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On Lorentz nuclear homogeneous polynomials between Banach spaces

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Abstract. In this paper we introduce the classes of Lorentz nuclear homogeneous polynomials and characterize their duals. These type of results are related to the study of convolution equations and Malgrange-type theorems on infinite-dimensional Banach spaces.

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1. Introduction

The investigation of dual theory of polynomials in Banach spaces plays an important role in Functional Analysis (for example, characterizations of the duals of approximable n-homogeneous polynomials and nuclear n-homogeneous polynomials can be found in [6], [12], respectively). In this paper we introduce the notion of Lorentz nuclear homogeneous polynomials and characterize the dual of these classes, extending results from [20]. Besides its own interest, we believe that our results can contribute to the investigation of convolution equations, as it will be explained in the next paragraph.

The study of polynomial and holomorphic nuclear mappings between Banach spaces appears in 1966 with Gupta's work [12] on an infinite-dimensional extension of a theorem due to Malgrange on the existence and approximation of convolution equations (see [13]). One year later, Martineau [14] investigated partial differential equations (of order infinity) from the point of view of convolution

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equations in certain spaces of entire functions. About ten years later, the second named author, in [16], [18], [19] extended [13], [14] for nuclear entire mappings and a decisive role is played in [16], [18], [19] by the characterizations of the duals of some classes of nuclear mappings. Recent works [9], [21] in the same direction also reinforce the importance played by the characterization of duals of spaces of mappings of nuclear type. We believe that the duality results obtained in the present article can lead to more general extensions of Gupta's and Malgrange's results, following the lines of [9], [21].

2. Lorentz sequence spaces

Henceforth, $\mathbb{N}$ represents the set of positive integers and if $m \in \mathbb{N}$, $I_m := \{1, \ldots, m\}$. The letters E and F will always denote Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. As usual, E' represents the topological dual of E and $B_{E'}$ denotes its unit ball. We denote by $c_0(E)$ the sequence space (with the sup norm $\|.\|_\infty$) composed by the sequences $(x_j)_{j=1}^\infty$ in E so that $\lim_{n\to\infty} x_n = 0$ and $c_{00}(E)$ is the subspace of $c_0(E)$ formed by the sequences $(x_j)_{j=1}^\infty$ for which there is a N_0 such that $x_n = 0$ for all $n \geq N_0$. When $E = \mathbb{K}$ we write c_0 and c_{00} instead of $c_0(\mathbb{K})$ and $c_{00}(\mathbb{K})$, respectively. If $u = (u_j) \in c_{00}(E)$, the symbol $\mathrm{card}(u)$ denotes the cardinality of the set $\{j; u_j \neq 0\}$.

As usual $l_\infty(E)$ represents the vector space of bounded sequences in E, with the sup norm and $l_\infty := l_\infty(\mathbb{K})$. If $m \in \mathbb{N}$, $(x_j)_{j=1}^m$ denotes $(x_1, \ldots, x_m, 0, 0, \ldots)$, and when $(x_j)_{j=1}^\infty$ is a sequence of non-negative real numbers, we say that $(x_j)_{j=1}^\infty$ admits a non-increasing rearrangement if there is an injection $\pi : \mathbb{N} \to \mathbb{N}$ such that $x_{\pi(1)} \geq x_{\pi(2)} \geq \cdots \geq 0$ and $\pi^{-1}(j) \neq \emptyset$ whenever $x_j \neq 0$. If $p \geq 1$, then p' denotes the conjugate of p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$.

Definition 2.1. Let E be a Banach space, $x = (x_j)_{j=1}^\infty \in l_\infty(E)$ and

$$a_{E,n}(x) := \inf\{\|x - u\|_\infty; u \in c_{00}(E) \text{ and } \mathrm{card}(u) < n\}.$$

For $0 < r, q < +\infty$, the Lorentz sequence space $l_{(r,q)}(E)$ consists of all sequences $x = (x_j)_{j=1}^\infty \in l_\infty(E)$ such that

$$\left(n^{1/r-1/q} a_{E,n}(x)\right)_{n=1}^\infty \in l_q.$$

For $x \in l_{(r,q)}(E)$ we define the quasi-norm

$$\|x\|_{(r,q)} = \left\|\left(n^{1/r-1/q} a_{E,n}(x)\right)_{n=1}^\infty\right\|_q.$$

It is well known that $l_{(r,q)}(E) \subset c_0(E)$ and it is also easy to prove that if $x = (x_j)_{j=1}^\infty \in c_0(E)$, then the sequence $(\|x_j\|)_{j=1}^\infty$ admits a non-increasing rearrangement.

2.1. Estimates for Lorentz sequence spaces. Usually, we are investigating the space $l_{(r,q)}(E)$ with the quasi-norm

$$\|x\|_{(r,q)} = \left\| \left(n^{1/r-1/q} a_{E,n}(x)\right)_{n=1}^{\infty} \right\|_q.$$

However, if $1 < r < \infty$ and $1 \le q < \infty$,

$$\|x\|_{(r,q)}^{\mathrm{norm}} = \left[\sum_{n=1}^{\infty} \left(n^{1/r-1/q} \frac{1}{n} \sum_{k=1}^{n} a_{E,k}(x) \right)^q \right]^{1/q}$$

defines an equivalent norm on $l_{(r,q)}(E)$. This result, for the case $E = \mathbb{K}$, appears in [27], 13.9.5. The general case is a simple consequence; in fact, if $x = (x_j)_{j=1}^{\infty} \in l_{(r,q)}(E)$, then

$$a_{E,n}(x) = a_{\mathbb{K},n}(z),$$

where

$$z = (\|x_1\|, \|x_2\|, \ldots).$$

From now on, will write $a_n(x)$ instead of $a_{E,n}(x)$.

Proposition 2.2. *Let $1 < r < \infty$ and $1 \le q < \infty$. If $x = (x_j)_{j=1}^{\infty} \in l_{(r,q)}(E)$ and $y = (y_j)_{j=1}^{\infty} \in l_{(r,q)}(E)$, then $z = (z_j)_{j=1}^{\infty} \in l_{(r,q)}(E)$, where $z_{2k-1} = x_k$ and $z_{2k} = y_k$, for $k \in \mathbb{N}$, and*

$$(\|z\|_{(r,q)}^{\mathrm{norm}})^q \le 2^q [(\|x\|_{(r,q)}^{\mathrm{norm}})^q + (\|y\|_{(r,q)}^{\mathrm{norm}})^q].$$

Proof. It is enough to show that

$$\left\| \left(a_j(z)\right)_{j=1}^{n} \right\|_1^q \le 2^q \left[\left\| \left(a_j(x)\right)_{j=1}^{n} \right\|_1^q + \left\| \left(a_j(y)\right)_{j=1}^{n} \right\|_1^q \right] \tag{1}$$

for all $n \in \mathbb{N}$.

In fact, if we prove (1) we can conclude that

$$\begin{aligned}
(\|z\|_{(r,q)}^{\mathrm{norm}})^q &= \sum_{n=1}^{\infty} \left[n^{1/r-1/q} \frac{1}{n} \sum_{k=1}^{n} a_k(z) \right]^q \\
&= \sum_{n=1}^{\infty} \left[\left(n^{1/r-1/q} \frac{1}{n} \right)^q \left(\sum_{k=1}^{n} a_k(z) \right)^q \right] \\
&\overset{(1)}{\le} \sum_{n=1}^{\infty} \left[\left(n^{1/r-1/q} \frac{1}{n} \right)^q 2^q \left(\left(\sum_{k=1}^{n} a_k(x) \right)^q + \left(\sum_{k=1}^{n} a_k(y) \right)^q \right) \right] \\
&= 2^q [(\|x\|_{(r,q)}^{\mathrm{norm}})^q + (\|y\|_{(r,q)}^{\mathrm{norm}})^q].
\end{aligned}$$

So, let us prove (1).

Since $l_{(r,q)}(E) \subset c_0(E)$, we have $x \in c_0(E)$ and $y \in c_0(E)$. So, $z \in c_0(E)$ and there is an injection π such that $(\|z_{\pi(j)}\|)_{j=1}^{\infty}$ is non-increasing. We denote by σ and δ the injections such that $(\|x_{\sigma(j)}\|)_{j=1}^{\infty}$ and $(\|y_{\delta(j)}\|)_{j=1}^{\infty}$ are non-increasing. We know that

$$a_j(z) = \|z_{\pi(j)}\|, \quad a_j(x) = \|x_{\sigma(j)}\| \quad \text{and} \quad a_j(y) = \|y_{\delta(j)}\|,$$

for every positive integer j. We define

$$I_{n,\sigma} = \{j \in \{1,\dots,n\}; \|z_{\pi(j)}\| = \|x_{\sigma(k)}\| \text{ for some } k = 1,\dots,n\}$$
$$I_{n,\delta} = \{j \in \{1,\dots,n\}; \|z_{\pi(j)}\| = \|y_{\delta(k)}\| \text{ for some } k = 1,\dots,n\}.$$

Now we have

$$\Big(\sum_{j\in I_{n,\sigma}} \|z_{\pi(j)}\|\Big)^q \le \Big(\sum_{j=1}^n \|x_{\sigma(j)}\|\Big)^q \le \Big(\sum_{j=1}^n \|x_{\sigma(j)}\|\Big)^q + \Big(\sum_{j=1}^n \|y_{\delta(j)}\|\Big)^q$$

and

$$\Big(\sum_{j\in I_{n,\delta}} \|z_{\pi(j)}\|\Big)^q \le \Big(\sum_{j=1}^n \|y_{\delta(j)}\|\Big)^q \le \Big(\sum_{j=1}^n \|x_{\sigma(j)}\|\Big)^q + \Big(\sum_{j=1}^n \|y_{\delta(j)}\|\Big)^q.$$

Then

$$\sum_{j\in I_{n,\sigma}} \|z_{\pi(j)}\| + \sum_{j\in I_{n,\delta}} \|z_{\pi(j)}\| \le 2\Big[\Big(\sum_{j=1}^n \|x_{\sigma(j)}\|\Big)^q + \Big(\sum_{j=1}^n \|y_{\delta(j)}\|\Big)^q\Big]^{1/q}$$

and hence

$$\begin{aligned}\Big(\sum_{j=1}^n \|z_{\pi(j)}\|\Big)^q &\le \Big(\sum_{j\in I_{n,\sigma}} \|z_{\pi(j)}\| + \sum_{j\in I_{n,\delta}} \|z_{\pi(j)}\|\Big)^q \\ &\le 2^q\Big[\Big(\sum_{j=1}^n \|x_{\sigma(j)}\|\Big)^q + \Big(\sum_{j=1}^n \|y_{\delta(j)}\|\Big)^q\Big].\end{aligned}$$ □

The following proposition appears, in essence, in [27], Remark 13.9.5:

Proposition 2.3 ([27], Remark 13.9.5). *If* $1 < r < \infty$ *and* $1 \leq q < \infty$, *then*

$$\|x\|_{(r,q)} \leq \|x\|_{(r,q)}^{\text{norm}} \leq \frac{r}{r-1}\|x\|_{(r,q)}$$

for all $x \in l_{(r,q)}(E)$.

Proof. In [27], Remark 13.9.5, the result appears for $E = \mathbb{K}$. The extension to the general case is straightforward. □

We end this section with an useful inequality for the quasi-norm $\|.\|_{(r,q)}$.

Proposition 2.4. *Let* $1 < r < \infty$ *and* $1 \leq q < \infty$. *If* $x = (x_j)_{j=1}^{\infty} \in l_{(r,q)}(E)$ *and* $y = (y_j)_{j=1}^{\infty} \in l_{(r,q)}(E)$ *then* $z = (z_j)_{j=1}^{\infty}$, *defined by* $z_{2k-1} = x_k$ *and* $z_{2k} = y_k$, *is in* $l_{(r,q)}(E)$ *and*

$$(\|z\|_{(r,q)})^q \leq 2^q\left(\frac{r}{r-1}\right)^q[(\|x\|_{(r,q)})^q + (\|y\|_{(r,q)})^q].$$

Proof. Using Propositions 2.2 and 2.3, we get

$$\begin{aligned}(\|z\|_{(r,q)})^q &\leq (\|z\|_{(r,q)}^{\text{norm}})^q \leq 2^q[(\|x\|_{(r,q)}^{\text{norm}})^q + (\|y\|_{(r,q)}^{\text{norm}})^q] \\ &\leq 2^q\left[\left(\frac{r}{r-1}\|x\|_{(r,q)}\right)^q + \left(\frac{r}{r-1}\|y\|_{(r,q)}\right)^q\right] \\ &= 2^q\left(\frac{r}{r-1}\right)^q[(\|x\|_{(r,q)})^q + (\|y\|_{(r,q)})^q].\end{aligned}$$

□

3. Lorentz summing polynomials—preparatory results

The concept of Lorentz summing polynomials was recently introduced and explored in [23]; in this section, for the sake of completeness, we will recall the results concerning Lorentz summing polynomials that will be needed in the present paper.

The space of continuous linear operators from E to F will be denoted by $\mathcal{L}(E;F)$ and the space of continuous n-homogeneous polynomials from E to F will be represented by $\mathcal{P}(^nE;F)$. In both classes, we consider the sup norm. We also represent the space of finite type polynomials from E to F by $\mathcal{P}_f(^nE;F)$. For the theory of polynomials and multilinear mappings in Banach spaces we refer to [5], [24].

Let $p > 0$. Recall that $l_p^w(E)$ denotes the vector space composed by the sequences $(x_j)_{j=1}^{\infty}$ in E such that $(\varphi(x_j))_{j=1}^{\infty} \in l_p$ for all $\varphi \in E'$. We define, for $0 < s, q < \infty$, $l_{(s,q)}^w(E)$ as the space of the sequences $(x_j)_{j=1}^{\infty}$ in E such that

$\|(\varphi(x_n))_{n=1}^{\infty}\|_{(s,q)} < \infty$ for every $\varphi \in E'$. An application of the Closed Graph Theorem assures that

$$\sup_{\|\varphi\| \leq 1} \|(\varphi(x_n))_{n=1}^{\infty}\|_{(s,q)} < \infty \tag{2}$$

and one can verify that

$$\|(x_n)_{n=1}^{\infty}\|_{w,(s,q)} := \sup_{\|\varphi\| \leq 1} \|(\varphi(x_n))_{n=1}^{\infty}\|_{(s,q)}$$

defines a quasi-norm on $l_{(s,q)}^{w}(E)$.

Definition 3.1. If $0 < p, q, r, s < \infty$, an n-homogeneous polynomial $P \in \mathscr{P}(^nE; F)$ is Lorentz $((s,p);(r,q))$-summing if $(P(x_j))_{j=1}^{\infty} \in l_{(s,p)}(F)$ for each $(x_j)_{j=1}^{\infty} \in l_{(r,q)}^{w}(E)$.

The vector space composed by the Lorentz $((s,p);(r,q))$-summing n-homogeneous polynomials from E to F is denoted by $\mathscr{P}_{as((s,p);(r,q))}(^nE; F)$. When $n = 1$ we write $\mathscr{L}_{as((s,p);(r,q))}(E; F)$.

When $s = p$, we write $\mathscr{L}_{as(s;(r,q))}$ instead of $\mathscr{L}_{as((s,s);(r,q))}$; when $r = q$, we denote $\mathscr{L}_{as((s,p);q)}$ instead of $\mathscr{L}_{as((s,p);(q,q))}$.

Note that when $n = 1$, $s = p$ and $r = q$ we have the usual concept of absolutely $(p;q)$-summing operator. The space of absolutely $(p;q)$-summing operators from E to F is represented by $\mathscr{L}_{as(p;q)}(E; F)$. When $p = q$, we simply write $\mathscr{L}_{as,p}$ instead of $\mathscr{L}_{as(p;q)}$. For the theory of absolutely summing linear operators we refer to [4].

The next characterization of Lorentz summing mappings was proved in [23]:

Theorem 3.2. *For $P \in \mathscr{P}(^nE; F)$, the following conditions are equivalent:*

(1) *P is Lorentz $((s,p);(r,q))$-summing.*

(2) *There is $C \geq 0$ such that*

$$\|(P(x_j))_{j=1}^{m}\|_{(s,p)} \leq C\|(x_j)_{j=1}^{m}\|_{w,(r,q)}^{n}$$

for all $m \in \mathbb{N}$ and $x_1, \ldots, x_m \in E$.

(3) *There is $C \geq 0$ such that*

$$\|(P(x_j))_{j=1}^{\infty}\|_{(s,p)} \leq C\|(x_j)_{j=1}^{\infty}\|_{w,(r,q)}^{n}$$

for all $(x_j)_{j=1}^{\infty} \in \ell_{(r,q)}^{w}(E)$.

The infimum of the constants C for which the above inequalities hold is a quasi-norm (denoted by $\|.\|_{as((s,p);(r,q))}$) for $\mathscr{P}_{as((s,p);(r,q))}(^nE; F)$ and under this quasi-norm, $\mathscr{P}_{as((s,p);(r,q))}(^nE; F)$ is complete.

4. Lorentz nuclear polynomials

The following lemma is simple but useful, and appears in [28], p. 111:

Lemma 4.1. *If* $r, q, p, s \in [1, \infty[$ *are such that* $r \leq q$, $s \leq p$, *then*

$$l_{(r,q)} \subset l_q, \quad \text{with } \|.\|_q \leq \|.\|_{(r,q)}, \qquad l_{(s,p)} \subset l_p, \quad \text{with } \|.\|_p \leq \|.\|_{(s,p)}. \tag{3}$$

Let E and F be Banach spaces, $n \in \mathbb{N}$ and $r, q, s, p \in]1, \infty[$ such that $r \leq q$, $s' \leq p'$ and

$$1 \leq \frac{1}{q} + \frac{n}{p'}.$$

If $(\lambda_j)_{j=1}^{\infty} \in l_{(r,q)}$, $(\varphi_j)_{j=1}^{\infty} \in l_{(s',p')}^{w}(E')$ and $(y_j)_{j=1}^{\infty} \in l_{\infty}(F)$, note that for every $x \in E$ we have

$$\left(\left\|\lambda_j\left(\varphi_j(x)\right)^n y_j\right\|\right)_{j=1}^{\infty} \in l_1.$$

Thus, denoting by J the canonical inclusion of E into E'', we have

$$\begin{aligned}
\sum_{j=1}^{\infty}\left\|\lambda_j\left(\varphi_j(x)\right)^n y_j\right\| &\leq \|(y_j)_{j=1}^{\infty}\|_{\infty} \sum_{j=1}^{\infty}\left|\lambda_j\left(\varphi_j(x)\right)^n\right| \\
&\leq \|(y_j)_{j=1}^{\infty}\|_{\infty}\left\|\left(\varphi_j(x)\right)_{j=1}^{\infty}\right\|_{p'}^{n}\|(\lambda_j)_{j=1}^{\infty}\|_q \\
&\overset{\text{Lemma 4.1}}{\leq} \|(y_j)_{j=1}^{\infty}\|_{\infty}\left\|\left(\varphi_j(x)\right)_{j=1}^{\infty}\right\|_{(s',p')}^{n}\|(\lambda_j)_{j=1}^{\infty}\|_{(r,q)} \\
&= \|(y_j)_{j=1}^{\infty}\|_{\infty}\left\|\left(Jx(\varphi_j)\right)_{j=1}^{\infty}\right\|_{(s',p')}^{n}\|(\lambda_j)_{j=1}^{\infty}\|_{(r,q)} \\
&\leq \|x\|^n\|(y_j)_{j=1}^{\infty}\|_{\infty}\|(\varphi_j)_{j=1}^{\infty}\|_{w,(s',p')}^{n}\|(\lambda_j)_{j=1}^{\infty}\|_{(r,q)}.
\end{aligned}$$

We conclude that the map $P: E \to F$ given by

$$P(x) = \sum_{j=1}^{\infty} \lambda_j\left(\varphi_j(x)\right)^n y_j \tag{4}$$

is a well-defined n-homogeneous polynomial. Besides, P is also continuous, since

$$\|P(x)\| \leq \|x\|^n\|(y_j)_{j=1}^{\infty}\|_{\infty}\|(\varphi_j)_{j=1}^{\infty}\|_{w,(s',p')}^{n}\|(\lambda_j)_{j=1}^{\infty}\|_{(r,q)}. \tag{5}$$

So, we have

$$\|P\| \le \|(y_j)_{j=1}^{\infty}\|_{\infty}\|(\varphi_j)_{j=1}^{\infty}\|_{w,(s',p')}^{n}\|(\lambda_j)_{j=1}^{\infty}\|_{(r,q)}. \tag{6}$$

Definition 4.2. Let E and F be Banach spaces, $n \in \mathbb{N}$ and $r, q, s, p \in]1, \infty[$ such that $r \le q$, $s' \le p'$ and

$$1 \le \frac{1}{q} + \frac{n}{p'}.$$

An n-homogeneous polynomial $P : E \to F$ is Lorentz $((r,q);(s,p))$-nuclear if

$$P(x) = \sum_{j=1}^{\infty} \lambda_j (\varphi_j(x))^n y_j, \tag{7}$$

with $(\lambda_j)_{j=1}^{\infty} \in l_{(r,q)}$, $(\varphi_j)_{j=1}^{\infty} \in l_{(s',p')}^{w}(E')$ and $(y_j)_{j=1}^{\infty} \in l_{\infty}(F)$.

We denote by $\mathscr{P}_{N,((r,q);(s,p))}({}^nE;F)$ the subset of $\mathscr{P}({}^nE;F)$ composed by the n-homogeneous polynomials which are Lorentz $((r,q);(s,p))$-nuclear. We define

$$\|P\|_{N,((r,q);(s,p))} = \inf\|(\lambda_j)_{j=1}^{\infty}\|_{(r,q)}\|(\varphi_j)_{j=1}^{\infty}\|_{w,(s',p')}^{n}\|(y_j)_{j=1}^{\infty}\|_{\infty}, \tag{8}$$

where the infimum is considered for all representations of $P \in \mathscr{P}_{N,((r,q);(s,p))}({}^nE;F)$ of the form (7).

Note that from (6) and (8) we have

$$\|P\| \le \|P\|_{N,((r,q);(s,p))}. \tag{9}$$

From now on, unless stated otherwise, $r, q, s, p \in]1, \infty[$, with $r \le q$ and $s' \le p'$.

Proposition 4.3. *The space* $(\mathscr{P}_{N,((r,q);(s,p))}({}^nE;F), \|.\|_{N,((r,q);(s,p))})$ *is a quasi-normed space. Besides, for* t_n *given by*

$$\frac{1}{t_n} = \frac{1}{q} + \frac{n}{p'},$$

there is a $M \ge 0$ *so that*

$$\|P+Q\|_{N,((r,q);(s,p))}^{t_n} \le M(\|P\|_{N,((r,q);(s,p))}^{t_n} + \|Q\|_{N,((r,q);(s,p))}^{t_n}). \tag{10}$$

For this reason we call this quasi-norm by "quasi t_n*-norm".*

Proof. If $P = 0$ it is clear that $\|P\|_{N,((r,q);(s,p))} = 0$. Conversely, if $\|P\|_{N,((r,q);(s,p))} = 0$, then (9) implies that $P = 0$.

It is routine to verify that $\|\lambda P\|_{N,((r,q);(s,p))} = |\lambda|\,\|P\|_{N,((r,q);(s,p))}$.

The triangular inequality is the property that needs justification.

We consider $P, Q \in \mathscr{P}_{N,((r,q);(s,p))}({}^nE;F)$ of the form

$$P(x) = \sum_{j=1}^{\infty} \lambda_j \big(\varphi_j(x)\big)^n y_j \quad \text{and} \quad Q(x) = \sum_{j=1}^{\infty} \eta_j \big(\vartheta_j(x)\big)^n z_j. \tag{11}$$

Define

$$R(x) = \sum_{j=1}^{\infty} \gamma_j \big(\psi_j(x)\big)^n w_j,$$

with

$$\gamma_{2j-1} = \lambda_j \quad \text{and} \quad \gamma_{2j} = \eta_j, \qquad \psi_{2j-1} = \varphi_j \quad \text{and} \quad \psi_{2j} = \vartheta_j,$$
$$w_{2j-1} = y_j \quad \text{and} \quad w_{2j} = z_j.$$

Note that

$$\begin{aligned}
\lim_{m\to\infty}\Big[\sum_{j=1}^{2m-1} \gamma_j\big(\psi_j(x)\big)^n w_j\Big] &= \lim_{m\to\infty}\Big[\sum_{j=1}^{m} \lambda_j\big(\varphi_j(x)\big)^n y_j + \sum_{j=1}^{m-1} \eta_j\big(\vartheta_j(x)\big)^n z_j\Big] \\
&= P(x) + Q(x), \\
\lim_{m\to\infty}\Big[\sum_{j=1}^{2m} \gamma_j\big(\psi_j(x)\big)^n w_j\Big] &= \lim_{m\to\infty}\Big[\sum_{j=1}^{m} \lambda_j\big(\varphi_j(x)\big)^n y_j + \sum_{j=1}^{m} \eta_j\big(\vartheta_j(x)\big)^n z_j\Big] \\
&= P(x) + Q(x),
\end{aligned}$$

and hence R is well defined and

$$R(x) = \lim_{m\to\infty}\Big[\sum_{j=1}^{m} \gamma_j\big(\psi_j(x)\big)^n w_j\Big] = P(x) + Q(x).$$

It follows (from Proposition 2.4) that there are $\alpha > 0$ and $\beta > 0$ such that

$$\begin{cases} \|(\gamma_j)_{j=1}^{\infty}\|_{(r,q)}^{q} \le \alpha[\|(\lambda_j)_{j=1}^{\infty}\|_{(r,q)}^{q} + \|(\eta_j)_{j=1}^{\infty}\|_{(r,q)}^{q}], \\ \|(\psi_j)_{j=1}^{\infty}\|_{w,(s',p')}^{p'} \le \beta[\|(\varphi_j)_{j=1}^{\infty}\|_{w,(s',p')}^{p'} + \|(\vartheta_j)_{j=1}^{\infty}\|_{w,(s',p')}^{p'}]. \end{cases} \tag{12}$$

For a given $\varepsilon > 0$ we choose representations of P and Q as in (11) in such a way that

$$\begin{cases} \|(y_j)_{j=1}^{\infty}\|_{\infty} = \|(z_j)_{j=1}^{\infty}\|_{\infty} = 1, \\ \|(\lambda_j)_{j=1}^{\infty}\|_{(r,q)} \leq [(1+\varepsilon)\|P\|_{N,((r,q);(s,p))}]^{t_n/q}, \\ \|(\eta_j)_{j=1}^{\infty}\|_{(r,q)} \leq [(1+\varepsilon)\|Q\|_{N,((r,q);(s,p))}]^{t_n/q}, \\ \|(\varphi_j)_{j=1}^{\infty}\|_{w,(s',p')} \leq [(1+\varepsilon)\|P\|_{N,((r,q);(s,p))}]^{t_n/p'}, \\ \|(\vartheta_j)_{j=1}^{\infty}\|_{w,(s',p')} \leq [(1+\varepsilon)\|Q\|_{N,((r,q);(s,p))}]^{t_n/p'}. \end{cases} \tag{13}$$

From (12) and (13) we get

$$\begin{cases} \|(\gamma_j)_{j=1}^{\infty}\|_{(r,q)} \leq [\alpha(1+\varepsilon)^{t_n}(\|P\|_{N,((r,q);(s,p))}^{t_n} + \|Q\|_{N,((r,q);(s,p))}^{t_n})]^{1/q}, \\ \|(\psi_j)_{j=1}^{\infty}\|_{w,(s',p')} \leq [\beta(1+\varepsilon)^{t_n}(\|P\|_{N,((r,q);(s,p))}^{t_n} + \|Q\|_{N,((r,q);(s,p))}^{t_n})]^{1/p'}, \\ \|(w_j)_{j=1}^{\infty}\|_{\infty} = 1. \end{cases}$$

It follows that for $M = \max\{\alpha, \beta\}$, we have (since $\frac{1}{q} + \frac{n}{p'} = \frac{1}{t_n}$),

$$\|P+Q\|_{N,((r,q);(s,p))} \leq M^{1/t_n}(1+\varepsilon)(\|P\|_{N,((r,q);(s,p))}^{t_n} + \|Q\|_{N,((r,q);(s,p))}^{t_n})^{1/t_n}$$

and

$$\|P+Q\|_{N,((r,q);(s,p))}^{t_n} \leq M(1+\varepsilon)^{t_n}(\|P\|_{N,((r,q);(s,p))}^{t_n} + \|Q\|_{N,((r,q);(s,p))}^{t_n}).$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$\|P+Q\|_{N,((r,q);(s,p))}^{t_n} \leq M(\|P\|_{N,((r,q);(s,p))}^{t_n} + \|Q\|_{N,((r,q);(s,p))}^{t_n}).$$

A standard argument provides

$$\|P+Q\|_{N,((r,q);(s,p))} \leq (2M)^{1/t_n}(\|P\|_{N,((r,q);(s,p))} + \|Q\|_{N,((r,q);(s,p))})$$

and so $\|.\|_{N,((r,q);(s,p))}$ is a quasi-norm. □

The proof of the next result is standard.

Proposition 4.4. *$\mathcal{P}_{N,((r,q);(s,p))}(^nE;F)$ with the metrizable topology defined by $\|.\|_{N,((r,q);(s,p))}$ is complete.*

Example 4.5. It is clear that $\mathcal{P}_f(^nE;F)$ is contained in $\mathcal{P}_{N,((r,q);(s,p))}(^nE;F)$ and

$$\|\varphi^n b\|_{N,((r,q);(s,p))} = \|\varphi^n b\|.$$

In fact,

$$\|\varphi^n b\|_{N,((r,q);(s,p))} \le \|\varphi\|^n \|b\| = \|\varphi^n b\| \overset{(9)}{\le} \|\varphi^n b\|_{N,((r,q);(s,p))}.$$

The following result (ideal property) has a straightforward proof.

Proposition 4.6. *If $P \in \mathscr{P}_{N,((r,q);(s,p))}(^nE;F)$, $T \in \mathscr{L}(F,G)$ and $S \in \mathscr{L}(D,E)$, then $T \circ P \circ S \in \mathscr{P}_{N,((r,q);(s,p))}(^nD;G)$ and*

$$\|T \circ P \circ S\|_{N,((r,q);(s,p))} \le \|S\|^n \|P\|_{N,((r,q);(s,p))} \|T\|.$$

The following definition will be useful for technical reasons:

Definition 4.7. If $P \in \mathscr{P}_f(^nE;F)$ we set

$$\|P\|_{N_f,((r,q);(s,p))} = \inf \|(\lambda_j)_{j=1}^m\|_{(r,q)} [\|(\varphi_j)_{j=1}^m\|_{w,(s',p')}]^n \|(y_j)_{j=1}^m\|_\infty,$$

where the infimum is taken for all finite representations of P of the form

$$P(x) = \sum_{j=1}^m \lambda_j (\varphi_j(x))^n y_j.$$

Remark 4.8. As in Proposition 4.3, we can prove that $\|.\|_{N_f,((r,q);(s,p))}$ is a quasi t_n-norm. In this case we see that the constant that appears on the triangular inequality is the same for both quasi t_n-norms. We denote by M as we have done before. It is clear that

$$\|P\|_{N,((r,q);(s,p))} \le \|P\|_{N_f,((r,q);(s,p))}, \tag{14}$$

for all $P \in \mathscr{P}_f(^nE;F)$. The natural question is to know if these quasi t_n-norms are equivalent for some choices of E and F.

Since $(\mathscr{P}_f(^nE;F), \|.\|_N)$ and $(\mathscr{P}_f(^nE;F), \|.\|_{N_f})$ are not complete, we cannot use the Open Mapping Theorem in addition to (14) to conclude that these quasi-norms are equivalent.

The next lemmata are easy, but useful. We omit their proofs:

Lemma 4.9. *If $(\lambda_j)_{j=1}^\infty \in l_{(r;q)}$, then*

$$\lim_{t\to\infty} \|(\lambda_j)_{j=t}^\infty\|_{(r,q)} = 0.$$

Lemma 4.10. *$\mathcal{P}_f({}^nE;F)$ is dense in $\left(\mathcal{P}_{N,((r,q);(s,p))}({}^nE;F), \|.\|_{N,((r,q);(s,p))}\right)$. Precisely, if*

$$P = \sum_{j=1}^{\infty} \lambda_j(\varphi_j)^n y_j \in \mathcal{P}_{N,((r,q);(s,p))}({}^nE;F),$$

then $Q_m := \sum_{j=1}^{m} \lambda_j(\varphi_j)^n y_j$ converges to P in $\|.\|_{N,((r,q);(s,p))}$.

From now on, the symbols M and t_n will be the same as in Proposition 4.3.

Proposition 4.11. *If E is finite-dimensional, then*

$$\|P\|_{N_f,((r,q);(s,p))} \le M^{1/t_n}\|P\|_{N,((r,q);(s,p))} \tag{15}$$

for all $P \in \mathcal{P}({}^nE;F)$.

Proof. In this case we know that $\mathcal{P}_f({}^nE;F) = \mathcal{P}({}^nE;F)$ is complete for the quasi t_n-norms $\|.\|_{N,((r,q);(s,p))}$ and $\|.\|_{N_f,((r,q);(s,p))}$. By the Open Mapping Theorem, these quasi t_n-norms are equivalent. Hence, there is $C > 0$ such that

$$\|P\|_{N_f,((r,q);(s,p))} \le C\|P\|_{N,((r,q);(s,p))}$$

for each P. For $\varepsilon > 0$ we choose a representation

$$P(x) = \sum_{j=1}^{\infty} \lambda_j\big(\varphi_j(x)\big)^n y_j$$

such that

$$\|(\lambda_j)_{j=1}^{\infty}\|_{(r,q)}\|(\varphi_j)_{j=1}^{\infty}\|_{w,(s',p')}^n\|(y_j)_{j=1}^{\infty}\|_{\infty} \le (1+\varepsilon)\|P\|_{N,((r,q);(s,p))}. \tag{16}$$

From Lemma 4.10, there is a $m \in \mathbb{N}$ such that

$$\begin{aligned}\Big\|\sum_{j>m} \lambda_j(\varphi_j)^n y_j\Big\|_{N,((r,q);(s,p))} &= \Big\|P - \sum_{j=1}^{m} \lambda_j(\varphi_j)^n y_j\Big\|_{N,((r,q);(s,p))} \\ &\le \frac{\varepsilon}{C}\|P\|_{N,((r,q);(s,p))}.\end{aligned}$$

We thus have

$$\begin{aligned}\Big\|\sum_{j=1}^{m} \lambda_j(\varphi_j)^n y_j\Big\|_{N_f,((r,q);(s,p))} &\le \|(\lambda_j)_{j=1}^{m}\|_{(r,q)}[\|(\varphi_j)_{j=1}^{m}\|_{w,(s',p')}]^n\|(y_j)_{j=1}^{m}\|_{\infty} \\ &\le \|(\lambda_j)_{j=1}^{\infty}\|_{(r,q)}[\|(\varphi_j)_{j=1}^{\infty}\|_{w,(s',p')}]^n\|(y_j)_{j=1}^{\infty}\|_{\infty} \\ &\le (1+\varepsilon)\|P\|_{N,((r,q);(s,p))}.\end{aligned}$$

Now, from Remark 4.8 we get

$$\begin{aligned}
&\|P\|^{t_n}_{N_f,((r,q);(s,p))}\\
&\quad \le M\Big\|\sum_{j=1}^{m}\lambda_j(\varphi_j)^n y_j\Big\|^{t_n}_{N_f,((r,q);(s,p))} + M\Big\|\sum_{j>m}\lambda_j(\varphi_j)^n y_j\Big\|^{t_n}_{N_f,((r,q);(s,p))}\\
&\quad \le M(1+\varepsilon)^{t_n}\|P\|^{t_n}_{N,((r,q);(s,p))} + MC^{t_n}\Big\|\sum_{j>m}\lambda_j(\varphi_j)^n y_j\Big\|^{t_n}_{N,((r,q);(s,p))}\\
&\quad \le M(1+\varepsilon)^{t_n}\|P\|^{t_n}_{N,((r,q);(s,p))} + M\varepsilon^{t_n}\|P\|^{t_n}_{N,((r,q);(s,p))}\\
&\quad \le M[(1+\varepsilon)^{t_n}+\varepsilon^{t_n}]\|P\|^{t_n}_{N,((r,q);(s,p))}.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this proves our result. □

Using Proposition 4.6 and Proposition 4.11 we easily prove the following lemma:

Lemma 4.12. *If* $P \in \mathscr{P}_{N,((r,q);(s,p))}(^nE;F)$ *and* $S \in \mathscr{L}_f(D,E)$, *then*

$$\|P\circ S\|_{N_f,((r,q);(s,p))} \le M^{1/t_n}\|P\|_{N,((r,q);(s,p))}\|S\|^n. \tag{17}$$

Proposition 4.13. *If* E' *has the bounded approximation property, then*

$$\|P\|_{N_f,((r,q);(s,p))} \le M^{1/t_n}\|P\|_{N,((r,q);(s,p))}$$

for every $P \in \mathscr{P}_f(^nE;F)$.

Proof. We know that there is a symmetric n-linear mapping T of finite type on $E\times\cdots\times E$, with values in F such that $P(x) = T(x,\dots,x)$ for all $x \in E$. It is not difficult to see that $T_1 \in \mathscr{L}(E;\mathscr{L}(^{n-1}E;F))$ given by

$$T_1(x_1)(x_2,\dots,x_n) = T(x_1,\dots,x_n)$$

belongs to $\mathscr{L}_f(E;\mathscr{L}(^{n-1}E;F))$. Since E' has the bounded approximation property (and hence the λ-approximation property for some $\lambda > 0$), from an adaptation of [27], 10.2.6, for each $\varepsilon > 0$ we can find $S \in \mathscr{L}_f(E;E)$ such that $T_1\circ S = T_1$ and

$$\|S\| \le (1+\varepsilon)\lambda. \tag{18}$$

We have

$$T(S(x_1),x_2,\dots,x_n) = T(x_1,\dots,x_n)$$

for all $x_1, \dots, x_n \in E$. In fact,

$$\begin{aligned} T(x_1, \dots, x_n) &= T_1(x_1)(x_2, \dots, x_n) \\ &= (T_1 \circ S)(x_1)(x_2, \dots, x_n) \\ &= T_1(Sx_1)(x_2, \dots, x_n) \\ &= T(Sx_1, x_2, \dots, x_n). \end{aligned} \tag{19}$$

Since T is symmetric, by repeating the argument of (19), we can easily conclude that

$$P \circ S(x) = T(Sx, \dots, Sx) = T(x, \dots, x) = P(x)$$

for all $x \in E$. We thus have

$$\begin{aligned} \|P\|_{N_f,((r,q);(s,p))} &= \|P \circ S\|_{N_f,((r,q);(s,p))} \\ &\overset{(17)}{\leq} M^{1/t_n} \|P\|_{N,((r,q);(s,p))} \|S\|^n \\ &\overset{(18)}{\leq} M^{1/t_n} (1+\varepsilon)^n \lambda^n \|P\|_{N,((r,q);(s,p))}. \end{aligned}$$

Now, making $\varepsilon \to 0$, we get

$$\|P\|_{N_f,((r,q);(s,p))} \leq M^{1/t_n} \lambda^n \|P\|_{N,((r,q);(s,p))} \tag{20}$$

for every $P \in \mathscr{P}_f({}^nE; F)$.

Let $C = M^{1/t_n} \lambda^n$.

For $\varepsilon > 0$ we choose a representation

$$P(x) = \sum_{j=1}^{\infty} \lambda_j \big(\varphi_j(x)\big)^n y_j$$

such that

$$\|(\lambda_j)_{j=1}^{\infty}\|_{(r,q)} \|(\varphi_j)_{j=1}^{\infty}\|_{w,(s',p')}^n \|(y_j)_{j=1}^{\infty}\|_{\infty} \leq (1+\varepsilon) \|P\|_{N,((r,q);(s,p))}. \tag{21}$$

So, since

$$\|(\lambda_j)_{j=1}^{\infty}\|_{(r,q)} \|(\varphi_j)_{j=1}^{\infty}\|_{w,(s',p')}^n \|(y_j)_{j=1}^{\infty}\|_{\infty} < \infty,$$

using Lemma 4.9 we know that there exists an integer $m \in \mathbb{N}$ such that

$$\|(\lambda_j)_{j=m+1}^{\infty}\|_{(r,q)} \|(\varphi_j)_{j=m+1}^{\infty}\|_{w,(s',p')}^n \|(y_j)_{j=m+1}^{\infty}\|_{\infty} \leq \frac{\varepsilon}{C} \|P\|_{N,((r,q);(s,p))}.$$

Hence

$$C\Big\|\sum_{j>m}\lambda_j(\varphi_j)^n y_j\Big\|_{N,((r,q);(s,p))} \le \varepsilon\|P\|_{N,((r,q);(s,p))}. \tag{22}$$

On the other hand, note that

$$\begin{aligned}\Big\|\sum_{j=1}^{m}\lambda_j(\varphi_j)^n y_j\Big\|_{N_f,((r,q);(s,p))} &\le \|(\lambda_j)_{j=1}^m\|_{(r,q)}\|(\varphi_j)_{j=1}^m\|_{w,(s',p')}^n\|(y_j)_{j=1}^m\|_\infty \\ &\overset{(21)}{\le} (1+\varepsilon)\|P\|_{N,((r,q);(s,p))}.\end{aligned} \tag{23}$$

Now, from the triangular inequality, we get

$$\begin{aligned}\|P\|_{N_f,((r,q);(s,p))}^{t_n} &\le M\Big\|\sum_{j=1}^{m}\lambda_j(\varphi_j)^n y_j\Big\|_{N_f,((r,q);(s,p))}^{t_n} \\ &\quad + M\Big\|\sum_{j>m}\lambda_j(\varphi_j)^n y_j\Big\|_{N_f,((r,q);(s,p))}^{t_n} \\ &\overset{(20)\text{ and }(23)}{\le} M(1+\varepsilon)^{t_n}\|P\|_{N,((r,q);(s,p))}^{t_n} \\ &\qquad + MC^{t_n}\Big\|\sum_{j>m}\lambda_j(\varphi_j)^n y_j\Big\|_{N,((r,q);(s,p))}^{t_n} \\ &\overset{(22)}{\le} M[(1+\varepsilon)^{t_n}+\varepsilon^{t_n}]\|P\|_{N,((r,q);(s,p))}^{t_n},\end{aligned}$$

and, since $\varepsilon > 0$ is arbitrary, the proof is done. □

5. The topological dual of $\mathscr{P}_{N,((r,q);(s,p))}(^nE;F)$

In this section, we will show that the topological dual of $\mathscr{P}_{N,((r,q);(s,p))}(^nE;F)$ is isomorphic to the space $\mathscr{P}_{as((r',q');(s',p'))}(^nE';F')$ provided that E' has the bounded approximation property.

The following intuitive lemma, which is proved in [23], will be needed in the next results.

Lemma 5.1. *Let $(c_n)_{n=1}^\infty$ be a decreasing sequence of non-negative real numbers. If $(b_j)_{j=1}^\infty$ is a sequence of non-negative real numbers so that there exists a bijection $\pi : N \to N$ such that $(b_{\pi(n)})_{n=1}^\infty$ is a non-increasing rearrangement, then*

$$\sum_{j=1}^\infty c_j b_j \le \sum_{j=1}^\infty c_j b_{\pi(j)}.$$

The same holds for injection instead of bijection.

Theorem 5.2. *If E' has the bounded approximation property then, for each $P \in \mathscr{P}_{as((r',q');(s',p'))}({}^nE';F')$, we can associate a well defined continuous linear functional T_P on $\mathscr{P}_{N,((r,q);(s,p))}({}^nE;F)$ given by*

$$T_P\Big(\sum_{j=1}^{\infty}\lambda_j\varphi_j^n y_j\Big) = \sum_{j=1}^{\infty}\lambda_j P(\varphi_j)(y_j),$$

and

$$\|T_P\| \le M^{1/t_n}\|P\|_{as((r',q');(s',p'))}.$$

Proof. Let $P \in \mathscr{P}_{as((r',q');(s',p'))}({}^nE';F')$. Define

$$T_P^{(f)} : \big(\mathscr{P}_f({}^nE;F), \|.\|_{N,((r,q);(s,p))}\big) \to \mathbb{K}$$

by

$$T_P^{(f)}(Q) = \sum_{j=1}^{m}\lambda_j P(\varphi_j)(y_j)$$

for $Q = \sum_{j=1}^m \lambda_j\varphi_j^n y_j$. One can show, using tensor products, that $T_P^{(f)}(Q)$ does not depend on the particular representation of Q.

Since $r \le q$, we have $r' \ge q'$ and $(j^{q'/r'-1})_{j=1}^m$ is non-increasing. Let us choose a bijection $\delta : I_m \to I_m$ such that $\big(\|P(\varphi_{\delta(j)})\|\big)_{j=1}^m$ is non-increasing.

Let $\pi : I_m \to I_m$ denote a bijection such that $(|\lambda_{\pi(j)}|)_{j=1}^m$ is non-increasing. We have

$$\begin{aligned}
&|T_P^{(f)}(Q)| \\
&\quad \le \sum_{j=1}^m |\lambda_j|\,|P(\varphi_j)(y_j)| \le \sum_{j=1}^m |\lambda_j|\,\|P(\varphi_j)\|\,\|y_j\| \\
&\quad = \sum_{j=1}^m |\lambda_{\pi(j)}|\,\|P(\varphi_{\pi(j)})\|\,\|y_{\pi(j)}\| \\
&\quad = \sum_{j=1}^m j^{1/r-1/q}|\lambda_{\pi(j)}|j^{1/r'-1/q'}\|P(\varphi_{\pi(j)})\|\,\|y_{\pi(j)}\| \\
&\quad \overset{\text{Hölder Ineq.}}{\le} \|(j^{1/r-1/q}|\lambda_{\pi(j)}|)_{j=1}^m\|_q\Big(\sum_{j=1}^m j^{q'/r'-1}\|P(\varphi_{\pi(j)})\|^{q'}\Big)^{1/q'}\|(y_{\pi(j)})_{j=1}^m\|_\infty \\
&\quad \overset{\text{Lemma 5.1}}{\le} \|(j^{1/r-1/q}|\lambda_{\pi(j)}|)_{j=1}^m\|_q\Big(\sum_{j=1}^m j^{q'/r'-1}\|P(\varphi_{\delta(j)})\|^{q'}\Big)^{1/q'}\|(y_j)_{j=1}^m\|_\infty \\
&\quad \le \|(\lambda_j)_{j=1}^m\|_{(r,q)}\|P\|_{as((r',q');(s',p'))}\|(\varphi_j)_{j=1}^m\|_{w,(s',p')}^n\|(y_j)_{j=1}^m\|_\infty.
\end{aligned}$$

This implies that

$$|T_P^{(f)}(Q)| \le \|P\|_{as((r',q'),(s',p'))} \|Q\|_{N_f,((r,q);(s,p))}$$

for all $Q \in \mathscr{P}_f({}^nE;F)$. Hence, if we consider E' with the bounded approximation property, we use Proposition 4.13 in order to obtain

$$|T_P^{(f)}(Q)| \le M^{1/t_n} \|P\|_{as((r',q');(s',p'))} \|Q\|_{N,((r,q);(s,p))} \tag{24}$$

for all $Q \in \mathscr{P}_f({}^nE;F)$.We conclude that $T_P^{(f)}$ is continuous.

From Lemma 4.10, we define (in the obvious way)

$$T_P : \left(\mathscr{P}_{N,((r,q);(s,p))}({}^nE;F), \|.\|_{N,((r,q);(s,p))}\right) \to \mathbb{K}$$

by

$$T_P(Q) = \lim_{m\to\infty} T_P^{(f)}\Big(\sum_{j=1}^m \lambda_j \varphi_j^n y_j\Big)$$

when $Q = \sum_{j=1}^\infty \lambda_j \varphi_j^n y_j$.

From (24) we have

$$\left|T_P^{(f)}\Big(\sum_{j=1}^m \lambda_j \varphi_j^n y_j\Big)\right| \le M^{1/t_n} \|P\|_{as((r',q');(s',p'))} \left\|\sum_{j=1}^m \lambda_j \varphi_j^n y_j\right\|_{N,((r,q);(s,p))}$$

and from the definition of T_P we get (making $m \to \infty$)

$$|T_P(Q)| \le M^{1/t_n} \|P\|_{as((r',q');(s',p'))} \|Q\|_{N,((r,q);(s,p))}$$

for every $Q \in \mathscr{P}_{N,((r,q);(s,p))}({}^nE;F)$. □

Theorem 5.3. *For each $T \in \mathscr{P}_{N,((r,q);(s,p))}({}^nE;F)'$, the map $P_T : E' \to F'$ given by*

$$P_T(\varphi)(y) = T(\varphi^n y)$$

belongs to $\mathscr{P}_{as((r',q');(s',p'))}({}^nE';F')$ and satisfies

$$\|P_T\|_{as((r',q');(s',p'))} \le \|T\|. \tag{25}$$

Proof. Let $T \in \mathcal{P}_{N,((r,q);(s,p))}({}^nE;F)'$. If $m \in \mathbb{N}$ and $\varphi_1, \dots, \varphi_m$ are in E', we consider a bijection $\pi : I_m \to I_m$, such that

$$(\|P_T(\varphi_{\pi(j)})\|)_{j=1}^m$$

is non-increasing.

For $\varepsilon > 0$ we can find $y_{\pi(j)} \in F$, with $\|y_{\pi(j)}\| = 1$, $j = 1, \dots, m$, such that

$$\begin{aligned}\|(\|P_T(\varphi_{\pi(j)})\|)_{j=1}^m\|_{(r',q')}^{q'} \\ \le \varepsilon + \sum_{j=1}^m [(j^{1/r'-1/q'})^{q'} \|P_T(\varphi_{\pi(j)})\|^{q'-1} \lambda_{\pi(j)} T(\varphi_{\pi(j)}^n y_{\pi(j)})] = (*),\end{aligned}$$

where we choose the $\lambda_{\pi(j)} \in \mathbb{K}$, with $|\lambda_{\pi(j)}| = 1$, $j = 1, \dots, m$, in a convenient way. Hence

$$\begin{aligned}(*) &= \varepsilon + \left| T\Big(\sum_{j=1}^m (j^{1/r'-1/q'})^{q'} \|P_T(\varphi_{\pi(j)})\|^{q'-1} \lambda_{\pi(j)} \varphi_{\pi(j)}^n y_{\pi(j)}\Big)\right| \\ &\le \varepsilon + \|T\| \, \|Q_m\|_{N,((r,q);(s,p))},\end{aligned}$$

where

$$Q_m = \sum_{j=1}^m [(j^{1/r'-1/q'})^{q'} \|P_T(\varphi_{\pi(j)})\|^{q'-1} \lambda_{\pi(j)} \varphi_{\pi(j)}^n y_{\pi(j)}].$$

Note that, since $\|y_{\pi(j)}\| = 1$, $j = 1, \dots, m$, we have

$$\begin{aligned}&\|Q_m\|_{N,((r,q);(s,p))} \\ &\le \|((j^{1/r'-1/q'})^{q'} \|P_T(\varphi_{\pi(j)})\|^{q'-1} \lambda_{\pi(j)})_{j=1}^m\|_{(r,q)} \|(\varphi_{\pi(j)})_{j=1}^m\|_{w,(s',p')}^n \|(y_{\pi(j)})_{j=1}^m\|_\infty \\ &= \|((j^{1/r'-1/q'})^{q'} \|P_T(\varphi_{\pi(j)})\|^{q'-1} \lambda_{\pi(j)})_{j=1}^m\|_{(r,q)} \|(\varphi_{\pi(j)})_{j=1}^m\|_{w,(s',p')}^n.\end{aligned}$$

Now we want to evaluate

$$(\#) = \|((j^{1/r'-1/q'})^{q'} \|P_T(\varphi_{\pi(j)})\|^{q'-1} \lambda_{\pi(j)})_{j=1}^m\|_{(r,q)}.$$

We recall that $|\lambda_{\pi(j)}| = 1$, $j = 1, \dots, m$ and that $(\|P_T(\varphi_{\pi(j)})\|)_{j=1}^m$ is non-increasing. Since $r \le q$, we have $r' \ge q'$ and $(j^{1/r'-1/q'})_{j=1}^m$ is non-increasing. In this case we have (using that $q + q' = qq'$ and $(q' - 1)q = q'$)

$$(\#) = \Big(\sum_{j=1}^{m} (j^{1/r-1/q})^{q} (j^{1/r'-1/q'})^{q'q} \|P_T(\varphi_{\pi(j)})\|^{(q'-1)q}\Big)^{1/q}$$
$$= \Big(\sum_{j=1}^{m} (j^{1/r-1/q})^{q} (j^{1/r'-1/q'})^{(q'+q)} \|P_T(\varphi_{\pi(j)})\|^{(q'-1)q}\Big)^{1/q}$$
$$= \Big(\sum_{j=1}^{m} (j^{1/r'-1/q'})^{q'} \|P_T(\varphi_{\pi(j)})\|^{q'}\Big)^{1/q}.$$

Now we replace this equality in the above inequalities in order to have

$$\big\|\big(\|P_T(\varphi_{\pi(j)})\|\big)_{j=1}^{m}\big\|_{(r',q')}^{q'}$$
$$\leq \varepsilon + \|T\| \Big(\sum_{j=1}^{m} (j^{1/r'-1/q'})^{q'} \|P_T(\varphi_{\pi(j)})\|^{q'}\Big)^{1/q} \|(\varphi_{\pi(j)})_{j=1}^{m}\|_{w,(s',p')}^{n}.$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\big\|\big(\|P_T(\varphi_{\pi(j)})\|\big)_{j=1}^{m}\big\|_{(r',q')}^{q'}$$
$$\leq \|T\| \Big(\sum_{j=1}^{m} (j^{1/r'-1/q'})^{q'} \|P_T(\varphi_{\pi(j)})\|^{q'}\Big)^{1/q} \|(\varphi_{\pi(j)})_{j=1}^{m}\|_{w,(s',p')}^{n}$$
$$= \|T\| \, \big\|\big(\|P_T(\varphi_{\pi(j)})\|\big)_{j=1}^{m}\big\|_{(r',q')}^{q'/q} \|(\varphi_{\pi(j)})_{j=1}^{m}\|_{w,(s',p')}^{n}$$

and hence

$$\big\|\big(\|P_T(\varphi_j)\|\big)_{j=1}^{m}\big\|_{(r',q')} = \big\|\big(\|P_T(\varphi_{\pi(j)})\|\big)_{j=1}^{m}\big\|_{(r',q')}$$
$$\leq \|T\| \, \|(\varphi_{\pi(j)})_{j=1}^{m}\|_{w,(s',p')}^{n} = \|T\| \, \|(\varphi_j)_{j=1}^{m}\|_{w,(s',p')}^{n}.$$

So, we conclude that $P_T \in \mathscr{P}_{as((r',q');(s',p'))}({}^nE';F')$ and

$$\|P_T\|_{as((r',q');(s',p'))} \leq \|T\|. \qquad \square$$

Theorem 5.4. *If E' has the bounded approximation property, then the linear mapping*

$$\Psi : \mathscr{P}_{N,((r,q);(s,p))}({}^nE;F)' \to \mathscr{P}_{as((r',q');(s',p'))}({}^nE';F')$$

given by $\Psi(T) = P_T$ is a topological isomorphism.

Proof. Note that if $\alpha \in \mathbb{K}$, $\varphi \in E'$ and $y \in F$, we have

$$\begin{aligned}\Psi(T_1 + \alpha T_2)(\varphi)(y) &= P_{T_1+\alpha T_2}(\varphi)(y) \\ &= (T_1 + \alpha T_2)(\varphi^n y) \\ &= T_1(\varphi^n y) + \alpha T_2(\varphi^n y) \\ &= \Psi(T_1)(\varphi)(y) + \alpha\Psi(T_2)(\varphi)(y)\end{aligned}$$

and so Ψ is linear. Theorem 5.3 asserts that

$$\|\Psi(T)\|_{as((r',q');(s',p'))} = \|P_T\|_{as((r',q');(s',p'))} \leq \|T\|$$

and hence Ψ is continuous.

Note that Ψ is onto. In fact, if $P \in \mathscr{P}_{as((r',q');(s',p'))}({}^nE'; F')$, consider

$$T_P : \mathscr{P}_{N,((r,q);(s,p))}({}^nE; F) \to \mathbb{K}$$

as defined in Theorem 5.2. So,

$$\Psi(T_P) = P_{T_P}$$

and

$$P_{T_P}(\varphi)(y) = T_P(\varphi^n y) = P(\varphi)(y).$$

We conclude that

$$P_{T_P} = P$$

and hence

$$\Psi(T_P) = P.$$

Note that Ψ in injective. In fact,

$$\Psi(T) = 0 \implies \Psi(T)(\varphi)(y) = 0 \implies T(\varphi^n y) = 0$$

for all $\varphi \in E'$ and $y \in F$. So, T is null in $\mathscr{P}_f({}^nE; F)$ and since the space $\left(\mathscr{P}_f({}^nE; F), \|.\|_{N,((r,q);(s,p))}\right)$ is dense in $\mathscr{P}_{N,((r,q);(s,p))}({}^nE; F)$ it follows that $T = 0$.

We conclude (using Theorem 5.2) that Ψ^{-1} is continuous and $\|\Psi^{-1}\| \leq M^{1/t_n}$. □

A natural continuation of this paper leads us to introduce the spaces of entire scalar-valued functions of Lorentz $\left((r,q);(s,p)\right)$-nuclear bounded type defined on a Banach space E (following the works of Nachbin [25], Gupta [12] and Matos [21]). Then it seems reasonable to study duality results between the dual of this space of entire functions and a suitable space of entire functions, following the lines of Gupta [12], Malgrange [13] and Matos [21]). So, it is possible to try to investigate existence and approximation results for convolution equations on the spaces of entire functions of Lorentz $((r,q);(s,p))$-nuclear bounded type (for related results we mention [1], [2], [3], [8], [7], [9], [10], [15], [16], [17], [18], [19], [21]).

The main obstacle in introducing the space of entire functions of Lorentz $((r,q);(s,p))$-nuclear bounded type is that the topology defined on the space $\mathscr{P}_{N,((r,q);(s,p))}(^nE)$ fails to be a locally convex topology and thus does not generate a locally convex topology on the spaces of entire functions of this type. We believe that it is possible follow Matos [21] to by-pass this obstacle by considering a suitable Banach space of n-homogeneous polynomials that contains $\mathscr{P}_{N,((r,q);(s,p))}(^nE)$ and that shares the same dual with $\mathscr{P}_{N,((r,q);(s,p))}(^nE)$. In this new context the related space of entire functions will be a Fréchet space; so a friendly environment to study existence and approximation results for convolution equations comes out.

References

[1] P. J. Boland, Malgrange theorem for entire functions on nuclear spaces. In *Proceedings on infinite dimensional holomorphy*, Lecture Notes in Math. 364, Springer, Berlin 1974, 135–144. Zbl 0282.46019 MR 0420271

[2] J.-F. Colombeau and M. C. Matos, Convolution equations in spaces of infinite-dimensional entire functions. *Nederl. Akad. Wetensch. Indag. Math.* **42** (1980), 375–389. Zbl 0453.46033 MR 597996

[3] J.-F. Colombeau and B. Perrot, Convolution equations in spaces of infinite-dimensional entire functions of exponential and related types. *Trans. Amer. Math. Soc.* **258** (1980), 191–198. Zbl 0426.46015 MR 554328

[4] J. Diestel, H. Jarchow, and A. Tonge, *Absolutely summing operators*. Cambridge Stud. Adv. Math. 43, Cambridge University Press, Cambridge 1995. Zbl 0855.47016 MR 1342297

[5] S. Dineen, *Complex analysis on infinite-dimensional spaces*. Springer Monogr. Math., Springer-Verlag London Ltd., London 1999. Zbl 1034.46504 MR 1705327

[6] S. Dineen, Holomorphy types on a Banach space. *Studia Math.* **39** (1971), 241–288. Zbl 0235.32013 MR 0304705

[7] T. A. W. Dwyer, III, Partial differential equations in Fischer-Fock spaces for the Hilbert-Schmidt holomorphy type. *Bull. Amer. Math. Soc.* **77** (1971), 725–730. Zbl 0222.46019 MR 0290103

[8] T. A. W. Dwyer, III, Convolution equations for vector-valued entire functions of nuclear bounded type. *Trans. Amer. Math. Soc.* **217** (1976), 105–119. Zbl 0299.46036 MR 0487450

[9] V. V. Fávaro, Convolution equations on spaces of quasi-nuclear functions of a given type and order. Bull. Belg. Math. Soc. Simon Stevin **17** (2010), 535–569.

[10] V. V. Fávaro, The Fourier–Borel transform between spaces of entire functions of a given type and order. *Port. Math.* **65** (2008), 285–309. Zbl 1152.46033 MR 2428422

[11] C. P. Gupta, Malgrange Theorem for nuclearly entire functions of bounded type. Thesis, University of Rochester 1966.

[12] C. P. Gupta, On the Malgrange theorem for nuclearly entire functions of bounded type on a Banach space. *Nederl. Akad. Wetensch. Indag. Math.* **73** (1970), 356–358. Zbl 0201.44605 MR 0290104

[13] B. Malgrange, Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution. *Ann. Inst. Fourier* (*Grenoble*) **6** (1955–1956), 271–355. Zbl 0071.09002 MR 0086990

[14] A. Martineau, Equations différentielles d'ordre infini. *Bull. Soc. Math. France* **95** (1967), 109–154. Zbl 0167.44202 MR 1507968

[15] M. C. Matos, Sur le théorème d'approximation et d'existence de Malgrange-Gupta. *C. R. Acad. Sci. Paris Sér. A* **271** (1970), 1258–1259. Zbl 0205.40902 MR 0285888

[16] M. C. Matos, On Malgrange theorem for nuclear holomorphic functions in open balls of a Banach space. *Math. Z.* **162** (1978), 113–123. Zbl 0402.46029 MR 0632054

[17] M. C. Matos, Correction to "On Malgrange Theorem for nuclear holomorphic functions in open balls of a Banach space". *Math. Z.* **171** (1980), 289–290. Zbl 0429.46028 MR 575246

[18] M. C. Matos, On the Fourier-Borel transformation and spaces of entire functions in a normed space. In *Functional analysis, holomorphy and approximation theory, II* (Rio de Janeiro, 1981), North-Holland Math. Stud. 86, North-Holland, Amsterdam 1984, 139–169. Zbl 0568.46036 MR 0771327

[19] M. C. Matos, On convolution operators in spaces of entire functions of a given type and order. In *Complex analysis, functional analysis and approximation theory* (Campinas, 1984), North-Holland Math. Stud. 125, North-Holland, Amsterdam 1986, 129–171. Zbl 0658.46016 MR 0893415

[20] M. C. Matos, On multilinear mappings of nuclear type. *Rev. Mat. Univ. Complut. Madrid* **6** (1993), 61–81. Zbl 0807.46022 MR 1245025

[21] M. C. Matos, Absolutely summing mappings, nuclear mappings and convolution equations. IMECC-UNICAMP, 2007.

[22] M. C. Matos and L. Nachbin, Entire functions on locally convex spaces and convolution operators. *Compositio Math.* **44** (1981), 145–181. Zbl 0485.46022 MR 662461

[23] M. C. Matos and D. Pellegrino, Lorentz summing mappings. *Math. Nachr.* **283** (2010), 1409–1427.

[24] J. Mujica, *Complex analysis in Banach spaces*. North-Holland Math. Stud. 120, North-Holland Publishing, Amsterdam 1986. Zbl 0586.46040 MR 0842435

[25] L. Nachbin, *Topology on spaces of holomorphic mappings*. Ergeb. Math. Grenzgeb. 47, Springer-Verlag, New York 1969. Zbl 0172.39902 MR 0254579

[26] L. Nachbin, Recent developments in infinite dimensional holomorphy. *Bull. Amer. Math. Soc.* **79** (1973), 625–640. Zbl 0279.32017 MR 0322509

[27] A. Pietsch, *Operator ideals*. North-Holland Math. Library 20, North-Holland Publishing, Amsterdam 1980. Zbl 0434.47030 MR 0582655

[28] A. Pietsch, *Eigenvalues and s-numbers*. Cambridge Stud. Adv. Math. 13, Cambridge University Press, Cambridge 1987. Zbl 0615.47019 MR 0917067

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