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# Diminishing functionals for nonclassical entropy solutions selected by kinetic relations

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**Abstract.** We consider nonclassical entropy solutions to hyperbolic conservation laws with concave-convex flux functions, whose undercompressive shocks are selected by a kinetic function $\varphi^\flat$. Extending earlier work of Baiti, LeFloch and Piccoli, we reinterpret their construction of the (generalized) strength of classical and nonclassical shocks, allowing us to simplify it, highlight its true nature and identify new degrees of freedom. Relying mainly upon the natural assumption that the composite function $\varphi^\flat \circ \varphi^\flat$ is uniformly contracting, we establish that the generalized total variation of front-tracking approximations is non-increasing in time, and we conclude with the existence of nonclassical solutions to the initial value problem. We also propose a definition of a generalized interaction potential, and investigate its monotonicity properties. In particular, we established that the interaction functional is globally non-increasing along a splitting-merging interaction pattern.

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## 1. Introduction

Consider the following initial value problem associated with a conservation law in one-space variable:

$$\begin{aligned} u_t + f(u)_x &= 0, \\ u(0,\cdot) &= u_0, \end{aligned} \tag{1.1}$$

where $u_0 : \mathbb{R} \to \mathbb{R}$ is a function with bounded variation (BV) on $\mathbb{R}$, and the (smooth) flux $f : \mathbb{R} \to \mathbb{R}$ is a *concave-convex function* in the sense that

$$\begin{gathered} uf''(u) > 0 \qquad (u \neq 0), \qquad f'''(0) \neq 0, \\ \lim_{|u| \to +\infty} f'(u) = +\infty. \end{gathered} \tag{1.2}$$

Following LeFloch [13], we consider nonclassical entropy solutions to this problem. Recall that, in many applications, only a single entropy inequality can be imposed on the solutions, i.e.,

$$U(u)_t + F(u)_x \leq 0, \tag{1.3}$$

where the so-called entropy $U$ is a given, strictly convex function and the entropy flux $F(u) := \int^u U'(v)f'(v)\,dv$ is determined by $U$. It is not difficult to construct multiple weak solutions to the initial value problem (1.1)–(1.3), so that one realizes that the single entropy inequality is too lax to determine a unique weak solution. In fact, for initial data restricted to lie in one region of concavity or convexity, the classical theory applies and leads to a unique entropy solution. Non-uniqueness arises when weak solutions contain transitions from positive to negative values, or vice-versa.

The above non-uniqueness property is closely related to the fact that discontinuous solutions, in general, depend upon their regularization, that is, different regularizations or approximations to the conservation law (1.1) may lead to different solutions in the limit. This, in particular, is true for solutions to the Riemann problem, corresponding to the initial data

$$u_0(x) = \begin{cases} u_l, & x < 0, \\ u_r, & x > 0, \end{cases} \tag{1.4}$$

where $u_l$, $u_r$ are constant states. Indeed, for a wide class of regularizations, including regularizations by nonlinear diffusion-dispersion terms, there exist nonclassical shocks which satisfy (1.3) yet violate Oleinik's entropy inequalities. The selection of nonclassical solutions is based on a *kinetic function* $\varphi^\flat : \mathbb{R} \to \mathbb{R}$ which, by definition, provides a characterization of admissible nonclassical shocks connecting two states $u_-$, $u_+$, that is,

$$u_+ = \varphi^\flat(u_-). \tag{1.5}$$

For scalar conservation laws and, more generally, nonlinear hyperbolic systems, LeFloch and co-authors initiated the development of a theory of nonclassical entropy solutions selected by the kinetic relation (1.5); many analytical and numerical issues have been covered. We refer to [12], [13], [14] for a review of the theory and to [16] for recent developments on the numerical approximation. On the other hand, the kinetic relation was originally introduced in the context of a hyperbolic-elliptic model describing the dynamics of phase transitions in liquids or solids, for which we refer the reader to Slemrod [20], Truskinovsky [21], Abeyaratne and Knowles [1], and LeFloch [11]. In particular, in [11] an existence theorem based on the Glimm scheme was established for a class of kinetic relations arising in phase dynamics. This subject has developed extensively since then, and we will not try to review this literature here.

Our objective in the present paper is to design functionals measuring the (generalized) total variation and wave interaction potential of nonclassical entropy solutions, in light of the earlier works [11], [2], [3], [4], [13], [17]. We consider solutions generated by Dafermos' front-tracking method [7], [8], when the local Riemann solutions are nonclassical and are determined by a given kinetic relation. We are interested in deriving uniform estimates for the total variation of solutions and showing that the scheme converges to global-in-time, nonclassical entropy solutions to the initial value problem (1.1).

This paper can be interpreted, in part, as a re-examination of the general definition of wave strength introduced by Baiti, LeFloch, and Piccoli [3]. The definition proposed there was somehow too abstract to be usable for systems of conservation laws while other simpler and more explicit definitions, such as those in [2], [4], [17] were somehow too simple to extend to systems of conservation laws. Therefore, in the present paper, we re-interpret the general definition of wave strength given in [3] as a straightforward change of variable in the $u$-variable. We then show that the change of variable can be constructed so as to satisfy additional identities and certain Lipschitz bounds. These results considerably clarify the analysis of BV bounds for solutions to conservation laws in that the arguments are simpler and allow the kinetic function to appear explicitly in the functionals. Most importantly, it appears that our new arguments are robust and may be generalized to tackle systems of equations. Yet, the notion of generalized wave strength under consideration possesses the same weaknesses as the one in [3] since it does not appear to be of use for kinetic functions defined by degenerate regularizations [3], [6]; see Example 2.3 below.

An outline of this paper follows. In Section 2, we begin with a brief review of the theory of kinetic relations and emphasize what we will need in the rest of this paper. We then introduce the definition of generalized wave strength from [3] and show that it can be constructed so as to satisfy two new properties. In Section 3, we establish that the proposed generalized total variation functional is non-increasing along a sequence of Dafermos' front-tracking solutions; cf. Theorem 3.1. In Section 4, we turn to the construction of an interaction functional based on Glimm's original construction. In Theorem 4.2 we show that Glimm's definition leads to an interaction functional which is non-increasing in all but four interaction patterns. Next, in Section 5 we establish that, despite the caveats of Theorem 4.2, the proposed interaction functional is actually globally non-increasing, at least in the significant case of merging-splitting wave patterns originally introduced by LeFloch and Shearer [17]. Hence, Theorem 5.1 below demonstrates the relevance of the proposed interaction functional to handle nonclassical entropy solutions.

## 2. Kinetic functions and generalized wave strength

**2.1. Assumptions on the kinetic function.** We begin by discussing some aspects of the theory of kinetic functions $\varphi^\flat$ for nonclassical entropy solutions to (1.1). This theory provides a synthetic description of the class of solutions generated by regularizations.

As a starting point, we consider the problem of describing the set of solutions $u$ to the Riemann problem (1.4) that can be realized as limits $\hat{u} \to u$ of solutions to the regularized conservation law

$$\hat{u}_t + f(\hat{u})_x = \beta\big(b(\hat{u}, \hat{u}_x)\hat{u}_x\big)_x + \gamma\big(c_1(\hat{u})\big(c_2(\hat{u})\hat{u}_x\big)_x\big)_x, \tag{2.1}$$

where $\hat{u} \to u$ as $\beta, \gamma \to 0$, with $\alpha := \beta^2/\gamma$ fixed. The purpose of the kinetic function $\varphi^\flat$ is to describe this class of solutions $u$ without requiring the explicit evaluation of the limit (2.1). The general theory [13] shows that for each conservation law (1.1), each entropy condition (1.3), and each regularization (including but not limited to those of the form (2.1)), the admissible undercompressive shocks are those whose right- and left- and hand states satisfy $u_+ = \varphi^\flat(u_-)$, for some function $\varphi^\flat : \mathbb{R} \to \mathbb{R}$.

The fundamental conditions required on the kinetic function are the following ones:

(A1) The map $\varphi^\flat : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous and one-to-one.

(A2) $\varphi^\flat(0) = 0$ and $\varphi^\flat$ is monotone decreasing.

(A3) There exists a compact neighborhood $I_1$ of the origin in which $\mathrm{Lip}_{I_1}(\varphi^\flat) < 1$.

(A4) The second iterate $\varphi^\flat \circ \varphi^\flat$ is a strict contraction, i.e., for some $C_1 \in (0,1)$,

$$|\varphi^\flat \circ \varphi^\flat(u)| \le C_1 |u|, \qquad u \in \mathbb{R}. \tag{2.2}$$

These assumptions do hold for a large class of (non-degenerate) regularizations, two of which are discussed in Examples 2.1 and 2.2, below. Natural analogues of the properties (A1), (A2), (A3) and (A4) are known to hold for *systems*, when $\varphi^\flat$ is properly defined in this more general setting. The same conditions were assumed in [3], and we refer the reader to the monograph [13] and the references cited therein for more detailed information.

We now proceed to describe, with the help of the kinetic function, the set of admissible states $u_-$, $u_+$ arising on the left- and the right-hand sides of a discontinuity. When $u_- > 0$ ($u_- < 0$, respectively), nonclassical shocks are required when the amplitude of the shock is large enough and the threshold $u_+ < \varphi^\sharp(u_-)$ ($\varphi^\sharp(u_-) < u_+$, resp.) is reached. This threshold function $\varphi^\sharp : \mathbb{R} \to \mathbb{R}$ is defined from $\varphi^\flat$ as the unique value $\varphi^\sharp(u_-) \notin \{u_-, \varphi^\flat(u_-)\}$ such that

$$\frac{f(u_-) - f\big(\varphi^\flat(u_-)\big)}{u_- - \varphi^\flat(u_-)} = \frac{f(u_-) - f\big(\varphi^\sharp(u_-)\big)}{u_- - \varphi^\sharp(u_-)}. \tag{2.3}$$

Geometrically, $\big(\varphi^\sharp(u_-), f\big(\varphi^\sharp(u_-)\big)\big)$ is the point at which the line connecting the points $\big(u_-, f(u_-)\big)$ and $\big(\varphi^\flat(u_-), f\big(\varphi^\flat(u_-)\big)\big)$ crosses the graph of $f$ and, therefore, for instance when $u_- > 0$ the following inequalities must hold

$$\varphi^\flat(u_-) \leq \varphi^\sharp(u_-) \leq u_-. \tag{2.4}$$

For concave-convex flux (1.2), the nonclassical entropy solution to the Riemann problem with data (1.4) and $u_l > 0$, is

i) a shock if $\varphi^\sharp(u_l) \leq u_r$, or
ii) a nonclassical shock connecting $u_l$ to $\varphi^\flat(u_l)$ followed by a classical shock connecting $\varphi^\flat(u_l)$ to $u_r$ if $\varphi^\flat(u_l) < u_r < \varphi^\sharp(u_l)$, or else
iii) a nonclassical shock connecting $u_l$ to $\varphi^\flat(u_l)$ followed by a rarefaction connecting $\varphi^\flat(u_l)$ to $u_r$ if $u_r \leq \varphi^\flat(u_l)$.

When $u_l < 0$, the nonclassical Riemann solver is similar.

Given two states $u_-$ and $u_+$ separating a discontinuity, the rate of entropy production, say $D(u_-, u_+)$ (associated with a given entropy) can be used to characterize nonclassical solutions. It is proven in [13] that, say for $u_- > 0$, $D$ admits two distinct roots $\varphi_0^\flat(u_-)$ and $u_-$, between which $D$ is negative, and admits a unique global minimum at $\varphi^\natural(u_-)$, where $\varphi^\natural(u)$ is the function defined by the tangency condition

$$f'\big(\varphi^\natural(u)\big) = \frac{f\big(\varphi^\natural(u)\big) - f(u)}{\varphi^\natural(u) - u}. \tag{2.5}$$

Given that all shocks with $\varphi^\natural(u_-) \leq u_+ \leq u_-$ are classical (i.e., would satisfy Oleinik's entropy conditions), the nonclassical entropy admissible shocks are those corresponding to $\varphi_0^\flat(u_-) \leq u_+ \leq \varphi^\natural(u_-)$. In fact, for all $u \neq 0$ one can check that

$$|\varphi^\natural(u)| \leq |\varphi^\flat(u)| < |\varphi_0^\flat(u)|. \tag{2.6}$$

Our assumptions on $f$ and $U$ (and independently of the chosen regularization) imply that $\varphi_0^\flat$ satisfies (A1) and (A2) as well as the stronger condition

$$\varphi_0^\flat \circ \varphi_0^\flat(u) = u. \tag{2.7}$$

A key point here is that these properties of $\varphi^\natural$ and $\varphi_0^\flat$ are independent of the regularization which defines $\varphi^\flat$. One may also show that $\varphi^\natural$ and $\varphi_0^\flat$ are piecewise $C^1$ [13].

We conclude the discussion of kinetic functions with the description of three important regularizations, defined in terms of the functions $b$, $c_1$ and $c_2$ appearing in (2.1).

**Example 2.1** (Linear diffusion and dispersion). As shown in [17] for the flux $f(u) = u^3$, the entropy $U(u) = u^2/2$, and the regularization $b = c_1 = c_2 \equiv 1$, the zero dissipation kinetic function is $\varphi_0^\flat(u) = -u$ while $\varphi^\natural(u) = -u/2$ and

$$\varphi^\flat(u) = \begin{cases} -u + \frac{1}{2}A(\alpha), & u \geq A(\alpha), \\ \varphi^\natural(u), & |u| \leq A(\alpha), \\ -u - \frac{1}{2}A(\alpha), & u \leq -A(\alpha), \end{cases} \tag{2.8}$$

where $\alpha := \beta^2/\gamma$ in (2.1) and $A(\alpha) = 2\alpha\sqrt{2}/3$.

**Example 2.2** (General non-degenerate diffusion and dispersion). Bedjaoui and LeFloch [5] considered $f$ satisfying (1.2), $b(u,v) = b(u)$ smooth and bounded $0 < \underline{B} \leq b(u,v) \leq \bar{B}$ and the entropy $U'' = c_1/c_2$. For each value of $\alpha := \beta/\gamma$, they showed that there exists thresholds $A_-(\alpha)$ and $A_+(\alpha)$ such that $\varphi_\alpha^\flat$ satisfies either

$$\varphi_\alpha^\flat(u) = \varphi^\natural(u) \qquad \text{for } A_-(\alpha) \leq u \leq A_+(\alpha), \tag{2.9}$$

or, the inequalities

$$0 < |\varphi^\natural(u)| < |\varphi_\alpha^\flat(u)| < |\varphi_0^\flat(u)|, \tag{2.10}$$

when either $u < A_-(\alpha)$ or $A_+(\alpha) < u$. The thresholds vanish only if $\alpha$ vanishes. Moreover, for any fixed $u \neq 0$, $\varphi_\alpha^\flat(u) \to \varphi_0^\flat(u)$ as $\alpha \to 0$.

**Example 2.3** (Degenerate diffusion-dispersion model). Bedjaoui and LeFloch [6] also analyzed regularizations of the form (2.1) where $b$ vanished nonlinearly as $u_x \to 0$, namely $c_1 = c_2 = 1$ and $b(u,v) = |v|^p v$ in (2.1). Such nonlinear diffusion-dispersion models occurs in certain models of fluids and in particular when using Von Neumann-Richtmyer artificial viscocity. For $f(u) = u^3$ and $U(u) = u^2/2$, then we again have $\varphi_0^\flat(u) = -u$ and $\varphi^\natural(u) = -u/2$. The kinetic function depends strongly on both $\alpha$ and the exponent $p$. It was shown that there still exists thresholds $A_-$ and $A_+$ such that both (2.9) and (2.10) hold. However, these thresholds may degenerate in the following manner:

(i) For $0 < p \leq \frac{1}{3}$, we have

$$(\varphi_{p,\alpha}^\flat)'(0) = -\frac{1}{2}, \qquad A_\pm(0) = 0, \qquad A_\pm'(0\pm) = \pm\infty.$$

(ii) For $\frac{1}{3} < p < \frac{1}{2}$, we have $(\varphi_{p,\alpha}^\flat)'(0) = -\frac{1}{2}$ and $A_\pm(\alpha) = 0$.

(iii) For $\frac{1}{2} < p$, we have $(\varphi_{p,\alpha}^\flat)'(0) = -1$ and $A_\pm(\alpha) = 0$.

This description provides a portrait of the complexity that may be found in degenerate regularizations. We note that when $p > 1/2$ the kinetic function fails to satisfy assumption (A4). See Theorems 2.2 and 2.3 in [6] for more detailed statements.

**2.2. Generalized wave strength.** Building on the pioneering work by Baiti, LeFloch, and Piccoli [3], we simplify here the original notion of generalized wave strength introduced therein, and we relate it directly to the fundamental contraction property (A4) of the kinetic function. Whereas the definition of wave strength in [3] was the most general possible, we show that a specialization of their argument allows one to recover an explicit and rather natural form of the wave strength. We begin by taking a fresh look at the results in [3] using a slightly different notation.

Consider a general change of variable defined by

$$\tilde{u} = \psi(u) := \begin{cases} \psi_+(u), & 0 \leq u, \\ \psi_-(u), & u < 0, \end{cases} \tag{2.11}$$

in which (to begin with) we assume only that $\psi$ is monotone increasing. Given any function $g = g(u)$ we write $\tilde{g}(\tilde{u}) = \psi \circ g \circ \psi^{-1}(\tilde{u})$ to distinguish its equivalent in the variables $\tilde{u}$ from the original mapping $g$ in the variables $u$. In [3], the authors essentially established the following result.

**Lemma 2.4.** *For $\varphi^\flat$ satisfying* (A1)–(A4) *and $M := \|u_0\|_{L^\infty}$ ($u_0$ being the initial data in* (1.1)), *there exists a change of variable* (2.11) *for which*

$$\tilde{u} + \tilde{\varphi}^\flat(\tilde{u}) \tag{2.12}$$

*is monotone increasing for all $\tilde{u} \in [-M, M]$. Moreover, $\tilde{\varphi}^\flat$ satisfies* (A1)–(A4) *with respect to $\tilde{u}$, and the change of variable $\psi$ can be chosen to satisfy the following four properties:*

(B1) *$\psi$ is Lipschitz continuous and one-to-one.*

(B2) *$\psi(0) = 0$ and $\psi$ is monotone increasing.*

(B3) *There exists a Lipschitz constant such that*

$$0 < \underline{\mathrm{Lip}}(\psi) := \inf_{u \neq v} \left| \frac{\psi(u) - \psi(v)}{u - v} \right|. \tag{2.13}$$

(B4) *Moreover,* (2.12) *is uniformly increasing in the sense that, for some constants $0 < a_1 < a_2$,*

$$a_1 < \underline{\mathrm{Lip}}\big(\tilde{u} + \tilde{\varphi}^\flat(\tilde{u})\big) \leq \mathrm{Lip}\big(\tilde{u} + \tilde{\varphi}^\flat(\tilde{u})\big) < a_2. \tag{2.14}$$

*Proof.* In fact, [3] proves that there exists a Lipschitz continuous and monotone increasing change of variable (2.11) such that

$$\psi_+(u) - (-\psi_- \circ \varphi^\flat)(u) \tag{2.15}$$

is monotone increasing for $u \geq 0$, while

$$-\psi_-(u) - (\psi_+ \circ \varphi^\flat)(u) \tag{2.16}$$

is monotone decreasing for $u < 0$. Writing $\tilde{u} = \psi(u)$, then for $u \geq 0$ expression (2.15) becomes

$$\psi_+(u) + \psi_- \circ \varphi^\flat \circ \psi_+^{-1} \circ \psi_+(u) = \tilde{u} + \tilde{\varphi}^\flat(\tilde{u}),$$

while for $u < 0$, expression (2.16) becomes

$$-\psi_-(u) - \psi_+ \circ \varphi^\flat \circ \psi_-^{-1} \circ \psi_-(u) = -\big(\tilde{u} + \tilde{\varphi}^\flat(\tilde{u})\big).$$

Therefore the fact that functions (2.15) and (2.16) are respectively monotone increasing and decreasing is equivalent to the fact that (2.12) is monotone increasing.

The properties (B1)–(B4) are consequences of the construction described in [3] although (B3) and (B4) are somewhat hidden in the proof. Equation (5.11) in [3] implies property (B3), while equation (5.6) states that

$$\psi(u) + \psi \circ \varphi^\flat(u) = u, \qquad u \in \mathbb{R}.$$

To check (B4), we use the previous identity to compute, for every $\tilde{u} \neq \tilde{v}$,

$$\left|\frac{\tilde{u} + \tilde{\varphi}^\flat(\tilde{u}) - \tilde{v} - \tilde{\varphi}^\flat(\tilde{v})}{\tilde{u} - \tilde{v}}\right| = \left|\frac{u - v}{\psi(u) - \psi(v)}\right|.$$

Therefore, it is easy to see that

$$\begin{aligned}
\underline{\mathrm{Lip}}\big(\tilde{u} + \tilde{\varphi}^\flat(\tilde{u})\big) &= 1/\mathrm{Lip}(\psi) > 0, \\
\mathrm{Lip}\big(\tilde{u} + \tilde{\varphi}^\flat(\tilde{u})\big) &= 1/\underline{\mathrm{Lip}}(\psi) < \infty.
\end{aligned}$$

This completes the proof of Lemma 2.4. □

Relying on the previous lemma, the authors in [3] proposed a definition of wave strength and showed that the resulting total variation functional is strictly decreasing at interactions in front-tracking approximations; see Section 3 below. We state here the definition from [3].

**Definition 2.5** (Notion of generalized wave strength). For each choice of change of variable $\tilde{u} = \psi(u)$, one defines the generalized strength $\sigma(u_-, u_+)$ of a classical or nonclassical wave $(u_-, u_+)$ as follows:

$$\sigma(u_-, u_+) = \begin{cases} |\tilde{u}_- - \tilde{u}_+|, & u_- u_+ \geq 0, \\ |\tilde{u}_- + \tilde{u}_+|, & u_- u_+ < 0. \end{cases} \tag{2.17}$$

We note immediately that the change of variable $\psi$ and its properties (B1)–(B4) guarantee that $\sigma$ has two important properties.

- *The proposed generalized strength is continuous* when $u_+$ crosses $\varphi^\sharp(u_-)$ and the solution of the Riemann problem goes from a single *crossing shock* (i.e., $u_- u_+ < 0$) to a nonclassical shock followed by a classical shock.

  For $u_- > 0$, this follows from inequalities $\varphi^\flat(u_-) < \varphi^\sharp(u_-) < 0$ and assumption (B3)

  $$\begin{aligned} \sigma\big(u_-, \varphi^\sharp(u_-)\big) &= |\tilde{u}_- - \tilde{\varphi}^\sharp(\tilde{u}_-)| \\ &= |\tilde{u}_- - \tilde{\varphi}^\flat(\tilde{u}_-)| + |\tilde{\varphi}^\flat(\tilde{u}_-) - \tilde{\varphi}^\sharp(\tilde{u}_-)| \\ &= \sigma\big(u_-, \varphi^\flat(u_-)\big) + \sigma\big(\varphi^\flat(u_-), \varphi^\sharp(u_-)\big). \end{aligned}$$

  In fact, if $\varphi^\sharp(u_-)$ were positive then the monotonicity of $\psi$ would imply that $\tilde{\varphi}^\sharp(\tilde{u}_-) < \tilde{\varphi}^\flat \circ \tilde{\varphi}^\flat(\tilde{u}_-) < \tilde{\varphi}^\flat(\tilde{u}_-)$ and a similar computation would show that continuity still holds.

- *The generalized strength is its equivalence with the usual notion of strength.* When the rarefaction and the non-crossing shocks have two neighboring states of the same sign, then assumption (B3) implies

  $$\sigma(u_-, u_+) = |\tilde{u}_- - \tilde{u}_+| \geq \underline{\mathrm{Lip}}(\psi)|u_- - u_+|. \tag{2.18}$$

  For crossing shocks, it suffices to use assumptions (A4), (B4), and the property (2.6) to show that the definition is equivalent to the usual notion of strength

  $$\sigma\big(u_-, \varphi^\flat(u_-)\big) = |\tilde{u}_- + \tilde{\varphi}^\flat(\tilde{u}_-)| > a_1 \frac{\mathrm{Lip}(\psi)}{1 + \underline{\mathrm{Lip}}(\varphi^\flat)} |u_- - \varphi^\flat(u_-)|. \tag{2.19}$$

In view of the presentation in the previous section, the results in [3] already provides a generalization of the contraction property (A4) of the kinetic function. Since $\varphi^\flat(0) = 0$ and $\psi(0) = 0$, then the positivity of (2.12) implies that $|\tilde{\varphi}^\flat(\tilde{u})| < |\tilde{u}|$ for all $u \neq 0$ and the strict positivity of $\underline{\mathrm{Lip}}\big(\tilde{u} + \tilde{\varphi}^\flat(\tilde{u})\big)$ implies that the kinetic function satisfies

$$\max_{[-M, M]} g\left|\frac{d\tilde{\varphi}^\flat}{d\tilde{u}} g\right| < 1. \tag{2.20}$$

In fact, the existence of the lower Lipschitz constant for $\tilde{u} + \tilde{\varphi}^\flat(\tilde{u})$ is equivalent to the above result. As a matter of fact, this simple observation was not mentioned explicitly in [3]. In this vein, we will now show that even the Lipschitz constant of the kinetic function can be manipulated with the help of a well-chosen change of variable. This will be particularly important in Section 5 when we study splitting-merging solutions.

**Proposition 2.6** (General class of changes of variable). *Consider a kinetic function that satisfies* (A1)–(A4) *and the condition*

$$0 < \underline{\mathrm{Lip}}(\varphi^\flat). \tag{2.21}$$

*Then, for any $\varepsilon > 0$ and $M = \|u_0\|_{L^\infty}$, there exists a change of variable $\tilde{u} = \psi(u)$ for which $\tilde{u} + \tilde{\varphi}^\flat(\tilde{u})$ is monotone increasing,* (B1)–(B4) *are satisfied, and, moreover,*

$$\mathrm{Lip}(\tilde{\varphi}^\flat) \in (1/2, 1) \qquad \textit{in } [-M, M] \backslash [-\varepsilon, \varepsilon]. \tag{2.22}$$

*Proof.* We start with a change of variable provided by Lemma 2.4. To simplify the notation, this new variable is simply denoted as $u$. From any fixed $\varepsilon > 0$ and any parameters $\lambda, p > 0$, we construct a new (smooth) change of variable

$$\tilde{u} = \psi(u) := \mathrm{sign}(u)\lambda|u|^p, \tag{2.23}$$

which we consider away from $[-\varepsilon, \varepsilon]$. Near the origin, it is always possible to modify the above expression of $\psi$ so that condition (B3) is satisfied.

The kinetic function $\varphi^\flat$ is Lipschitz continuous, only. Away from the origin, we can compute the new Lipschitz constant

$$\begin{aligned} \frac{d\tilde{\varphi}^\flat}{d\tilde{u}} &= \frac{d}{d\tilde{u}}\left(\psi \circ \varphi^\flat \circ \psi^{-1}(\tilde{u})\right) = \frac{(\psi \circ \varphi^\flat)'(u)}{\psi'(u)} \\ &= \frac{\mathrm{sign}\left(\varphi^\flat(u)\right)^2 \lambda p |\varphi^\flat(u)|^{p-1} (\varphi^\flat)'(u)}{\mathrm{sign}(u)^2 \lambda p |u|^{p-1}} = g\left|\frac{\varphi^\flat(u)}{u} g\right|^{p-1} \frac{d\varphi^\flat}{du}(u). \end{aligned}$$

Given that $\varphi^\flat$ satisfies (2.20) and (2.21), there exist positive constants $\varepsilon_0$ and $\varepsilon_1$ such that

$$-1 + \varepsilon_1 < (\varphi^\flat)'(u) < -\varepsilon_0.$$

Then, picking $p > 0$ sufficiently small so that

$$1/2 < \varepsilon_0^p,$$

we get

$$g\left|\frac{d\tilde{\varphi}^\flat}{d\tilde{u}} g\right| > |(\varphi^\flat)'(\xi_u)|^{p-1}(\varphi^\flat)'(u) > \varepsilon_0^p > 1/2$$

for some $\xi_u$. On the other hand, the supremum of the Lipschitz constant is

$$g\left|\frac{d\tilde{\varphi}^\flat}{d\tilde{u}} g\right| < |(\varphi^\flat)'(\xi_u)|^{p-1}(\varphi^\flat)'(u) < (1-\varepsilon_1)^p < 1,$$

which implies that $\tilde{u} + \tilde{\varphi}^\flat(\tilde{u})$ is monotone increasing. The other conditions (B1), (B2) and (B4) follow easily from the fact that the change of variable is smooth. □

The main difficulty within the previous lemma was not to decrease the (absolute) value of the derivative of $\varphi^\flat$ but to increase it and guarantee that it falls within a given interval. For example, it is an easy exercise to show using a change of variable of the form $\psi = u + \lambda\mu(u)$, with $\mu$ odd and concave-convex, that the (absolute) value of the derivative can be decreased arbitrarily close to zero.

**Example 2.7** (Cubic flux with linear diffusion and linear dispersion). Given the importance of the analysis of the kinetic function for a hyperbolic conservation law with a constant ratio of linear diffusion and dispersion (Example 2.1), we present an explicit example of a change of variable satisfying (2.22). At the moment, the derivative of $\varphi^\flat$ is either $-1/2$ or $-1$ and therefore the Lipschitz constant is just outside the interval $(1/2, 1)$. Recall that for each value of $\alpha := \beta^2/\gamma$ in (2.1), there exists a threshold $A = A(\alpha)$ on the strength of waves below which all waves are classical.

For a fixed $\alpha$, we consider the piecewise smooth change of variable

$$\psi(u) := \begin{cases} u, & |u| < A/2, \\ (u + A/2)/2 + \eta(u - A/2)^2, & A/2 \le u, \\ (u - A/2)/2 - \eta(u + A/2)^2, & u \le -A/2. \end{cases} \tag{2.24}$$

Simple calculations show that

$$\psi \circ \varphi^\flat(u) = \begin{cases} -u/2, & |u| < A, \\ -u/2 - \eta(-u + A)^2, & A \le u, \\ -u/2 + \eta(-u - A)^2, & u \le -A, \end{cases}$$

and

$$(\psi\circ\varphi^\flat)'(u)=\begin{cases}-1/2, & |u|<A,\\ -1/2+2\eta(-u+A), & A\le u,\\ -1/2-2\eta(-u-A), & u\le -A.\end{cases}$$

Therefore

$$\begin{aligned}\frac{d\tilde{\varphi}^\flat}{d\tilde{u}}(u)&=\frac{(\psi\circ\varphi^\flat)'(u)}{\psi'(u)}\\ &=\begin{cases}-1/2, & |u|<A/2,\\ -\big(1+2\eta(2u-A)\big)^{-1}, & A/2\le u<A,\\ -\big(1-2\eta(2u+A)\big)^{-1}, & -A<u\le -A/2,\\ -\big(1+4\eta(u-A)\big)/\big(1+2\eta(2u-A)\big), & A\le u,\\ -\big(1-4\eta(u+A)\big)/\big(1-2\eta(2u+A)\big), & u\le -A.\end{cases}\end{aligned}$$

When nonclassical shocks appear, the derivative is $|(\tilde{\varphi}^\flat)'(\pm\tilde{A})|=|-1/(1+2\eta A)|<1$. In fact, if $\eta A<1/2$ the absolute value of this derivative is superior to 1/2. Since the derivative is increasing with respect to $u$, for any bound on the size of the initial data $u_0$ in (1.1), the condition (2.22) will be satisfied if it is satisfied when $u=A$. For any fixed $\alpha$, it suffices to take $\eta$ sufficiently small to guarantee that the kinetic function uniformly satisfies (2.22).

**2.3. Specific choice of interest.** We now introduce an explicit and natural change of variable that one might want to consider. The purpose of the following change of variable would be to remove any asymmetry present in $\varphi_0^\flat$ due to the flux or the entropy. Define the Lipschitz continuous change of variable

$$\psi_0(u):=\begin{cases}u, & u\ge 0,\\ -\varphi_0^\flat(u), & u<0.\end{cases}\tag{2.25}$$

Although this explicit change of variable will not be required in Sections 3, 4 and 5, we have included a discussion of it because it is an obvious candidate and much of this work was initially motivated by the desire to understand to what extent the choice $\psi_0$ was valid.

We begin by looking at $\varphi_0^\flat$ in the new coordinates (2.25). With respect to $\tilde{u}=\psi_0(u)$, we have when $\tilde{u}\ge 0$

$$\tilde{\varphi}_0^\flat(\tilde{u})=\psi_-\circ\varphi_0^\flat\circ\psi_+^{-1}(\tilde{u})=-\varphi_0^\flat\circ\varphi_0^\flat(u)=-\psi_0(u)=-\tilde{u},$$

and

$$\tilde{\varphi}_0^\flat(\tilde{u})=\psi_+\circ\varphi_0^\flat\circ\psi_-^{-1}(\tilde{u})=\varphi_0^\flat\circ\varphi_0^\flat(-\tilde{u})=-\tilde{u}.$$

Rather than looking directly at the change of variable (2.25), we will construct a change of variable satisfying (2.22) and $\tilde{\varphi}_0^\flat(\tilde{u}) = -\tilde{u}$.

**Proposition 2.8.** *Assume that the kinetic function $\varphi^\flat$ and the zero-dissipation function $\varphi_0^\flat$ satisfy*

$$0 < \min\big(\underline{\mathrm{Lip}}(\varphi^\flat), \underline{\mathrm{Lip}}(\varphi_0^\flat)\big), \qquad \max\big(\mathrm{Lip}(\varphi_0^\flat \circ \varphi^\flat), \mathrm{Lip}(\varphi^\flat \circ \varphi_0^\flat)\big) < 1. \quad (2.26)$$

*There exists a change of variable $\tilde{u} = \psi(u)$ satisfying* (B1)–(B4) *and* (2.22), *for which $\tilde{u} + \tilde{\varphi}^\flat(\tilde{u})$ is monotone increasing and such that the identity*

$$\tilde{\varphi}_0^\flat(\tilde{u}) = -\tilde{u}. \quad (2.27)$$

*holds.*

*Proof.* We begin by studying the change of variable $\psi_0$ before showing that the technique in Proposition 2.6 can also be used to satisfy (2.22).

Using the Lipschitz change of variable $\psi$, (B1)–(B2) hold because of the equivalent to (A1)–(A2) holds for $\varphi_0^\flat$. The property (B3) holds because of our assumption (2.26). To check (B4), we begin by assuming that $\tilde{u} = \psi_0(u) = u > 0$ and compute

$$\tilde{u} + \tilde{\varphi}^\flat(\tilde{u}) = \tilde{u} + \psi_- \circ \varphi^\flat \psi_+^{-1}(\tilde{u}) = \tilde{u} - \varphi_0^\flat \circ \varphi^\flat(\tilde{u}),$$

while for $\tilde{u} = \psi_0(u) = -\varphi_0^\flat(u) < 0$ we find

$$\tilde{u} + \tilde{\varphi}^\flat(\tilde{u}) = \tilde{u} + \psi_+ \circ \varphi^\flat \psi_-^{-1}(\tilde{u}) = \tilde{u} + \varphi^\flat \circ \varphi_0^\flat(-\tilde{u}).$$

We can now see that for $\mathbb{R}^+$

$$\underline{\mathrm{Lip}}\big(\tilde{u} + \tilde{\varphi}^\flat(\tilde{u})\big) \leq 1 - \mathrm{Lip}(\varphi_0^\flat \circ \varphi^\flat),$$

and for $\mathbb{R}^-$

$$\underline{\mathrm{Lip}}\big(\tilde{u} + \tilde{\varphi}^\flat(\tilde{u})\big) \leq 1 - \mathrm{Lip}(\varphi^\flat \circ \varphi_0^\flat).$$

Since $\tilde{u} + \tilde{\varphi}^\flat(\tilde{u})$ is already Lipschitz continuous, we have proved (B4).

In the proof of Proposition 2.6, we introduced a change of variable of the form $\psi(u) = \mathrm{sign}(u)\lambda|u|^p$, at least away from the origin. Assume that the correction near the origin is such that $\psi$ becomes an odd function. Then for $\tilde{\tilde{u}} = \psi \circ \psi_0(u) > 0$,

$$\tilde{\tilde{\varphi}}^\flat = \psi_- \circ \tilde{\varphi}_0^\flat \circ \psi_+^{-1}(\tilde{\tilde{u}}) = \psi_- \circ \tilde{\varphi}_0^\flat(\tilde{u}) = \psi_-(-\tilde{u}) = -\psi_+(\tilde{u}) = -\tilde{\tilde{u}},$$

and for $\tilde{\tilde{u}} = \psi \circ \varphi_0^\flat(u) < 0$, we find

$$\tilde{\tilde{\varphi}}^\flat = \psi_+ \circ \tilde{\varphi}_0^\flat \circ \psi_-^{-1}(\tilde{\tilde{u}}) = \psi_+ \circ \tilde{\varphi}_0^\flat(\tilde{u}) = \psi_+(-\tilde{u}) = -\psi_-(\tilde{u}) = -\tilde{\tilde{u}}.$$

Therefore, the technique used in Proposition 2.6 preserves the property (2.27). □

**Remark 2.9.** In the case of a regularization defined by linear diffusion and dispersion, Example 2.1, $\varphi_0^\flat(u) = -u$ is already equal to the condition (2.27). However, $\varphi^\flat$ and $\varphi_0^\flat$ do not satisfy conditions (2.26). Moreover, for $\tilde{u} > 0$

$$\tilde{u} + \tilde{\varphi}^\flat(\tilde{u}) = u + \left(-u + \frac{1}{2}A(\alpha)\right) = \frac{1}{2}A(\alpha) \tag{2.28}$$

is not monotone increasing.

Nonetheless, there could be cases where such a change of variable might be convenient. In fact, Proposition 2.8 can be interpreted as saying that in some variable $\tilde{u}$, the correct way to measure the strength of nonclassical waves is with the quantities $u - \varphi_0^\flat \circ \varphi^\flat(u)$ when $u \geq 0$ and $\varphi_0^\flat(u) - \varphi^\flat(u)$ when $u < 0$.

**Example 2.10.** For a kinetic function corresponding to a regularization with linear diffusion and dispersion, as in Example 2.1, property (2.27) is already satisfied but conditions (2.26) are not, and neither are (B4). We propose to look at a slight perturbation of $\psi = \varphi_0^\flat$ which will asymptotically satisfy condition (2.27). In this example, the change of variable will be a smooth and arbitrarily small perturbation of $\varphi_0^\flat$ near the origin. (This example suggests that a theory exists for changes of variable based on higher-order perturbations of the zero diffusion kinetic function $\varphi_0^\flat$. Such a theory will not be necessary in this paper.) Consider

$$\psi(u) = \begin{cases} \psi_+(u) := u + \frac{1}{\lambda}\Delta(e^{-\lambda u} - 1), & u \geq 0, \\ \psi_-(u) := u - \frac{1}{\lambda}\Delta(e^{\lambda u} - 1), & u \leq 0, \end{cases} \tag{2.29}$$

where $\lambda > 0$ and $\Delta \in (0, 1/2)$. Using (2.8), we find

$$(\psi_+)'(u) = 1 - \Delta e^{-\lambda u}, \qquad (\psi_-)'(u) = 1 - \Delta e^{\lambda u},$$

$$(\psi_- \circ \varphi^\flat)'(u) = \begin{cases} -\frac{1}{2} + \frac{\Delta}{2} e^{-\lambda u/2}, & 0 < u < A, \\ -1 + \Delta e^{-\lambda u + \lambda A/2}, & A \leq u. \end{cases}$$

For simplicity, we will only show that $\tilde{u} + \tilde{\varphi}^\flat(\tilde{u})$ is monotone increasing for $\tilde{u} > 0$. We begin by computing the quantity

$$\begin{aligned}
\frac{d}{d\tilde{u}}\left(\tilde{\varphi}^\flat(\tilde{u})\right) &= \frac{d}{d\tilde{u}}\left(\psi_- \circ \varphi^\flat \circ (\psi_+)^{-1}(\tilde{u})\right) \\
&= (\psi_- \circ \varphi^\flat)'(u) \cdot [(\psi_+)'(u)]^{-1} \\
&= \begin{cases} -\frac{1}{2} \cdot (1 - \Delta e^{-\lambda u/2})/(1 - \Delta e^{-\lambda u}) & \text{for } 0 < u < A, \\ -(1 - \Delta e^{\lambda A/2} e^{-\lambda u})/(1 - \Delta e^{-\lambda u}) & \text{for } A \le u. \end{cases}
\end{aligned}$$

When $u < A$, $1 > e^{-\lambda u/2} > e^{-\lambda u}$ and therefore the quotient is always positive and less than 1 and the derivative of the kinetic function is greater than $-1/2$. When $u > A$, then the quotient is again less than one and the derivative is greater than $-1$. By taking $\lambda$ positive and large, one may make this derivative less than $-1/2$ over any compact interval. This change of variable therefore satisfies (2.27) asymptotically and properties (2.22) and (B4) exactly.

## 3. Diminishing total variation functional

In this section, we establish that the total variation functional associated with the generalized wave strengths (cf. Definition 2.5) is non-increasing in time for solutions generated by front-tracking approximations. Although a proof of such a result was already provided in [3], we establish here more detailed estimates of the decrease of the total variation and, in this manner, provide basic estimates required later in Sections 4 and 5. In particular, we measure the change in the total variation functional associated with the wave strength (2.5), and show that at interactions involving rarefactions the change is proportional to the strength of the incoming rarefaction and, if required, involves the Lipschitz constant of $\tilde{u} + \tilde{\varphi}^\flat(\tilde{u})$. These estimates are new, easy to interpretn and will be important for the analysis of systems.

We now introduce front-tracking approximate solutions to (1.1) based on a nonclassical Riemann solver, following Dafermos [7] in the classical setting. These approximations are piecewise constant in space and are determined from the nonclassical Riemann solver described in the previous section.

The first step of the construction is to build a piecewise constant approximation of the initial data $u_0$ which admits finitely many discontinuities and approaches $u_0$ in the $L^1$ norm with an error $\varepsilon$, for some small $\varepsilon$. The Rankine–Hugoniot condition can be used to propagate, in a conservative manner, the discontinuities of the initial data. When the Riemann solver calls for continuous waves, one replaces them by a sequence of small discontinuities $(u_-, u_+)$ whose strength satisfy $\sigma(u_-, u_+) < \varepsilon$.

When two discontinuities meet, the nonclassical Riemann solver is used, but we continue to enforce that all outgoing waves be discontinuities. One can check (see [13] for details) that for a kinetic function satisfying (A1)–(A4), the total number of discontinuities remains bounded for all times, so that the front-tracking approximation can be defined for all times.

For a front-tracking approximation $u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, formed entirely of propagating discontinuities (each denoted by $\alpha$), the inequalities (2.18) and (2.19), as well as the fact that $\varphi^\flat$ and $\psi$ are Lipschitz continuous, imply that

$$V\big(u(t)\big) := \sum_\alpha \sigma(u_-^\alpha, u_+^\alpha) \tag{3.1}$$

is equivalent to the total variation norm

$$\mathrm{TV}\big(u(t)\big) := \sum_\alpha |u_-^\alpha - u_+^\alpha|, \tag{3.2}$$

where $u_\pm^\alpha$ denote the left- and right-hand states of the discontinuity $\alpha$.

As described in Section 4.3 of [13], when the kinetic function satisfies (A1)–(A4) then the set of wave interactions appearing in front-tracking approximations can be classified in 16 different categories, depending on the type and strength of the incoming and outgoing waves. Classical shocks and rarefactions joining two positive states are denoted by $C_+^\downarrow$ and $R_+^\uparrow$, respectively. Similarly, when both neighboring states are negative, we write $C_-^\uparrow$ and $R_-^\downarrow$. When a shock joins a positive state with a negative state, we write $C_\pm^\downarrow$ or $N_\pm^\downarrow$ depending on whether or not the shock is classical or nonclassical, respectively. When the signs of the neighboring states are reversed, we simply write $C_\mp^\uparrow$ and $N_\mp^\uparrow$.

**Theorem 3.1** (Diminishing total variation functional; see also [3], [4]). *Let $\varphi^\flat$ be a kinetic function satisfying the properties* (A1)–(A4) *and let $\psi$ be a change of variable satisfying* (B1)–(B4) *and used to define the wave strength in Definition* 2.5. *Then, for every front-tracking approximation $u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ to the conservation law* (1.1) *based on the nonclassical Riemann solver associated with $\varphi^\flat$, the generalized total variation functional $V\big(u(t)\big)$ is non-increasing. Precisely, the change in $V$ during an interaction is given by*

$$[V] \le \begin{cases} -2\sigma(R^{\mathrm{in}}), & \textit{cases } \text{RC-1, RC-3, CR-1, CR-2, CR-4}, \\ -2\,\underline{\mathrm{Lip}}\big(\tilde{u} + \tilde{\varphi}^\flat(\tilde{u})\big)\sigma(R^{\mathrm{in}}), & \textit{cases } \text{RC-2, RN}, \\ -2\big(\sigma(R^{\mathrm{in}}) - \sigma(R^{\mathrm{out}})\big), & \textit{case } \text{CR-3}, \\ 0, & \textit{all other cases}. \end{cases} \tag{3.3}$$

*Here, $R^{\mathrm{in}}$ and $R^{\mathrm{out}}$ denote the incoming and outgoing rarefactions at an interaction, respectively.* (*The list of interactions is specified in the proof below.*)

From this theorem, standard techniques allow one to deduce the following existence result.

**Corollary 3.2** ([3], [4] Existence of nonclassical entropy solutions). *For any initial data* $u_0 \in L^\infty(\mathbb{R}) \cap \mathrm{BV}(\mathbb{R})$ *of* (1.1) *and any sequence of front-tracking approximations* $u^\varepsilon$ *such that* $u^\varepsilon(\cdot, 0)$ *converges to* $u_0(\cdot)$ *in* $L^1(\mathbb{R})$, *there exists a subsequence of front-tracking approximations that converge in* $\mathrm{Lip}([0,T), L^1(\mathbb{R})) \cap L^\infty([0,T), \mathrm{BV}(\mathbb{R}))$ *to a solution of the initial value problem* (1.1).

*Proof of Theorem* 3.1. We need here to compute the variation of our functional $V$ by distinguishing between 16 possible interactions, after assuming $u_l > 0$ for definiteness. We will assume that the change of variable $\psi$ has been applied, all quantities are to be considered in the coordinates $\tilde{u}$ and therefore it will be convenient to omit the tilde superscripts.

The subscript $'$ is used to indicate that a wave is outgoing and with some abuse of notation for the wave strength we write, for example, $\sigma(N_\pm^\downarrow)$ for the shock strength of a nonclassical wave. During a generic interaction between two waves, we denote the states on both sides of the left-hand wave by $u_l$ and $u_m$ while those associated with the right-hand wave are denoted by $u_m$ and $u_r$. Finally, the bounds on the change of $V$ depend only on the properties (2.4) and (2.12) which are therefore used freely throughout.

*Case* RC-1: $(R_+^\uparrow C^\downarrow)$-$(C^{\downarrow\prime})$. This case is determined by the constraints

$$\max\big(\varphi^\sharp(u_l), \varphi^\sharp(u_m)\big) < u_r < u_l, \qquad 0 < u_l < u_m.$$

This is further subdivided into two subcases depending on the sign of $u_r$. When $u_r > 0$, then the interactions are entirely classical $(R_+^\uparrow C_+^\downarrow)$-$(C_+^{\downarrow\prime})$ and the inequalities $0 < u_r < u_l < u_m$ suffice to check that

$$\begin{aligned}[V] &= \sigma(C_+^{\downarrow\prime}) - \sigma(R_+^\uparrow) - \sigma(C_+^\downarrow)\\ &= |u_l - u_r| - |u_m - u_l| - |u_m - u_r| = -2\sigma(R_+^\uparrow).\end{aligned}$$

When $u_r < 0$, then the interaction involves crossing shocks $(R_+^\uparrow C_\pm^\downarrow)$-$(C_\pm^{\downarrow\prime})$ and the states involved in measuring the strengths of the waves are

$$0 < -u_r < u_l < u_m, \tag{3.4}$$

since $-u_r < -\varphi^\flat(u_l) < u_l$. The conclusion is therefore the same.

*Case* RC-2: $(R_+^\uparrow C_\pm^\downarrow)$-$(N_\pm^{\downarrow\prime} R_-^{\downarrow\prime})$. This case is defined by

$$\varphi^\sharp(u_m) < u_r \le \varphi^\flat(u_l) < 0 < u_l < u_m.$$

In the first subcase, we assume that $-u_r < u_l$ and use the previous conditions to deduce

$$-\varphi^\flat(u_l) < -u_r < u_l < u_m. \tag{3.5}$$

The analysis when $u_l < -u_r$ will not be treated since $-u_r < u_m$ continues to hold and the conclusions remain the same.

$$\begin{aligned}[V] &= \sigma(N_\pm^{\downarrow\prime}) + \sigma(R_-^{\downarrow\prime}) - \sigma(R_+^\uparrow) - \sigma(C_\pm^\downarrow)\\ &= |u_l + \varphi^\flat(u_l)| + |\varphi^\flat(u_l) - u_r| - |u_m - u_l| - |u_m + u_r|\\ &= 2|\varphi^\flat(u_l) - u_r| - 2|u_m - u_l|.\end{aligned}$$

Since $\varphi^\sharp(u_m) < u_r < 0$, from properties (2.4) and (2.12) we deduce $0 < -u_r < -\varphi^\sharp(u_m) < \varphi^\flat(u_m)$. Combining this with (3.5), we find

$$[V] \leq 2|\varphi^\flat(u_l) - \varphi^\flat(u_m)| - 2|u_m - u_l| = -2\,\underline{\mathrm{Lip}}(u + \varphi^\flat)\sigma(R_+^\uparrow).$$

*Case* RC-3: $(R_+^\uparrow C^\downarrow)$-$(N_\pm^{\downarrow\prime} C^{\uparrow\prime})$. The conditions initially satisfied by the neighboring states of the incoming waves are

$$\max\big(\varphi^\flat(u_l), \varphi^\sharp(u_m)\big) < u_r < \varphi^\sharp(u_l), \qquad 0 < u_l < u_m.$$

In the first subcase, we assume $u_r < 0$ and the interaction is $(R_+^\uparrow C_\pm^\downarrow)$-$(N_\pm^{\downarrow\prime} C_-^{\uparrow\prime})$ with the following states appearing in the strength of the waves

$$-u_r < -\varphi^\flat(u_l) < u_l < u_m. \tag{3.6}$$

Therefore, the change in $V$ is

$$\begin{aligned}[V] &= \sigma(N_\pm^{\downarrow\prime}) + \sigma(C_-^{\uparrow\prime}) - \sigma(R_+^\uparrow) - \sigma(C_\pm^\downarrow)\\ &= |u_l + \varphi^\flat(u_l)| + |\varphi^\flat(u_l) - u_r| - |u_m - u_l| - |u_m + u_r|\\ &= -2|u_m - u_l| = -2\sigma(R_+^\uparrow).\end{aligned}$$

In the second subcase with $0 < u_r$, we have $u_r < \varphi^\sharp(u_l) < \varphi^\flat \circ \varphi^\flat(u_l) < -\varphi^\flat(u_l)$ (because given two successive nonclassical shocks, the speed of the second must be less than the first) and therefore

$$u_r < -\varphi^\flat(u_l) < u_l < u_m. \tag{3.7}$$

The change in $V$ is then

$$\begin{aligned}[V] &= \sigma(N_{\pm}^{\downarrow\prime}) + \sigma(C_{\mp}^{\uparrow\prime}) - \sigma(R_{+}^{\uparrow}) - \sigma(C_{+}^{\downarrow}) \\ &= |u_l + \varphi^\flat(u_l)| + |-\varphi^\flat(u_l) - u_r| - |u_m - u_l| - |u_m - u_r| \\ &= -2|u_m - u_l| = -2\sigma(R_{+}^{\uparrow}).\end{aligned}$$

*Case* RN: $(R_{+}^{\uparrow}N_{\pm}^{\downarrow})$-$(N_{\pm}^{\downarrow\prime}R_{-}^{\downarrow\prime})$. The states on both sides of the waves satisfy

$$0 < u_l < u_m \qquad \text{and} \qquad u_r = \varphi^\flat(u_m).$$

Two cases again occur depending on the relative order of $u_l$ with respect to $-u_r = -\varphi^\flat(u_m)$. We consider only the case where

$$-\varphi^\flat(u_l) < -\varphi^\flat(u_m) < u_l < u_m, \tag{3.8}$$

the other, $-\varphi^\flat(u_l) < u_l < -\varphi^\flat(u_m) < u_m$, being similar. In this case, the change in $V$ is

$$\begin{aligned}[V] &= \sigma(N_{\pm}^{\downarrow\prime}) + \sigma(R_{-}^{\downarrow\prime}) - \sigma(R_{+}^{\uparrow}) - \sigma(N_{\pm}^{\downarrow}) \\ &= |u_l + \varphi^\flat(u_l)| + |\varphi^\flat(u_l) - \varphi^\flat(u_m)| - |u_m - u_l| - |u_m + \varphi^\flat(u_m)| \\ &= 2|\varphi^\flat(u_l) - \varphi^\flat(u_m)| - 2|u_m - u_l| \\ &\le -2\,\underline{\mathrm{Lip}}(u + \varphi^\flat)|u_m - u_l| \le -2\,\underline{\mathrm{Lip}}(u + \varphi^\flat)\sigma(R_{+}^{\uparrow}).\end{aligned}$$

*Case* CR-1: $(C_{\pm}^{\downarrow}R_{-}^{\downarrow})$-$(C_{\pm}^{\downarrow\prime})$. This is a simple case where the states are initially ordered as

$$\varphi^\sharp(u_l) < u_r < u_m \le 0 < u_l.$$

This provides $-u_m < -u_r < -\varphi^\sharp(u_l) < -\varphi^\flat(u_l) < u_l$, and we get

$$\begin{aligned}[V] &= \sigma(C_{\pm}^{\downarrow\prime}) - \sigma(C_{\pm}^{\downarrow}) - \sigma(R_{-}^{\downarrow}) \\ &= |u_l + u_r| - |u_l + u_m| - |u_m - u_r| = -2|u_m - u_r| = -2\sigma(R_{-}^{\downarrow}).\end{aligned}$$

*Case* CR-2: $(C_{+}^{\downarrow}R_{+}^{\uparrow})$-$(C_{+}^{\downarrow\prime})$. The waves are entirely classical since $0 \le u_m < u_r < u_l$. There is nothing new to check and the change in $V$ is immediately found to be

$$[V] = -2|u_r - u_m| = -2\sigma(R_{+}^{\uparrow}).$$

*Case* CR-3: $(C_{\pm}^{\downarrow}R_{-}^{\downarrow})$-$(N_{\pm}^{\downarrow\prime}R_{-}^{\downarrow\prime})$. The states begin in the order

$$u_r \le \varphi^\flat(u_l) < \varphi^\sharp(u_l) < u_m \le 0 < u_l.$$

One subcase is obtained when we assume that $u_l < -u_r$. The important states appearing in the definition of the strength of waves are then ordered as

$$-u_m < -\varphi^\flat(u_l) < u_l < -u_r. \tag{3.9}$$

The change in $V$ is then found to be

$$\begin{aligned}[V] &= \sigma(N_\pm^{\downarrow\prime}) + \sigma(R_-^{\downarrow\prime}) - \sigma(C_\pm^{\downarrow}) - \sigma(R_-^{\downarrow}) \\ &= |u_l + \varphi^\flat(u_l)| + |\varphi^\flat(u_l) - u_r| - |u_l + u_m| - |u_m - u_r| \\ &= -2|u_m - \varphi^\flat(u_l)|.\end{aligned}$$

In the second subcase, when $-u_r < u_l$, we use

$$-u_m < -\varphi^\flat(u_l) < -u_r < u_l \tag{3.10}$$

to deduce the same equality:

$$[V] = |u_l + \varphi^\flat(u_l)| + |\varphi^\flat(u_l) - u_r| - |u_l + u_m| - |u_m - u_r| = -2|u_m - \varphi^\flat(u_l)|.$$

We now observe that in both subcases, the change is equal to a physically relevant quantity

$$[V] = -2\big(\sigma(R_-^{\downarrow}) - \sigma(R_-^{\downarrow\prime})\big).$$

*Case* CR-4: $(C_\pm^{\downarrow} R_-^{\downarrow})$-$(N_\pm^{\downarrow\prime} C_-^{\uparrow\prime})$. The states of the incoming waves satisfy

$$\varphi^\flat(u_l) < u_r < \varphi^\sharp(u_l) < u_m \le 0 < u_l. \tag{3.11}$$

This leads us to the inequalities

$$-u_m < -u_r < -\varphi^\flat(u_l) < u_l. \tag{3.12}$$

The change in our functional is therefore

$$\begin{aligned}[V] &= \sigma(N_\pm^{\downarrow\prime}) + \sigma(C_-^{\uparrow\prime}) - \sigma(C_\pm^{\downarrow}) - \sigma(R_-^{\downarrow}) \\ &= |u_l + \varphi^\flat(u_l)| + |u_r - \varphi^\flat(u_l)| - |u_l + u_m| - |u_m - u_r| \\ &= -2|u_m - u_r| = -2\sigma(R_-^{\downarrow}).\end{aligned}$$

*Case* CC-1: $(C_+^{\downarrow} C^{\downarrow})$-$(C^{\downarrow\prime})$. This is another simple case. We begin with

$$\max\big(\varphi^\sharp(u_l), \varphi^\sharp(u_m)\big) < u_r < u_m < u_l, \quad \text{and} \quad 0 \le u_m.$$

When $0 \leq u_r$, all the waves are classical and it is easy to show that $[V] = 0$. When $u_r < 0$, we still have $\varphi^\flat(u_m) < u_r$ and therefore the important states are ordered

$$-u_r < -\varphi^\flat(u_m) < u_m < u_l.$$

It then easy to check that even in this case, $[V] = 0$.

*Case* CC-2: $(C_\pm^\downarrow C^\uparrow)$-$(C^{\downarrow\prime})$. This interaction is constrained by the initial states such that

$$\varphi^\sharp(u_l) < u_m < u_r < \varphi^\sharp(u_m) < u_l, \qquad \text{and} \qquad u_m < 0.$$

Two subcases appear depending on the sign of $u_r$. When $u_r > 0$, the interaction is $(C_\pm^\downarrow C_\mp^\uparrow)$-$(C_+^{\downarrow\prime})$. Since $\varphi^\flat(u_l) < u_m$, we have $-u_m < -\varphi^\flat(u_l) < u_l$. Combining this with the fact that $u_r < \varphi^\sharp(u_m) < \varphi^\flat(u_m) < -u_m$, we deduce that the states appearing in $[V]$ are ordered,

$$u_r < -u_m < u_l.$$

The change in $V$ is then

$$[V] = \sigma(C_+^{\downarrow\prime}) - \sigma(C_\pm^\downarrow) - \sigma(C_\mp^\uparrow) = |u_l - u_r| - |u_l + u_m| - |u_r + u_m| = 0.$$

The second subcase treats $u_r < 0$ and interactions $(C_\pm^\downarrow C_-^\uparrow)$-$(C_\pm^{\downarrow\prime})$. The states used in our definition of the strength of the waves are

$$-u_r < -u_m < u_l.$$

A short computation shows that

$$[V] = \sigma(C_\pm^{\downarrow\prime}) - \sigma(C_\pm^\downarrow) - \sigma(C_-^\uparrow) = 0.$$

*Case* CC-3: $(C_+^\downarrow C^\downarrow)$-$(N_\pm^{\downarrow\prime} C^{\uparrow\prime})$. This interaction represents a typical transition from one crossing shock to a nonclassical shock. The states are

$$\varphi^\flat(u_l) < \varphi^\sharp(u_m) < u_r < \varphi^\sharp(u_l) < u_m < u_l, \qquad 0 \leq u_m.$$

The first (and most common) subcase occurs when $u_r < 0$ and the interaction is $(C_+^\downarrow C_\pm^\downarrow)$-$(N_\pm^{\downarrow\prime} C_-^{\uparrow\prime})$. The first subcase needs to further subdivided into two cases. Assuming $u_m < -\varphi^\flat(u_l)$, and observing that $-u_r < -\varphi^\flat(u_m) < u_m$, we find that the important states can be ordered

$$-u_r < u_m < -\varphi^\flat(u_l) < u_l. \tag{3.13}$$

Under these circumstances, the functional $V$ does not change.

$$\begin{aligned}[V] &= \sigma(N_{\pm}^{\downarrow\prime}) + \sigma(C_{-}^{\uparrow\prime}) - \sigma(C_{+}^{\downarrow}) - \sigma(C_{\pm}^{\downarrow}) \\ &= |u_l + \varphi^\flat(u_l)| + |u_r - \varphi^\flat(u_l)| - |u_l - u_m| - |u_m + u_r| = 0.\end{aligned}$$

On the other hand, when $u_r < 0$ and $-\varphi^\flat(u_l) < u_m$, we have

$$-u_r < -\varphi^\flat(u_l) < u_m < u_l. \tag{3.14}$$

It is easy to see that we again have $[V] = 0$.

In the second subcase, $0 < u_r$, we again need to introduce two additional subcases to handle the interactions $(C_{+}^{\downarrow}C_{+}^{\downarrow})$-$(N_{\pm}^{\downarrow\prime}C_{\mp}^{\uparrow\prime})$. When $u_m < -\varphi^\flat(u_l)$, then the states used in the definition of wave strengths are

$$u_r < u_m < -\varphi^\flat(u_l) < u_l.$$

The change in $[V]$ is

$$\begin{aligned}[V] &= \sigma(N_{\pm}^{\downarrow\prime}) + \sigma(C_{\mp}^{\uparrow\prime}) - \sigma(C_{+}^{\downarrow}) - \sigma(C_{+}^{\downarrow}) \\ &= |u_l + \varphi^\flat(u_l)| + |-\varphi^\flat(u_l) - u_r| - |u_l - u_m| - |u_m - u_r| = 0.\end{aligned}$$

When $0 < u_r$ and $-\varphi^\flat(u_l) < u_m$, we obtain the same result.

*Case* CN-1: $(C_{+}^{\downarrow}N_{\pm}^{\downarrow})$-$(C_{\pm}^{\downarrow\prime})$. The states defining the waves are characterized by the inequalities

$$0 < u_m < u_l \qquad \text{and} \qquad \varphi^\sharp(u_l) \le u_r = \varphi^\flat(u_m).$$

This implies that

$$-\varphi^\flat(u_m) < u_m < u_l.$$

We deduce

$$[V] = \sigma(C_{\pm}^{\downarrow\prime}) - \sigma(C_{+}^{\downarrow}) - \sigma(N_{\pm}^{\downarrow}) = |u_l + \varphi^\flat(u_m)| - |u_l - u_m| - |u_m + \varphi^\flat(u_m)| = 0.$$

*Case* CN-2: $(C_{\pm}^{\downarrow}N_{\mp}^{\uparrow})$-$(C_{+}^{\downarrow\prime})$. We begin with states satisfying

$$\varphi^\sharp(u_l) < u_m < 0 \qquad \text{and} \qquad u_r = \varphi^\flat(u_m).$$

To measure the wave strengths, we observe that $\varphi^\flat(u_l) < u_m$ implies $-u_m < -\varphi^\flat(u_l) < u_l$ and therefore

$$u_r = \varphi^\flat(u_m) < -u_m < u_l.$$

The jump in $V$ is now

$$\begin{aligned}[V] &= \sigma(C_{+}^{\downarrow\prime}) - \sigma(C_{\pm}^{\downarrow}) - \sigma(N_{\mp}^{\uparrow}) \\ &= |u_l - \varphi^\flat(u_m)| - |u_l + u_m| - |-u_m - \varphi^\flat(u_m)| = 0.\end{aligned}$$

*Case* CN-3: $(C_{+}^{\downarrow}N_{\pm}^{\downarrow})$-$(N_{\pm}^{\downarrow\prime}C_{-}^{\uparrow\prime})$. The states are initially ordered as

$$0 < u_m < u_l \qquad \text{and} \qquad u_r = \varphi^\flat(u_m) < \varphi^\sharp(u_l).$$

A first subcase occurs when $u_m < -\varphi^\flat(u_l)$. To compute the change in $V$, we can then use the inequalities

$$-\varphi^\flat(u_m) < u_m < -\varphi^\flat(u_l) < u_l, \tag{3.15}$$

to deduce that

$$\begin{aligned}[V] &= \sigma(N_{\pm}^{\downarrow\prime}) + \sigma(C_{-}^{\uparrow\prime}) - \sigma(C_{+}^{\downarrow}) - \sigma(N_{\pm}^{\downarrow}) \\ &= |u_l + \varphi^\flat(u_l)| + |\varphi^\flat(u_l) - \varphi^\flat(u_m)| - |u_l - u_m| - |u_m + \varphi^\flat(u_m)| = 0.\end{aligned}$$

Similarly, if $-\varphi^\flat(u_l) < u_m$, then the important inequalities become

$$-\varphi^\flat(u_m) < -\varphi^\flat(u_l) < u_m < u_l, \tag{3.16}$$

and $[V] = 0$.

*Case* NC: $(N_{\pm}^{\downarrow}C^{\uparrow})$-$(C^{\downarrow\prime})$. This interaction is constrained by the states

$$u_m = \varphi^\flat(u_l) \qquad \text{and} \qquad \varphi^\sharp(u_l) < u_r < \varphi^\sharp(u_m) < u_l.$$

The first subcase occurs when $u_r < 0$ and the interaction is precisely $(N_{\pm}^{\downarrow}C_{-}^{\uparrow})$-$(C_{\pm}^{\downarrow\prime})$. The important states are then ordered as

$$-u_r < -\varphi^\flat(u_l) < u_l.$$

With these observations, we find that

$$[V] = \sigma(C_{\pm}^{\downarrow\prime}) - \sigma(N_{\pm}^{\downarrow}) - \sigma(C_{-}^{\uparrow}) = |u_l + u_r| - |u_l + \varphi^\flat(u_l)| - |\varphi^\flat(u_l) - u_r| = 0.$$

In the second subcase, $u_r > 0$, it is easy to check that

$$u_r < -\varphi^\sharp(u_l) < -\varphi^\flat(u_l) < u_l,$$

and $[V] = 0$.

*Case* NN: $(N_{\pm}^{\downarrow} N_{\mp}^{\uparrow})$-$(C_{+}^{\downarrow\prime})$. This interaction is the limiting case $u_r \to \varphi^\flat(u_m)$ of Case NC. By continuity of wave strengths, we must also have $[V] = 0$. □

## 4. Quadratic interaction potential

In the rest of this paper, we investigate Glimm's quadratic interaction potentials, keeping in mind from experience with classical shock waves, that different functionals may be of particular interest in different circumstances. We begin by searching for a functional of the form

$$Q(u(t)) := \sum_{\alpha \text{ approaches } \beta} \sigma(u_l^\alpha, u_r^\alpha)\sigma(u_l^\beta, u_r^\beta), \tag{4.1}$$

where the proper definition of "pairs of approaching waves" is essential and is now specified.

In Glimm's original paper [9] for systems of conservation laws, a definition is proposed which, in the scalar case, imposes that two waves are always approaching unless both are rarefactions. The purpose of this section is to investigate the original definition of Glimm in the context of nonclassical shocks.

**Definition 4.1.** A wave $\alpha$ is said to *weakly approach* a wave $\beta$, unless both are rarefaction waves. As far as this definition is concerned, waves $R_{+}^{\uparrow}$, $R_{-}^{\downarrow}$ are both to be considered as rarefaction waves and $C_{+}^{\downarrow}$, $C_{\pm}^{\downarrow}$, $N_{\pm}^{\downarrow}$, $C_{-}^{\uparrow}$, $C_{\mp}^{\uparrow}$, $N_{\mp}^{\uparrow}$ are all to be considered as shock waves.

Our main result in the present section is as follows.

**Theorem 4.2** ("Weak interaction" potential for nonclassical shocks). *Let $\varphi^\flat$ be a kinetic function satisfying the properties* (A1)–(A4) *and a change of variable $\psi$ satisfying* (B1)–(B4). *Consider the functional $Q_{weak}$ defined by* (4.1) *where the summation is made over all weakly interacting waves in the sense of Definition* 4.1. *Then, when evaluated on a sequence of front-tracking solutions, $Q_{weak}$ is strictly decreasing during all interactions except in the cases* RC-3, CR-4, CC-3 *and* CN-3. *In fact, for each of these exceptional interactions, there exist initial data for which $V + C_0 Q_{weak}$ is increasing during the interaction for every positive $C_0$.*

We will actually show below that the potential decreases even in Case CN-3 provided the incoming wave is sufficiently small, so that in this regime our interaction potential *increases only when a nonclassical shock is generated* at an interaction between classical waves.

In contrast, in Chapter 8 of [13] different definitions of wave strengths and approaching waves are used and the resulting Glimm functional $V + KQ$ is strictly decreasing for some $K$. In this sense, the interaction functional $Q_{\text{weak}}$ above may appear to be less satisfactory. However, our assumptions on the kinetic function are *completely natural*—a major advantage toward a future extension to systems—and, furthermore, an analysis of "splitting/merging" solutions (in the following section) will show that *globally in time* the functional $Q_{\text{weak}}$ does decrease.

Several justifications for our definition of potential are now provided, the strongest argument being the requirement of continuity:

1. Given that $V$ is continuous in $\mathrm{BV}(\mathbb{R})$ (endowed with its usual total variation semi-norm), it is tempting to assume that any reasonable interaction potential $Q$ should also be continuous in $\mathrm{BV}(\mathbb{R})$. We observe that any shock $C_+^{\downarrow}$ can be continuously deformed (as measured by Definition 2.5) into, first, a crossing shock $C_{\pm}^{\downarrow}$ and, then, a pair of shocks $N_{\pm}^{\downarrow} C_{-}^{\uparrow}$. It is easy to see that imposing continuity and an interaction potential of the form (4.1) would imply Definition 4.1.
2. Another argument can be found by looking at a class of solutions called splitting/merging solutions, introduced in [17] and discussed further in Section 5. These solutions illustrate that some initial data can go through a nearly periodic process of creation and cancellation of nonclassical shocks. In particular, nonclassical shocks can indirectly have non-trivial interactions with shocks on their right-hand side and such interactions cannot be excluded a priori.
3. In [13], nonclassical shocks are precluded from interacting with their right-hand neighbours, and it is argued that nonclassical shocks are (slow) undercompressive and, thus, move away from their right-hand neighbors. However, this definition of approaching waves ignores the above-mentioned possibility of nonclassical shocks having indirect interactions with shocks on their right-hand side. In any case, such an interaction functional would not be continuous in $\mathrm{BV}(\mathbb{R})$.

*Proof of Theorem* 4.2. During any isolated interaction between two waves in an approximation, the change in $Q = Q_{\text{weak}}$ is of two types

$$[Q] = [Q]_1 + [Q]_2. \tag{4.2}$$

In this decomposition, $[Q]_1$ denotes the change in the products of the strengths of waves either incoming or exiting the interaction and $[Q]_2$ denotes the change in products of strengths of waves where only *one* of the waves was directly involved in the interaction. Moreover, if a wave $C$ is involved in an interaction, we define

$$W(C) := \sum_{\substack{B \text{ approaches } C \\ B \text{ did not interact}}} \sigma(B).$$

According to Definition 4.1, if the incoming wave $C_{\text{in}}$ and the outgoing wave $C_{\text{out}}$ are of the same type (i.e. both of rarefaction or shock type), then $W(C_{\text{in}}) = W(C_{\text{out}})$. Throughout the proof, we use liberally the estimates derived in the proof of Theorem 3.1 and omit the superscripts tilde since the change of variable $\psi$ will be assumed to have been to every quantity and function.

*Case* RC-1: $(R_+^{\uparrow} C^{\downarrow})$-$(C^{\downarrow\prime})$. After examining (3.4), it is immediate that $\sigma(C^{\downarrow}) > \sigma(C^{\downarrow\prime})$ and $W(C^{\downarrow}) = W(C^{\downarrow\prime})$. Therefore,

$$[Q]_1 = -\sigma(R_+^{\uparrow})\sigma(C^{\downarrow}) < 0$$

and

$$\begin{aligned}[Q]_2 &= W(C^{\downarrow\prime})\sigma(C^{\downarrow\prime}) - W(R_+^{\uparrow})\sigma(R_+^{\uparrow}) - W(C^{\downarrow})\sigma(C^{\downarrow}) \\ &= -W(R_+^{\uparrow})\sigma(R_+^{\uparrow}) - W(C^{\downarrow})\big(\sigma(C^{\downarrow}) - \sigma(C^{\downarrow\prime})\big) < 0.\end{aligned}$$

*Case* RC-2: $(R_+^{\uparrow} C_{\pm}^{\downarrow})$-$(N_{\pm}^{\downarrow\prime} R_-^{\downarrow\prime})$. The conditions defining this case imply two subcases:

$$\text{either } -\varphi^{\flat}(u_l) < -u_r < u_l < u_m, \qquad \text{or} \qquad -\varphi^{\flat}(u_l) < u_l < -u_r < u_m.$$

Given that $\varphi^{\sharp}(u_m) < u_r < 0$, then property (2.12) leads to

$$0 < -u_r < -\varphi^{\sharp}(u_m) < -\varphi^{\flat}(u_m) < u_m.$$

This means that in the first subcase,

$$\sigma(R_-^{\downarrow\prime}) = \big|-u_r - \big(-\varphi^{\flat}(u_l)\big)\big| < \big|-\varphi^{\flat}(u_m) - \big(-\varphi^{\flat}(u_l)\big)\big| < |u_m - u_l| = \sigma(R_+^{\uparrow}).$$

In that subcase, we also have

$$\sigma(C_{\pm}^{\downarrow}) = \sigma(R_+^{\uparrow}) + |u_l + u_r|, \qquad \sigma(N_{\pm}^{\downarrow\prime}) = \sigma(R_-^{\downarrow\prime}) + |u_l + u_r|,$$

so $\sigma(N_{\pm}^{\downarrow\prime}) < \sigma(C_{\pm}^{\downarrow})$. Similar arguments imply the same inequalities in the second case. These inequalities therefore imply that

$$[Q]_1 = \sigma(N_{\pm}^{\downarrow\prime})\sigma(R_-^{\downarrow\prime}) - \sigma(C_{\pm}^{\downarrow})\sigma(R_+^{\uparrow}) < 0.$$

Using the proposed definition of *weakly approaching waves*, we also deduce

$$\begin{aligned}[Q]_2 &= W(N_{\pm}^{\downarrow\prime})\sigma(N_{\pm}^{\downarrow\prime}) + W(R_{-}^{\downarrow\prime})\sigma(R_{-}^{\downarrow\prime}) - W(R_{+}^{\uparrow})\sigma(R_{+}^{\uparrow}) - W(C_{\pm}^{\downarrow})\sigma(C_{\pm}^{\downarrow}) \\ &= -W(R_{-}^{\downarrow\prime})\big(\sigma(R_{+}^{\uparrow}) - \sigma(R_{-}^{\downarrow\prime})\big) - W(N_{\pm}^{\downarrow\prime})\big(\sigma(C_{\pm}^{\downarrow}) - \sigma(N_{\pm}^{\downarrow\prime})\big) < 0.\end{aligned}$$

*Case* RC-3: $(R_{+}^{\uparrow}C^{\downarrow})$-$(N_{\pm}^{\downarrow\prime}C^{\uparrow\prime})$. Two subcases occur depending on the sign of $u_r$. If $u_r < 0$, then (3.6) holds and when $u_r > 0$, then (3.7) holds. In both of these cases

$$\sigma(C^{\downarrow}) - \sigma(R_{+}^{\uparrow}) = \sigma(C^{\uparrow\prime}) + \sigma(N_{\pm}^{\downarrow\prime}). \tag{4.3}$$

The change in $[Q]_1$ is

$$[Q]_1(u_l, u_m, u_r) = \sigma(N_{\pm}^{\downarrow\prime})\sigma(C^{\uparrow\prime}) - \sigma(R_{+}^{\uparrow})\sigma(C_{\pm}^{\downarrow}).$$

Clearly, the RC-3 interaction can still occur even as the strength $\sigma(R_{+}^{\uparrow}) \to 0$ and the strengths $\sigma(N_{\pm}^{\downarrow\prime})$, $\sigma(C^{\uparrow\prime})$ approach non-zero values. These observations imply that $[Q]_1 > 0$ in that limiting case. For the other interaction term, we use (4.3) to check that

$$\begin{aligned}[Q]_2 &= W(N_{\pm}^{\downarrow\prime})\sigma(N_{\pm}^{\downarrow\prime}) + W(C^{\uparrow\prime})\sigma(C^{\uparrow\prime}) - W(R_{+}^{\uparrow})\sigma(R_{+}^{\uparrow}) - W(C^{\downarrow})\sigma(C^{\downarrow}) \\ &= -W(N_{\pm}^{\downarrow\prime})\big(\sigma(C^{\downarrow}) - \sigma(N_{\pm}^{\downarrow\prime}) - \sigma(C^{\uparrow\prime})\big) - W(R_{+}^{\uparrow})\sigma(R_{+}^{\uparrow}) < 0.\end{aligned}$$

*Case* RN: $(R_{+}^{\uparrow}N_{\pm}^{\downarrow})$-$(N_{\pm}^{\downarrow\prime}R_{-}^{\downarrow\prime})$. The states appearing in the shock strengths describe two subcases:

$$\text{either } -\varphi^{\flat}(u_l) < -\varphi^{\flat}(u_m) < u_l < u_m, \qquad \text{or} \qquad -\varphi^{\flat}(u_l) < u_l < -\varphi^{\flat}(u_m) < u_m.$$

For both sets of inequalities $\sigma(R_{-}^{\downarrow\prime}) < \sigma(R_{+}^{\uparrow})$ and $\sigma(N_{\pm}^{\downarrow\prime}) < \sigma(N_{\pm}^{\downarrow})$. As a result,

$$[Q]_1 = \sigma(N_{\pm}^{\downarrow\prime})\sigma(R_{-}^{\downarrow\prime}) - \sigma(R_{+}^{\uparrow})\sigma(N_{\pm}^{\downarrow}) < 0.$$

For the second term, using the bounds on the wave strengths given in the previous lemma, we again have a negative contribution

$$\begin{aligned}[Q]_2 &= W(N_{\pm}^{\downarrow\prime})\sigma(N_{\pm}^{\downarrow\prime}) + W(R_{-}^{\downarrow\prime})\sigma(R_{-}^{\downarrow\prime}) - W(R_{+}^{\uparrow})\sigma(R_{+}^{\uparrow}) - W(N_{\pm}^{\downarrow})\sigma(N_{\pm}^{\downarrow}) \\ &= -W(N_{\pm}^{\downarrow})\big(\sigma(N_{\pm}^{\downarrow}) - \sigma(N_{\pm}^{\downarrow\prime})\big) - W(R_{+}^{\uparrow})\big(\sigma(R_{+}^{\uparrow}) - \sigma(R_{-}^{\downarrow\prime})\big) < 0.\end{aligned}$$

*Case* CR-1: $(C_{\pm}^{\downarrow}R_{-}^{\downarrow})$-$(C_{\pm}^{\downarrow\prime})$. In this case, the states satisfy $-u_m < -u_r < -\varphi^{\sharp}(u_l) < u_l$ and the wave strengths satisfy $\sigma(C_{\pm}^{\downarrow\prime}) < \sigma(C_{\pm}^{\downarrow})$. Since only one wave is outgoing, $[Q]_1 = -\sigma(C_{\pm}^{\downarrow})\sigma(R_{-}^{\downarrow}) < 0$. On the other hand, it is easy to check that

$$\begin{aligned}[Q]_2 &= W(C_{\pm}^{\downarrow\prime})\sigma(C_{\pm}^{\downarrow\prime}) - W(C_{\pm}^{\downarrow})\sigma(C_{\pm}^{\downarrow}) - W(R_{-}^{\downarrow})\sigma(R_{-}^{\downarrow}) \\ &= -W(C_{\pm}^{\downarrow})\big(\sigma(C_{\pm}^{\downarrow}) - \sigma(C_{\pm}^{\downarrow\prime})\big) - W(R_{-}^{\downarrow})\sigma(R_{-}^{\downarrow}) < 0.\end{aligned}$$

*Case* CR-2: $(C_+^{\downarrow} R_+^{\uparrow})$-$(C_+^{\downarrow\prime})$. This case is entirely classical, so it is easy to check that

$$[Q]_1 = -\sigma(C_+^{\downarrow})\sigma(R_+^{\uparrow}) < 0,$$
$$[Q]_2 = -W(C_+^{\downarrow})\big(\sigma(C_+^{\downarrow}) - \sigma(C_+^{\downarrow\prime})\big) - W(R_+^{\uparrow})\sigma(R_+^{\uparrow}) < 0.$$

*Case* CR-3: $(C_\pm^{\downarrow} R_-^{\downarrow})$-$(N_\pm^{\downarrow\prime} R_-^{\downarrow\prime})$. A first subcase is defined by the additional condition $u_l < \varphi^\flat(u_r)$ which provides (3.9). The relative strengths of the waves are then seen to be $\sigma(R_-^{\downarrow\prime}) < \sigma(R_-^{\downarrow})$ and $\sigma(N_\pm^{\downarrow\prime}) < \sigma(C_\pm^{\downarrow})$. In the second subcase, given by (3.10), these two inequalities still hold. It is now easy to conclude

$$[Q]_1 = \sigma(N_\pm^{\downarrow\prime})\sigma(R_-^{\downarrow\prime}) - \sigma(C_\pm^{\downarrow})\sigma(R_-^{\downarrow}) < 0,$$
$$[Q]_2 = -W(C_\pm^{\downarrow})\big(\sigma(C_\pm^{\downarrow}) - \sigma(N_\pm^{\downarrow\prime})\big) - W(R_-^{\downarrow})\big(\sigma(R_-^{\downarrow}) - \sigma(R_-^{\downarrow\prime})\big) < 0.$$

*Case* CR-4: $(C_\pm^{\downarrow} R_-^{\downarrow})$-$(N_\pm^{\downarrow\prime} C_-^{\uparrow\prime})$. The states satisfy (3.12),

$$-u_m < -u_r < -\varphi^\flat(u_l) < u_l,$$

and therefore

$$\begin{aligned}[Q]_1(u_l, u_m, u_r) &= \sigma(N_\pm^{\downarrow\prime})\sigma(C_-^{\uparrow\prime}) - \sigma(C_\pm^{\downarrow})\sigma(R_-^{\downarrow}) \\ &= \big(u_l + \varphi^\flat(u_l)\big)\big(-\varphi^\flat(u_l) + u_r\big) - (u_l + u_m)(-u_r + u_m).\end{aligned}$$

If $u_r$ and $u_m$ approach each other while maintaining the condition

$$u_r < \varphi^\sharp(u_l) < u_m,$$

then $-u_r + u_m \to 0$ while $u_l + u_m$ remains bounded. Upon inspection, it is clear that in this limit $[Q]_1 \to \sigma(N_\pm^{\downarrow\prime})\sigma(C_-^{\uparrow\prime}) > 0$. The inequalities defining this case suffice to show that $\sigma(N_\pm^{\downarrow\prime}) + \sigma(C_-^{\uparrow\prime}) \le \sigma(C_\pm^{\downarrow})$, and therefore that the second term is negative

$$[Q]_2 = -W(C_\pm^{\downarrow})\big(\sigma(C_\pm^{\downarrow}) - \sigma(N_\pm^{\downarrow\prime}) - \sigma(C_-^{\uparrow\prime})\big) - W(R_-^{\downarrow})\sigma(R_-^{\downarrow}) < 0.$$

*Case* CC-1: $(C_+^{\downarrow} C^{\downarrow})$-$(C^{\downarrow\prime})$. When $u_r$ is positive, all waves are classical and it is easy to show that $[Q]_1 < 0$ and $[Q]_2 = 0$. It is an exercise to see that when $u_r < 0$, then $[Q]_1 < 0$ is still negative and $[Q]_2$ vanishes.

*Case* CC-2: $(C_\pm^{\downarrow} C^{\uparrow})$-$(C^{\downarrow\prime})$. Two subcases appear depending on the sign of $u_r$, but each time we have

$$[V] = \sigma(C^{\downarrow\prime}) - \sigma(C_\pm^{\downarrow}) - \sigma(C^{\uparrow}) = 0.$$

Only one wave is outgoing, so $[Q]_1 < 0$ and the previous identity implies that $[Q]_2 = 0$.

*Case* CC-3: $(C_+^{\downarrow} C^{\downarrow})$-$(N_{\pm}^{\downarrow\prime} C^{\uparrow\prime})$. As mentioned in the proof of Theorem 3.1, the set of states are subdivided into two subcases depending on the sign of $u_r$. When $u_r < 0$, we identify two possibilities (3.13) and (3.14), namely,

$$\text{either } -u_r < u_m < -\varphi^\flat(u_l) < u_l, \qquad \text{or} \qquad -u_r < -\varphi^\flat(u_l) < u_m < u_l,$$

where the second set of inequalities corresponds to a weak $C_+^{\downarrow}$. The change in $[Q]_1$ is then

$$\begin{aligned}[Q]_1(u_l, u_m, u_r) &= \sigma(N_{\pm}^{\downarrow\prime})\sigma(C_-^{\uparrow\prime}) - \sigma(C_+^{\downarrow})\sigma(C_{\pm}^{\downarrow}) \\ &= \big(u_l - (-\varphi^\flat(u_l))\big)\big(-\varphi^\flat(u_l) - (-u_r)\big) - (u_l - u_m)\big(u_m - (-u_r)\big).\end{aligned}$$

Consider the function $B(\lambda) := \lambda(1 - \lambda)$ which is symmetric about its maximum at $\lambda = 1/2$. There exists constants $\lambda, \lambda' \in [0, 1]$ such that the change can be rewritten as

$$[Q]_1 = (u_l + u_r)^2 \big(B(\lambda) - B(\lambda')\big).$$

It is therefore clear that $[Q]_1$ will be negative if and only if $|\lambda' - 1/2| < |\lambda - 1/2|$. As far as $[Q]_2$ is concerned, since all waves are shocks, then $[V] = 0$ and $[Q]_2 = 0$.

*Case* CN-1: $(C_+^{\downarrow} N_{\pm}^{\downarrow})$-$(C_{\pm}^{\downarrow\prime})$. This is another simple case where the fact that $[V] = 0$ and that $W(\cdot)$ is equal for all waves involved implies that $[Q]_2 = 0$.

*Case* CN-2: $(C_{\pm}^{\downarrow} N_{\mp}^{\uparrow})$-$(C_+^{\downarrow\prime})$. Same as Case CN-1.

*Case* CN-3: $(C_+^{\downarrow} N_{\pm}^{\downarrow})$-$(N_{\pm}^{\downarrow\prime} C_-^{\uparrow\prime})$. We identify two subcases:

$$\text{either } -\varphi^\flat(u_m) < u_m < -\varphi^\flat(u_l) < u_l, \qquad \text{or} \qquad -\varphi^\flat(u_m) < -\varphi^\flat(u_l) < u_m < u_l,$$

where the second one occurs when the incoming shock $C_+^{\downarrow}$ is weak. The change can be written as

$$\begin{aligned}[Q]_1(u_l, u_m, u_r) &= \sigma(N_{\pm}^{\downarrow\prime})\sigma(C_-^{\uparrow\prime}) - \sigma(C_+^{\downarrow})\sigma(N_{\pm}^{\downarrow}) \\ &= \big(u_l + \varphi^\flat(u_l)\big)\big(-\varphi^\flat(u_l) + \varphi^\flat(u_m)\big) - (u_l - u_m)\big(u_m + \varphi^\flat(u_m)\big).\end{aligned}$$

Using a bit of algebra, we rewrite the change as

$$\begin{aligned}
[Q]_1 &= \big(u_l + \varphi^\flat(u_l)\big)\big(-\varphi^\flat(u_l) + \varphi^\flat(u_m)\big) \\
&\quad - \big(u_l + \varphi^\flat(u_l) - \varphi^\flat(u_l) - u_m\big)\big(u_m + \varphi^\flat(u_m)\big) \\
&= \big(u_l + \varphi^\flat(u_l)\big)\big(-\varphi^\flat(u_l) - u_m\big) \\
&\quad + \big(u_m + \varphi^\flat(u_l)\big)\big(u_m + \varphi^\flat(u_m)\big) \\
&= -\big(u_m + \varphi^\flat(u_l)\big)\big((u_l - u_m) + \big(\varphi^\flat(u_l) - \varphi^\flat(u_m)\big)\big),
\end{aligned} \tag{4.4}$$

thus concluding that $[Q]_1 < 0$ as long as $C_+^\downarrow$ is weak. Clearly, $[Q]_1$ will be positive when $C_+^\downarrow$ is strong. To show that $[Q]_2 = 0$, it suffices to observe that $[V] = 0$.

*Case* NC: $(N_\pm^\downarrow C^\uparrow)$-$(C^{\downarrow\prime})$. Same as Case CN-1.

*Case* NN: $(N_\pm^\downarrow N_\mp^\uparrow)$-$(C_+^{\downarrow\prime})$. Same as Case CN-1. □

## 5. Global diminishing property for splitting-merging patterns

We will now show that, despite the fact that the quadratic interaction potential $Q_{\text{weak}}$ increases at interactions CR-3, RC-4, CN-3 and CC-3, this potential is indeed *strictly decreasing* globally in time for a large class of perturbations of crossing shocks. Hence, with the bound we describe below, we provide the first step towards an analysis of the global-in-time change of $Q_{\text{weak}}$ for arbitrary nonclassical entropy solutions. This section, therefore, provides a strong justification for the potential proposed in the previous section.

The *splitting-merging solutions* considered now were introduced in LeFloch and Shearer [17], where a modification of the total variation functional [13] was shown to be strictly decreasing along the evolution of such splitting-merging solutions. The total variation functional $V$ presented in Section 3 also accomplishes this, but here we improve on those results by establishing a similar monotonicity result for the quadratic functional $Q_{\text{weak}}$. Our analysis also brings to light some interesting aspects of splitting-merging solutions that were not seen in [17].

Splitting-merging solutions are, roughly speaking, perturbations of crossing shocks that lead to the creation and cancellation of a nonclassical shock. Such solutions contain two (classical and nonclassical) big waves that may merge together (as a classical shock) and also interact with (classical) small waves. A typical initial data for splitting-merging patterns is formed of

(i) an isolated crossing shock with left- and right-hand states $u_-$, $u_+$ satisfying $u_- > 0$ and $\varphi^\sharp(u_-) < u_+$, but $u_+ - \varphi^\sharp(u_-)$ small,

(ii) followed, on the right-hand side, by a small rarefaction and a small shock.

The rarefaction is sufficiently strong that it has an interaction of type CR-4 with the crossing shock, thereby leading to the creation of a pair of shock waves $N_{\pm}^{\downarrow}$, $C^{\uparrow}$. If the right-most shock is sufficiently strong, then when it eventually interacts with $C^{\uparrow}$ and the resulting shock will begin to approach $N_{\pm}^{\downarrow}$. The final interaction of type NC will involve $N_{\pm}^{\downarrow}$ and the shock just described, thereby eliminating the nonclassical $N_{\pm}^{\downarrow}$. By adding more waves to the left and the right, this process of creation and cancellation of $N_{\pm}^{\downarrow}$ can be repeated indefinitely.

We consider a slightly more general configuration in the sense that we do not explicitly demand that a small shock on the right be responsible for the penultimate NC interaction. Fix some value $u^* > 0$ and define

$$u_0^*(x) = \begin{cases} u^*, & x < 0, \\ \varphi^\sharp(u^*), & x > 0. \end{cases} \tag{5.1}$$

Let $\theta_\varepsilon$ be some function of locally bounded total variation and of oscillation bounded by some small positive $\varepsilon$, i.e.,

$$\sigma\big(\theta_\varepsilon(x), 0\big) < \varepsilon, \qquad x \in \mathbb{R}.$$

Furthermore, assume that $\varepsilon$ is small with respect to the quantities

$$|u^* + \varphi^\sharp(u^*)| \quad \text{and} \quad |\varphi^\sharp(u^*) - \varphi^\flat(u^*)|, \tag{5.2}$$

which will be the generic strength of a nonclassical and classical shock to be defined below. Without loss of generality, we may assume that $\theta_\varepsilon$ is piecewise constant. Let $u_\varepsilon$ be the nonclassical solution to the conservation law (1.1) with initial data $u_0^* + \theta_\varepsilon$, as generated by the front-tracking method. Assuming the solution initially possesses a single isolated crossing shock located at the origin $x = 0$, that is, assuming that $\varphi^\sharp\big(u_\varepsilon(0-)\big) < u_\varepsilon(0+)$, we see that the crossing shock will be adjacent to many small classical shocks and rarefactions. After an interaction of type RC-3, CR-4, or CC-3, the small waves neighboring $C_{\pm}^{\downarrow}$ may lead to the creation of a pair of waves, a nonclassical shock $N_{\pm}^{\downarrow}$ and a classical shock $C^{\uparrow}$. After the creation of $N_{\pm}^{\downarrow}$, the only types of interaction involving small waves incoming from the left of $N_{\pm}^{\downarrow}$ are RN and CN-3. The only types of interaction involving small waves and the shock $C^{\uparrow}$, coming from either the left or the right, are entirely classical (CC-1, RC-1 or CR-1). Moreover, no waves can cross $C^{\uparrow}$ from the right or the left although the small waves that reach $N_{\pm}^{\downarrow}$ from the left, will cross and eventually reach $C^{\uparrow}$. Therefore, the only way that the nonclassical shock $N_{\pm}^{\downarrow}$ can be destroyed is if the shocks $N_{\pm}^{\downarrow}$ and $C^{\uparrow}$ change their speeds and eventually interact back together, leading us back to (a perturbation of) the original crossing shock.

These observations imply, among other things, that *no waves can exit the domain* $\Omega$ bounded by the trajectories of $N_{\pm}^{\downarrow}$ and $C^{\uparrow}$. Our goal, in the present section, is to obtain a local bound on the change of the potential $Q_{\text{weak}}$ relative only to the waves entering the domain $\Omega$. It should already be clear that the key here is to compare the total strength of the waves crossing $N_{\pm}^{\downarrow}$ to the total strength of the waves terminating at $C^{\uparrow}$.

Before stating our main result, we introduce some additional notation. Let $t_0$ be a time of creation of a nonclassical wave $N_{\pm}^{\downarrow}$ and denote by $t_1, t_2, \dots, t_m$ the times of the next $m$ interactions between $N_{\pm}^{\downarrow}$ and small waves $W_i$ on the left, and let $t_{m+1}$ be the time at which $N_{\pm}^{\downarrow}$ is destroyed from an interaction with the shock $C^{\uparrow}$. Similarly, let $\tilde{t}_i$ and $\bar{t}_i$ be the times at which an interaction occurs between $C^{\uparrow}$ and left-incoming waves $\tilde{W}_i$ or right-incoming waves $\overline{W}_i$, respectively. We define the *total variation along the trajectory* $N_{\pm}^{\downarrow}$ to be

$$\mathrm{TV}(N_{\pm}^{\downarrow}) := \sum_{i=1}^{m} \left| \sigma\big(N_{\pm}^{\downarrow}(t_i+)\big) - \sigma\big(N_{\pm}^{\downarrow}(t_i-)\big) \right|, \tag{5.3}$$

and its *signed variation*

$$\mathrm{SV}(N_{\pm}^{\downarrow}) := \sigma\big(N_{\pm}^{\downarrow}(t_{m+1}-)\big) - \sigma\big(N_{\pm}^{\downarrow}(t_0+)\big). \tag{5.4}$$

Completely similar definitions also apply to the wave $C^{\uparrow}$, but an additional decomposition is introduced by separating the contributions from the left- and the right-hand sides:

$$\mathrm{SV}(C^{\uparrow}) = g\Big(\sum_{i=1}^{\tilde{m}} \tilde{s}_i \sigma(\tilde{W}_i) g\Big) + g\Big(\sum_{i=1}^{\bar{m}} \bar{s}_i \sigma(\overline{W}_i) g\Big) =: \mathrm{SV}_L(C^{\uparrow}) + \mathrm{SV}_R(C^{\uparrow}),$$

where $\tilde{s}_i$ ($\bar{s}_i$) is $+1$ if $\tilde{W}_i$ ($\overline{W}_i$) is a shock and $-1$ otherwise. For convenience, the strengths of the small wave $W_i$, before and after it has crossed $N_{\pm}^{\downarrow}$ at some time $t_i$, are denoted by $W_i^-$ and $W_i^+$, respectively.

**Theorem 5.1** (Global diminishing property for splitting-merging patterns). *Suppose that* $\varphi^\flat$ *is a kinetic function satisfying conditions* (A1)–(A4) *and property* (2.22). *Moreover, assume that* $\varphi^\flat$ *is a* $C^1$ *function over the open interval* $\{u \mid \varphi^\flat(u) \neq \varphi^\natural\}$*, that is, for the set of* $u$ *for which nonclassical shocks exist.*

*Let* $u_\varepsilon$ *be the nonclassical solution to the conservation law* (1.1) *with initial data* $u_0^* + \theta_\varepsilon$*, where* $u_0^*$ *is defined in* (5.1) *and the perturbation* $\theta_\varepsilon$ *is of locally bounded variation, of small amplitude and satisfies* (5.2). *Suppose that* $u_\varepsilon$ *exhibits a splitting-merging pattern on the time interval* $[t_0, t_{m+1}]$ *along successive interac-*

*tions with small waves (cf. the notation above). If $\varepsilon$ is sufficiently small and the total effect of all waves on the classical shock $C^{\uparrow}$ increases its total strength, that is,*

$$\mathrm{SV}(C^{\uparrow}) > 0, \tag{5.5}$$

*then the variation of the interaction potential* (4.1) *is negative,*

$$[Q_{weak}]|_{[t_0,\, t_{m+1}]} < 0.$$

Our main assumption $\mathrm{SV}(C^{\uparrow}) > 0$ requires that the total effect of the interaction of all waves on $C^{\uparrow}$ is to increase its strength. In fact, this is always the case for the perturbations of splitting-merging solutions within the setting [17]. In our slightly more general setting though, waves crossing through $N^{\downarrow}_{\pm}$ will change the critical state $\varphi^{\sharp}(u_l)$ and therefore could conceivably lead to an NC interaction even if $C^{\uparrow}$ interacted only with rarefactions on the right ($\mathrm{SV}(C^{\uparrow}) < 0$). A precise statement can be found in the following.

**Lemma 5.2** (Necessary condition for merging). *Consider a splitting-merging solution under the same assumptions as Theorem* 5.1. *Let $u_l$ be the left-hand state of the nonclassical shock $N^{\downarrow}_{\pm}$ and define the strength of the splitting to be*

$$\gamma_0 := \left|\tilde{u}_l(t_0+) + \tilde{\varphi}^{\sharp}\big(\tilde{u}_l(t_0+)\big)\right| - \sigma\big(N^{\downarrow}_{\pm}(t_0+)\big) - \sigma\big(C^{\uparrow}(t_0+)\big).$$

*Under these conditions,*

$$\gamma_0 + \tilde{\varphi}^{\sharp}(\tilde{u}^*) \sum_{i=1}^{m} s_i \sigma(W_i^-) < \mathrm{SV}_R(C^{\uparrow}) \tag{5.6}$$

*up to a constant of order $\mathcal{O}(\varepsilon)\,\mathrm{TV}(N^{\downarrow}_{\pm})$.*

*Proof.* Throughout, we will assume that the wave strength and the states are measured in the variables $\tilde{u} = \psi(u)$, if such a change of variable was necessary, and we will therefore omit the tilde superscript. If the waves $N^{\downarrow}_{\pm}(t_{m+1}-)$ and $C^{\uparrow}(t_{m+1}-)$ meet at time $t_{m+1}$, then there are no waves between them and the speed of $N^{\downarrow}_{\pm}$ must be greater than the speed of $C^{\uparrow}$. Graphically, if the states are $u_l(t_{m+1}-)$, $u_m(t_{m+1}-)$ and $u_r(t_{m+1}-)$ then we have

$$\varphi^{\sharp}\big(u_l(t_{m+1}-)\big) < u_r(t_{m+1}-).$$

Note that $\varphi^{\flat}$ is $C^1$ in a neighborhood of $u^*$ and therefore $\varphi^{\sharp}$, which satisfies (2.3), must also be $C^1$. Exploiting the smoothness of $\varphi^{\sharp}$ and the identity $u_r(t_0+) + \gamma_0 = \varphi^{\sharp}\big(u_l(t_0+)\big)$, we can therefore deduce that, for some $\xi$,

$$\gamma_0 + (\varphi^{\sharp})'(\xi)\big(u_l(t_{m+1}-) - u_l(t_0+)\big) < u_r(t_{m+1}-) - u_r(t_0+).$$

The right-hand side is clearly $\mathrm{SV}_R(C^{\uparrow})$ and

$$u_l(t_{m+1}-) - u_l(t_0+) = \sum_{i=1}^{m} s_i \sigma(W_i^-).$$

To complete the proof, it suffices to observe that, because $\xi \in \big(u_l(t_{m+1}-), u_l(t_0+)\big)$ is within $\varepsilon$ of $u^*$, the continuity of the derivative of $\varphi^\sharp$ and the fact that $(\varphi^\flat)'(u^*) < 1$ together imply that

$$(\varphi^\sharp)'(\xi)\Big(\sum_{i=1}^{m} s_i\sigma(W_i^-)\Big) = (\varphi^\sharp)'(u^*)\Big(\sum_{i=1}^{m} s_i\sigma(W_i^-)\Big) + \mathcal{O}(\varepsilon)\,\mathrm{TV}(N_\pm^\downarrow). \qquad \square$$

**Example 5.3.** Despite the conclusion of Lemma 5.2 that condition (5.5) is not optimal, it is interesting to study the application of condition (5.6) in the case of a simple interaction. Quite surprisingly, the bound (5.6) is too tight to imply $[Q] < 0$. The analysis will show that the speed at which the interactions occur (i.e., whether $t_{m+1} - t_0$ is large or small) plays a role in ultimately making $Q_{\mathrm{weak}}$ globally decreasing. This should not come entirely as a shock since it is well known that in the presence of nonclassical shocks, the asymptotic profile is not entirely determined by the BV structure of the initial data but also by the relative positions of these waves.

We consider initial data involving a single rarefaction $R$ crossing the nonclassical shock $N_\pm^\downarrow$ and we assume that $\varphi^\sharp$ is monotonically increasing (or else a single rarefaction would be insufficient). Suppose that the strength of the rarefaction wave after crossing $N_\pm^\downarrow$ is $v_R$ and that the change in $N_\pm^\downarrow$ during this interaction is $-v_L$. In Lemma 5.5 below, we will check that $v_L < v_R$ and that $v_L + v_R$ is the strength of the original rarefaction. Suppose the initial splitting is created by a CR-4 interaction, where the incoming waves are

$$C_\pm^\downarrow = \big(u_l(t_0-), u_r(t_0-)\big), \qquad R_-^\downarrow = \big(u_r(t_0-), u_r(t_0+)\big),$$

and the outgoing waves are

$$N_\pm^\downarrow = \big(u_l(t_0-), \varphi^\flat\big(u_l(t_0-)\big)\big), \qquad C^\uparrow = \big(\varphi^\flat\big(u_l(t_0-)\big), u_r(t_0+)\big).$$

The strength of the splitting is, as defined in the statement of Lemma 5.2,

$$\gamma_0 = \varphi^\sharp\big(u_l(t_0-)\big) - u_r(t_0+).$$

The change in $Q_{\mathrm{weak}}$ during the first interaction CR-4 is therefore

$$[Q]_1(t_0) = \sigma(N_\pm^\downarrow)\sigma(C^\uparrow) - \sigma(C_\pm^\downarrow)\sigma(R_-^\downarrow) \le \sigma(N_\pm^\downarrow)\sigma(C^\uparrow) - \big(\sigma(N_\pm^\downarrow) + \sigma(C^\uparrow)\big)\gamma_0.$$

The above bound is optimal in the sense that if the rarefaction $R_-^{\downarrow}$ in the front-tracking approximation splits into two rarefactions $\big(u_r(t_0-), \varphi^\sharp\big(u_l(t_0-)\big)\big)$ and $\big(\varphi^\sharp\big(u_l(t_0-)\big), u_r(t_0+)\big)$ just prior to the interaction at time $t_0$, then this is exactly the change which would be computed (and also the largest possible).

Assuming that the interactions are CR-4, to create $N_\pm^{\downarrow}$ and $C^{\uparrow}$, followed by RN when the rarefaction crosses $N_\pm^{\downarrow}$, RC-1 when the rarefaction reaches $C^{\uparrow}$ and finally NC that returns a crossing shock, we have

$$\begin{aligned}
[Q]_1|_\Omega &= \sigma(N_\pm^{\downarrow})\sigma(C^{\uparrow}) \qquad \text{(due to CR-4)}\\
&\quad - v_L\big(\sigma(N_\pm^{\downarrow}) + v_R\big) \qquad \text{(due to RN)}\\
&\quad - \sigma(C^{\uparrow})v_R \qquad \text{(due to RC-1)}\\
&\quad - \big(\sigma(N_\pm^{\downarrow}) - v_L\big)\big(\sigma(C^{\uparrow}) - v_R\big) \qquad \text{(due to NC)}\\
&= -2v_Lv_R + \sigma(N_\pm^{\downarrow})(v_R - v_L + \gamma_0) + \sigma(C^{\uparrow})(-v_R + v_L - \gamma_0).
\end{aligned}$$

We immediately note that condition (5.6) is now written as

$$\gamma_0 < (\varphi^\sharp)'(\xi)(v_L + v_R), \tag{5.7}$$

and that $v_L$, $v_R$ and $\gamma_0$ are small with respect to $\sigma(N_\pm^{\downarrow})$ and $\sigma(C^{\uparrow})$. Generically, the only way $[Q]_1$ can be negative is if $v_R - v_L < \gamma_0$. Moreover, Proposition 2.6 allows us to make $v_L$ arbitrarily small, so that the true condition is $v_R < \gamma_0$.

In the best possible scenario, $|(\varphi^\sharp)'(\xi)| < 1$ (geometrically we expect something like $|(\varphi^\sharp)'| < |(\varphi^\flat)'|$) and (5.7) becomes an equality. In other words, the rarefaction is just strong enough to bring $N_\pm^{\downarrow}$ and $C^{\uparrow}$ back together. This would imply that $[Q]_1 < 0$. However, if the interaction period is large then the waves $N_\pm^{\downarrow}$ and $C^{\uparrow}$ can move arbitrarily far apart and the total strength of the rarefactions entering $\Omega$ can be made much larger than $\gamma_0$. In conclusion, the condition (5.6) is insufficient to prove that the interaction potential is globally decreasing.

**Remark 5.4.** We make two observations concerning Theorem 5.1.

1. Although the condition (5.5) is stronger than (5.6), in contrast to the later, it appears to be sufficient to prove that the weak interaction potential is globally decreasing.

2. We have assumed that $\varphi^\flat$ is $C^1$ away from the threshold where only classical shocks exist. This rather strong assumption is verified in the examples presented in Section 2 but does not appear naturally as a result of the general theory [13]. Nonetheless, it is used often in the proof and appears to be necessary, at least to the analysis in this section.

For the proof of Theorem 5.1 we will need the following two lemmas. The interest of the first lemma is to make more precise the (mainly linear) dependence of the change $[Q]_1$ in terms of the incoming wave $\sigma(W_i^-)$. The second lemma provides an estimate which closely relates the signed variations of $C^\uparrow$ and $N_\pm^\downarrow$.

**Lemma 5.5** (Interactions with the nonclassical shock). *Consider interactions* RN *and* CN-3 *at the time $t_i$ involving a left-incoming weak wave $W_i$ and the nonclassical shock $N_\pm^\downarrow$. Then there exists a positive constant $L_i < 1$ such that*

$$\begin{aligned} \sigma(W_i^+) &= L_i \sigma(W_i^-), \\ \sigma\big(N_\pm^\downarrow(t_i+)\big) &= \sigma\big(N_\pm^\downarrow(t_i-)\big) + s_i(1-L_i)\sigma(W_i^-), \\ [Q]_1 &= -(1-L_i)\sigma(W_i^-)\big(\sigma\big(N_\pm^\downarrow(t_i-)\big) - s_i L_i \sigma(W_i^-)\big) < 0, \end{aligned}$$

*where $s_i$ is $+1$ if $W_i$ is a shock and $-1$ otherwise. Moreover, one has*

$$L_i = (\varphi^\flat)'(u^*) + \mathcal{O}(\varepsilon). \tag{5.8}$$

*Proof.* We consider only an RN interaction, since the calculations for an incoming shock are similar and have been essentially treated in Theorem 4.2; see equation (4.4). When $W_i$ is small, then the states are ordered as in (3.8), namely,

$$-\varphi^\flat(u_l^i) < -\varphi^\flat(u_m^i) < u_l^i < u_m^i.$$

Then we have

$$\sigma(W_i^+) = |-\varphi^\flat(u_l^i) + \varphi^\flat(u_m^i)| =: L_i |u_l^i - u_m^i| = L_i \sigma(W_i^-),$$

with $L_i < 1$ since $\varphi^\flat$ is a strict contraction with a Lipschitz constant uniformly below 1. On the other hand, relation (5.8) is obvious since $\varphi^\flat$ is $C^1$ in a neighborhood of $u^*$.

Finally, the outgoing nonclassical shock has strength

$$\begin{aligned} \sigma\big(N_\pm^\downarrow(t_i+)\big) &= |u_l^i + \varphi^\flat(u_l^i)| \\ &= -|u_l^i - u_m^i| + |u_m^i + \varphi^\flat(u_m^i)| + |\varphi^\flat(u_m^i) - \varphi^\flat(u_l^i)| \\ &= \sigma\big(N_\pm^\downarrow(t_i-)\big) - (1-L_i)\sigma(W_i^-), \end{aligned}$$

while the change $[Q]_1$ takes the form

$$\begin{aligned} [Q]_1 &= \sigma\big(N_\pm^\downarrow(t_i+)\big)\sigma(W_i^+) - \sigma\big(N_\pm^\downarrow(t_i-)\big)\sigma(W_i^-) \\ &= \big(\sigma\big(N_\pm^\downarrow(t_i-)\big) - (1-L_i)\sigma(W_i^-)\big)L_i\sigma(W_i^-) - \sigma\big(N_\pm^\downarrow(t_i-)\big)\sigma(W_i^-) \\ &= -(1-L_i)\sigma(W_i^-)\big(\sigma\big(N_\pm^\downarrow(t_i-)\big) + L_i\sigma(W_i^-)\big) < 0. \end{aligned}$$

□

**Lemma 5.6** (Property of the signed variations). *If* $L^* := (\varphi^\flat)'(u^*) \in (1/2, 1)$, *then* $\lambda^* := (1 - L^*)/L^* \in (0, 1)$ *and one has*

$$|\lambda^* \, \mathrm{SV}_L(C^\uparrow) - \mathrm{SV}(N^\downarrow_\pm)| \le \mathcal{O}(\varepsilon) \, \mathrm{TV}(N^\downarrow_\pm). \tag{5.9}$$

*Proof.* Using Lemma 5.5, we compute

$$\begin{aligned} \mathrm{SV}(N^\downarrow_\pm) &= \sum_{i=1}^{m} s_i(1 - L_i)\sigma(W_i^-) \\ &= \sum_{i=1}^{m} s_i(1 - L^*)\sigma(W_i^-) + \sum_{i=1}^{m} s_i(L^* - L_i)\sigma(W_i^-) \\ &= \lambda^* \sum_{i=1}^{m} s_i L^* \sigma(W_i^-) + \sum_{i=1}^{m} s_i(L^* - L_i)\sigma(W_i^-) \\ &= \lambda^* \sum_{i=1}^{m} s_i L_i \sigma(W_i^-) + (1 + \lambda^*) \sum_{i=1}^{m} s_i(L^* - L_i)\sigma(W_i^-). \end{aligned} \tag{5.10}$$

Observe that the waves $W_i^+$ which crossed $N^\downarrow_\pm$, but have yet to reach $C^\uparrow$, may interact in $\Omega$. All these waves are weak and therefore the potential interactions are all classical and will preserve the signed variation. This provides the identity

$$\sum_{i=1}^{m} s_i L_i \sigma(W_i^-) = \sum_{i=1}^{m} s_i \sigma(W_i^+) = \sum_{i=1}^{\tilde{m}} \tilde{s}_i \sigma(\tilde{W}_i) = \mathrm{SV}_L(C^\uparrow),$$

which, when substituted into (5.10), proves

$$\mathrm{SV}(N^\downarrow_\pm) = \lambda^* \, \mathrm{SV}_L(C^\uparrow) + (1 + \lambda^*) \sum_{i=1}^{m} s_i(L^* - L_i)\sigma(W_i^-).$$

If $L^* < 1$, a difficulty appears because the signs of $L^* - L_i$ are uncorrelated to the signs $s_i$. On the other hand, because the kinetic function is smooth in a neighborhood of $u^*$, for each index $i$, $L^* - L_i = \mathcal{O}_i(\varepsilon)$. If we take $\mathcal{O}(\varepsilon) = \max_i \mathcal{O}_i(\varepsilon)$ and use $L := \max_i L_i$, then the difference is

$$\begin{aligned} |\lambda^* \, \mathrm{SV}_L(C^\uparrow) - \mathrm{SV}(N^\downarrow_\pm)| &\le 2\mathcal{O}(\varepsilon) \sum_{i=1}^{m} |\sigma(W_i^-)| \\ &\le \frac{2}{1 - L} \mathcal{O}(\varepsilon) \, \mathrm{TV}(N^\downarrow_\pm) = \mathcal{O}(\varepsilon) \, \mathrm{TV}(N^\downarrow_\pm). \end{aligned}$$

□

*Proof of Theorem* 5.1. To help the reader understand the wave interactions (and cancellations) in this proof, we begin with a few preliminary comments. Simply put, the final NC interaction should provide a quadratic term which is, later, cancelled by a similar quadratic term when the two waves merge. One may naïvely expect the sum

$$-\sigma\big(N_{\pm}^{\downarrow}(t_{m+1}-)\big)\sigma\big(C^{\uparrow}(t_{m+1}-)\big) + \sigma\big(N_{\pm}^{\downarrow}(t_0+)\big)\sigma\big(C^{\uparrow}(t_0+)\big)$$

to be negative. Of course, the cumulative strength of the changes during the interactions with the small waves must also be taken into account. The difference between the strength of the initial and final waves $N_{\pm}^{\downarrow}$, $C^{\uparrow}$ is measured by the signed variation SV along those two shocks. Our proof below shows that, along the trajectories $N_{\pm}^{\downarrow}$ and $C^{\uparrow}$, the change $[Q]_1$ is negative and proportional to the total variation $\mathrm{TV}(N_{\pm}^{\downarrow}) + \mathrm{TV}(C^{\uparrow})$. The total variation being larger than the signed variation, after further analysis one can conclude that $[Q]_1|_{\Omega} < 0$. Given that only classical shocks exist away from $\Omega$, the change $[Q]_2$, and therefore $[Q]$, is globally negative over the time interval $[t_0, t_{m+1}]$.

The key technical information is provided by Lemma 5.6, which implies that, up to a quantity of order $\mathcal{O}(\varepsilon)\,\mathrm{TV}(N_{\pm}^{\downarrow})$, we have $\mathrm{SV}(N_{\pm}^{\downarrow}) = \lambda^*\,\mathrm{SV}_L(C^{\uparrow})$ and, in particular, that the signed variations have the same sign. We now have all the tools necessary to proceed with the proof of Theorem 5.1.

The perturbation $\theta_\varepsilon$ has bounded oscillation and therefore it can only alter the right-hand state of $C^{\uparrow}$ by an amount $\varepsilon$. The small waves entering $\Omega$ through $N_{\pm}^{\downarrow}$ only alter its strength by $|(1-L_i)\sigma(W_i^-)| < \varepsilon$. In both cases, we expect that everywhere along their trajectories,

$$\sigma\big(N_{\pm}^{\downarrow}(t)\big) > \sigma\big(N_{\pm}^{\downarrow}(t_0+)\big) - 2\varepsilon, \qquad \sigma\big(C^{\uparrow}(t)\big) > \sigma\big(C^{\uparrow}(t_0+)\big) - 2\varepsilon.$$

Ignoring the negative contribution to $[Q]_1$ coming from the interaction that generated $N_{\pm}^{\downarrow}$ (which, anyway, can be arbitrarily small), and neglecting also all classical interactions inside $\Omega$ (for which $[Q]_1 \le 0$ and possibly 0), we have

$$\begin{aligned}[Q]_1|_{\Omega} \le\ & +\sigma\big(N_{\pm}^{\downarrow}(t_0+)\big)\sigma\big(C^{\uparrow}(t_0+)\big) \\ & - \sum_{i=1}^{m}(1-L_i)\sigma(W_i^-)\Big(\sigma\big(N_{\pm}^{\downarrow}(t_i-)\big) - s_i L_i \sigma(W_i^-)\Big) \\ & - \sum_{i=1}^{\tilde m}\sigma(\tilde W_i)\sigma\big(C^{\uparrow}(\tilde t_i-)\big) - \sum_{i=1}^{\bar m}\sigma(\overline{W}_i)\sigma\big(C^{\uparrow}(\bar t_i-)\big) \\ & - \sigma\big(N_{\pm}^{\downarrow}(t_{m+1}-)\big)\sigma\big(C^{\uparrow}(t_{m+1}-)\big). \end{aligned} \tag{5.11}$$

Using the previous bounds on the strength of shocks bounding $\Omega$, we rewrite the previous expression as

$$
\begin{aligned}
[Q]_1|_\Omega \le\ & +\sigma\big(N_\pm^\downarrow(t_0+)\big)\sigma\big(C^\uparrow(t_0+)\big) \\
& - \big(\sigma\big(N_\pm^\downarrow(t_0+)\big) - 3\varepsilon\big)\,\mathrm{TV}(N_\pm^\downarrow) - \big(\sigma\big(C^\uparrow(t_0+)\big) - 2\varepsilon\big)\,\mathrm{TV}(C^\uparrow) \\
& - \big(\sigma\big(N_\pm^\downarrow(t_0+)\big) + \mathrm{SV}(N_\pm^\downarrow)\big)\big(\sigma\big(C^\uparrow(t_0+)\big) + \mathrm{SV}(C^\uparrow)\big) \\
=\ & -\big(\sigma\big(N_\pm^\downarrow(t_0+)\big) - 3\varepsilon\big)\,\mathrm{TV}(N_\pm^\downarrow) - \big(\sigma\big(C^\uparrow(t_0+)\big) - 2\varepsilon\big)\,\mathrm{TV}(C^\uparrow) \\
& - \sigma\big(N_\pm^\downarrow(t_0+)\big)\,\mathrm{SV}(C^\uparrow) - \sigma\big(C^\uparrow(t_0+)\big)\,\mathrm{SV}(N_\pm^\downarrow) - \mathrm{SV}(N_\pm^\downarrow)\,\mathrm{SV}(C^\uparrow). \qquad (5.12)
\end{aligned}
$$

Clearly, the main difficulty now lies in the sign of $\mathrm{SV}(N_\pm^\downarrow)$ and $\mathrm{SV}(C^\uparrow)$. If both are positive, then all the terms are negative and the proof is completed. Recall that $\mathrm{SV}(C^\uparrow) \ge 0$, by assumption, so that it suffices to consider the (only possibly unfavorable) case $\mathrm{SV}(N_\pm^\downarrow) < 0$ which makes the fourth and fifth terms negative.

We immediately note that any correction terms of order $\mathcal{O}(\varepsilon)\,\mathrm{TV}(N_\pm^\downarrow)$ can be included into the first term of the decomposition (5.12) and, therefore, taking a smaller $\varepsilon$ if necessary, we can make the new term negative:

$$
-\big(\sigma\big(N_\pm^\downarrow(t_0+)\big) - \mathcal{O}(\varepsilon)\big)\,\mathrm{TV}(N_\pm^\downarrow) < 0.
$$

This fact is used below without further comment.

Using Lemma 5.6 we have $|\mathrm{SV}(N_\pm^\downarrow)| \approx \lambda^*|\mathrm{SV}_L(C^\uparrow)| \le \lambda^*\,\mathrm{TV}(C^\uparrow)$, and the fourth term in (5.12) can be written as

$$
-\sigma\big(C^\uparrow(t_0+)\big)\,\mathrm{SV}(N_\pm^\downarrow) \le -\lambda^*\sigma\big(C^\uparrow(t_0+)\big)\,\mathrm{TV}(C^\uparrow) + \mathcal{O}(\varepsilon)\,\mathrm{TV}(N_\pm^\downarrow). \qquad (5.13)
$$

Taking $\eta_C = \lambda^*$ suffices to guarantee that the term vanishes.

The fifth term in (5.12) can be bounded as follows

$$
-\mathrm{SV}(N_\pm^\downarrow)\,\mathrm{SV}(C^\uparrow) \le \mathrm{SV}(C^\uparrow)\,\mathrm{TV}(N_\pm^\downarrow) \le \varepsilon\,\mathrm{TV}(N_\pm^\downarrow). \qquad (5.14)
$$

To conclude, we remark that the last term in (5.13) can be included in the first term of (5.12) while the upper bound (5.14) can be included in the second term of (5.12). This completes the proof of Theorem 5.1. □

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