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Bounded lattice structured discriminator varieties

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Introduction

This paper is concerned with bounded lattice structured discriminator varieties, i.e., discriminator varieties interpreting the variety $\mathscr{L}_{01}$ of bounded lattices. The many examples of such varieties found in the literature are in fact varieties interpreting bounded distributive lattices or even Heyting algebras. The theory of discriminator varieties, developed by many authors, started with a paper by A. Pixley in 1971 [21]. It relies on the existence of a term defining the discriminator function on all the subdirectly irreducible algebras of a variety. It plays a unifying role for a large class of varieties, providing for some of them the results previously obtained by many authors.

Our search of all bounded lattice structured discriminator varieties covering the variety $\mathscr{TM}$ of tetravalued modal algebras, aroused our interest in studying the class of varieties in the title, with tools from the general theory of discriminator varieties. The variety $\mathscr{TM}$, generated by a four-element quasi-primal algebra, covers the variety $\mathscr{LM}_3$ of three-valued Łukasiewicz–Moisil algebras. The latter was introduced and studied by Gr. C. Moisil in 1940, 1941; the former was introduced by A. A. Monteiro in 1978, and studied by I. Loureiro for her thesis [16], under the guidance of A. A. Monteiro.

Our work started by recognizing that, for the special discriminator varieties under study, the role played by the ternary discriminator may be played by a binary function. This function was called a $(0,1)$-switching function in [23], where some results concerning these varieties were given without proofs, and examples of discriminator varieties covering $\mathscr{TM}$ were presented.

This paper is an expanded version of the first part of [23]. A version of the remaining part will be given in another paper. To avoid repetitions when we speak of a variety $\mathscr{V}$ we mean a variety interpreting the variety $\mathscr{L}_{01}$. In Section 1, basic facts from discriminator varieties are recalled and some facts regarding known varieties, which are generalized here, are focused. In Section 2, discriminator varieties $\mathscr{V}$ are characterized by equational conditions on a binary term. The proof of the equational characterization theorem provides a description of the principal congruences on each algebra from $\mathscr{V}$, and a characterization of its subdirectly irreducible (simple) algebras. In Section 3, a discriminator variety $\mathscr{S}w\mathscr{L}_{01}$ interpreting $\mathscr{L}_{01}$ which is interpretable in every variety $\mathscr{V}$ is considered. The study of some questions regarding discriminator varieties $\mathscr{V}$ amounts to studing them for the variety $\mathscr{S}w\mathscr{L}_{01}$. Four special unary terms for a variety $\mathscr{V}$ are derived from the special binary term: a term defining an existential quantifier, a term defining a universal quantifier, a term defining a weak pseudocomplementation, and a term defining a weak dual pseudocomplentation. All these terms determine in any algebra the same subreduct which is a Boolean algebra, and is the core of the algebra. It is also shown that the variety $\mathscr{T}\mathscr{M}$ as well as $\mathscr{L}\mathscr{M}_3$ are termwise equivalent to subvarieties of $\mathscr{S}w\mathscr{L}_{01}$. In Section 4, it is shown that the Boolean subreduct $B(\mathbb{A})$ of an algebra $\mathbb{A} \in \mathscr{V}$ is isomorphic to the lattice of principal congruences on $\mathbb{A}$, and it determines all congruences of $\mathbb{A}$. Moreover, $\mathbb{A}$ and $B(\mathbb{A})$ have isomorphic congruence lattices. Descriptions of congruences are presented, and the 1-cosets and 0-cosets are characterized. Finally, in Section 5, we consider a term operation of weak implication for any variety $\mathscr{V}$, generalizing the one introduced and studied by A. A. Monteiro for the variety $\mathscr{L}\mathscr{M}_3$; the deductive systems relative to this operation are precisely the 1-cosets.

1. Preliminaries

The *ternary discriminator function* (the *discriminator*, for short), and the *quaternary switching function* (also called the *normal transform*) on a nonempty set A are functions $d : A^3 \to A$, and $s : A^4 \to A$ defined by

$$d(x,y,z) = \begin{cases} x & \text{if } x \neq y \\ z & \text{if } x = y \end{cases}, \qquad s(x,y,z,v) = \begin{cases} v & \text{if } x \neq y \\ z & \text{if } x = y \end{cases}.$$

The two functions are interrelated by

$$d(x,y,z) = s(x,y,z,x); \qquad s(x,y,z,v) = d\big(d(x,y,z), d(x,y,v), v\big).$$

The *ternary switching function* on A is derived as follows:

$$s(x,y,z) = s\big(x,y,z,d(x,z,y)\big) = d\big(d(x,y,z), x, d(x,z,y)\big).$$

A *discriminator* [*quasi-primal*] algebra is a nontrivial [finite] algebra $\mathbb{A}$ having a term which induces the discriminator on A (equivalentely, having a term which induces the quaternary switching function on A); any such term is called a discriminator term (a switching term) for $\mathbb{A}$. Any discriminator algebra is simple. A class of nontrivial algebras having a common discriminator term is called a discriminator class.

A *discriminator variety* (D-*variety*, for short) is a variety generated by a discriminator class of algebras. For any variety $\mathscr{V}$, $\mathrm{Si}\,\mathscr{V}$ denotes the class of all its nontrivial subdirectly irreducible algebras. A variety $\mathscr{V}$ is a D-variety iff $\mathrm{Si}\,\mathscr{V}$ is a discriminator class. Some properties of a discriminator variety $\mathscr{V}$ are recalled next.

$\mathscr{V}$ is semisimple, i.e., each $\mathbb{A} \in \mathrm{Si}\,\mathscr{V}$ is simple; each finite algebra of $\mathscr{V}$ is a direct product of finite simple algebras; $\mathscr{V}$ is arithmetical (i.e., congruence-distributive (CD), and congruence-permutable (CP)); $\mathscr{V}$ is congruence-regular i.e., any congruence of any algebra is determined by any of its equivalence classes: $\theta = \Theta([a]\theta) = \{(x, y) : s(x, y, a) \in [a]\theta\}$; each algebra $\mathbb{A} \in \mathscr{V}$ has equationally definable principal congruences (EDPC): $\Theta(a, b) = \{(x, y) : d(a, b, x) = d(a, b, y)\}$; the congruence lattice of each algebra $\mathbb{A} \in \mathscr{V}$, $\mathrm{Con}\,\mathbb{A}$, has as a sublattice the set $\mathrm{Con}_p\,\mathbb{A}$ of principal congruences of $\mathbb{A}$, and $\mathrm{Con}_p\,\mathbb{A}$ is a generalized Boolean lattice; $\mathrm{Con}_p\,\mathbb{A}$ is a Boolean lattice iff the largest congruence ∇ is principal. From [14], Lemma 5.3, using the congruence regularity (see also [24], and [3]):

$$\Theta(a, b) \vee \Theta(a, c) = \Theta\big(a, s(a, b, c, b)\big),$$
$$\Theta(a, b) \wedge \Theta(a, c) = \Theta\big(a, s(a, b, b, c)\big),$$
$$\Theta(a, b) \dot{-} \Theta(a, c) = \Theta\big(a, d(a, c, b)\big),$$

where $\dot{-}$ denotes the dual relative pseudocomplement of $\Theta(a, c)$ in $\Theta(a, b)$ (the complement of $\Theta(a, c)$ in $[\Delta, \Theta(a, b) \vee \Theta(a, c)]$).

A variety $\mathscr{W}$ is said to be *interpretable* (or *representable*) into a variety $\mathscr{V}$ if there exists a system of $\mathscr{V}$-terms $\mathbf{t} = \langle t_j : j \in J\rangle$ such that for each algebra $\mathbb{A} \in \mathscr{V}$ the algebra $\mathbb{A}_{\mathbf{t}} = \langle A; t_j, j \in J\rangle \in \mathscr{W}$. We also say that $\mathscr{V}$ interprets $\mathscr{W}$ via the system $\mathbf{t}$. This system is called an *interpretation* (or a *representation*) of $\mathscr{W}$ into $\mathscr{V}$; $\mathbb{A}_{\mathbf{t}}$ is said to be a $\mathbf{t}$-reduct of $\mathbb{A}$, and any subalgebra of $\mathbb{A}_{\mathbf{t}}$ is said to be a $\mathbf{t}$-subreduct of $\mathbb{A}$. If $\mathbf{t}$ is formed by basic operations of $\mathscr{V}$, this variety is said to be an expansion of $\mathscr{W}$.

Two algebras $\mathbb{A}$ and $\mathbb{B}$ (not necessarily of the same type) are said to be *equivalent* (or *term equivalent*) if they have the same universe and the same n-ary term operations for every n, i.e., the same clone of operations. This means that each basic operation of $\mathbb{A}$ is a term operation of $\mathbb{B}$ and vice-versa.

Two varieties $\mathscr{V}$ and $\mathscr{W}$ are (*termwise*) *equivalent* if there is an interpretation $\mathbf{t}$ of $\mathscr{W}$ into $\mathscr{V}$ and an interpretation $\mathbf{q}$ of $\mathscr{V}$ into $\mathscr{W}$ such that $\mathbb{A}_{\mathbf{tq}} = \mathbb{A}$ for all

$\mathbb{A} \in \mathscr{V}$, and $\mathbb{B}_{\mathbf{qt}} = \mathbb{B}$ for all $\mathbb{B} \in \mathscr{W}$. It is worth remembering that: *Two varieties $\mathscr{V}$ and $\mathscr{W}$ are equivalent iff $\mathscr{V} = \mathscr{V}(\mathbb{A})$ and $\mathscr{W} = \mathscr{V}(\mathbb{B})$, for some equivalent algebras $\mathbb{A}$ and $\mathbb{B}$* ([19], 4.140).

The notion of an existential (and a universal) quantifier for a Boolean algebra was considered by P. Halmos in [13]; it was extended to bounded special lattices by many authors. For a bounded lattice an *existential quantifier* D is an additive, semi-multiplicative, and normal closure operator. Equivalently, it can be defined by the following conditions [8]:

Q0. $D(0) = 0$; Q1. $x \wedge D(x) = x$;

Q2. $D(x \wedge D(y)) = D(x) \wedge D(y)$; Q3. $D(x \vee y) = D(x) \vee D(y)$.

Dually, a *universal quantifier* E is a semi-additive, multiplicative, normal interior operator. Equivalently, it can be defined by

$\tilde{\mathrm{Q}}$0. $E(1) = 1$; $\tilde{\mathrm{Q}}$1. $x \vee E(x) = x$;

$\tilde{\mathrm{Q}}$2. $E(x \wedge y) = E(x) \wedge E(y)$; $\tilde{\mathrm{Q}}$3. $E(x \vee E(y)) = E(x) \vee E(y)$.

An existential [universal] quantifier is *simple* if $D(x) = 1$, for all $x \neq 0$ [$E(x) = 0$, for all $x \neq 1$].

An algebra having two distinct constant term operations, denoted $0, 1$, is usually called a $(0,1)$-*pointed* algebra.

Let $\mathbb{A}$ be any $(0,1)$-pointed discriminator algebra. Then $(A; d, 0, 1)$ is a reduct of $\mathbb{A}$, and $(\{0,1\}; d, 0, 1)$ is a subreduct. We note that the algebra $(\{0,1\}; d, 0, 1)$ is equivalent to the Boolean algebra $\mathbb{B}_2 = (\{0,1\}; \wedge, \vee, -, 0, 1)$. The binary terms

$$s(x,y) = s(x,y,1,0) = d\big(d(x,y,1), d(x,y,0), 0\big) = d\big(0, d(x,y,0), d(x,y,1)\big), \text{ and}$$
$$\tilde{s}(x,y) = s(x,y,0,1) = d\big(d(x,y,0), d(x,y,1), 1\big) = d\big(1, d(x,y,1), d(x,y,0)\big)$$

induce on A the following functions:

$$s^{\mathbb{A}}(x,y) = \begin{cases} 0 & \text{if } x \neq y \\ 1 & \text{if } x = y \end{cases}, \qquad \tilde{s}^{\mathbb{A}}(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}.$$

These functions [terms] will be called the $(0,1)$-*switching function* [*term*], and the *dual* $(0,1)$-*switching function* [*term*], respectively; they are related to each other by:

$$\tilde{s}(x,y) = s\big(0, s(x,y)\big), \qquad \text{and} \qquad s(x,y) = \tilde{s}\big(1, \tilde{s}(x,y)\big).$$

Thus, $(A; s, 0, 1)$ and $(A; \tilde{s}, 0, 1)$ are equivalent reducts of $\mathbb{A}$. The operations s and $\tilde{s}$ characterize equality in any algebra from a $(0,1)$-pointed discriminator variety.

We now present some facts concerning known discriminator varieties for comparison with facts, given for the discriminator varieties $\mathscr{V}$ under study, in the forthcoming sections, where the role played by the binary switching functions is emphasized.

The variety $\mathscr{B}$ of Boolean algebras, the most important $(0,1)$-pointed discriminator variety, is generated by the (quasi-)primal algebra $\mathbb{B}_2$. Besides the basic operations $\wedge$, and $\vee$, the term operations $x \to y := -x \vee y$, and its dual $x \leftarrow y := -x \wedge y$ allow one to equationally characterize the partial order on each algebra $\mathbb{B} \in \mathscr{B}$:

$$x \leq y \quad \text{iff} \quad x \wedge y = x \quad \text{iff} \quad x \vee y = y \quad \text{iff} \quad x \to y = 1 \quad \text{iff} \quad x \leftarrow y = 0.$$

Hence equality can be equationally characterized by:

$$x = y \quad \text{iff} \quad x \wedge y = x \vee y \quad \text{iff} \quad (x \to y) \wedge (y \to x) = 1 \quad \text{iff} \\ (x \leftarrow y) \vee (y \leftarrow x) = 0.$$

In $\{0,1\}$, the switching functions defined above are induced by the following terms:

$$s(x,y) = (x \to y) \wedge (y \to x) = (-x \vee y) \wedge (x \vee -y) = -(x \vee y) \vee (x \wedge y),$$
$$\tilde{s}(x,y) = (x \leftarrow y) \vee (y \leftarrow x) = (-x \wedge y) \vee (x \wedge -y) = -(x \wedge y) \wedge (x \vee y).$$

Note that $\tilde{s}(x,y)$ is the symmetric difference, usually denoted $x \oplus y$, and its dual $s(x,y) = -\tilde{s}(x,y)$ is sometimes called equivalence and denoted by $\leftrightarrow$.

Among the discriminator terms for $\mathbb{B}_2$ we choose:

$$\begin{aligned} d(x,y,z) &= [(-x \vee y) \wedge (x \vee -y) \wedge z] \vee (x \wedge -y) \\ &= [(x \to y) \wedge (y \to x) \wedge z] \vee \{-[(x \to y) \wedge (y \to x)] \wedge x\} \\ &= [s(x,y) \wedge z] \vee [-s(x,y) \wedge x] = [\tilde{s}(x,y) \vee z] \wedge [-\tilde{s}(x,y) \vee x]. \end{aligned}$$

For $\mathbb{B} \in \mathscr{B}$, $\mathrm{Con}_p\, \mathbb{B} \cong \mathbb{B}$. For $a, b \in B$, we choose some ways of giving the principal congruence generated by (a,b) by means of binary operations:

$$\begin{aligned} \Theta(a,b) &= \Theta\big((-a \vee b) \wedge (a \vee -b), 1\big) = \Theta\big((a \to b) \wedge (b \to a), 1\big) = \Theta\big(s(a,b), 1\big) \\ &= \{(x,y) : s(a,b) \wedge x = s(a,b) \wedge y\} = \Theta_{\langle \wedge \rangle}\big(s(a,b), 1\big) \\ &= \Theta\big(0, (-a \wedge b) \vee (a \wedge -b)\big) = \Theta\big(0, (a \leftarrow b) \vee (b \leftarrow a)\big) = \Theta\big(0, -s(a,b)\big) \\ &= \{(x,y) : -s(a,b) \vee x = -s(a,b) \vee y\} = \Theta_{\langle \vee \rangle}\big(0, -s(a,b)\big), \end{aligned}$$

where $\Theta_{\langle\wedge\rangle}(C)$ $[\Theta_{\langle\vee\rangle}(C)]$ denotes the congruence of $\mathbb{B}_{\langle\wedge\rangle}$ $[\mathbb{B}_{\langle\vee\rangle}]$, the $\langle\wedge\rangle$-reduct $[\langle\vee\rangle$-reduct$]$ of $\mathbb{B}$, generated by C.

For $\theta \in \operatorname{Con}\mathbb{B}$, the 1-coset, $[1]\theta$, is a filter, the 0-coset, $[0]\theta$, is an ideal, and

$$\begin{aligned}\theta &= \Theta([1]\theta) = \{(x,y) : s(x,y) \in [1]\theta\} = \{(x,y) : x \to y \in [1]\theta, y \to x \in [1]\theta\}\\ &= \Theta_{\langle\wedge\rangle}([1]\theta) = \{(x,y) : \exists f \in [1]\theta, x \wedge f = y \wedge f\}\\ &= \Theta([0]\theta) = \{(x,y) : -s(x,y) \in [0]\theta\} = \{(x,y) : x \leftarrow y \in [0]\theta, y \leftarrow x \in [0]\theta\}\\ &= \Theta_{\langle\vee\rangle}([0]\theta) = \{(x,y) : \exists i \in [0]\theta, x \vee i = y \vee i\}.\end{aligned}$$

The variety $\mathscr{LM}_3$ of three-valued Łukasiewicz–Moisil algebras is another important $(0,1)$-pointed discriminator variety generated by the quasi-primal algebra $\mathbb{L}_3 = (\{0,a,1\}; \wedge, \vee, \sim, D, 0, 1)$, where $(\{0,a,1\}; \wedge, \vee, \sim, 0, 1)$ is a simple Kleene algebra, and D is the unary operation: $D(0) = 0$, $D(a) = D(1) = 1$. Among the discriminator terms for $\mathbb{L}_3$ we point out (see [24] for another):

$$d(x,y,z) = [\sim s(x,y) \vee z] \wedge [s(x,y) \vee x] = [s(x,y) \wedge z] \vee [\sim s(x,y) \wedge x]$$

where $s(x,y) := [\sim D(x \vee y) \vee D(x \wedge y)] \wedge [D \sim (x \vee y) \vee \sim D \sim (x \wedge y)]$ is a $(0,1)$-switching term for $\mathbb{L}_3$.

The operation D determines an endomorphism on the bounded lattice reduct of each algebra from $\mathscr{LM}_3$. The variety $\mathscr{B}$ may be considered as a subvariety of $\mathscr{LM}_3$, by adding the trivial unary operation D. Then $\mathscr{B}$ is the subvariety of $\mathscr{LM}_3$ characterized by the identity $Dx = x$.

The variety $\mathscr{LM}_3$ interprets the variety $\mathscr{HA}$ of Heyting algebras (see [20], Ch. VII, §3, and [2], Ch. 4, §3). More precisely, the variety $\mathscr{LM}_3$ is equivalent to the variety generated by a 3-element Heyting algebra $\mathbb{H}_3 = (\{0,a,1\}; \wedge, \vee, \Rightarrow, \sim, 0, 1)$ with a dual automorphism $\sim$ of period 2 ([20], Ch. VII, Th. 3.5). The Heyting implication $\Rightarrow$ is given by the $\mathscr{LM}_3$-term $x \Rightarrow y = \sim Dx \vee y \vee (D \sim x \wedge Dy)$, and D is given by $Dx = (\sim x \Rightarrow x)$.

$\mathscr{LM}_3$-algebras (and $\mathscr{LM}_n$-algebras, $n > 3$), were introduced and studied by Gr. C. Moisil in 1940, 1941, as an algebraic counterpart of three-valued propositional logic.

The variety $\mathscr{TM}$ of tetravalent modal algebras is generated by the algebra $\mathbb{T} = (\{0,a,b,1\}; \wedge, \vee, \sim, D, 0, 1)$, where $(\{0,a,b,1\}; \wedge, \vee, \sim, 0, 1)$ is a simple De Morgan algebra, and D is the unary operation: $D(0) = 0$, $D(a) = D(b) = D(1) = 1$. Discriminator terms given above for $\mathbb{L}_3$ are also discriminator terms for $\mathbb{T}$, as also is the $(0,1)$-switching term.

The lattice of subvarieties of $\mathscr{TM}$ is the chain: $\mathscr{B} \subset \mathscr{LM}_3 \subset \mathscr{TM}$. The variety $\mathscr{LM}_3$ is the subvariety of $\mathscr{TM}$ characterized by D being an endomorphism, i.e. characterized by the equation: $D(x \wedge y) = D(x) \wedge D(y)$. Unlike what happens for $\mathscr{LM}_3$, the variety $\mathscr{HA}$ is not interpretable in $\mathscr{TM}$, and the operation D determines an existential quantifier on the bounded lattice reduct of any algebra from $\mathcal{TM}$.

The variety $\mathcal{TM}$ was defined by A. A. Monteiro in 1978, and studied by I. Loureiro in [16].

The variety $\mathcal{MB}$ of monadic Boolean algebras is another important example of a discriminator variety, introduced and studied by P. Halmos in [13]. Monadic Boolean algebras are Boolean algebras with an existential quantifier D. The simple algebras are those having a simple quantifier. The proper subvarieties of $\mathcal{MB}$ form an ω-chain of discriminator varieties $\mathcal{MB}_n$, $n \in \mathbb{N}$. Each $\mathcal{MB}_n$ is generated by a simple algebra having n atoms; $\mathcal{MB}_1$, the least nontrivial subvariety of $\mathcal{MB}$, is equivalent to the variety $\mathcal{B}$, since its simple algebra is a two-element Boolean algebra with the trivial quantifier D. Discriminator terms for the simple algebras of $\mathcal{MB}$ are found in [3], and [24]:

$$d(x, y, z) = [-s(x, y) \vee z] \wedge [s(x, y) \vee x] = [s(x, y) \wedge z] \vee [-s(x, y) \wedge x]$$

where $s(x, y) = -D[(x \vee y) \wedge -(x \wedge y)] = -D - [-(x \vee y) \vee (x \wedge y)]$ are $(0, 1)$-switching terms.

All these examples are discriminator varieties interpreting the variety $\mathcal{D}_{01}$ of bounded distributive lattices. All but one, $\mathcal{TM}$, interpret the variety $\mathcal{HA}$ of Heyting algebras.

General references for notions and facts on universal algebra, lattices and varieties are [19], [4], [12], and [1]. For discriminator varieties we refer the reader to [24], [4], [14] and [3] and the references given therein. For interpretations and equivalence of algebras and varieties the reader may consult [11] and [19]. For the main results about $\mathcal{LM}_3$, the reader is referred to [1], and to [20], where A. A. Monteiro gives an excellent account of his own works and those of other members of his School in Bahia Blanca. Also the book [2], published in 1991, is an excellent survey of all the results about these algebras and their generalizations, obtained until 1988, without tools from discriminator varieties.

2. Characterizations of discriminator varieties interpreting $\mathcal{L}_{01}$

As was observed in the previous section, any $(0, 1)$-pointed discriminator algebra $\mathbb{A}$ has necessarily a $(0, 1)$-switching term and a dual $(0, 1)$-switching term. The converse is not true. For instance, the algebra $(\{0, 1\}; \oplus, 0, 1)$ is not a discriminator algebra. It was also observed that the existence of a $(0, 1)$-switching term for any algebra is equivalent to the existence of a dual $(0, 1)$-switching term.

In this section we are concerned with varieties $\mathcal{V}$, for which there is an interpretation of the variety $\mathcal{L}_{01}$ into $\mathcal{V}$, which will be denoted by $(\wedge, \vee, 0, 1)$.

As is suggested by the examples recalled in the previous section, it is easily checked that for any algebra having a $(\vee, \wedge, 0, 1)$-reduct in $\mathscr{L}_{01}$, the existence of a $(0,1)$-switching term $s(x, y)$ implies the existence of a discriminator term:

$$\big(s(x, y) \wedge z\big) \vee \big(s\big(0, s(x, y)\big) \wedge x\big).$$

Hence, discriminator varieties interpreting $\mathscr{L}_{01}$ may be characterized by the existence of a binary [dual] $(0,1)$-switching term for the class of their subdirectly irreducibles, as stated in our first theorem.

Theorem 2.1. *For a variety $\mathscr{V}$ interpreting $\mathscr{L}_{01}$, the following are equivalent:*

(1) *There exists a discriminator term for* $\mathrm{Si}\,\mathscr{V}$.

(2) *There exists a* $(0,1)$*-switching term for* $\mathrm{Si}\,\mathscr{V}$.
[*There exists a dual* $(0,1)$*-switching term for* $\mathrm{Si}\,\mathscr{V}$.]

Theorem 2.2. *Let $\mathscr{V}$ be a variety and $(\wedge, \vee, 0, 1)$ be a family of terms giving an interpretation of $\mathscr{L}_{01}$ into $\mathscr{V}$. For a binary term $s(x, y)$, the following are equivalent:*

(1) $s(x, y)$ *is a* $(0,1)$*-switching term for* $\mathrm{Si}\,\mathscr{V}$.

(2) $\big(s(x, y) \wedge z\big) \vee \big(s\big(0, s(x, y)\big) \wedge x\big)$ *is a discriminator term for* $\mathrm{Si}\,\mathscr{V}$.
$\big[\big(s\big(0, s(x, y)\big) \vee z\big) \wedge \big(s(x, y) \vee x\big)$ *is a discriminator term for* $\mathrm{Si}\,\mathscr{V}$.]

Proof. (2) $\Rightarrow$ (1) Let $\mathbb{A} \in \mathrm{Si}\,\mathscr{V}$, and $d(x, y, z) := \big(s(x, y) \wedge z\big) \vee \big(s\big(0, s(x, y)\big) \wedge x\big)$ be a discriminator term. We first show that $s(a, a) = 1$, for all $a \in A$. We obtain from (2): $0 = d(a, a, 0) = s\big(0, s(a, a)\big) \wedge a$, for every $a \in A$. Hence, $z = d(a, a, z) = s(a, a) \wedge z$, for every $z \in A$, so $1 = d(a, a, 1) = s(a, a)$. For $a, b \in A$, with $a \neq b$, we obtain from (2): $a = \big(s(a, b) \wedge c\big) \vee \big(s\big(0, s(a, b)\big) \wedge a\big)$. For $c = 0$, we obtain

$$a = s\big(0, s(a, b)\big) \wedge a. \tag{$*$}$$

Hence $a = \big(s(a, b) \wedge c\big) \vee a$, for all $c \in A$. But, if $c = 1$, we obtain $a = s(a, b) \vee a$. Thus, taking $a = 0$,

$$b \neq 0 \quad \Rightarrow \quad s(0, b) = 0. \tag{$**$}$$

If $a \neq 0$, and $b \neq a$, we have $s(a, b) = 0$, since otherwise, by $(**)$, we would have $s\big(0, s(a, b)\big) = 0$ and, by $(*)$, $a = 0$, a contradiction. □

Discriminator varieties were characterized in [18] (see also [22]) by equational conditions on a ternary term. In the following theorem discriminator varieties interpreting the variety $\mathscr{L}_{01}$ are characterized by equational conditions on a binary term.

Theorem 2.3. *For a variety* $\mathscr{V}$ *with a family of terms* $(\wedge, \vee, 0, 1)$ *giving an interpretation of* $\mathscr{L}_{01}$ *into* $\mathscr{V}$, *and a binary term* $s(x, y)$, *the following are equivalent:*

(1) $s(x, y)$ *is a* $(0, 1)$*-switching term for* $\mathrm{Si}\,\mathscr{V}$.

(2) $\mathscr{V}$ *satisfies the following identities:*
S1. $s(x, x) = 1$,
S2. $s(x, y) \wedge x = s(x, y) \wedge y$,
S3. $s(x, y) \vee s(0, s(x, y)) = 1$,
and, for each n-ary operation symbol f *of* $\mathscr{V}$,
S$_f$. $s(x, y) \wedge f(z_1, \ldots, z_n) = s(x, y) \wedge f(s(x, y) \wedge z_1, \ldots, s(x, y) \wedge z_n)$
$(s(x, y) \wedge f(z_1, \ldots, z_n) = f(s(x, y) \wedge z_1, \ldots, s(x, y) \wedge z_n)$, *whenever* $f(0, \ldots, 0) = 0)$.

(3) $\mathscr{V}$ *satisfies the identity*
S1. $s(x, x) = 1$,
and, for each $\mathbb{A} \in \mathscr{V}$, *and any* $a, b \in A$, *with* $a \neq b$, $(a, b) \notin \Theta(0, s(a, b))$.

Proof. (1) $\Rightarrow$ (2) It is easily checked that these identities hold in $\mathrm{Si}\,\mathscr{V}$, hence they hold in $\mathscr{V}$.

(2) $\Rightarrow$ (3) We first observe that properties of bounded semilattices together with S1, S2 ensure that for $\mathbb{A} \in \mathscr{V}$, and $\theta \in \mathrm{Con}\,\mathbb{A}$,

$$(a, b) \in \theta \quad \Leftrightarrow \quad (s(a, b), 1) \in \theta \quad \Leftrightarrow \quad (s(a, b) \wedge x, x) \in \theta, \qquad \text{for all } x \in A. \quad (*)$$

In fact,

$$\begin{aligned}
(a, b) \in \theta \quad &\Rightarrow \quad (s(a, b), 1) = (s(a, b), s(b, b)) \in \theta && \text{by S1} \\
&\Rightarrow \quad (s(a, b) \wedge x, x) \in \theta, \quad \text{for all } x \in A && \\
&\Rightarrow \quad (s(a, b) \wedge a, a) \in \theta,\ (s(a, b) \wedge b, b) \in \theta && \\
&\Rightarrow \quad (a, b) \in \theta. && \text{by S2 and transitivity}
\end{aligned}$$

Let $\mathbb{A} \in \mathscr{V}$, and $a, b \in A$. The equivalence relation

$$\tau = \{(x, y) \in A \times A : s(a, b) \wedge x = s(a, b) \wedge y\}$$

is a congruence on $\mathbb{A}$, since it is compatible with each basic operation f of $\mathscr{V}$, by the identities S$_f$. By S2, $(a, b) \in \tau$. We shall now prove that τ is the least congruence containing (a, b).

Let $\theta \in \mathrm{Con}\,\mathbb{A}$ be such that $(a, b) \in \theta$. We want to show that $\tau \subseteq \theta$. Let $(x, y) \in \tau$, i.e., $s(a, b) \wedge x = s(a, b) \wedge y$. As $(s(a, b) \wedge x, x) \in \theta$, and $(s(a, b) \wedge y, y) \in \theta$ by $(*)$, we conclude that $(x, y) \in \theta$ by the transitivity of θ. Thus, for any $a, b \in A$, τ is the congruence of $\mathbb{A}$ generated by (a, b):

$$\Theta(a, b) = \{(x, y) \in A \times A : s(a, b) \wedge x = s(a, b) \wedge y\}. \qquad (**)$$

Now, let $a, b \in A$, with $a \neq b$. We have that $(a, b) \notin \Theta(0, s(a, b))$, since otherwise we would obtain a contradiction:

$$
\begin{aligned}
(a,b) \in \Theta(0, s(a,b)) &\Leftrightarrow (s(a,b), 1) \in \Theta(0, s(a,b)) && \text{by } (*) \\
&\Leftrightarrow s(0, s(a,b)) \wedge s(a,b) = s(0, s(a,b)) && \text{by } (**) \\
&\Rightarrow 0 = s(0, s(a,b)) && \text{by S2} \\
&\Rightarrow s(a,b) = 1 && \text{by S3} \\
&\Rightarrow a = b. && \text{by S2}
\end{aligned}
$$

(3) $\Rightarrow$ (1) By S1, $s(x,x) = 1$, for all $x \in A$. It remains to show that for any $\mathbb{A} \in \mathrm{Si}\,\mathscr{V}$, $s(x,y) = 0$ for all $x, y \in A$, with $x \neq y$. Let $\mathbb{A} \in \mathrm{Si}\,\mathscr{V}$, with monolith $\Theta(a,b)$, $a \neq b$. The hypothesis $(a,b) \notin \Theta(0, s(a,b))$ implies that $\Theta(0, s(a,b)) = \Delta$. Hence $s(a,b) = 0$. By S1, $(s(a,b), 1) \in \Theta(a,b)$. So $(0,1) \in \Theta(a,b)$, i.e., $\Theta(a,b) = \nabla$, and $\mathbb{A}$ is a simple algebra. Let $x, y \in A$, with $x \neq y$. By the hypothesis, $\Theta(0, s(x,y)) \neq \nabla$. Then, by the simplicity of $\mathbb{A}$, $\Theta(0, s(x,y)) = \Delta$, and $s(x,y) = 0$. □

Another equational characterization, dual to the one in Theorem 2.3 is given next. It can be stated in terms of $\tilde{s}(x,y)$.

Theorem 2.3′. *For a variety $\mathscr{V}$, such that $(\wedge, \vee, 0, 1)$ is an interpretation of $\mathscr{L}_{01}$ into $\mathscr{V}$, and a binary term $s(x,y)$ $[\tilde{s}(x,y)]$, the following are equivalent:*

(1) *$s(x,y)$ $[\tilde{s}(x,y)]$ is a [dual] $(0,1)$-switching term for* $\mathrm{Si}\,\mathscr{V}$.

(2) *$\mathscr{V}$ satisfies the following identities:*
S1′. $s(0, s(x,x)) = 0$, $[\tilde{s}(x,x) = 0]$,
S2′. $s(0, s(x,y)) \vee x = s(0, s(x,y)) \vee y$, $[\tilde{s}(x,y) \vee x = \tilde{s}(x,y) \vee y]$,
S3′. $s(x,y) \wedge s(0, s(x,y)) = 0$, $[\tilde{s}(1, \tilde{s}(x,y)) \wedge \tilde{s}(x,y) = 0]$,
and, for each n-ary operation symbol f of $\mathscr{V}$,
S'_f. $s(0, s(x,y)) \vee f(z_1, \ldots, z_n)$
$= s(0, s(x,y)) \vee f(s(0, s(x,y)) \vee z_1, \ldots, s(0, s(x,y)) \vee z_n)$
$[\tilde{s}(x,y) \vee f(z_1, \ldots, z_n) = \tilde{s}(x,y) \vee f(\tilde{s}(x,y)) \vee z_1, \ldots, \tilde{s}(x,y) \vee z_n]$
$(s(0, s(x,y)) \vee f(z_1, \ldots, z_n) = f(s(0, s(x,y)) \vee z_1, \ldots, s(0, s(x,y)) \vee z_n)$
$[\tilde{s}(x,y) \vee f(z_1, \ldots, z_n) = f(\tilde{s}(x,y) \vee z_1, \ldots, \tilde{s}(x,y) \vee z_n)]$
whenever $f(1, \ldots, 1) = 1)$.

Remarks. (1) The proof of Theorem 2.3 provides us with the following facts: principal congruences of algebras from $\mathscr{V}$ can be equationally described as principal congruences of semilattices; the subdirectly irreducible algebras are simple (known fact), and are those for which $s(x,y)$ is a $(0,1)$-switching function. The proof of Theorem 2.3′, with arguments dual to those in the proof of Theorem 2.3, would provide us with the facts:

$$(a,b) \in \theta \quad \Leftrightarrow \quad \bigl(0, s\bigl(0, s(a,b)\bigr)\bigr) \in \theta \quad \Leftrightarrow \quad \bigl(s\bigl(0, s(a,b)\bigr) \vee x, x\bigr) \in \theta, \quad \text{for all } x \in A$$

$$\Theta(a,b) = \bigl\{(x,y) \in A \times A : s\bigl(0, s(a,b)\bigr) \vee x = s\bigl(0, s(a,b)\bigr) \vee y\bigr\}.$$

(2) The axiom S2 [S2′], and properties of bounded semilattices ensure that:
  (a) $s(0,x) \wedge x = 0$ [(a′) $s\bigl(0, s(x,1)\bigr) \vee x = 1$]. This implies that: $s(0,1) = 0$.
  (b) $s(x,y) \wedge x = s(x,y) \wedge x \wedge y$ [(b′) $s\bigl(0, s(x,y)\bigr) \vee x = s\bigl(0, s(x,y)\bigr) \vee x \vee y$]
  (c) $s(x,y) = 1 \Rightarrow x = y$ [(c′) $s\bigl(0, s(x,y)\bigr) = 0 \Rightarrow x = y$].

(3) The commutativity of $s(x,y)$ follows from the commutativity of $\wedge$ together with S1, S2, and S_s:

$$s(x,y) \wedge s(y,x) = s(x,y) \wedge s\bigl(s(x,y) \wedge y, s(x,y) \wedge x\bigr) = s(x,y), \quad \text{and}$$
$$s(y,x) \wedge s(x,y) = s(y,x) \wedge s\bigl(s(y,x) \wedge x, s(y,x) \wedge y\bigr) = s(y,x).$$

(4) $x \leq y$ iff $s(x \wedge y, x) = 1$ iff $s(x, x \vee y) = 1$; $x = y$ iff $s(x \wedge y, x \vee y) = 1$ iff $s(x \wedge y, x) \wedge s(x \wedge y, y) = 1$ iff $s(x, x \vee y) \wedge s(y, x \vee y) = 1$.

A variety which is interpretable in each discriminator variety $\mathscr{V}$ in which the variety $\mathscr{L}_{01}$ is interpretable is next defined.

Definition 2.4. Let $\mathscr{S}w\mathscr{L}_{01}$ denote the class of all algebras $(A; \wedge, \vee, s, 0, 1)$, with $(A; \wedge, \vee, 0, 1) \in \mathscr{L}_{01}$, satisfying the following identities:

S1. $s(x,x) = 1$,

S2. $s(x,y) \wedge x = s(x,y) \wedge y$,

S3. $s(x,y) \vee s\bigl(0, s(x,y)\bigr) = 1$,

$S_\vee$. $s(x,y) \wedge (z_1 \vee z_2) = \bigl(s(x,y) \wedge z_1\bigr) \vee \bigl(s(x,y) \wedge z_2\bigr)$,

S_s. $s(x,y) \wedge s(u,v) = s(x,y) \wedge s\bigl(s(x,y) \wedge u, s(x,y) \wedge v\bigr)$.

By Theorem 2.3, $\mathscr{S}w\mathscr{L}_{01}$ is a discriminator variety. Theorem 2.3′ provides another equational basis for $\mathscr{S}w\mathscr{L}_{01}$:

S1′. $s\bigl(0, s(x,x)\bigr) = 0$,

S2′. $s\bigl(0, s(x,y)\bigr) \vee x = s\bigl(0, s(x,y)\bigr) \vee y$,

S3′. $s(x,y) \wedge s\bigl(0, s(x,y)\bigr) = 0$,

$S_\wedge$. $s\bigl(0, s(x,y)\bigr) \vee (z_1 \wedge z_2) = \bigl(s(x,y) \vee z_1\bigr) \vee \bigl(s(x,y) \vee z_2\bigr)$,

S_s. $s\bigl(0, s(x,y)\bigr) \vee s(u,v) = s\bigl(0, s(x,y)\bigr) \vee s\bigl(s\bigl(0, s(x,y)\bigr) \vee u, s\bigl(0, s(x,y)\bigr) \vee v\bigr)$.

Theorem 2.3 yields now the following:

Theorem 2.5. *A variety $\mathscr{V}$ interpreting $\mathscr{L}_{01}$ is a discriminator variety iff it interprets the variety $\mathscr{S}w\mathscr{L}_{01}$, and satisfies the identities* S_f *(or* S'_f*), for each basic operation symbol f.*

3. The variety $\mathscr{S}w\mathscr{L}_{01}$

In this section, every theorem established for the variety $\mathscr{S}w\mathscr{L}_{01}$ also holds for any discriminator variety $\mathscr{V}$ interpreting $\mathscr{L}_{01}$ (equivalently, interpreting $\mathscr{S}w\mathscr{L}_{01}$). Other identities holding in $\mathscr{S}w\mathscr{L}_{01}$ and therefore also holding in $\mathscr{V}$ are given in the following proposition. It is routine to check that they hold in each $\mathbb{A} \in \mathrm{Si}\,\mathscr{V}$.

Proposition 3.1. *The following identities hold in the variety $\mathscr{S}w\mathscr{L}_{01}$:*

S4. $s(x,1) \vee x = x$;

S4′. $s(0, s(0,x)) \wedge x = x$;

S5. $s(s(x,y),1) = s(x,y)$;

S5′. $s(0, s(0, s(x,y))) = s(x,y)$;

S6. $s(x \wedge y, 1) = s(x,1) \wedge s(y,1)$;

S6′. $s(0, s(0, (x \vee y))) = s(0, s(0,x)) \vee s(0, s(0,y))$;

S7. $s(0, x \vee y) = s(0,x) \wedge s(0,y)$;

S8. $s(s(x,1) \vee y, 1) = s(x,1) \vee s(y,1)$;

S8′. $s(0, s(v,x) \wedge y) = s(0, s(v,x)) \vee s(0,y)$.

Proposition 3.2. *For any $\mathbb{A} \in \mathscr{S}w\mathscr{L}_{01}$, and $v, x, y, z \in A$, the set $\{s(v,x), y, z\}$ generates a distributive sublattice.*

Proof. Let $\mathbb{A} \in \mathrm{Si}\,\mathscr{S}w\mathscr{L}_{01}$. As on $\mathbb{A}$ $s(v,x) \in \{0,1\}$, it is obvious that $\{s(v,x), y, z\}$ generates a distributive sublattice. Hence the same holds for any algebra from $\mathscr{S}w\mathscr{L}_{01}$. Alternatively, it is readily seen that $\mathbb{A}$ satisfies the identity:

S9. $(s(v,x) \vee y) \wedge (y \vee z) \wedge (z \vee s(v,x)) = (s(v,x) \wedge y) \vee (y \wedge z) \vee (z \wedge s(v,x))$,

i.e., the elements $s(v,x)$ are neutral. Then the claim follows from [12], pg 140. □

For shorter notation in what follows, we will write:
$s_0(x)$ for $s(0,x)$; $s_1(x)$ for $s(x,1)$; $s_{00}(x)$ for $s_0 s_0(x)$; $s_{01}(x)$ for $s_0 s_1(x)$.

These four unary term operations form a semigroup generated by $\{s_0, s_1\}$ as well as by $\{s_{01}, s_{00}\}$, and $\{s_{01}, s_0\}$.

	s_0	s_1	s_{00}	s_{01}
s_0	s_{00}	s_{01}	s_0	s_1
s_1	s_0	s_1	s_{00}	s_{01}
s_{00}	s_0	s_1	s_{00}	s_{01}
s_{01}	s_{00}	s_{01}	s_0	s_1

Proposition 3.3. *For each $\mathbb{A} \in \mathcal{S}w\mathcal{L}_{01}$,*

$$s_{01}(A) = s_0(A) = s_1(A) = s_{00}(A) = \{s(a,b) : a, b \in A\}.$$

Proof. Let $\mathbb{A} \in \mathcal{S}w\mathcal{L}_{01}$. As can be seen by the above table, $s_{01}(a) = s_0 s_1(a)$, $s_0(a) = s_1(s_0(a))$, $s_1(a) = s_{00}s_1(a)$, for all $a \in A$. Thus,

$$s_{01}(A) \subseteq s_0(A) \subseteq s_1(A) \subseteq s_{00}(A) \subseteq \{s(a,b) : a, b \in A\}.$$

The equalities hold since, by S5′, $s(a,b) = s_{00}(s(a,b)) = s_{01}(s_0 s(a,b)) \in s_{01}(A)$, for all $a, b \in A$. □

Theorem 3.4. *For any $\mathbb{A} \in \mathcal{S}w\mathcal{L}_{01}$, the algebra*

$$B(\mathbb{A}) := (s_0(A); \wedge, \vee, s_0, 0, 1)$$

is the largest Boolean $(\wedge, \vee, s_0, 0, 1)$-subreduct of $\mathbb{A}$.

Proof. Let $\mathbb{A} \in \mathcal{S}w\mathcal{L}_{01}$. Taking into account Proposition 3.3, $B(\mathbb{A})$ is closed under $\wedge$ by S6, it is closed under $\vee$ by S6′, and it is closed under s_0 by S5′. So, $B(\mathbb{A})$ is a subalgebra of $(A; \wedge, \vee, s_0, 0, 1)$. Moreover, by Proposition 3.2, $B(\mathbb{A})$ is a distributive lattice, and the operation s_0 is the complementation, by S5′, S7, and S3.

Let $(B; \vee, \wedge, s_0, 0, 1)$ be a Boolean subalgebra of $(A; \wedge, \vee, s_0, 0, 1)$. Then, for any $b \in B$, $s_0(b) \in B$, and $b = s_0 s_0(b) \in s_0(A)$. So, $B \subseteq s_0(A)$. □

Proposition 3.5. *For any $\mathbb{A} \in \mathcal{S}w\mathcal{L}_{01}$,*

(a) s_0 [s_{01}] *induces a weak* [*dual*] *pseudocomplementation, and* $s_0(x) \leq s_{01}(x)$.

(b) s_1 [s_{00}] *induces a universal* [*existential*] *quantifier. Moreover,* $s_{00} = s_0 s_1 s_0$, *and* $s_1 = s_{01} s_{00} s_{01}$.

Proof. (a) We have $s_0(x) \wedge x = 0$, by Remark 2(a); $x \leq y$ implies $s_0(y) \leq s_0(x)$, by S7; $x \leq s_0(s_0(x))$, by S4′. [Dually for s_{01}.]

(b) We have $s_1(1) = 1$, by S1; $s_1(x) \le x$, by S4; $s_1\big(s_1(x) \vee y\big) = s_1(x) \vee s_1(y)$, by S8; $s_1(x \wedge y) = s_1(x) \wedge s_1(y)$, by S6. [Dually for s_{00}.] □

Notice that in $B(\mathbb{A})$, $s_0 = s_{01}$, $s_1 = s_{00} = \mathrm{id}$.

As a consequence of Theorem 3.4, Proposition 3.5 and Remark 1, in §2, we have:

Corollary 3.6. *Let $\mathbb{A} \in \mathscr{S}w\mathscr{L}_{01}$. Then, $\mathbb{A}$ is simple iff $B(\mathbb{A})$ is simple iff s_{00} is a simple existential quantifier iff s_1 is a simple universal quantifier.*

Theorem 3.7. *For a subvariety $\mathscr{W}$ of $\mathscr{S}w\mathscr{L}_{01}$, the following are equivalent:*

(a) *s_{00} $[s_1]$ is an endomorphism of $\mathbb{A}_{\langle\wedge,\vee,0,1\rangle}$, for each $\mathbb{A} \in \mathscr{W}$.*

(b) *s_0 $[s_{01}]$ is a dual endomorphism of $\mathbb{A}_{\langle\wedge,\vee,0,1\rangle}$, for each $\mathbb{A} \in \mathscr{W}$.*

(c) *In each simple algebra of $\mathscr{W}$, 0 [1] is $\wedge$-irreducible [$\vee$-irreducible].*

Proof. (a) ⇒ (b) Let $\mathbb{A} \in \mathscr{W}$. We have $s_0(0) = 1$, and $s_0(1) = 0$, and by S7, for all $a, b \in A$, $s_0(a \vee b) = s_0(a) \wedge s_0(b)$. Supposing that s_{00} is an endomorphism we obtain:

$$\begin{aligned} s_0(a \wedge b) &= s_0 s_{00}(a \wedge b) = s_0\big(s_{00}(a) \wedge s_{00}(b)\big) = s_0 s_0\big(s_0(a) \vee s_0(b)\big) \\ &= s_{00}\big(s_0(a) \vee s_0(b)\big) = s_{00}\big(s_0(a) \vee s_{00}s_0(b)\big) = s_0(a) \vee s_0(b). \end{aligned}$$

(b) ⇒ (c) Let $\mathbb{A} \in \mathrm{Si}\,\mathscr{W}$, and assume that s_0 is a dual endomorphism of $\mathbb{A}_{\langle\wedge,\vee,0,1\rangle}$. If $0 = a \wedge b$, with $a, b \in A\backslash\{0\}$, then $1 = s_0(a \wedge b) = s_0(a) \vee s_0(b) = 0 \vee 0 = 0$, a contradiction. Thus 0 is $\wedge$-irreducible.

(c) ⇒ (a) Let us suppose that in every simple algebra of $\mathscr{W}$, 0 is $\wedge$-irreducible. As, by Corollary 3.6, s_{00} is a simple existential quantifier, it is easily seen that s_{00} is an endomorphism of $\mathbb{A}_{\langle\wedge,\vee,0,1\rangle}$. □

The subvariety of $\mathscr{S}w\mathscr{L}_{01}$ consisting of all algebras $(A; \wedge, \vee, s, 0, 1)$ in which $(A; \wedge, \vee, 0, 1) \in \mathscr{D}_{01}$, and satisfying S1, S2, S3, and S_s will be denoted by $\mathscr{S}w\mathscr{D}_{01}$.

We will now focus the bottom of the lattice of subvarieties of $\mathscr{S}w\mathscr{L}_{01}$. The least subvariety of $\mathscr{S}w\mathscr{L}_{01}$ is the variety $\mathscr{V}(\mathbb{S}_2)$, generated by the simple algebra $\mathbb{S}_2 = (\{0,1\}; \wedge, \vee, s, 0, 1)$ equivalent to $\mathbb{B}_2$. As $\mathbb{B}_2$ is a primal algebra, the equivalence of $\mathbb{S}_2$ and $\mathbb{B}_2$ follows from the observation that the negation in $\mathbb{B}_2$, $-$, is the term operation $s(0, x)$ of $\mathbb{S}_2$. Obviously, $\mathscr{V}(\mathbb{S}_2)$ is the subvariety of $\mathscr{S}w\mathscr{D}_{01}$ characterized by the equation $s(x, 1) = x$ (or by the equation $s\big(0, s(0, x)\big) = x$).

Theorem 3.8. *The variety $\mathscr{T}\mathscr{M} = \mathscr{V}(\mathbb{T})$ is equivalent to the variety $\mathscr{V}(\mathbb{A}_4)$, where $\mathbb{A}_4 = (\{0, a, b, 1\}; \wedge, \vee, s, 0, 1) \in \mathscr{S}w\mathscr{L}_{01}$, with a and b atoms.*

Proof. It suffices to show that $\mathbb{A}_4 = (\{0, a, b, 1\}; \wedge, \vee, s, 0, 1)$ is equivalent to $\mathbb{T} = (\{0, a, b, 1\}; \wedge, \vee, \sim, D, 0, 1)$. They have the same bounded lattice reduct and, as pointed out in §1, the operation s of $\mathbb{A}_4$ is a term operation of $\mathbb{T}$:

$$s(x, y) = \big(\sim D(x \vee y) \vee D(x \wedge y)\big) \wedge \big(D \sim (x \vee y) \vee \sim D \sim (x \wedge y)\big).$$

On the other side, the De Morgan negation $\sim$, and the operator D of $\mathbb{T}$ are term operations for $\mathbb{A}_4$:

$$\sim x = \big(x \vee s_0(x)\big) \wedge s_{01}(x) = \big(x \wedge s_{01}(x)\big) \vee s_0(x), \qquad D(x) = s_{00}(x). \qquad \square$$

Corollary 3.9. *The variety* $\mathscr{LM}_3 = \mathscr{V}(\mathbb{L}_3)$ *is equivalent to the variety* $\mathscr{V}(\mathbb{A}_3)$, *where* $\mathbb{A}_3 = (\{0, a, 1\}; \wedge, \vee, s, 0, 1) \in \mathscr{S}w\mathscr{L}_{01}$.

Remarks. (1) The four operators in Proposition 3.5 are as follows for $\mathscr{TM}$:

$$s_0(x) = \sim D(x); \quad s_{00}(x) = D(x); \quad s_1(x) = \sim D \sim (x); \quad s_{01}(x) = D \sim (x).$$

Thus, we obtain

$$\begin{aligned} s(x, y) &= \big(s_0(x \vee y) \vee s_{00}(x \wedge y)\big) \wedge \big(s_{01}(x \vee y) \vee s_1(x \wedge y)\big) \\ &= \big(s_0(x) \vee s_{00}(x \wedge y)\big) \wedge \big(s_0(y) \vee s_{00}(x \wedge y)\big) \\ &\quad \wedge \big(s_{01}(x \vee y) \vee s_1(x)\big) \wedge \big(s_{01}(x \vee y) \vee s_1(y)\big) \\ &= \big(s_0(x) \vee s_{00}(x \wedge y)\big) \wedge \big(s_0(y) \vee s_{00}(x \wedge y)\big) \\ &\quad \wedge \big(s_{01}(x) \vee s_1(y)\big) \wedge \big(s_{01}(y) \vee s_1(x)\big). \end{aligned}$$

the last equality, because in the algebra $\mathbb{T}$ the following identity holds:

$$\begin{aligned} &\big(s_0(x) \vee s_{00}(x \wedge y)\big) \wedge \big(s_{01}(x \vee y) \vee s_1(y)\big) \\ &\quad = \big(s_0(x) \vee s_{00}(x \wedge y)\big) \wedge \big(s_{01}(x) \vee s_1(y)\big). \end{aligned}$$

(2) Taking into account that, for $\mathscr{LM}_3$, the operator $s_{00} = D$ is an endomorphism, we obtain from (1)

$$s(x, y) = \big(s_0(x) \vee s_{00}(y)\big) \wedge \big(s_0(y) \vee s_{00}(x)\big) \wedge \big(s_{01}(x) \vee s_1(y)\big) \wedge \big(s_{01}(x) \vee s_1(y)\big).$$

By Theorem 2.5, the variety $\mathscr{S}w\mathscr{L}_{01}$ is interpretable into the discriminator variety $\mathscr{MB}$ of monadic Boolean algebras; $(0, 1)$-switching terms for Si $\mathscr{MB}$ were given in §1.

Theorem 3.10. *For each* $n \in \mathbb{N}$, *the variety generated by the simple algebra* $(2^n; \wedge, \vee, s, 0, 1) \in \mathscr{S}w\mathscr{L}_{01}$ *is interpretable in the variety* $\mathscr{MB}_n$. *In particular, the variety* $\mathscr{TM}$ *is interpretable in the variety* $\mathscr{MB}_2$.

Proof. As seen above, $(2^n; \wedge, \vee, s, 0, 1)$ is a reduct of the simple algebra $\mathbb{MB}_n = (2^n; \wedge, \vee, -, D, 0, 1)$ which generates $\mathscr{MB}_n$. By Theorem 3.8, the variety $\mathscr{TM}$ is equivalent to the variety generated by the simple algebra $(2^2; \wedge, \vee, s, 0, 1)$. Hence the conclusion is obvious. We only point out that the De Morgan negation $\sim$ is given by the unary term $\big(x \vee -D(x)\big) \wedge D(-x)$ for $\mathbb{MB}_2$. □

However the variety $\mathscr{MB}_n$ is not interpretable in $\mathscr{S}w\mathscr{L}_{01}$; hence $\mathscr{MB}_2$ is not equivalent to $\mathscr{TM}$, because $\mathbb{L}_3$ cannot be a reduct of any monadic Boolean algebra.

4. Descriptions of congruences

As recalled in Section 1, any discriminator variety is congruence regular. This means that for each algebra $\mathbb{A}$ and any $a \in A$, $\theta \mapsto [a]\theta$ defines an order isomorphism between $\mathrm{Con}\,\mathbb{A}$ and the closure system of all a-cosets $\{[a]\theta : \theta \in \mathrm{Con}\,\mathbb{A}\}$, being $\theta = \Theta([a]\theta)$ the congruence generated by $[a]\theta$.

Throughout this Section $\mathscr{V}$ denotes any discriminator variety interpreting the variety $\mathscr{S}w\mathscr{L}_{01}$. We shall present descriptions of congruences on algebras from these special discriminator varieties, and we shall characterize the 1-cosets and the 0-cosets.

As was shown in the proof of Theorem 2.3, for $\mathbb{A} \in \mathscr{V}$ and $\theta \in \mathrm{Con}\,\mathbb{A}$,

$$\begin{aligned}(x, y) \in \theta \quad &\Rightarrow \quad \big(s(x, y), 1\big) \in \theta \quad \Leftrightarrow \quad \big(s(x, y) \wedge z, z\big) \in \theta, \qquad \text{for all } z \in A \\ &\Rightarrow \quad (x, y) \in \theta. \end{aligned} \tag{4.1}$$

From these equivalences we obtain that for any $a, b \in A$:

$$\Theta(a, b) = \Theta\big(s(a, b), 1\big) \tag{4.2}$$

and, since (4.1) also holds for any $\theta \in \mathrm{Con}\,\mathbb{A}_{\langle \wedge, s\rangle}$, $\mathbb{A}_{\langle \wedge, s\rangle} = (A; \wedge, s)$, we have:

$$\Theta_{\langle \wedge, s\rangle}(a, b) = \Theta_{\langle \wedge, s\rangle}\big(s(a, b), 1\big). \tag{4.3}$$

But the description of $\Theta(a, b)$, given in the proof of Theorem 2.3, means that $\Theta(a, b)$ is precisely the congruence generated by $\big(s(a, b), 1\big)$ on the $\langle \wedge \rangle$-reduct of $\mathbb{A}$. Hence, it is also the congruence generated by $\big(s(a, b), 1\big)$ on any reduct of $\mathbb{A}$ having reduct $\mathbb{A}_{\langle \wedge \rangle}$. So, taking into account (4.2) and (4.3), we obtain:

$$\Theta(a, b) = \Theta\big(s(a, b), 1\big) = \Theta_{\langle \wedge \rangle}\big(s(a, b), 1\big) = \Theta_{\langle \wedge, s\rangle}\big(s(a, b), 1\big) = \Theta_{\langle \wedge, s\rangle}(a, b).$$

Dually, by applying s_0, we obtain:

$$
\begin{aligned}
(x, y) \in \Theta \quad &\Leftrightarrow \quad \big(0, s\big(0, s(x, y)\big)\big) \in \theta \\
&\Leftrightarrow \quad \big(z, z \vee s\big(0, s(x, y)\big)\big) \in \theta, \qquad \forall z \in A
\end{aligned} \tag{4.4}
$$

and the other dual facts summarized in (a) of the following theorem.

Theorem 4.1. *Let $\mathbb{A} \in \mathscr{V}$. Then:*

(a) *For any $a, b \in A$,*

$$
\begin{aligned}
\Theta(a,b) &= \{(x, y) \in A \times A : s(a,b) \wedge x = s(a,b) \wedge y\} = \Theta_{\langle\wedge\rangle}\big(s(a,b), 1\big) \\
&= \Theta\big(s(a,b), 1\big) = \Theta_{\langle\wedge,s\rangle}\big(s(a,b), 1\big) = \Theta_{\langle\wedge,\vee,s\rangle}\big(s(a,b), 1\big) \\
&= \Theta_{\langle\wedge,s\rangle}(a,b) = \Theta_{\langle\wedge,\vee,s\rangle}(a,b) \\
&= \{(x, y) \in A \times A : s_0 s(a,b) \vee x = s_0 s(a,b) \vee y\} = \Theta_{\langle\vee\rangle}\big(0, s_0 s(a,b)\big) \\
&= \Theta\big(0, s_0 s(a,b)\big) = \Theta_{\langle\vee,s\rangle}\big(0, s_0 s(a,b)\big) = \Theta_{\langle\wedge,\vee,s\rangle}\big(0, s_0 s(a,b)\big) \\
&= \Theta_{\langle\vee,s\rangle}(a,b).
\end{aligned}
$$

(b) *$[1]\Theta(a,b)$ is the principal filter $\big[s(a,b)\big) = \{x \in A : s(a,b) \leq x\}$.*
$[0]\Theta(a,b)$ is the principal ideal $\big(s_0 s(a,b)\big] = \{x \in A : x \leq s_0 s(a,b)\}$.
$[c]\Theta(a,b) = [s(a,b) \wedge c, s_0 s(a,b) \vee c]$.

Proof. (b) Follows immediately from (a). □

Lemma. *For any algebra $\mathbb{A}$ and any reduct $\mathbb{A}_{\mathbf{t}}$,*

$$\operatorname{Con} \mathbb{A} = \operatorname{Con} \mathbb{A}_{\mathbf{t}} \quad \text{iff} \quad \operatorname{Con}_p \mathbb{A} = \operatorname{Con}_p \mathbb{A}_{\mathbf{t}}.$$

Proof. Obviously $\operatorname{Con} \mathbb{A} \subseteq \operatorname{Con} \mathbb{A}_{\mathbf{t}}$. If $\operatorname{Con} \mathbb{A} = \operatorname{Con} \mathbb{A}_{\mathbf{t}}$, then $\operatorname{Con}_p \mathbb{A} = \operatorname{Con}_p \mathbb{A}_{\mathbf{t}}$. If $\operatorname{Con}_p \mathbb{A} = \operatorname{Con}_p \mathbb{A}_{\mathbf{t}}$, then for any $\theta \in \operatorname{Con} \mathbb{A}_{\mathbf{t}}$, we also have $\theta \in \operatorname{Con} \mathbb{A}$, since θ is the join of principal congruences, and joins in $\mathbb{A}$ and $\mathbb{A}_{\mathbf{t}}$ are the same. □

Thus, we have as a consequence of Theorem 4.1(a) together with this Lemma:

Corollary 4.2. *Let $\mathbb{A} \in \mathscr{V}$. Then,*

$$\operatorname{Con} \mathbb{A} = \operatorname{Con} \mathbb{A}_{\langle\wedge,\vee,s,0,1\rangle} = \operatorname{Con} \mathbb{A}_{\langle\wedge,s,0,1\rangle} = \operatorname{Con} \mathbb{A}_{\langle\vee,s,0,1\rangle}.$$

The first equivalence in (4.1) and its dual in (4.4) yield, respectively, the following descriptions of a congruence θ:

$$\theta = \Theta([1]\theta) = \{(x, y) \in A \times A : s(x, y) \in [1]\theta\}. \tag{4.5}$$

$$\theta = \Theta([0]\theta) = \{(x, y) \in A \times A : s_0 s(x, y) \in [0]\theta\}. \tag{4.6}$$

For $\mathbb{A} \in \mathscr{V}$ and $\theta \in \operatorname{Con} \mathbb{A}$, the 1-coset $[1]\theta$ is a filter of $\mathbb{A}_{\langle \wedge \rangle}$ [the 0-coset $[0]\theta$ is an ideal of $\mathbb{A}_{\langle \vee \rangle}$]. By (4.1) and (4.4), the filter $[1]\theta$ and the ideal $[0]\theta$ have the following properties:

$$x \in [1]\theta \quad \Leftrightarrow \quad (x, 1) \in \theta \quad \Leftrightarrow \quad \big(s(x, 1), 1\big) \in \theta \quad \Leftrightarrow \quad s_1(x) \in [1]\theta. \tag{4.7}$$

$$x \in [0]\theta \quad \Leftrightarrow \quad (0, x) \in \theta \quad \Leftrightarrow \quad \big(0, s\big(0, s(0, x)\big)\big) \in \theta \quad \Leftrightarrow \quad s_{00}(x) \in [0]\theta. \tag{4.8}$$

Thus $s_1([1]\theta) \subseteq [1]\theta$, and $s_{00}([0]\theta) \subseteq [0]\theta$.

Theorem 4.3. *Let $\mathbb{A} \in \mathscr{V}$. Then, for any $\theta \in \operatorname{Con} \mathbb{A}$,*

$$\begin{aligned}
\theta &= \Theta([1]\theta) = \{(x, y) \in A \times A : s(x, y) \in [1]\theta\} = \Theta_{\langle \wedge, s \rangle}([1]\theta) = \Theta_{\langle \wedge, \vee, s \rangle}([1]\theta) \\
&= \Theta_{\langle \wedge \rangle}([1]\theta) = \{(x, y) \in A^2 : f \wedge x = f \wedge y, \text{ for some } f \in [1]\theta\} \\
&= \{(x, y) \in A^2 : s_1(f) \wedge x = s_1(f) \wedge y, \text{ for some } f \in [1]\theta\} \\
&= \Theta([0]\theta) = \{(x, y) \in A \times A : s_0 s(x, y) \in [0]\theta\} = \Theta_{\langle \vee, s \rangle}([0]\theta) = \Theta_{\langle \wedge, \vee, s \rangle}([0]\theta) \\
&= \Theta_{\langle \vee \rangle}([0]\theta) = \{(x, y) \in A^2 : i \vee x = i \vee y, \text{ for some } i \in [0]\theta\} \\
&= \{(x, y) \in A^2 : s_{00}(i) \vee x = s_{00}(i) \vee y, \text{ for some } i \in [0]\theta\}.
\end{aligned}$$

Proof. The two first equalities are (4.5). That $\theta = \Theta_{\langle \wedge, s \rangle}([1]\theta) = \Theta_{\langle \wedge, \vee, s \rangle}([1]\theta)$ is a consequence of Corollary 4.2, since $\Theta([1]\theta) = \bigcap\{\alpha \in \operatorname{Con} \mathbb{A} : [1]\theta \times [1]\theta \subseteq \alpha\}$. To show that $\theta = \Theta([1]\theta) = \Theta_{\langle \wedge \rangle}([1]\theta)$, we first recall that

$$\Theta_{\langle \wedge \rangle}([1]\theta) = \{(x, y) \in A^2 : f \wedge x = f \wedge y, \text{ for some } f \in [1]\theta\}.$$

Then we observe that:

$$\begin{aligned}
&\{(x, y) \in A^2 : f \wedge x = f \wedge y, \text{ for some } f \in [1]\theta\} \\
&\qquad = \{(x, y) \in A^2 : s_1(f) \wedge x = s_1(f) \wedge y, \text{ for some } f \in [1]\theta\}.
\end{aligned}$$

In fact, by (4.7), the second member is contained in the first. Conversely, if $f \wedge x = f \wedge y$ for some $f \in [1]\theta$, then $s_1(f) \wedge x = s_1(f) \wedge y$, since $s_1(f) \leq f$, and $s_1(f) \in [1]\theta$ by (4.7).

The identities S_f ensure that

$$\Theta_{\langle \wedge \rangle}([1]\theta) = \{(x, y) \in A^2 : s_1(f) \wedge x = s_1(f) \wedge y \text{ for some } s_1(f) \in [1]\theta\} \in \operatorname{Con} \mathbb{A}.$$

As the only congruence of $\mathbb{A}$ having 1-coset $[1]\theta$ is $\Theta([1]\theta)$, we must have $\Theta([1]\theta) = \Theta_{\langle \wedge \rangle}([1]\theta)$.

The subsequent equalities are proved dually. □

Definition 4.4. For any $\mathbb{A} \in \mathscr{V}$, a *strong filter* or s_1-*filter* [*strong ideal* or s_{00}-*ideal*] is a filter F [an ideal I] of $\mathbb{A}_{\langle\wedge\rangle}$ [$\mathbb{A}_{\langle\vee\rangle}$] such that:

$$s_1(F) \subseteq F \qquad [s_{00}(I) \subseteq I].$$

The poset of all strong filters [ideals] of $\mathbb{A}$ will be denoted by $\mathscr{F}il_s\,\mathbb{A}$ [$\mathscr{I}d_s\,\mathbb{A}$].

We note that if F is a strong filter, then F is the order filter generated by $s_1(F)$, since $s_1(F)$ is closed under $\wedge$. Dually for strong ideals.

In the following theorem it is shown that the strong filters [ideals] of $\mathbb{A}$ are precisely the 1-cosets [0-cosets].

Theorem 4.5. *Let $\mathbb{A} \in \mathscr{V}$ with $B(\mathbb{A})$ its largest Boolean subreduct. Then, for $F \subseteq A$ $[I \subseteq A]$, the following are equivalent:*

(a) $F = [1]\theta$ $[I = [0]\theta]$, *for some* $\theta \in \operatorname{Con}\mathbb{A}$.

(b) *F is a strong filter* [*I is a strong ideal*].

(c) *F is the filter of $\mathbb{A}_{\langle\wedge,\vee,0,1\rangle}$ generated by a filter of $B(\mathbb{A})$ [I is the ideal of $\mathbb{A}_{\langle\wedge,\vee,0,1\rangle}$ generated by an ideal of $B(\mathbb{A})$].*

Proof. (a) $\Rightarrow$ (b) Shown in (4.7).

(b) $\Rightarrow$ (c) Let F be a strong filter of $\mathbb{A}$. Then $s_1(F)$ is a filter of $B(\mathbb{A})$, since s_1 is a $(\wedge, 1)$-endomorphism of $\mathbb{A}_{\langle\wedge,1\rangle}$ with range $B(\mathbb{A})$. Moreover, F is the (order) filter of $\mathbb{A}$ generated by $s_1(F)$. From $s_1(F) \subseteq F$ we obtain $\left[s_1(F)\right) \subseteq F$. Since for any $f \in F$, we have $s_1(f) \leq f$, we conclude that $f \in \left[s_1(F)\right)$. So, $F = \left[s_1(F)\right)$.

(c) $\Rightarrow$ (a) The filter F of $\mathbb{A}$ generated by a filter $\bar{F}$ of $B(\mathbb{A})$ is the order filter generated by $\bar{F}$, $F = \{x \in A : \bar{x} \leq x,$ for some $\bar{x} \in \bar{F}\}$. The equivalence relation on A, defined by

$$\theta := \{(x, y) \in A^2 : \bar{f} \wedge x = \bar{f} \wedge y, \text{ for some } \bar{f} \in \bar{F}\}.$$

is compatible with every basic operation of $\mathbb{A}$ by the identities S_f. So $\theta \in \operatorname{Con}\mathbb{A}$. Moreover, $[1]\theta = F$, since

$$(x, 1) \in \theta \quad\Leftrightarrow\quad \exists\bar{f} \in \bar{F} : \bar{f} \wedge x = \bar{f} \quad\Leftrightarrow\quad \exists\bar{f} \in \bar{F} : \bar{f} \leq x \quad\Leftrightarrow\quad x \in F. \qquad \square$$

Theorem 4.6. *Let $\mathbb{A} \in \mathscr{V}$, and $B(\mathbb{A})$ be its largest Boolean subreduct. Then, there is an order isomorphim between the lattice of filters [ideals] of $B(\mathbb{A})$ and the lattice of strong filters [ideals] of $\mathbb{A}$.*

Proof. By the proof of Theorem 4.5 (b) $\Rightarrow$ (c), we have an onto mapping

$$\begin{aligned} &\mathscr{F}il\,B(\mathbb{A}) \to \mathscr{F}il_s\,\mathbb{A} \\ &\qquad \bar{F} \mapsto [\bar{F}) = \{x \in A : \bar{x} \leq x, \text{ for some } \bar{x} \in \bar{F}\} \end{aligned}$$

For any $\bar{F}_1, \bar{F}_2 \in \mathscr{Fil}\, B(\mathbb{A})$, it is clear that $\bar{F}_1 \subseteq \bar{F}_2 \Rightarrow [\bar{F}_1) \subseteq [\bar{F}_2)$. To show the converse implication, assume that $[\bar{F}_1) \subseteq [\bar{F}_2)$. Then, $\bar{x} \in \bar{F}_1 \Rightarrow \bar{x} \in [\bar{F}_1) \subseteq [\bar{F}_2) \Rightarrow \bar{x}_2 \leq \bar{x}$, for some $\bar{x}_2 \in \bar{F}_2 \Rightarrow \bar{x} \in \bar{F}_2$. Thus $\bar{F}_1 \subseteq \bar{F}_2$. □

By this fact strong filters [ideals] in any algebra $\mathbb{A} \in \mathscr{V}$ have the same properties as filters [ideals] in Boolean algebras. For instance, we have immediately that each proper strong filter [ideal] is an intersection of maximal strong filters [ideals]. This could also follow from the congruence regularity and the semisimplicity of $\mathbb{A}$, since every $\theta \in \mathrm{Con}\, \mathbb{A} \backslash \{\nabla\}$ is an intersection of maximal congruences.

Theorem 4.7. *Let $\mathbb{A} \in \mathscr{V}$ with $B(\mathbb{A})$ its largest Boolean subreduct. Then,*

$$\mathrm{Con}\, \mathbb{A} \cong \mathrm{Con}\, B(\mathbb{A}).$$

Proof. By the congruence regularity of $\mathbb{A}$ and of $B(\mathbb{A})$, together with Theorem 4.5 and Theorem 4.6, we have $\mathrm{Con}\, \mathbb{A} \cong \mathscr{Fil}_s\, \mathbb{A} \cong \mathscr{Fil}\, B(\mathbb{A}) \cong \mathrm{Con}\, B(\mathbb{A})$. □

Some of these general results provide, under a different approach, results for algebras from $\mathscr{MB}$ proved by P. Halmos in [13], results for algebras from $\mathscr{LM}_3$ proved by A. Monteiro, in the early sixties of the last century (see Chap. VII of [20]), and similar results for algebras from $\mathscr{TM}$ proved by I. Loureiro in [15] and [17].

5. 1-Cosets and 0-cosets as deductive systems

Generalizing the Boolean implication, A. A. Monteiro introduced, in 1963, the notion of a weak implication for algebras from $\mathscr{LM}_3$ (see [20]; Ch. VII). This operation, denoted $\rightarrow$, is induced by the term $D \sim x \vee y$, which also induces a weak implication on algebras from $\mathscr{TM}$ [15]. The weak implication was extended to algebras from $\mathscr{LM}_n$, $n > 3$, by R. Cignoli in [7]. Gr. C. Moisil considered in 1942 and 1963, the intuitionistic or Heyting implication for algebras from $\mathscr{LM}_n$ which gives for any two elements a, b the greatest element in $\{z \in A : z \wedge a \leq b\}$. As is pointed out in §1, the Heyting implication for $\mathscr{LM}_3$, denoted $\Rightarrow$, is induced by the terms $\sim Dx \vee y \vee (D \sim x \wedge Dy) = (D \sim x \vee y) \wedge (\sim Dx \vee Dy)$. By Remark 1, in §3, we can write:

$$x \Rightarrow y = (s_{01}(x) \vee y) \wedge (s_0(x) \vee s_{00}(y)) = (-s_1(a) \vee y) \wedge (-s_{00}(x) \vee s_{00}(y));$$
$$x \rightarrow y := s_{01}(x) \vee y = -s_1(x) \vee y.$$

We will show that for any discriminator variety $\mathscr{V}$ interpreting $\mathscr{S}w\mathscr{L}_{01}$ (via $(\wedge, \vee, s, 0, 1)$) the weak implication induced by the term $s_{01}(x) \vee y$ allows us to extend to these varieties the characterization of 1-cosets, given for algebras from

$\mathscr{LM}_3$ in [20], for algebras from $\mathscr{LM}_n$, $n > 3$, in [7], and for algebras from $\mathscr{TM}$ in [15].

For $\mathbb{A} \in \mathscr{V}$, and $a, b \in A$, we shall see in the next proposition that there exists a greatest element in $\{z \in A : z \wedge a \leq b\}$ whenever $a \in B(\mathbb{A})$; in this case, the greatest element is $a \Rightarrow b = -a \vee b$. This is not the case if $a \notin B(\mathbb{A})$. Noticing that for $a \notin B(\mathbb{A})$, $s_1(a) < a < s_{00}(a)$, and $[s_1(a), s_{00}(a)] \cap B(\mathbb{A}) = \{s_1(a), s_{00}(a)\}$, we could consider two term operations as weak implications: $s_0(x) \vee y = -s_{00}(x) \vee y$ and $s_{01}(x) \vee y = -s_1(x) \vee y$. For $\mathbb{A} \in \mathscr{V}$, $y = b \in A$, and $x = a \in B(\mathbb{A})$, both coincide with the Heyting implication, so they give the Boolean implication if $a, b \in B(\mathbb{A})$. As $s_1(x) \leq s_{00}(x)$, the weak implication, $\rightarrow$, induced by $-s_1(x) \vee y$ is more convenient than the other.

Proposition 5.1. *For* $\mathbb{A} \in \mathscr{V}$, *and* $a, b \in A$, *let* $a \rightarrow b := s_{01}(a) \vee b = -s_1(a) \vee b$, *and* $a \leftarrow b := s_0(a) \wedge b = -s_{00}(a) \wedge b$. *Then*

$$a \rightarrow b = s_1(a) \Rightarrow b = \max\{z \in A : z \wedge s_1(a) \leq b\},$$
$$a \leftarrow b = s_{00}(a) \Leftarrow b = \min\{z \in A : b \leq z \vee s_{00}(a)\}.$$

Proof. To prove that $-s_1(a) \vee b = \max\{z \in A : z \wedge s_1(a) \leq b\}$, first note that:

$$\big(-s_1(a) \vee b\big) \wedge s_1(a) = \big(-s_1(a) \wedge s_1(a)\big) \vee \big(b \wedge s_1(a)\big) = b \wedge s_1(a) \leq b.$$

Now let $z \in A$ be such that $z \wedge s_1(a) \leq b$. Then

$$\begin{aligned} z \wedge s_1(a) \leq b \quad &\Leftrightarrow \quad -s_1(a) \vee \big(z \wedge s_1(a)\big) \leq -s_1(a) \vee b \\ &\Rightarrow \quad \big(-s_1(a) \vee z\big) \wedge \big(-s_1(a) \vee s_1(a)\big) \leq -s_1(a) \vee b \\ &\Rightarrow \quad -s_1(a) \vee z \leq -s_1(a) \vee b \\ &\Rightarrow \quad z \leq -s_1(a) \vee b. \end{aligned}$$

Dually for $a \leftarrow b$. □

In order to give another characterization of the 1-cosets [0-cosets], we need the notion of a deductive system, introduced by A. Monteiro in 1963 (see [5]; Ch. 9, §4).

Definition 5.2. A [*dual*] *deductive system* of an algebra $\mathbb{A} \in \mathscr{V}$ relative to the binary term operation $x \rightarrow y := s_{01}(x) \vee y$ [$x \leftarrow y := s_0(x) \wedge y$] and a constant 1 [0] is a subset S [$\tilde{S}$] of A such that:

D1. $1 \in S$ [$\tilde{\text{D}}$1. $0 \in \tilde{S}$];

D2. $(\forall a \in A)$ $a \in S$, $a \rightarrow b \in S$ imply $b \in S$. [$\tilde{\text{D}}$2. $(\forall a \in A)$ $a \in \tilde{S}$, $a \leftarrow b \in \tilde{S}$ imply $b \in \tilde{S}$.]

Theorem 5.3. *For $\mathbb{A} \in \mathscr{V}$, and $F \subseteq A$ $[I \subseteq A]$, the following are equivalent:*

(a) *F is a strong filter $[I$ is a strong ideal$]$.*

(b) *F $[I]$ is a deductive system relative to $\to$ and 1 $[\leftarrow$ and $0]$.*

Proof. (a) $\Rightarrow$ (b) Let F be a strong filter of $\mathbb{A}$. Then $F = \big[s_1(F)\big)$. To show that F is a deductive system relative to $\to$ and 0, let $a \in F$, and $-s_1(a) \vee b \in F$. Then, $s_1(a) \in F$, and $s_1(a) \wedge \big(-s_1(a) \vee b\big) \in F$. By Proposition 3.2 and Theorem 3.4, we obtain $s_1(a) \wedge b \in F$, which implies $b \in F$.

(b) $\Rightarrow$ (a) Let F be a deductive system of $\mathbb{A}$. Let $d \in F$. As $-s_1(d) \vee s_1(d) = 1 \in F$ by S3, we have $s_1(d) \in F$, by D2. Thus F is closed under s_1. We show now that F is a filter, i.e., for $d_1, d_2 \in A$, $d_1, d_2 \in F \Leftrightarrow d_1 \wedge d_2 \in F$. Let $d_1, d_2 \in F$. By Remark 2(a′), in §2,

$$1 = -s_1(d_2 \wedge d_1) \vee (d_1 \wedge d_2) = -s_1(d_2) \vee -s_1(d_1) \vee (d_1 \wedge d_2) \in F$$
$$\Rightarrow_{D2} -s_1(d_1) \vee (d_1 \wedge d_2) \in F \Rightarrow_{D2} d_1 \wedge d_2 \in F.$$

If $d_1 \wedge d_2 \in F$, then as $-s_1(d_1 \wedge d_2) \vee d_1 = 1 \in F$, and $-s_1(d_1 \wedge d_2) \vee d_2 = 1 \in F$, we obtain by D2, $d_1, d_2 \in F$. □

For any $\mathbb{A} \in \mathscr{V}$, any $F \in \mathscr{Fil}_s\,\mathbb{A}$, and $a, b \in A$, we have seen in Theorem 4.3, that $(a, b) \in \Theta(F)$ iff $s(a, b) \in F$. For the varieties $\mathscr{LM}_3$ and $\mathscr{TM}$, the terms given for $s(a, b)$, in Remarks 1, and 2, in §3, will provide descriptions of congruences in terms of the weak implication, similar to the one for Boolean algebras in terms of implication.

Theorem 5.4. *For $\mathbb{A} \in \mathscr{LM}_3$, and $F \in \mathscr{Fil}_s\,\mathbb{A}$ (i.e. F a deductive system),*

$$\Theta(F) = \{(x, y) : x \to y \in F, y \to x \in F, Dx \to Dy \in F, Dy \to Dx \in F\}.$$

Proof. From the description of $s(x, y)$ given in Remark 2, in §3, we obtain:

$$\begin{aligned}(x, y) \in \Theta(F) \quad \Leftrightarrow \quad & s_0(x) \vee s_{00}(y) \in F \quad \& \quad s_0(y) \vee s_{00}(x) \in F \\ \& \quad & s_{01}(x) \vee s_1(y) \in F \quad \& \quad s_{01}(y) \vee s_1(x) \in F.\end{aligned}$$

Now, it suffices to notice that: $s_0(x) \vee s_{00}(y) = s_{00}(x) \to s_{00}(y) = Dx \to Dy$, since $s_0(x) = s_{01}s_{00}(x)$ (see table in §3); $s_{01}(x) \vee s_1(y) \in F \Leftrightarrow s_{01}(x) \vee y \in F \Leftrightarrow x \to y \in F$, by the strongness of F, the fact that s_1 is an endomorphism, and $s_1 s_{01}(x) = s_{01}(x)$. □

A characterization of congruences in terms of the weak implication on algebras from $\mathscr{TM}$ was proved by A. Figallo in [10]. We can now easily obtain such a characterization, the second in the following theorem.

Theorem 5.5. *For* $\mathbb{A} \in \mathscr{TM}$, *and* $F \in \mathscr{Fil}_s \mathbb{A}$ (*i.e. F a deductive system*),

$$\begin{aligned}\Theta(F) = \{(x, y) : (x \vee y) \to y \in F, (x \vee y) \to x \in F, \\ Dx \to D(x \wedge y) \in F, Dy \to D(x \wedge y) \in F\} \\ = \{(x, y) : x \to y \in F, y \to x \in F, \\ Dx \to D(x \wedge y) \in F, Dy \to D(x \wedge y) \in F\}.\end{aligned}$$

Proof. From the two ways of describing $s(x, y)$ in Remark 1 in §3, we obtain

$$\begin{aligned}(x, y) \in \Theta(F) \quad &\Leftrightarrow \quad s_0(x) \vee s_{00}(x \wedge y) \in F \quad \& \quad s_0(y) \vee s_{00}(x \wedge y) \in F \\ &\& \quad s_{01}(x \vee y) \vee s_1(y) \in F \quad \& \quad s_{01}(x \vee y) \vee s_1(x) \in F \\ &\Leftrightarrow \quad s_0(x) \vee s_{00}(x \wedge y) \in F \quad \& \quad s_0(y) \vee s_{00}(x \wedge y) \in F \\ &\& \quad s_{01}(x) \vee s_1(y) \in F \quad \& \quad s_{01}(y) \vee s_1(x) \in F.\end{aligned}$$

As in the proof of Theorem 5.4, we have $s_0(x) \vee s_{00}(x \wedge y) = Dx \to D(x \wedge y)$, and $s_{01}(x \vee y) \vee s_1(y) = s_1(s_{01}(x \vee y) \vee y)$. So $s_{01}(x \vee y) \vee s_1(y) \in F \Leftrightarrow s_{01}(x \vee y) \vee y \in F \Leftrightarrow (x \vee y) \to y \in F$. This proves the first description of $\Theta(F)$.

The second description of $\Theta(F)$ is obtained analogously. □

Remarks. The quasi-primal tetravalent modal algebra $\mathbb{T}$ has height 2. We point out that for any variety $\mathscr{V}$ generated by a set of quasi-primal algebras with underlying bounded lattice of height 2, we would obtain the same results as for $\mathscr{TM} = \mathscr{V}(\mathbb{T})$. An example of such a variety is the discriminator variety generated by the diamond $\mathbb{M}_3$ with the additional $(0, 1)$-switching operation s. This will be seen in a forthcoming paper.

The author would like to thank the anonymous referee who provided two references concerning the possibility of expressing the discriminator by binary term functions: [9] for algebras with two constants 0 and 1, and [6] for algebras with only one constant 0.

In [9], a nontrivial algebra $\mathbb{A}$ with two distinct constants 0 and 1 is called a helau (with respect to 0 and 1) if it has two binary terms $\vee$ and $\wedge$, and a unary term $'$ such that $x \wedge 1 = x$, $0 \wedge x = 0 = x \wedge 0$, $x \vee 0 = x = 0 \vee x$, $0' = 1$, $1' = 0$. A helau has the two-element Boolean algebra $(\{0, 1\}; \wedge, ', 0, 1)$ as a subreduct. In that paper, a predicate is any term function with values in $\{0, 1\}$; the predicate equal is our $(0, 1)$-switching term. By Proposition 1.1 in [9], any discriminator algebra with distinct constants 0 and 1 is a helau; by Theorem 1.6 (iv) $\Leftrightarrow$ (v) in [9], for any helau, the existence of a discriminator term is equivalent to the existence of a predicate equal. We observe that such an equivalence holds for algebras more general than helaus and algebras having a bounded lattice reduct, namely for algebras with two distinct constant terms 0 and 1, and two binary operations $\vee$ and $\wedge$ such that $0 \wedge x = 0$, $1 \wedge x = x$, $0 \vee x = x = x \vee 0$.

In [6], a nontrivial algebra $\mathbb{A}$ with a constant 0 is called a 0-semihelau if it has two binary terms $\vee$ and $\wedge$ satisfying $x \wedge x = x$, $0 \wedge x = 0 = x \wedge 0$, $x \vee 0 = x = 0 \vee x$, and $x \vee y = 0 \Rightarrow x = y$. An algebra having a bounded lattice reduct is a 0-semihelau. By Proposition 1 in [6], for any nontrivial 0-semihelau, the existence of a discriminator term is equivalent to the existence of two specific binary terms.

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