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The poset of closure systems on an infinite poset: detachability and semimodularity

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Abstract. Closure operators on a poset can be characterized by the corresponding closure systems. It is known that in a directed complete partial order (DCPO), in particular in any finite poset, the collection of all closure systems is closed under arbitrary intersection and has a "detachability" or "anti-matroid" property, which implies that the collection of all closure systems is a lower semimodular complete lattice (and dually, the closure operators form an upper semimodular complete lattice).

After reviewing the history of the problem, we generalize these results to the case of an infinite poset where closure systems do not necessarily constitute a complete lattice; thus the notions of lower semimodularity and detachability are extended accordingly. We also give several examples showing that many properties of closure systems on a complete lattice do not extend to infinite posets.

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1. Introduction: history of the problem

Let P be a partially ordered set (from now on *poset*). Following [2], for any $x \in P$, let us write

$$x^{\uparrow} = \{y \in P \mid y \geq x\} \quad \text{and} \quad x^{\downarrow} = \{y \in P \mid y \leq x\}.$$

For $X \subseteq P$, we say that X is a *down-set* if $x^{\downarrow} \subseteq X$ for all $x \in X$, [2]. Following [14], we call an *operator on* P a map $P \to P$, and for $x \in P$ and an operator ψ on P, the image of x by ψ is written $\psi(x)$; given two operators ξ and ψ on P the operator $P \to P : x \mapsto \psi(\xi(x))$ is the *composition* of ξ followed by ψ, following [14] we write it $\psi\xi$ rather than the usual $\psi \circ \xi$ (as in [2]).

A *closure operator* on P is an operator φ that is *isotone* ($x \leq y \Rightarrow \varphi(x) \leq \varphi(y)$), *extensive* ($x \leq \varphi(x)$) and *idempotent* ($\varphi\big(\varphi(x)\big) = \varphi(x)$). Equivalently [9], [20]:

$$\forall x, y \in P, \qquad x \leq \varphi(y) \iff \varphi(x) \leq \varphi(y).$$

For any closure operator φ, let

$$\mathsf{Inv}(\varphi) = \{x \in P \mid \varphi(x) = x\} = \{\varphi(x) \mid x \in P\}$$

be the *invariance domain* of φ [14] (it is also called the *range* of φ [12]). Let us write $\Phi(P)$ for the set of all closure operators on P.

A *closure system* on P is a subset S of P such that for any $x \in P$, $x^{\uparrow} \cap S$ is nonvoid and has a least element. In [12] such a set is called a *closure range*. Let us write $\Sigma(P)$ for the set of closure systems on P. It has been known at least since [18] that there is a bijection between $\Phi(P)$ and $\Sigma(P)$: to a closure operator φ we associate the closure system $\mathsf{Inv}(\varphi)$, and conversely to a closure system S we associate the closure operator mapping each $x \in P$ to the least element of $x^{\uparrow} \cap S$.

Closure systems can be ordered by set inclusion, while closure operators can be ordered elementwise: $\varphi_1 \leq \varphi_2$ iff $\varphi_1(x) \leq \varphi_2(x)$ for all $x \in P$. It is easily shown (cf. [14]) that two closure operators φ_1 and φ_2 satisfy

$$\varphi_1 \leq \varphi_2 \iff \mathsf{Inv}(\varphi_1) \supseteq \mathsf{Inv}(\varphi_2) \iff \varphi_2\varphi_1 = \varphi_2 \iff \varphi_1\varphi_2 = \varphi_2. \tag{1}$$

Thus the bijection $\varphi \mapsto \mathsf{Inv}(\varphi)$ between the posets $\Phi(P)$ and $\Sigma(P)$ is a dual isomorphism.

A classical result of Ward [24] shows that in case P is a complete lattice (whose universal bounds are written $\mathbf{0}$ and $\mathbf{1}$), a subset S of P is a closure system iff it is closed under arbitrary infima (in particular for the empty infimum, $\mathbf{1} \in S$). Furthermore, $\Phi(P)$ and $\Sigma(P)$ are dually isomorphic complete lattices. More precisely, $\Phi(P)$ is closed under elementwise infimum: for $\varphi_i \in \Phi(P)$ $(i \in I)$, $\bigwedge_{i \in I} \varphi_i : x \mapsto \bigwedge_{i \in I} \varphi_i(x)$ belongs to $\Phi(P)$ (in particular, for $I = \emptyset$, $x \mapsto \mathbf{1}$ is the greatest closure operator); on the other hand, $\Sigma(P)$ is closed under arbitrary intersection: for $S_i \in \Sigma(P)$ $(i \in I)$, $\bigcap_{i \in I} S_i \in \Sigma(P)$ (in particular, for $I = \emptyset$, P is the greatest closure system).

Given a complete lattice P (with greatest element $\mathbf{1}$), $\{\mathbf{1}\}$ is the least element of $\Sigma(P)$, and the $\{\mathbf{1}, x\}$ $(x \in P\backslash\{\mathbf{1}\})$ are the *atoms* of $\Sigma(P)$. Thus every $S \in \Sigma(P)$ is a union of atoms, and the complete lattice $\Sigma(P)$ is *atomistic*.

Manara [17] showed that for a complete lattice P, given two closure systems S_1 and S_2 on P such that $S_1 \subset S_2$, in the lattice $\Sigma(P)$ we have $S_2 \succ S_1$ (i.e., S_2 covers S_1) iff $S_2\backslash S_1$ is a singleton. It follows then that the lattice $\Sigma(P)$ is lower semimodular:

$$\forall S, S' \in \Sigma(P), \qquad S \vee S' \succ S' \implies S \succ S \cap S';$$

hence dually the lattice $\Phi(P)$ is upper semimodular:

$$\forall \varphi, \varphi' \in \Phi(P), \qquad \varphi' \succ \varphi \wedge \varphi' \implies \varphi \vee \varphi' \succ \varphi.$$

(Here $\varphi \vee \varphi'$ denotes the join in the lattice $\Phi(P)$, *not* the elementwise join $x \mapsto \varphi(x) \vee \varphi'(x)$). As remarked in [4], Ore [21] had shown these two results in the case where the complete lattice P is $\mathscr{P}(E)$, the Boolean lattice of all subsets of a set E.

Note that [17] contains several erroneous statements, arising mainly because of a misquote of previous results by Dwinger [10], [11]: Manara confused the notion of a "well-ordered complete lattice" (i.e., a successor ordinal) with that of a "totally ordered complete lattice" (i.e., a complete chain). For example [10] showed that $\Phi(P)$ is distributive iff P is a complete chain, and that $\Phi(P)$ is Boolean iff P is a successor ordinal; however Theorem 1.12 of [17] states (citing [11]) that $\Phi(P)$ is distributive iff it is Boolean, iff P is a complete chain.

What can we say in case P is not a complete lattice? Several authors [1], [19], [13] presented proofs that if the poset P satisfies the *ACC* (ascending chain condition, or equivalently, every directed subset of P has a greatest element), then $\Sigma(P)$ is closed under arbitrary intersection (and has P as greatest element), hence it is a complete lattice. This holds in particular if P is finite. The two proofs in [13] are correct. We have not checked the correctness of the proof in [1], but [23] showed that the argument of [19] is flawed. The result can in fact be obtained by a simple argument. Let φ_i ($i \in I$) be closure operators with associated closure systems S_i; let $\mathbf{M}$ be the monoid generated by the φ_i ($i \in I$); then for any $x \in P$, the set of $\mu(x)$ for $\mu \in \mathbf{M}$ is directed (for $\mu, \mu' \in \mathbf{M}$, $\mu\mu' \in \mathbf{M}$ and $\mu(x), \mu'(x) \leq \mu\mu'(x)$), hence it has a greatest element $\psi(x)$; it is easily seen that $\psi(x)$ is the least element of $x^{\uparrow} \cap \bigcap_{i \in I} S_i$, hence $\bigcap_{i \in I} S_i$ is a closure system; then ψ is a closure operator, namely the supremum in $\Phi(P)$ of the φ_i ($i \in I$).

The second proof in [13] relies on a generalization of Ward's [24] characterization of a closure system on a complete lattice as a subset closed under arbitrary infima (and containing the greatest element $\mathbf{1}$). A subset S of a poset P is called *MLB-closed* [22] iff for any $X \subseteq S$, every maximal lower bound of X must belong to S (in particular for $X = \emptyset$, every maximal element of P must belong to S). It is easily seen that every closure system is MLB-closed, and that the collection of all MLB-closed subsets of P is closed under arbitrary intersection (in particular for the empty intersection: P is MLB-closed). Now [13] showed that in a poset P satisfying the ACC, a subset S of P is a closure system iff it is MLB-closed, hence the collection $\Sigma(P)$ of all closure systems is closed under arbitrary intersection.

In [22] the equivalence between MLB-closed sets and closure systems was extended to the case where P is a DCPO (a poset where every directed subset has a supremum). It was then indicated that it is also possible to prove directly that $\Sigma(P)$ is closed under arbitrary intersection. Indeed, if we return to the above argument with the monoid $\mathbf{M}$ generated by the closure operators φ_i ($i \in I$), for any $x \in P$, the directed set of all $\mu(x)$, for $\mu \in \mathbf{M}$, has a supremum $\psi(x)$. This gives an extensive and isotone operator ψ, and the least closure operator $\varphi \geq \psi$ can be constructed by several methods, see [6], in particular by transfinite induction, as did [16] in the case where P is a complete lattice. Then we have $\mathsf{Inv}(\varphi) = \bigcap_{i \in I} \mathsf{Inv}(\varphi_i)$.

Manara's [17] results about covers and semimodularity extend also to any poset P where $\Sigma(P)$ is closed under arbitrary intersection (in particular if P is a DCPO). For any $X \subseteq P$, the set of all $S \in \Sigma(P)$ such that $X \subseteq S$, is closed under intersection, hence there is a least closure system containing X, we write it $\sigma(X)$ and call it the *closure system generated by* X. In [13] it was shown that if P has ACC, then $\Sigma(P)$ satisfies the *anti-matroid exchange property*:

$$\forall S \in \Sigma(P),\ \forall x, y \in P \setminus S,\ x \neq y, \qquad y \in \sigma(S \cup \{x\}) \implies x \notin \sigma(S \cup \{y\}). \tag{2}$$

Erné [12] showed that for any poset P with $\Sigma(P)$ closed under arbitrary intersection, $\Sigma(P)$ is *detachable*:

$$\forall S \in \Sigma(P),\ \forall x \in P \setminus S, \qquad \sigma(S \cup \{x\}) \setminus \{x\} \in \Sigma(P). \tag{3}$$

This concept comes from [15] (where one says *extremally detachable*), and obviously detachability (3) implies the anti-matroid exchange property (2); in [15] conditions are given under which both properties are equivalent. From any of (2,3) follow Manara's result stating that for $S_1, S_2 \in \Sigma(P)$ such that $S_1 \subset S_2$, S_2 covers S_1 in the lattice $\Sigma(P)$ iff $S_2 \setminus S_1$ is a singleton, hence that $\Sigma(P)$ is lower semimodular.

In case the poset P is not a DCPO, $\Sigma(P)$ does not necessarily constitute a complete lattice, and even in case $\Sigma(P)$ is a complete lattice, the infimum operation in $\Sigma(P)$ is not necessarily the intersection, see Section 3.

The goal of this paper is to generalize the notions of semimodularity and detachability to the case where $\Sigma(P)$ is not necessarily a complete lattice, and to prove the corresponding extensions of Manara's [17] and Erné's [12] results. We obtain the following detachability property:

$$\begin{aligned} &\forall S_0, S_1 \in \Sigma(P), \qquad S_0 \subset S_1, \qquad \forall x \in S_1 \setminus S_0, \\ &\exists S_2 \in \Sigma(P), \qquad S_0 \cup \{x\} \subseteq S_2 \subseteq S_1, \qquad S_2 \setminus \{x\} \in \Sigma(P). \end{aligned} \tag{4}$$

It is easily seen that if $\Sigma(P)$ is closed under arbitrary intersection, then (4) is equivalent to (3). Next we show that the poset $\Sigma(P)$ is lower semimodular in the following sense:

$\forall S_1, S_2 \in \Sigma(P)$, if in $\Sigma(P)$ the join $S_1 \vee S_2$ exists and $S_1 \vee S_2 \succ S_1$, then the meet $S_1 \wedge S_2$ exists and $S_2 \succ S_1 \wedge S_2$.

Note that in this case we will have $S_1 \vee S_2 = S_1 \cup S_2$ and $S_1 \wedge S_2 = S_1 \cap S_2$. By duality, we obtain that the poset $\Phi(P)$ is upper semimodular in the following sense:

> $\forall \varphi_1, \varphi_2 \in \Phi(P)$, if in $\Phi(P)$ the meet $\varphi_1 \wedge \varphi_2$ exists and $\varphi_1 \succ \varphi_1 \wedge \varphi_2$, then the join $\varphi_1 \vee \varphi_2$ exists and $\varphi_1 \vee \varphi_2 \succ \varphi_2$.

Here we have no indication on the explicit values of $(\varphi_1 \wedge \varphi_2)(x)$ and $(\varphi_1 \vee \varphi_2)(x)$ for $x \in P$.

Concerning the other properties of $\Sigma(P)$ in the case where P is a complete lattice (e.g., $\Sigma(P)$ is atomistic, $\Sigma(P)$ is closed under arbitrary intersection, closure systems coincide with MLB-closed sets, etc.), we will see in Section 3 that they are generally lost in the case where P is an arbitrary infinite poset. However, some of these properties may hold if one makes appropriate assumptions on P.

2. The main argument

In this section we shall fix a poset P. At the basis of detachability is the following generalization of some constructions made in [19]:

Proposition 2.1. *Let $S_0, S_1 \in \Sigma(P)$ such that $S_0 \subseteq S_1$, and let A be a down-set in P. Let $S_2 = (S_0 \backslash A) \uplus (S_1 \cap A)$. Then $S_2 = S_0 \cup (S_1 \cap A) = (S_0 \cup A) \cap S_1$, $S_0 \subseteq S_2 \subseteq S_1$ and $S_2 \in \Sigma(P)$.*

The closure operators $\varphi_0, \varphi_1, \varphi_2$ corresponding to S_0, S_1, S_2 satisfy $\varphi_1 \leq \varphi_2 \leq \varphi_0$, and for every $x \in P$ we have

$$\varphi_2(x) = \begin{cases} \varphi_1(x) & \text{if } \varphi_1(x) \in A, \\ \varphi_0(x) & \text{if } \varphi_1(x) \notin A. \end{cases}$$

Proof. As $S_0 \subseteq S_1$, we get $S_2 = (S_0 \backslash A) \cup (S_0 \cap A) \cup (S_1 \cap A) = S_0 \cup (S_1 \cap A)$; by the modular equality, we obtain $S_2 = (S_0 \cup A) \cap S_1$, and $S_0 \subseteq S_2 \subseteq S_1$. For any $x \in P$ we have two cases:

(1) $\varphi_1(x) \in A$. Then $\varphi_1(x) \in S_1 \cap A \subseteq S_2$. As $\varphi_1(x)$ is the least element of $x^\uparrow \cap S_1$ and $\varphi_1(x) \in x^\uparrow \cap S_2$, with $x^\uparrow \cap S_2 \subseteq x^\uparrow \cap S_1$, we deduce that $\varphi_1(x)$ is the least element of $x^\uparrow \cap S_2$. Set $\varphi_2(x) = \varphi_1(x)$.

(2) $\varphi_1(x) \notin A$. For any $y \in x^\uparrow \cap S_1$, we have $\varphi_1(x) \leq y$, and as A is a down-set and $\varphi_1(x) \notin A$, we get $y \notin A$. Hence $x^\uparrow \cap S_1 \cap A = \emptyset$, and as $S_0 \subseteq S_1$, $x^\uparrow \cap S_0 \cap A = \emptyset$. Hence

$$\begin{aligned} x^\uparrow \cap S_0 &= (x^\uparrow \cap S_0 \backslash A) \cup (x^\uparrow \cap S_0 \cap A) = x^\uparrow \cap S_0 \backslash A \\ &= (x^\uparrow \cap S_0 \backslash A) \cup (x^\uparrow \cap S_1 \cap A) = x^\uparrow \cap S_2. \end{aligned}$$

Now $\varphi_0(x)$ is the least element of $x^\uparrow \cap S_0 = x^\uparrow \cap S_2$. Set $\varphi_2(x) = \varphi_0(x)$.

Thus in both cases $\varphi_2(x)$ is the least element of $x^\uparrow \cap S_2$, so S_2 is a closure system and φ_2 is a closure operator. We have $\varphi_2(x) = \varphi_1(x)$ for $\varphi_1(x) \in A$, and $\varphi_2(x) = \varphi_0(x)$ for $\varphi_1(x) \notin A$. □

An interesting particular case is for $S_1 = P$ (that is, φ_1 is the identity operator):

Corollary 2.2. *Given a closure system S on P and a down-set A in P, then $S \cup A$ is a closure system. If φ is the closure operator corresponding to S, then the closure operator φ_A corresponding to $S \cup A$ is given by*

$$\varphi_A(x) = \begin{cases} x & \text{for } x \in A, \\ \varphi(x) & \text{for } x \notin A. \end{cases}$$

We obtain from Proposition 2.1 the detachability condition (4):

Corollary 2.3. *For any $S_0, S_1 \in \Sigma(P)$ such that $S_0 \subset S_1$ and for any $x \in S_1 \setminus S_0$, there exists $S_2 \in \Sigma(P)$ such that $S_0 \cup \{x\} \subseteq S_2 \subseteq S_1$ and $S_2 \setminus \{x\} \in \Sigma(P)$.*

Proof. $A = x^\downarrow$ and $A' = x^\downarrow \setminus \{x\} = \{y \in P \mid y < x\}$ are down-sets. Let

$$S_2 = (S_0 \setminus A) \uplus (S_1 \cap A) \quad \text{and} \quad S_2' = (S_0 \setminus A') \uplus (S_1 \cap A').$$

By Proposition 2.1, $S_2, S_2' \in \Sigma(P)$, $S_2 = S_0 \cup (S_1 \cap A)$, $S_2' = S_0 \cup (S_1 \cap A')$, $S_2 \subseteq S_1$ and $S_0 \subseteq S_2'$; as $x \in S_1 \cap A$, we have $x \in S_2$, and as $x \notin S_0$, $S_2 \setminus \{x\} = (S_0 \setminus \{x\}) \cup (S_1 \cap A \setminus \{x\}) = S_0 \cup (S_1 \cap A') = S_2'$. Hence $S_0 \cup \{x\} \subseteq S_2 \subseteq S_1$ and $S_2' = S_2 \setminus \{x\}$, with $S_2, S_2' \in \Sigma(P)$. □

This gives thus a generalization of Manara's first result:

Corollary 2.4. *For $S_0, S_1 \in \Sigma(P)$ such that $S_0 \subset S_1$, we have $S_1 \succ S_0$ (in $\Sigma(P)$) iff $S_1 \setminus S_0$ is a singleton.*

Proof. Take $x \in S_1 \setminus S_0$, then by Corollary 2.3 there is $S_2 \in \Sigma(P)$ such that $S_2 \setminus \{x\} \in \Sigma(P)$ and $S_0 \subseteq S_2 \setminus \{x\} \subset S_2 \subseteq S_1$. If S_1 covers S_0 in $\Sigma(P)$, we must necessarily have $S_2 \setminus \{x\} = S_0$ and $S_2 = S_1$; hence $S_1 \setminus S_0$ is a singleton. Conversely, if $S_1 \setminus S_0$ is a singleton, then S_1 covers S_0 in $\mathscr{P}(E)$, so obviously it covers S_0 in the smaller poset $\Sigma(P)$. □

It follows by induction that for $S_0, S_1 \in \Sigma(P)$ such that $S_0 \subset S_1$, the interval $[S_0, S_1]$ in $\Sigma(P)$ has finite height iff $S_1 \setminus S_0$ is finite, and then the height of $[S_0, S_1]$ in $\Sigma(P)$ equals the size of $S_1 \setminus S_0$.

Returning to Corollary 2.3, writing $S_3 = S_2 \setminus \{x\}$, we obtain thus:

$$\forall S_0, S_1 \in \Sigma(P), \quad S_0 \subset S_1 \implies \exists S_2, S_3 \in \Sigma(P), \quad S_0 \subseteq S_3 \prec S_2 \subseteq S_1.$$

In other words every interval in $\Sigma(P)$ contains a cover; one says then that $\Sigma(P)$ is *weakly atomic* [6].

In order to prove the lower semimodularity of $\Sigma(P)$, previous works [17], [13], [12] required $\Sigma(P)$ to be closed under arbitrary intersection; since we do not make this assumption, we need to show that in some particular cases the intersection of two closure systems is a closure system:

Proposition 2.5. *Let $S_1, S_2 \in \Sigma(P)$ such that $S_2 \backslash S_1$ is finite; then $S_1 \cap S_2 \in \Sigma(P)$.*

Given φ_1, φ_2 the closure operators corresponding to S_1, S_2, the closure operator corresponding to $S_1 \cap S_2$ is $(\varphi_2\varphi_1)^n\varphi_2$, where $n = |S_2 \backslash S_1|$.

Proof. Let $n = |S_2 \backslash S_1|$. For any $x \in P$, $\varphi_2(x) \in S_2$. For any $y \in S_2$, we have two cases:

(1) If $y \in S_1 \cap S_2$, then $\varphi_1(y) = y = \varphi_2(y)$, so $\varphi_2\varphi_1(y) = y$.

(2) If $y \in S_2 \backslash S_1$, then $y \notin \mathsf{Inv}(\varphi_1)$, so $y < \varphi_1(y)$, and $\varphi_1(y) \leq \varphi_2\varphi_1(y)$, thus $y < \varphi_2\varphi_1(y)$.

Let $x \in P$; for any $t \in \mathbf{N}$, $(\varphi_2\varphi_1)^t\varphi_2(x) \in S_2$. If there is some $t < n$ such that $(\varphi_2\varphi_1)^t\varphi_2(x) \in S_1 \cap S_2$, case 1 yields that

$$(\varphi_2\varphi_1)^{t+1}\varphi_2(x) = \varphi_2\varphi_1\big((\varphi_2\varphi_1)^t\varphi_2(x)\big) = (\varphi_2\varphi_1)^t\varphi_2(x),$$

and thus $(\varphi_2\varphi_1)^n\varphi_2(x) \in S_1 \cap S_2$. If $(\varphi_2\varphi_1)^t\varphi_2(x) \in S_2 \backslash S_1$ for every $t < n$, by case 2 we get

$$\varphi_2(x) < \cdots < (\varphi_2\varphi_1)^t\varphi_2(x) < \cdots < (\varphi_2\varphi_1)^n\varphi_2(x),$$
$$\text{with } \varphi_2(x), \ldots, (\varphi_2\varphi_1)^{n-1}\varphi_2(x) \in S_2 \backslash S_1;$$

as $|S_2 \backslash S_1| = n$, this means that $(\varphi_2\varphi_1)^n\varphi_2(x) \notin S_2 \backslash S_1$, that is $(\varphi_2\varphi_1)^n\varphi_2(x) \in S_1 \cap S_2$. Thus we have shown that for any $x \in P$, $(\varphi_2\varphi_1)^n\varphi_2(x) \in S_1 \cap S_2$. Now for $x \in S_1 \cap S_2$, by induction case 1 gives $(\varphi_2\varphi_1)^n\varphi_2(x) = x$; hence $(\varphi_2\varphi_1)^n\varphi_2$ is idempotent. Clearly $(\varphi_2\varphi_1)^n\varphi_2$ inherits the isotony and extensivity of φ_1 and φ_2. Therefore $(\varphi_2\varphi_1)^n\varphi_2$ is a closure operator, with $\mathsf{Inv}\big((\varphi_2\varphi_1)^n\varphi_2\big) = S_1 \cap S_2$, so $S_1 \cap S_2$ is a closure system. □

We obtain thus the lower semimodularity of $\Sigma(P)$, the following argument is the same as in previous works [17], [13], [12]:

Corollary 2.6. *For $S_1, S_2 \in \Sigma(P)$, if in $\Sigma(P)$ the join $S_1 \vee S_2$ exists and $S_1 \vee S_2 \succ S_1$, then the meet $S_1 \wedge S_2$ exists and $S_2 \succ S_1 \wedge S_2$. We have then $S_1 \vee S_2 = S_1 \cup S_2$ and $S_1 \wedge S_2 = S_1 \cap S_2$.*

Proof. Suppose that $S_1 \vee S_2$ exists and $S_1 \vee S_2 \succ S_1$; thus $S_1 \subseteq S_1 \cup S_2 \subseteq S_1 \vee S_2$. We may not have $S_1 \cup S_2 = S_1$, otherwise $S_2 \subseteq S_1$ and then $S_1 \vee S_2 = S_1$. Hence $S_1 \subset S_1 \cup S_2 \subseteq S_1 \vee S_2$, and by Corollary 2.4, $(S_1 \vee S_2)\backslash S_1$ is a singleton; thus we get $S_1 \vee S_2 = S_1 \cup S_2$. Now $(S_1 \vee S_2)\backslash S_1 = (S_1 \cup S_2)\backslash S_1 = S_2\backslash S_1$; as this is a singleton, Proposition 2.5 gives $S_1 \cap S_2 \in \Sigma(P)$. Then obviously $S_1 \cap S_2$ is the meet of S_1 and S_2 in $\Sigma(P)$. We have $S_2\backslash(S_1 \cap S_2) = S_2\backslash S_1$; as this is a singleton, $S_2 \succ S_1 \cap S_2$ by Corollary 2.4. □

By duality, $\Phi(P)$ is upper semimodular in the following sense: for $\varphi_1, \varphi_2 \in \Phi(P)$, if in $\Phi(P)$ the meet $\varphi_1 \wedge \varphi_2$ exists and $\varphi_1 \succ \varphi_1 \wedge \varphi_2$, then the join $\varphi_1 \vee \varphi_2$ exists and $\varphi_1 \vee \varphi_2 \succ \varphi_2$.

3. Some counterexamples to other properties

We have shown that the results of Erné and Manara, namely that $\Sigma(P)$ is detachable and lower semimodular, can be extended to the case where P is an arbitrary poset. What about the other properties satisfied by $\Sigma(P)$ in case P is a complete lattice? It is easy to show that the following hold:

(Pa) Every closure system is MLB-closed, and the collection of all MLB-closed sets is closed under arbitrary intersections.

(Pb) If the poset P has a greatest element $\mathbf{1}$, then $\Sigma(P)$ has least element $\{\mathbf{1}\}$ and atoms $\{\mathbf{1}, x\}$ for all $x \in P\backslash\{\mathbf{1}\}$, so it is atomistic.

(Pc) If P is a meet-semilattice (i.e., every two elements of P have a meet), then for any $S_1, S_2 \in \Sigma(P)$, the join $S_1 \vee S_2$ exists in $\Sigma(P)$, it is the set of all $x_1 \wedge x_2$ for $x_1 \in S_1$ and $x_2 \in S_2$ (NB: elements of $S_1 \cup S_2$ take this form: $x_1 = x_1 \wedge \varphi_2(x_1)$ for all $x_1 \in S_1$, where $\varphi_2(x_1) \in S_2$); in particular, if S_1 and S_2 are finite, then $S_1 \vee S_2$ is finite.

(Pd) If P is a complete meet-semilattice (i.e., every non-void subset of P has an infimum), then for any non-void family $S_i \in \Sigma(P)$ $(i \in I)$, its supremum $\bigvee_{i \in I} S_i$ exists in $\Sigma(P)$, it is the set of $\bigwedge_{i \in I} x_i$ for all choices $x_i \in S_i$; in particular, $\Sigma(P)$ is a complete join-semilattice (i.e., every non-void subset of P has a supremum).

(Pe) If $P = \mathscr{P}(E)$, the Boolean lattice of all subsets of a finite set E, then [5] $\Sigma(P)$ is *join-semidistributive*:

$$\forall X, Y, Z \in \Sigma(P), \quad \text{if } X \vee Y = X \vee Z, \quad \text{then } X \vee Y = X \vee (Y \wedge Z);$$

hence it is *join-pseudocomplemented*:

$$\forall X \in \Sigma(P), \quad \{Y \in \Sigma(P) \mid X \vee Y = P\} \text{ has a least element.}$$

However, we will see through counterexamples that in the general case of an arbitrary poset P, none of these properties can be extended. More precisely:

(F1) $\Sigma(P)$ does not necessarily constitute a complete lattice, nor even a lattice.

(F2) Even in case $\Sigma(P)$ is a complete lattice, the infimum operation in $\Sigma(P)$ is not necessarily the intersection.

(F3) Even in case $\Sigma(P)$ is a complete lattice with the infimum operation given by the intersection, $\Sigma(P)$ can be larger than the collection of all MLB-closed sets.

(F4) In case P does not have a greatest element, $\Sigma(P)$ can have elements that are not joins of atoms; in some cases it may have no atoms.

(F5) There can be finite $S_1, S_2 \in \Sigma(P)$ with an infinite join $S_1 \vee S_2$ in $\Sigma(P)$, or even having no join in $\Sigma(P)$.

(F6) Even in case $\Sigma(P)$ is a complete lattice, it can be neither join-semidistributive nor join-pseudocomplemented.

These failures of expected properties are shown through several counterexamples. The first ones are posets without greatest element:

A. Naturals: Let $P = \mathbf{N}$, the set of natural integers. Every subset of $\mathbf{N}$ (including $\emptyset$) is MLB-closed; on the other hand a subset of $\mathbf{N}$ is a closure system if and only if it is *infinite*. Then $\Sigma(\mathbf{N})$ is closed under non-void unions and has $\mathbf{N}$ as greatest element; it is thus a complete join-semilattice. However two closure systems on $\mathbf{N}$ do not necessarily have a meet in $\Sigma(\mathbf{N})$; for example let $S_0 = 2\mathbf{N}$ (the set of even naturals) and $S_1 = 2\mathbf{N} + 1$ (the set of odd naturals), as $S_0 \cap S_1 = \emptyset$, there is no closure system contained in both S_0 and S_1; seen otherwise, S_0 and S_1 correspond to the two closure operators φ_0 and φ_1 that map every natural to the least even (resp., odd) natural above it, and iterating $\varphi_0\varphi_1$ on any $n \in \mathbf{N}$, the sequence $(\varphi_0\varphi_1)^t(n)$ ($t \in \mathbf{N}$) goes to infinity. Hence $\Sigma(\mathbf{N})$ is not a lattice, cf. (F1). Furthermore, there are no atoms nor any least element in $\Sigma(\mathbf{N})$, cf. (F4).

B. Fence over naturals: This example comes from [19], [22]. Let $P = \mathbf{N} \cup \{a, b, c, d\}$, where $\{a, b, c, d\}$ is a fence ($a > c$, $b > c$, $b > d$) standing above $\mathbf{N}$ ($\forall n \in \mathbf{N}$, $c, d > n$), with the usual order on $\mathbf{N}$; the Hasse diagram of P is shown in Figure 1 (a). The MLB-closed subsets of P are all those containing $\{a, b, c\}$. On the other hand, the closure systems on P can be of two forms: either $\{a, b, c\} \cup X$ for any $X \subseteq \mathbf{N}$, or $\{a, b, c, d\} \cup Y$ for an *infinite* $Y \subseteq \mathbf{N}$. It is easily checked that $\Sigma(P)$ is closed under non-void unions and constitutes a complete lattice with universal bounds $\{a, b, c\}$ and P. However given $S_0 = \{a, b, c, d\} \cup 2\mathbf{N}$ and $S_1 = \{a, b, c, d\} \cup (2\mathbf{N} + 1)$, we have $S_0, S_1 \in \Sigma(P)$, with $S_0 \cap S_1 = \{a, b, c, d\}$, but in $\Sigma(P)$ we have $S_0 \wedge S_1 = \{a, b, c\}$. Thus the infimum operation in $\Sigma(P)$ is not the intersection, cf. (F2). Moreover, as there is no least

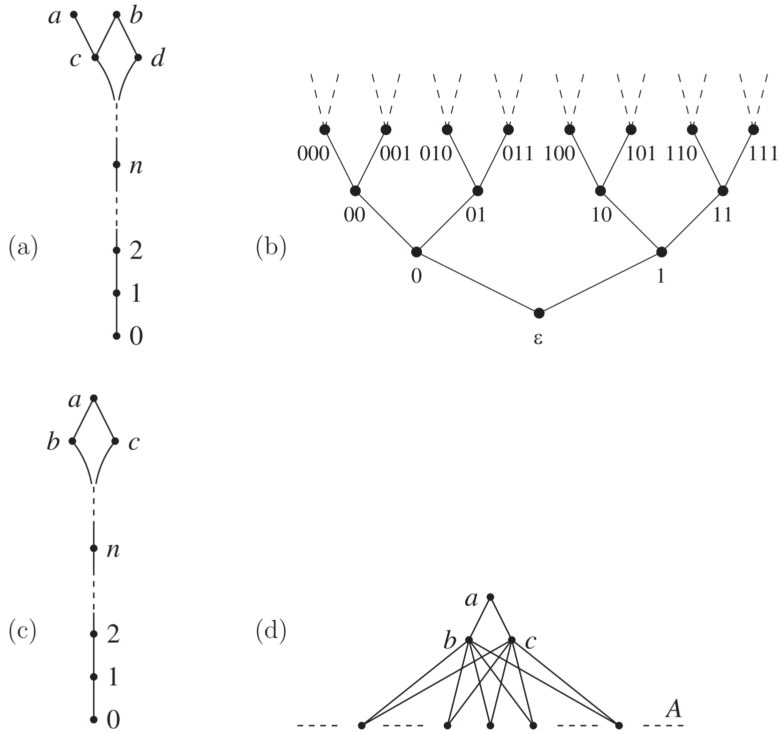

Figure 1. Hasse diagrams of posets: (a) fence over naturals; (b) 01-words; (c) hat over naturals; (d) hat over antichain.

closure system containing $\{a, b, c, d\}$, the operator σ is not defined and the properties (2,3) make no sense here. The atoms of $\Sigma(P)$ are all $\{a, b, c, n\}$ for $n \in \mathbf{N}$, thus joins of atoms have the form $\{a, b, c\} \cup X$ for $X \subseteq \mathbf{N}$; hence $\Sigma(P)$ is not atomistic, cf. (F4); it is however *atomic* (i.e., every nonzero element of $\Sigma(P)$ contains an atom). Finally, let $S_2 = \{a, b, c\} \cup \mathbf{N} = P \setminus \{d\}$; then $S_2 \in \Sigma(P)$, $S_2 \vee S_0 = S_2 \vee S_1 = P$, but $S_2 \vee (S_0 \wedge S_1) = S_2 \vee \{a, b, c\} = S_2$; in fact there is no least element in the set of all $S \in \Sigma(P)$ such that $S_2 \vee S = P$. Thus $\Sigma(P)$ is neither join-semidistributive nor join-pseudocomplemented, cf. (F6).

C. 01-words: Let P be the set of words on the alphabet $\{0, 1\}$, with prefix ordering, see the Hasse diagram in Figure 1 (b); equivalently, $P = \bigcup_{n \in \mathbf{N}} \{0, 1\}^n$, where for $x = (x_0, \ldots, x_{m-1}) \in \{0, 1\}^m$ and $y = (y_0, \ldots, y_{n-1}) \in \{0, 1\}^n$ we have $x \leq y$ iff $m \leq n$ and $x_i = y_i$ for $i = 0, \ldots, m-1$. Then P is a complete meet-semilattice; the least element is the empty word. It is easily checked that the only closure operator on P is the identity, thus P is the only closure system and $\Sigma(P)$ is trivially closed under arbitrary intersection. On the other hand the MLB-closed sets are all sets closed under non-empty infimum, there are infinitely many such sets, cf. (F3).

Our next counterexamples are posets having a greatest element; we will see that this property does not bring much more than the fact that every closure system is a join of atoms (cf. property (Pb) above):

D. Hat over naturals: This example comes from [19]. Let $P = \mathbf{N} \cup \{a, b, c\}$, where $\{a, b, c\}$ is a hat ($a > b$, $a > c$) standing above $\mathbf{N}$ ($\forall n \in \mathbf{N}$, $b, c > n$), with the usual order on $\mathbf{N}$; the Hasse diagram of P is shown in Figure 1 (c). The MLB-closed subsets of P are all those containing a. On the other hand, the closure systems on P are the sets $\{a\} \cup X$, $\{a, b\} \cup X$ and $\{a, c\} \cup X$ for any $X \subseteq \mathbf{N}$, and the sets $\{a, b, c\} \cup Y$ for an *infinite* $Y \subseteq \mathbf{N}$. Now given $S_0 = \{a, b, c\} \cup 2\mathbf{N}$ and $S_1 = \{a, b, c\} \cup (2\mathbf{N} + 1)$, we have $S_0, S_1 \in \Sigma(P)$, with $S_0 \cap S_1 = \{a, b, c\} \notin \Sigma(P)$, so in $\Sigma(P)$ the pair $\{S_0, S_1\}$ has two maximal lower bounds $\{a, b\}$ and $\{a, c\}$. Thus $\Sigma(P)$ is not a lattice, cf. (F1). Note that both $\{a, b\}$ and $\{a, c\}$ are finite elements of $\Sigma(P)$, but they have no join in $\Sigma(P)$, cf. (F5).

E. Hat over antichain: Let $P = A \cup \{a, b, c\}$, where A is an infinite antichain ($\forall x, y \in A$, $x \not< y$ and $y \not< x$) and $\{a, b, c\}$ is a hat ($a > b$, $a > c$) standing above A ($\forall x \in A$, $b, c > x$); the Hasse diagram of P is shown in Figure 1 (d). Since P has finite height, it satisfies the ACC condition, thus closure systems on P coincide with MLB-closed subsets of P [13], they are: the sets $\{a\} \cup X$, $\{a, b\} \cup X$ and $\{a, c\} \cup X$ for any $X \subseteq A$, and P. Note that the two pairs $\{a, b\}$ and $\{a, c\}$ are atoms, with join $\{a, b\} \vee \{a, c\} = P$ which is infinite, cf. (F5).

F. Finite/co-finite: Let $\mathbf{P}$ be the collection consisting of all finite and co-finite (complement of finite) subsets of $\mathbf{N}$. Then $\mathbf{P}$, ordered by the inclusion relation, is a Boolean lattice, but not a complete lattice. Note that a subset $\mathbf{Q}$ of $\mathbf{P}$ is MLB-closed iff for any family $S_i \in \mathbf{Q}$ ($i \in I$), either $\bigcap_{i \in I} S_i \in \mathbf{Q}$ or $\bigcap_{i \in I} S_i \notin \mathbf{P}$ (i.e., $\bigcap_{i \in I} S_i$ is infinite with infinite complement); in other words $\mathbf{Q}$ is the trace on $\mathbf{P}$ of a closure system (Moore family) on $\mathscr{P}(E)$. Now $\Sigma(\mathbf{P})$ is not a lattice, cf. (F1). Indeed, for any $n \in \mathbf{N}$, let $A_n = 2\mathbf{N} \cap [0, n] = \{k \in 2\mathbf{N} \mid k \leq n\}$, and define $\mathbf{S}_0 \subseteq \mathbf{P}$ (resp., $\mathbf{S}_1 \subseteq \mathbf{P}$) to be the set of all co-finite subsets B of $\mathbf{N}$ such that $2\mathbf{N} \subseteq B$, and of all $A_n \cup X$ for $n \in 4\mathbf{N}$ (resp., $n \in 4\mathbf{N} + 2$) and X a finite subset of $2\mathbf{N} + 1$; we can check that $\mathbf{S}_0, \mathbf{S}_1 \in \Sigma(\mathbf{P})$, but $\mathbf{S}_0$ and $\mathbf{S}_1$ have no meet in $\Sigma(\mathbf{P})$, because $\mathbf{S}_0 \cap \mathbf{S}_1$ is the set of all co-finite subsets B of $\mathbf{N}$ containing $2\mathbf{N}$, and the closure systems contained in it are its finite parts closed under intersection, and none of them is maximal.

4. Discussion and conclusion

Our counter-examples in Section 3 indicate that in the general case, one probably cannot obtain more than the results given in Section 2. Nevertheless, in view of previous works [17], [13], [22], [12], [5], we can inquire about the most general properties that a poset P should satisfy in order to guarantee some properties for $\Sigma(P)$; thus we can raise the following questions:

- What are the conditions for $\Sigma(P)$ to be a complete lattice? to be closed under arbitrary intersection? to coincide with the collection of all MLB-closed subsets of P?
- If P does not have a greatest element, when is $\Sigma(P)$ an atomistic complete lattice?
- We showed that for $S_1, S_2 \in \Sigma(P)$ such that $S_2 \backslash S_1$ is finite, $S_1 \cap S_2 \in \Sigma(P)$. What are then the conditions for the join $S_1 \vee S_2$ to exist in $\Sigma(P)$?
- What are the conditions for $\Sigma(P)$ to be join-semidistributive? to be join-pseudocomplemented?

Another interesting question is the analysis of the properties of $\Sigma(P)$ in case P has a greatest element (cf. property (Pb) at the beginning of Section 3). It seems that adding a greatest element to a poset increases considerably the collection of all closure systems.

Let us now briefly discuss some related questions concerning closure operators and systems, in particular their characterizations.

If $P = \mathscr{P}(E)$, the Boolean lattice of all subsets of a finite set E, the fact that $\Sigma(P)$ is join-semidistributive and join-pseudocomplemented (cf. property (Pe) at the beginning of the previous section) follows from the fact that $\Sigma(P)$ is *lower bounded*, that is, the image of a free lattice under a homomorphism η such that for every $S \in \Sigma(P)$, $\eta^{-1}(S)$ has a least element. The latter property was obtained in [5] by showing that $\Sigma(P)$ satisfies a property on join-irreducible elements that was given in [7] as a necessary and sufficient condition for the lower boundedness of a *finite* lattice. It seems thus illusory to extend this result to the infinite case, but for a finite poset P, it is possible to investigate under what conditions $\Sigma(P)$ is lower bounded.

The first proof in [13] shows that in a poset P satisfying the ACC, if we associate to each closure operator φ the equivalence relation $\equiv$ on P defined by $x \equiv y$ iff $\varphi(x) = \varphi(y)$, given a family φ_i $(i \in I)$ of closure operators with corresponding equivalence relations $\equiv_i$, then the supremum of the $\equiv_i$ $(i \in I)$ in the complete lattice of equivalence relations on P will correspond to the supremum in $\Phi(P)$ of the φ_i $(i \in I)$. Now this supremum of the $\equiv_i$ $(i \in I)$ is the transitive closure of their union, so the result in [13] means that $(\bigvee_{i \in I} \varphi_i)(x) = (\bigvee_{i \in I} \varphi_i)(y)$ iff there is a sequence $x = z_0, \ldots, z_n = y$ such that for $t = 0, \ldots, n-1$ there is some $i(t) \in I$ with $\varphi_{i(t)}(z_i) = \varphi_{i(t)}(z_{i+1})$. It can be seen that this holds because for every $x \in P$, the set of $\varphi_{i_1} \ldots \varphi_{i_n}(x)$ $(i_1, \ldots, i_n \in I)$ has a greatest element. However, in case P does not have ACC, this result does not hold anymore (even if P is a DCPO).

Nevertheless there is a related result in case P is a complete lattice. Given a closure operator φ on P, since we have $\varphi(\bigvee_{i \in I} x_i) = \varphi(\bigvee_{i \in I} \varphi(x_i))$, the relation $\equiv$ on P defined by $x \equiv y$ iff $\varphi(x) = \varphi(y)$, is an equivalence relation closed under supremum: $x_i \equiv y_i$ $(i \in I)$ implies $\bigvee_{i \in I} x_i \equiv \bigvee_{i \in I} y_i$; such a relation is called a *complete join-congruence* [3]. Conversely, given a complete join-congruence $\equiv$, the operator φ defined by $\varphi(x) = \bigvee \{y \in P \mid x \equiv y\}$ (in fact, the greatest $y \in P$ such that $x \equiv y$) is a closure operator, and we obtain in this way a bijection between closure operators on P and complete join-congruences on P. Considering the inclusion order on complete join-congruences, by (1) we obtain that this bijection is an isomorphism between the complete lattice $\Phi(P)$ of closure operators on P, and the one of complete join-congruences on P [3]. This result appeared also in [8], where several lattice-theoretical properties of complete join-congruences are presented. The same reference gives also an isomorphism between complete join-congruences and *quotient systems*: a *quotient system* is a binary relation ρ on P that is contained in the relation $\leq$, reflexive, transitive, compatible with the supremum, and such that for $a \leq c \leq d \leq b$ with $a \, \rho \, b$, we must have $c \, \rho \, d$; here the quotient system corresponding to $\equiv$ is given by $a \, \rho \, b$ iff $a \leq b \equiv a$, that is, $a \leq b \leq \varphi(a)$.

In case $\Sigma(P)$ is closed under arbitrary intersection, it is interesting to consider the decomposition of a closure system as an intersections of some "elementary" closure systems. Let P be a join-semilattice (i.e., every two elements of P have a join). For any $a, b \in P$, $b^{\uparrow}$ is a closure system (corresponding to the closure operator $x \mapsto x \vee b$), and $P \backslash a^{\uparrow}$ is a down-set; by Corollary 2.2, their union

$$F_{a,b} = b^{\uparrow} \cup (P \backslash a^{\uparrow}) = \{x \in P \mid a \not\leq x \text{ or } b \leq x\} = \{x \in P \mid a \leq x \Rightarrow b \leq x\}, \quad (5)$$

is a closure system. The closure systems $F_{a,b}$ (for all $a, b \in P$) have been studied in depth in the particular case where P is $\mathscr{P}(E)$ for a finite set E [5], where they are called *implicational* closure systems. Given a closure system S corresponding to a closure operator φ, for any $a, b \in P$ we have $S \subseteq F_{a,b} \Leftrightarrow b \leq \varphi(a)$ (i.e., $\varphi(b) \leq \varphi(a)$), while for any $x \notin S$ we have $x \notin F_{x,\varphi(x)}$; hence [5]

$$S = \bigcap \{F_{a,b} \mid a, b \in P, b \leq \varphi(a)\} = \bigcap_{a \in P} F_{a,\varphi(a)}.$$

Now if $\Sigma(P)$ is closed under arbitrary intersection (for example, if P is a complete lattice), it follows that a subset S of P is a closure system iff it is an intersection of implicational closure systems $F_{a,b}$. In the case where $P = \mathscr{P}(E)$ for a finite set E, for any $A, B \in \mathscr{P}(E)$, $F_{A,B}$ is meet-irreducible iff $A \subset S$ and B is a singleton disjoint from A [5].

The lattice of closure operators on a complete lattice is isomorphic to that of *implication systems* [8] (also called *Armstrong systems*, *FD-systems* [8], or *full im-*

plicational systems [5]). This isomorphism extends to the case of a poset, provided that we modify the definition of an implication system:

Let P be a poset. An *implication system* on P is a binary relation σ on P (i.e., a subset of P^2) that:

(1) is transitive: for all $a, b, c \in P$, if $(a, b), (b, c) \in \sigma$, then $(a, c) \in \sigma$;

(2) contains the relation $\geq$: for all $a, b \in P$, if $a \geq b$, then $(a, b) \in \sigma$;

(3) is upper bounded on the right: for all $a \in P$, the set $\{b \in P \mid (a, b) \in \sigma\}$ has a greatest element.

Note that if P is a complete lattice, condition (3) can be replaced by the following, which is the original condition (3) from [8]:

(3′) given $\{(a_i, b_i) \mid i \in I\} \subseteq \sigma$, we have $(\bigvee_{i \in I} a_i, \bigvee_{i \in I} b_i) \in \sigma$.

Write $\mathbf{IS}(P)$ for the poset of implication systems on P (ordered by inclusion). Then it is easily shown that there is an isomorphism between the two posets $\Phi(P)$ and $\mathbf{IS}(P)$, where:

- to every closure operator φ corresponds the implication system $\sigma_\varphi = \{(a, b) \in P^2 \mid b \leq \varphi(a)\}$;
- to every implication system σ corresponds the closure operator φ_σ given by setting, for all $a \in P$, $\varphi_\sigma(a)$ equal to the greatest element of the set $\{b \in P \mid (a, b) \in \sigma\}$.

Note that [8] gave also, in case P is a complete lattice, an isomorphism between $\mathbf{IS}(P)$ and the complete lattice of complete join-congruences on P: $a \equiv b$ iff $(a, b), (b, a) \in \sigma$, and conversely $(a, b) \in \sigma$ iff $a \equiv a \vee b$. Composed with the isomorphism between $\Phi(P)$ and $\mathbf{IS}(P)$, we obtain the isomorphism given above between closure operators and complete join-congruences, namely $a \equiv b$ iff $\varphi(a) = \varphi(b)$.

In [5] this isomorphism between $\Phi(P)$ and $\mathbf{IS}(P)$ is obtained, for $P = \mathscr{P}(E)$, through a Galois connection between P and P^2. We show here how it can be generalized. We define the binary relation $\sim$ between P^2 and P by $(a, b) \sim f$ iff $a \not\leq f$ or $b \leq f$. We obtain thus a Galois connection made of the two maps:

$$\alpha : \mathscr{P}(P^2) \to \mathscr{P}(P) : \sigma \mapsto \{f \in P \mid (a, b) \sim f \text{ for all } (a, b) \in \sigma\};$$

$$\beta : \mathscr{P}(P) \to \mathscr{P}(P^2) : S \mapsto \{(a, b) \in P^2 \mid (a, b) \sim f \text{ for all } f \in S\}.$$

From (5) we see that for any $(a, b) \in P^2$, $\alpha\big(\{(a, b)\}\big) = F_{a,b}$, hence for any $\sigma \in \mathscr{P}(P^2)$ we have $\alpha(\sigma) = \bigcap_{(a,b) \in \sigma} F_{a,b}$. Thus if P is a join-semilattice and $\Sigma(P)$ is closed under arbitrary intersection (for example, if P is a complete lattice), it follows that $\Sigma(P) = \alpha\big(\mathscr{P}(P^2)\big)$, the image of α. By the Galois connection, the restriction of β to $\Sigma(P)$ constitutes a dual isomorphism between $\Sigma(P)$ and $\beta\big(\mathscr{P}(P)\big)$, the image of β. Now for a closure system S corresponding to a closure operator φ, $(a,b) \in \beta(S) \Leftrightarrow S \subseteq \alpha(\{(a,b)\}) = F_{a,b}$, hence

$$\beta(S) = \{(a,b) \in P^2 \mid S \subseteq F_{a,b}\} = \{(a,b) \in P^2 \mid b \leq \varphi(a)\} = \sigma_\varphi,$$

the implication system corresponding to φ.

Let us conclude. In Section 2 we proved that for a poset P, $\Sigma(P)$ is detachable and lower semimodular. Then Section 3 showed that many related properties, given in case P is a complete lattice, do not hold in general. In this section we have raised the question of the level of generality of these properties, namely of the minimal conditions on the poset P that guarantee such properties. We have also considered alternative characterizations of closure operators and systems, given in case P is a complete lattice: complete join-congruences, quotient systems and implication systems. Although the characterization by implication systems extends to the general case of an arbitrary poset P, we do not know if this can also be done for complete join-congruences and quotient systems. Thus further research on the above problems is amply justified.

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