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An extension of Minkowski's theorem to simply connected 2-step nilpotent groups

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Abstract. This note extends the classical theorem of Minkowski on lattice points and convex bodies in $\mathbb{R}^n$ to 2-step simply connected nilpotent Lie groups with a $\mathbb{Q}$-structure. This includes all groups of Heisenberg type. More generally (and more naturally), it works for any simply connected nilpotent Lie group with a $\mathbb{Q}$-structure whose Lie algebra admits a *grading* of length 2. Here a new invariant associated with the grading occurs which we call the *degree*. It explains why some directions are more equal than others.

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In this note we extend the classical theorem of Minkowski on convex bodies in $\mathbb{R}^n$ (see for example Chapter 8 of [1]) to simply connected nilpotent Lie groups, G, with a $\mathbb{Q}$-structure whose Lie algebra, $\mathfrak{g}$, admits a length 2 *grading*. This includes all groups of Heisenberg type. A new invariant associated with the grading occurs which we call the *degree*. This note is a continuation of a program, initiated in [10], of extending classical theorems on lattices in Euclidean space to more general groups. For another generalization of Minkowski's theorem to certain locally compact abelian groups see [9].

A Lie algebra $\mathfrak{g}$ is said to admit a grading (see [4]) if there is a finite family of subspaces $V_1, \ldots, V_r$ with $\mathfrak{g} = V_1 \oplus \cdots \oplus V_r$ satisfying $[V_i, V_j] \subseteq V_{i+j}$ for all i, j. The integer r is its length. Of course if $\mathfrak{g}$ is abelian, then we can just take $\mathfrak{g}$ itself as the sole V. Another example of a graded Lie algebra is the Heisenberg algebra, or more generally any 2-step nilpotent algebra, $\mathfrak{g}$. Here one takes V_1 to be any vector space complement to the center, $\mathfrak{z}(\mathfrak{g}) = V_2$. Since $[V_1, V_1] \subseteq \mathfrak{z}(\mathfrak{g})$, this is a grading.

*This paper is dedicated to the memory of Richard Sacksteder.

If $\mathfrak{g}$ is a graded Lie algebra, define for $t \in \mathbb{R}^\times$,

$$\alpha_t(v_1, \dots, v_r) = (tv_1, t^2 v_2, \dots, t^r v_r).$$

We leave to the reader the easy check that each α_t is a Lie algebra automorphism of $\mathfrak{g}$. For $t \to 0$ these are the so called *shrinking* automorphisms (see [8]). We note that because of these automorphisms, the theorem of [8] implies that if $\mathfrak{g}$ admits a grading it must be nilpotent.

The degree of a graded Lie algebra, $\mathfrak{g}$ (or the associated real Lie group) is defined by

$$\deg(\mathfrak{g}) = \sum_{1=1}^{r} i \dim V_i.$$

So, for example, if $\mathfrak{g}$ is abelian, then $\deg(\mathfrak{g}) = \dim(\mathfrak{g})$, while if $\mathfrak{g}$ is 2-step nilpotent, $\deg(\mathfrak{g}) = \dim(\mathfrak{g}) + \dim(\mathfrak{z}(\mathfrak{g}))$.

Of course a given nilpotent Lie algebra may have several gradings. For example, if $\mathfrak{g}$ is abelian one can take a basis, $\{X_1, \dots, X_n\}$, of $\mathfrak{g}$ and V_i the line through X_i. Then $\mathfrak{g}$ is the direct sum of the V_i and since $[V_i, V_j] = (0)$, this is in V_{i+j}. Here $\deg = \sum_{i=1}^n i = \frac{n(n+1)}{2}$ which is always bigger than $n = \dim(\mathfrak{g})$, unless $n = 1$. As we shall see, our extension of Minkowski's theorem, below, will be optimal when the grading produces a degree that is minimal.

Finally, we define a *homogeneous norm* on a graded Lie algebra $\mathfrak{g}$ as a function, $\|\cdot\| : \mathfrak{g} \to \mathbb{R}$, satisfying the following conditions.

(1) $\|\cdot\| \geq 0$ and is 0 only at 0.

(2) $\|X\| = \|-X\|$ for all $X \in \mathfrak{g}$.

(3) $\|\alpha_t(X)\| = |t| \, \|X\|$, for all $t \in \mathbb{R}$ and $X \in \mathfrak{g}$.

(4) $\|X + Y\| \leq \|X\| + \|Y\|$ for all X and $Y \in \mathfrak{g}$.

Given a homogeneous norm, $\|\cdot\|$ we have open and closed "balls" centered at the origin: $B(c) = \{X \in \mathfrak{g} : \|X\| < c\}$ and $B(c)^- = \{X \in \mathfrak{g} : \|X\| \leq c\}$. Evidently these are symmetric and have the same volume. We now calculate $\mathrm{vol}(B(c))$ as a function of $c > 0$.

Proposition 1. *For a ball* $B(t)$ *centered at* 0 *in* $\mathfrak{g}$, $\mathrm{vol}(B(t)) = t^{\deg G} \mathrm{vol}(B(1))$. *More generally,* $\mathrm{vol}(B(tc)) = t^{\deg G} \mathrm{vol}(B(c))$.

Proof. Observe that for all $t, c > 0$ we have $\alpha_t(B(c)) = B(tc)$. Hence $\mathrm{vol}(B(tc)) = |\det d(\alpha_t)| \, \mathrm{vol}(B(c))$. But $|\det d(\alpha_t)| = t^{n_1} \cdot t^{2n_2} \cdot \ldots \cdot t^{rn_r} = t^{n_1 + 2n_2 + \cdots + rn_r} = t^{\deg G}$. Thus $\mathrm{vol}(B(tc)) = t^{\deg G} \mathrm{vol}(B(c))$. Taking $c = 1$ gives the first conclusion. □

Definition 2. We shall say a closed ball $B(c)^-$ is convex in the *sense of shrinking automorphisms* if given X and $Y \in B(c)^-$, then $\alpha_s(X) + \alpha_{1-s}(Y) \in B(c)^-$ for all $s \in [0, 1]$.

Proposition 3. *Balls in* $\|\cdot\|$ *centered at* 0 *are convex in the sense of shrinking automorphisms.*

Proof. Suppose $\|X\|$ and $\|Y\|$ are both $\leq c$ and $0 \leq s \leq 1$. Then

$$\begin{aligned}\|\alpha_s(X) + \alpha_{1-s}(Y)\| &\leq \|\alpha_s(X)\| + \|\alpha_{1-s}(Y)\| \\ &= |s|\,\|X\| + |1-s|\,\|Y\| \leq sc + (1-s)c = c.\end{aligned}$$ □

We now compare volumes in G versus $\mathfrak{g}$. Let G be a simply connected nilpotent Lie group. Then G is unimodular so we can just speak of Haar measure. The center $Z(G)$ of G is connected and has positive dimension. Using induction on $\dim G$, and the formula $\int_G dg = \int_{G/Z(G)} dg^\circ \int_{Z(G)} dz$ shows that Haar measure, μ, is Lebesgue measure in appropriate global coordinates. A well-known formula for the derivative of the exponential map of a Lie group is:

$$d(\exp)_X = \sum_{n=0}^{\infty} \frac{(-1)^n \operatorname{ad}_X^n}{(n+1)!}.$$

Because G is nilpotent each ad_X is simultaneously nil-triangular, and this analytic function is actually a polynomial. Hence $d(\exp)_X$ is unipotent and $\det(d(\exp)_X) \equiv 1$. Since G is simply connected and nilpotent, $\exp : \mathfrak{g} \to G$ is a global diffeomorphism [5] and because $\det(d(\exp)_x) \equiv 1$, the change of variable formula for multiple integrals tells us that if S is a measurable set in $\mathfrak{g}$, then $\mu(\exp(S)) = \int_S |\det(d(\exp)_x)|\, dv = v(S)$, where v is Lebesgue measure on the Euclidean space, $\mathfrak{g}$. Thus

Corollary 4. *If* $B(c)^-$ *is a ball centered at* 0 *in* $\mathfrak{g}$, *then for all* $c > 0$,

$$\mu(\exp(B(c)^-)) = v(B(c)^-).$$

In general a simply connected nilpotent group G might not have any lattices at all so we shall have to assume G contains a lattice, Γ. By this we mean Γ is a discrete subgroup of G and G/Γ is compact. As G is simply connected and nilpotent the well known result of Malcev [6] tells us G contains a lattice if and only if $\mathfrak{g}$ has a $\mathbb{Q}$-structure. We call a lattice Γ in a simply connected nilpotent group a *log-lattice* if $\log(\Gamma)$ is a lattice in $\mathfrak{g}$, where log denotes the inverse of exp. In [7] it is proved that if G has a lattice it must have a log lattice. In fact if Γ is any lattice in G, then Γ sits in between two log lattices, $\Gamma_1 \subseteq \Gamma \subseteq \Gamma_2$.

Now we examine the details behind the classical Minkowski theorem. Let Γ^* be a log-lattice in G and $\Gamma = \log(\Gamma^*)$. Suppose $\mathrm{vol}\big(B(c)^-\big) \geq 2^{\deg G}\,\mathrm{vol}(G/\Gamma^*)$. Let $\pi : \mathfrak{g} \to \mathfrak{g}/\Gamma$. There are two possibilities: either π is injective on $\alpha_{1/2}\big(B(c)^-\big) = B(\frac{1}{2}c)^-$, or it is not. In the latter case there must be a $\gamma \neq 0$ so that $\gamma + x \in B(\frac{1}{2}c)^-$ and $x \in B(\frac{1}{2}c)^-$. But then using symmetry and subadditivity of balls we see $\gamma + x - x = \gamma \in B(\frac{1}{2}c)^- + B(\frac{1}{2}c)^- \subseteq B(c)^-$. Thus γ is a non trivial lattice point in $B(c)^-$.

We will show that the other alternative, namely that π is injective on $B(\frac{1}{2}c)^-$, is impossible. For if this were so, π would be injective on $B(\frac{1}{2}c)$. Now $\mathrm{vol}\big(B(\frac{1}{2}c)\big) = \mathrm{vol}\big(\pi\big(B(\frac{1}{2}c)\big)\big) \leq \mathrm{vol}(\mathfrak{g}/\Gamma)$. But since $\mathrm{vol}\big(B(c)\big) \geq 2^{\deg(\mathfrak{g})}\,\mathrm{vol}(\mathfrak{g}/\Gamma)$, we know by Proposition 1 that $\mathrm{vol}\big(\pi\big(B(\frac{1}{2}c)\big)\big) \geq \mathrm{vol}(\mathfrak{g}/\Gamma)$. That is, they are equal. It follows that π restricted to this set is surjective. For if the image were smaller, since $\mathfrak{g}/\Gamma$ is of finite (regular) measure, there would be an open set of positive measure left out, a contradiction. Because $\pi\big(\big(B(\frac{1}{2}c)\big)\big) = \mathfrak{g}/\Gamma$ it follows $\mathfrak{g} = \bigcup_{\gamma\in\Gamma} \gamma + B(\frac{1}{2}c)$. Since π is also injective and Γ is a subgroup of $\mathfrak{g}$ the union is *disjoint*. This violates the connectedness of $\mathfrak{g}$ and proves,

Theorem 5. *Let G be a simply connected nilpotent* Lie *group with a $\mathbb{Q}$-structure whose* Lie *algebra admits a grading and Γ be a log-lattice in G. Let $\|\cdot\|$ be any homogeneous norm on $\mathfrak{g}$ and $B(c)^-$ be a closed ball in this norm. If* $\mathrm{vol}\big(B(c)^-\big) \geq 2^{\deg G}\,\mathrm{vol}(G/\Gamma)$*, then $B(c)^-$ contains a non-trivial lattice point.*

In particular, by Proposition 1, $B(c)^-$ hits a non-trivial lattice point if $c \geq 2\left(\frac{\mathrm{vol}(G/\Gamma)}{\mathrm{vol}(B(1))}\right)^{1/\deg G}$.

This concludes our treatment of the general Minkowski theorem based on a homogeneous norm. However, we haven't yet seen an example of such a norm. Now a graded Lie algebra, $\mathfrak{g}$, possesses natural candidate for a homogeneous norm as follows:

For $X = (v_1, \dots, v_r)$ let

$$\|X\| = (\|v_1\|_1^{2r} + \|v_2\|_2^{2r-2} + \cdots + \|v_r\|_r^2)^{1/2r},$$

where $\|\cdot\|_i$ is the Euclidean norm on each V_i. (Henceforth we shall suppress the subscript.)

We leave to the reader to check that this norm has properties (1) and (2) above and that property (3) holds if and only if $r \leq 2$. Hence when $r = 2$ it possesses all the properties of a homogeneous norm save subadditivity. Of course when $r = 1$ this is just the Schwarz inequality.

Proposition 6. *When $r = 2$, $\|X + Y\| \leq \|X\| + \|Y\|$.*

Our proof below consists of a sequence of equivalent inequalities terminating with one which is self evidently correct.

Proof. Let $X = (v, z)$ and $Y = (w, \zeta)$. Then $X + Y = (v + w, z + \zeta)$ and, taking 4th powers, what we have to prove is

$$\|v + w\|^4 + \|z + \zeta\|^2 \leq (\|X\| + \|Y\|)^4.$$

Applying the binomial theorem it is sufficient to show

$$\|v + w\|^4 + \|z + \zeta\|^2 \leq \|v\|^4 + \|z\|^2 + \|w\|^4 + \|\zeta\|^2 + 4(*)^{3/4}(**)^{1/4} + 6(*)^{1/2}(**)^{1/2} + 4(*)^{1/4}(**)^{3/4},$$

where $* = \|v\|^4 + \|z\|^2$ and $** = \|w\|^4 + \|\zeta\|^2$, respectively.

Expanding the left side, applying the Schwarz inequality to the two norms, and cancelling appropriate terms yields

$$6\|v\|^2\|w\|^2 + 4\|v\|^3\|w\| + 4\|v\|\,\|w\|^3 + 2\|z\|\,\|\zeta\| \leq 4(*)^{3/4}(**)^{1/4} + 6(*)^{1/2}(**)^{1/2} + 4(*)^{1/4}(**)^{3/4}.$$

Estimating the first and last terms on the right by taking z and $\zeta = 0$ and cancelling the second and third terms on the left which they respectively dominate gives us after dividing by 2

$$3\|v\|^2\|w\|^2 + \|z\|\,\|\zeta\| \leq 3(*)^{1/2}(**)^{1/2}.$$

Now square both sides again getting

$$9\|v\|^4\|w\|^4 + 6\|v\|^2\|w\|^2\|z\|\,\|\zeta\| + \|z\|^2\|\zeta\|^2 \leq 9(\|v\|^4 + \|z\|^2)(\|w\|^4 + \|\zeta\|^2).$$

Then multiplying the right side out and again making appropriate cancellations gives,

$$6\|v\|^2\|w\|^2\|z\|\,\|\zeta\| \leq 9\|v\|^4\|\zeta\|^2 + 9\|w\|^4\|z\|^2 + 8\|z\|^2\|\zeta\|^2.$$

Evidently this inequality is true if any of the norms involved is zero. Hence we may assume they are all positive. Discarding the term involving 8 and dividing yields,

$$6 \leq 9\left(t + \frac{1}{t}\right)$$

where $t = \frac{\|v\|^2\|\zeta\|}{\|w\|^2\|z\|} > 0$ and since $t + \frac{1}{t} \geq 2$, this is true. $\square$

Our results apply, for example, to the Lie groups of Heisenberg type since they are all simply connected and 2-step nilpotent. Moreover, by [3] they each have a $\mathbb{Q}$-structure. Hence

Corollary 7. *Our extension of the Minkowski theorem holds for Lie groups of Heisenberg type.*

In particular, this is so for the N-part of the Iwasawa decomposition of any real rank 1, non-compact, simple group (see [2]).

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