn-filling conditions”, the simplest of which require that a pair of edges with a common vertex be fillable (perhaps in a unique way) by a 2-simplex of which they are faces. By weakening these conditions, one arrives at generalizations of the notions of category and groupoid, as in [32].

Let us now apply this idea to the composition of linear canonical relations. For simplicity, we limit our attention to linear canonical relations from a fixed symplectic vector space X to itself. These form a monoid, i.e., a category $E(X)$ with one object, and we form the usual nerve in which $E^{[k]}(X)$ is the cartesian power $\text{Lag}(X \times \bar{X})^k$. Although each $E^{[k]}(X)$ is a topological space, and even a smooth manifold, $E^\bullet(X)$ is not a simplicial topological space because the boundary operators are discontinuous.

The first way to build a simplicial space out of this one is based on the construction of Wehrheim and Woodward described in Section 4. Within each $E^{[k]}(X)$, there is an open dense subset $E_{\cap}^{[k]}(X)$ of “completely transversal” sequences, namely those for which the composition $f_1 \circ \cdots \circ f_k$ is transversal in the sense that $L_1 \times \cdots \times L_k$ is transversal in $(X \times \bar{X})^k$ to the multidagonal $X \times (\Delta_X)^{k-1} \times \bar{X}$. It is not hard to show that this collection of subsets is invariant under the boundary and degeneracy operators, and that the restricted operators are smooth, making of $E_{\cap}^\bullet(X)$ a simplicial manifold.

Other constructions might be based on Sabot’s multiple-valued composition. The simplest one would be to take the closures of the graphs of composition on the $E_{\cap}^{[k]}(X)$ defined in Section 7 above, but it is not clear that these form a simplicial object. It may also be useful to consider structures in which there is a whole family of 2-simplices whose edges are a pair (L_1, L_2) of nontransversely composable relations and an element of $L_1 \bullet L_2$. In any case, it should be possible to have the face and degeneracy operators be ordinary mappings of varieties rather than rational maps, since the multiple-valuedness of composition is incorporated in

the definition of the spaces of simplices. Associativity may be taken into account through identification of some simplices, or through the introduction of extra higher simplices.

To quantize these simplicial versions of the symplectic category (more precisely, enhancements thereof), one will need to look for similar simplicial structures derived from the composition of unbounded operators on function spaces. This is essentially the point of view taken in [17]. Since the Floer cohomology classes which give 2-morphisms in the 2-category described in Section 6 are themselves equivalence classes of cochains, Mau, Wehrheim, and Woodward replace the Donaldson category attached to any admissible symplectic manifold by a Fukaya-type A^∞ category. To any admissible canonical relation they attach an A^∞ functor, and to any Floer cocycle between such relations a natural transformation of A^∞ functors, in a way which is compatible with composition of strongly transversal pairs, up to homotopy of A^∞ functors.

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