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Denseness of ergodicity for a class of volume-preserving flows

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Abstract. We consider the class of C^1 partially hyperbolic volume-preserving flows with one-dimensional central direction endowed with the C^1-Whitney topology. We prove that, within this class, any flow can be approximated by an ergodic C^2 volume-preserving flow and so, as a consequence, ergodicity is dense.

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1. Introduction

To find the foundations of ergodic theory we must go back to the nineteen century and to the remarkable work of L. Boltzmann. In the context of the dynamic theory of gases he formulated a principle fundamental in statistical physics—*the ergodic hypothesis*. In roughly terms, this principle says that *time averages* equal *space averages* at least for typical points. This principle can be formalized by saying that the μ-invariant flow $\varphi^t : M \to M$ must satisfy the following equality

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t f\left(\varphi^s(x)\right) ds = \int_M f(x)\, d\mu(x),$$

for μ-a.e. $x \in M$ and any continuous observable $f : M \to \mathbb{R}$. Another equivalent definition of ergodicity says that any φ^t-invariant set, for all t, must have zero or full μ-measure.

A central question is to decide if a given system (flow or diffeomorphism) is ergodic and, even more, if the system is stably ergodic, that is it remains ergodic

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after small perturbations. It is well known that there are examples of open sets of systems such that

- all the elements of the open set are ergodic (Anosov [1], examples with hyperbolic behavior),
- there are no ergodic systems in the open set (KAM theory, see for instance [24], examples with elliptical behavior).

These examples are far from a complete description of the situation so it is relevant to obtain properties of systems that assure ergodicity or stable ergodicity. In this context *accessibility* and *partial hyperbolicity* have played an important role in the development of this theory.

Based on results related with this central question Pugh and Shub conjectured that, in broad terms, partial hyperbolicity should guarantee denseness of stable ergodicity among conservative systems [19]. For related results we refer the reader to the survey [22] and to the book [8] and the references there in.

In the context of partially hyperbolic discrete time systems which preserves a symplectic form, Avila, Bochi and Wilkinson [4], proved that ergodicity is C^1-generic.

We also mention the recent and remarkable results of Burns and Wilkinson [13] and of F. Rodriguez-Hertz, M. Rodriguez-Hertz and Ures [23] in the discrete time setting. Despite the fact that their results are more general (in the discrete case) than the one we get (for the continuous case), we observe that our approach is more direct, and to obtain the equivalent of their results it will be necessary to transpose or adapt to the framework of flows all the machinery developed there.

Settling the Pugh and Shub conjecture Bonatti, Matheus, Viana and Wilkinson [9] proved that there exists a C^1 open and dense subset, $\mathcal{U}$, of the partially hyperbolic and volume-preserving diffeomorphisms with one-dimensional central bundle, such that the C^2 diffeomorphisms of $\mathcal{U}$ are ergodic. In ([23], Theorem A and Theorem B) is obtained a generalization of the previous result to the C^r setting, $r \geq 2$. These results did not allow to deduce denseness of ergodic systems among the C^1 partially hyperbolic and volume-preserving diffeomorphisms with one-dimensional central bundle because it was unknown whether the C^r diffeomorphisms were C^1-dense in the set of C^1 volume-preserving diffeomorphisms. However, recently Avila [3] announced the proof of the C^1-density of C^∞ volume-preserving diffeomorphisms among C^1 volume-preserving diffeomorphisms.

In this paper, we obtain the counterpart of their result for the continuous-time setting, considering the space of C^1 partially hyperbolic divergence-free vector fields, $\mathcal{PH}^1_\mu(M)$ (where μ denotes the Lebesgue measure), endowed with the usual Whitney topology. In this context, combining our result with a theorem of Zuppa [25], we obtain that ergodic vector fields are C^1-dense.

Let us now state the fundamental result of this article.

Theorem 1. *Let $X \in \mathcal{PH}^1_\mu(M)$ be a vector field with one-dimensional central direction and $\mathcal{V}$ be an arbitrary neighborhood of X in the C^1-Whitney topology. There exists a C^1 open set $\mathcal{U} \subset \mathcal{V}$ such that any C^2 vector field $Z \in \mathcal{U} \cap \mathcal{PH}^1_\mu(M)$ is ergodic.*

We point out that by a result of Zuppa [25] the subset of C^2 divergence-free vector fields is C^1-dense in the space of C^1 divergence-free vector fields. In particular $\mathcal{U} \cap \mathcal{PH}^2_\mu(M)$ is nonempty, where $\mathcal{U}$ is the set given by Theorem 1.

This theorem has a global formulation. In fact define

$$\mathcal{A} = \bigcup_{X \in \mathcal{P}} \Big(\bigcup_{\mathcal{V} \in \mathcal{N}_X} \mathcal{U}(\mathcal{V}) \Big),$$

where $\mathcal{N}_X$ denotes the set of all neighborhoods of X, $\mathcal{P}$ is the subset of vector fields of $\mathcal{PH}^1_\mu(M)$ having unidimensional central direction, and $\mathcal{U}(\mathcal{V})$ is the open set given by the theorem applied to the pair X, $\mathcal{U}$. The set $\mathcal{A} \cap \mathcal{P}$ is open and dense in $\mathcal{P}$ and if $Z \in \mathcal{A} \cap \mathcal{P}$ is of class C^2 then Z is ergodic. In particular ergodicity is dense in $\mathcal{P}$.

One of the steps of the proof of the theorem consists in a continuous time version of ([23], Theorem A) and as a consequence we obtain the following corollary.

Corollary 1. *There exists an open and dense subset $\mathcal{T} \subset \mathcal{PH}^1_\mu(M)$ such that X is transitive, for all $X \in \mathcal{T}$.*

We observe that, in [6], the first author proved that the elements of a residual subset of the C^1-divergence-free vector fields are topological mixing.

This paper is organized as follows. In Section 2 we introduce some definitions and results. In Section 3 we obtain the main theorem as a consequence of another three theorems. In Sections 4, 5 and 6 we explain how these three results are obtained. Corollary 1 is proved in Section 5.

2. Preliminaries and basic results

Let M be a compact, connected and boundaryless smooth Riemannian manifold, with dimension $n \geq 4$, and let μ denote the Lebesgue measure induced by a fixed volume form on M. We say that the flow φ is volume-preserving if φ^t is a volume-preserving diffeomorphism for all $t \in \mathbb{R}$.

There exists a natural correspondence between flows and vector fields. Clearly given a C^r vector field $X : M \to TM$ the solution of the equation $x' = X(x)$ gives rise to a C^r flow, φ_X; by the other side given a C^r flow we can define a C^{r-1} vector field by $X_\varphi(x) = \frac{d\varphi^t(x)}{dt}\big|_{t=0}$. Observe that, by Liouville formula, a flow φ is volume-preserving if and only if the corresponding vector field, X_φ, is divergence-free.

Let $\mathfrak{X}^r(M)$ denote the space of C^r vector fields and we consider the usual C^r Whitney topology on this space. Let $\mathrm{Per}(X)$ denote the set of closed orbits of the flow φ_X.

Given a vector field X we denote by $\mathrm{Sing}(X)$ the set of *singularities* of X, say the points $x \in M$ such that $X(x) = \vec{0}$. Let $R := M \backslash \mathrm{Sing}(X)$ be the set of *regular* points. Given $x \in R$ we consider its normal bundle $N_x = X(x)^\perp \subset T_x M$ and define the associated *linear Poincaré flow* by $P_t^X(x) := \Pi_{\varphi_X^t(x)} \circ D\varphi_X^t(x)$ where $\Pi_{\varphi_X^t(x)} : T_{\varphi_X^t(x)}M \to N_{\varphi_X^t(x)}$ is the projection along the direction of $X(\varphi_X^t(x))$. A P_t^X-invariant splitting $N = N^1 \oplus \cdots \oplus N^k$ is called a *ℓ-dominated splitting* for the linear Poincaré flow if there exists $\ell \in \mathbb{N}$ such that, for all $x \in M$ and $0 \le i < j \le k$, we have

$$\frac{\|P_\ell^X(x)|_{N_x^j}\|}{\mathfrak{m}(P_\ell^X(x)|_{N_x^i})} \le \frac{1}{2},$$

where $\mathfrak{m}(\cdot)$ denotes the co-norm of an operator, that is $\mathfrak{m}(A) = \|A^{-1}\|^{-1}$. We say that the subbundle N^i is *hyperbolic* if there exists $k \in \mathbb{N}$ such that either $\|(P_k^X(x) \cdot u)^{-1}\| \le 1/2$ (expanding), for all $x \in M$ and any unit vector $u \in N^i(x)$, or $\|P_k^X(x) \cdot u\| \le 1/2$ (contracting), for all $x \in M$ and any unit vector $u \in N^i(x)$.

Given a vector field X, let $\Lambda \subseteq M \backslash \mathrm{Sing}(X)$ be a φ_X^t-invariant set. We say that X is (uniformly) *partially hyperbolic for the linear Poincaré flow on* Λ if there exists a P_t^X-invariant dominated splitting $N = N^u \oplus N^c \oplus N^s$ in Λ such that N^u is hyperbolic expanding and N^s is hyperbolic contracting; moreover these two subbundles are not trivial.

On the other hand, X is (uniformly) *partially hyperbolic on* Λ if there exists a $D\varphi_X^t$-invariant and a continuous splitting

$$T_x M = E_x^u \oplus E_x^c \oplus E_x^s,$$

being each subbundle of constant dimension with E_x^s and E_x^u nontrivial, and there exists $\ell \in \mathbb{N}$ such that for all $x \in \Lambda$ one has

- (domination)

$$\frac{\|D\varphi_X^\ell(x)|_{E_x^c}\|}{\mathfrak{m}(D\varphi_X^\ell(x)|_{E_x^u})} \le \frac{1}{2} \quad \text{and} \quad \frac{\|D\varphi_X^\ell(x)|_{E_x^s}\|}{\mathfrak{m}(D\varphi_X^\ell(x)|_{E_x^c})} \le \frac{1}{2},$$

- (hyperbolicity) $\|(D\varphi_X^\ell(x) \cdot u)^{-1}\| \le 1/2$ (expanding) for any unit vector $u \in E_x^u$, and $\|D\varphi_X^\ell(x) \cdot v\| \le 1/2$ (contracting) for any unit vector $v \in E_x^s$.

We observe that $\mathbb{R}X(x) \subset E^c_x$ for all $x \in \Lambda$. If this central bundle is unidimensional then Λ is a hyperbolic set. We also note that if X is partially hyperbolic on Λ then the diffeomorphism φ^t_X is *partially hyperbolic* on Λ for all $t \neq 0$ (for the definition of partial hyperbolicity in the diffeomorphisms context see, for example, [8]).

Definition 2.1. We say that X is *partially hyperbolic*, respectively *partially hyperbolic for the linear Poincaré flow*, if X is partially hyperbolic on M, respectively partially hyperbolic for the linear Poincaré flow on M.

Note that if the vector field X is partially hyperbolic then it does not have singularities. For each $x \in M$, we write $E^c_x = \mathscr{E}^c_x \oplus \mathbb{R}X(x)$, where the subbundle $\mathscr{E}^c_x$ is continuous on x.

Definition 2.2. We say that a partially hyperbolic vector field X has a *one-dimensional central direction* if $\dim E^c = 2$, that is, $\mathscr{E}^c$ is a one-dimensional subbundle.

We also observe that, when Λ is compact, the partial hyperbolicity of X on Λ implies the partial hyperbolicity for the linear Poincaré flow of X on Λ. This fact follows from the fact that the condition of domination also holds if we switch E^c by $\mathscr{E}^c$ and the fact that the continuity of the splitting and compactness of the set Λ guarantee that the angles between E^s, E^u or $\mathscr{E}^c$ and $\mathbb{R}X(x)$ are bounded away from zero, thus allowing to define N^σ_x as the orthogonal projection of E^σ_x onto $\mathbb{R}X(x)^\perp$, $\sigma = u, s$, and N^c_x as the orthogonal projection of $\mathscr{E}^c_x$.

Hyperbolic divergence-free vector fields or suspensions of volume-preserving partially hyperbolic diffeomorphisms are natural examples of partially hyperbolic divergence-free flows. Another kind of examples, using a weaker definition of partial hyperbolicity, are obtained in [14] motivated by the study of the Σ-geodesic flows corresponding to mechanical systems with constraints.

We denote by $\mathcal{PH}^k_\mu(M)$ the space of partially hyperbolic C^k divergence-free vector fields defined on M, $k \in \mathbb{N}$. This space is an open subset of the space of the C^k divergence-free vector fields. Also the condition of the central subbundle to have dimension equal to one is an open condition.

3. Proof of Theorem 1

The proof of the Theorem 1 is based in the strategy used by Bonatti, Matheus, Wilkinson and Viana [9] to obtain C^1-denseness of ergodicity for C^2 partially hyperbolic and conservative diffeomorphisms having one-dimensional central direction. This strategy is based in three results of Bonatti and Baraviera [5], followed by [7], of F. Rodriguez-Hertz, M. Rodriguez-Hertz and Ures ([23], Theorem A), and of Burns, Dolgopyat and Pesin [12], being the last two adapted to the flow setting.

In this section we present the flow formulation of these results and then deduce the main theorem. If the reader is not familiar with some notions involved we suggest the previous reading of the Sections 4, 5 and 6.

In the stable ergodic context the first result allows us to remove zero central Lyapunov exponents for flows for $X \in \mathcal{PH}^1_\mu(M)$ and with one-dimensional central direction.

Theorem 3.1. *Let $X \in \mathcal{PH}^1_\mu(M)$ be a vector field with one-dimensional central direction. Then, for every $\varepsilon > 0$, there exists $Y \in \mathcal{PH}^2_\mu(M)$ ε-C^1-close to X, such that*

$$\int_M \log\|D\varphi^1_Y|_{\mathcal{E}^c_x}\|\, d\mu(x) \neq 0.$$

The second result shows that accessibility is a C^1-generic property on the space of C^1-conservative partially hyperbolic flows equipped with the C^1 topology and with one-dimensional central direction.

Theorem 3.2. *Accessibility holds in a C^1-open and dense subset of $\mathcal{PH}^1_\mu(M)$, if the central manifold is one-dimensional.*

To obtain this result we adapt the main ingenious ideas of the proof of ([23], Theorem A), to the flow setting. We remark that we obtain this result only in the C^1 topology while their result holds for any C^k.

We notice that, for conservative diffeomorphisms, when the central direction has dimension equal to two, recent results (see [21]) guarantee abundance of ergodicity. Moreover, before that Dolgopyat and Wilkinson (see [15]) proved that accessibility is C^1-dense among the setting of volume-preserving partial hyperbolic diffeomorphisms with central bundle with arbitrary dimension. Their proof is very technical and intricate and, despite the fact that we have a powerful perturbation result (the pasting lemma [2]), its not clear for us how to adapt it to the flows context.

Finally, the third result allows us to obtain ergodicity from the so called *mostly contracting* condition, first introduced by Bonatti and Viana [10], and the accessibility property.

Theorem 3.3. *Let $Z \in \mathcal{PH}^2_\mu(M)$. Assume that Z has the accessibility property and that*

$$\int_M \log\|D\varphi^1_Z|_{\mathcal{E}^c_x}\|\, d\mu(x) < 0.$$

Then Z is ergodic.

Let us now explain how one gets the Theorem 1 from the three previous theorems.

Proof. Fix a vector field $X \in \mathcal{PH}^1_\mu(M)$ with unidimensional central direction and choose an arbitrary neighborhood of X in the C^1-Whitney topology, denoted by $\mathcal{V}$. Observe that if we take $\mathcal{V}$ small then every $Z \in \mathcal{V}$ is also partially hyperbolic with one-dimensional central direction. Theorem 3.1 applied to X guarantees that there exists $Y \in \mathcal{V} \cap \mathcal{PH}^2_\mu(M)$ such that

$$I(Y) := \int_M \log\|D\varphi^1_Y|_{\mathscr{E}^c_x}\|\, d\mu(x) \neq 0.$$

We assume that $I(Y) < 0$; otherwise we consider the vector field $-Y$ instead of Y. As the map

$$Z \in \mathcal{PH}^1_\mu(M) \mapsto \int_M \log|\det(D\varphi^1_Z|_{\mathscr{E}^c_x})|\, d\mu(x) = I(Z)$$

is continuous for the C^1 topology, we can fix a C^1-open subset $\mathcal{W}$ such that $Y \in \mathcal{W} \subset \mathcal{V}$ and $I(Z) < 0$, for every $Z \in \mathcal{W} \cap \mathcal{PH}^1_\mu(M)$.

Applying Theorem 3.2 to the pair Y and $\mathcal{W}$ we get a C^2 vector field $Z \in \mathcal{W}$ with the accessibility property. Moreover, we have that $I(Z) < 0$; hence, by Theorem 3.3, Z is ergodic, which ends the proof. □

In Section 4 we prove Theorem 3.1. In Section 5 we explain how to adapt the proof of ([23], Theorem A) in order to get Theorem 3.2. Finally, in Section 6 we deduce Theorem 3.3.

4. Proof of Theorem 3.1

In this section we derive Theorem 3.1. In [7], transposing to the vector field scenario a previous result of Baraviera and Bonatti [5], we proved that a divergence-free C^1 vector field X, which is partially hyperbolic for the linear Poincaré map and stably ergodic, can be C^1-perturbed in order to obtain a C^2 vector field whose sum of the central Lyapunov exponents is nonzero. The stable ergodicity hypothesis was only used to get that the sum of the central Lyapunov exponents is equal to

$$\int_M \log|\det P^X_1|_{N^c_x}|\, d\mu(x),$$

and then we proved that this integral becomes nonzero after a particular perturbation; hence, without the stable ergodicity assumption what we prove in fact is that

$$\int_M \log|\det P_1^Y|_{N_x^c}|\, d\mu(x) \neq 0$$

for a C^2 vector field Y C^1-arbitrary close to X.

The next lemma jointly with [7], Theorem 1, ends the proof of Theorem 3.1.

Lemma 4.1. *Let* $Y \in \mathcal{PH}_\mu^1(M)$ *be a vector field with one-dimensional central direction. Then one has*

$$\int_M \log|\det P_1^Y|_{N_x^c}|\, d\mu(x) = \int_M \log\|D\varphi_Y^1|_{\mathcal{E}_x^c}\|\, d\mu(x).$$

Proof. As the bundle N_x^c is unidimensional and using the definition of the linear Poincaré flow, we choose a unit vector $v \in N_x^c$ and we get

$$\begin{aligned}\int_M \log|\det P_1^Y|_{N_x^c}|\, d\mu(x) &= \int_M \log\|P_1^Y(v)\|\, d\mu(x)\\ &= \int_M \log\|\Pi_{\varphi_Y^1(x)} \circ D\varphi_Y^1(v)\|\, d\mu(x).\end{aligned}$$

Now we write $v = \alpha_x v^c + v^Y$, where v^c is a unit vector of the unidimensional space $\mathcal{E}_x^c$, $v^Y \in \mathbb{R}Y(x)$ and $\alpha \in \mathbb{R}$ is given by $\cos(\gamma_x) = \frac{1}{\alpha_x}$, where $\gamma_x = \measuredangle(\mathcal{E}_x^c, N_x^c)$ (partial hyperbolicity implies that this angle is always less and bounded away from $\frac{\pi}{2}$). Therefore

$$\begin{aligned}\int_M \log\|\Pi_{\varphi_Y^1(x)} \circ D\varphi_Y^1(v)\|\, d\mu(x) &= \int_M \big(\log(|\alpha_x|) + \log\|\Pi_{\varphi_Y^1(x)} \circ D\varphi_Y^1(v^c)\|\big)\, d\mu(x)\\ &= \int_M \left(\log(|\alpha_x|) + \log\left\|\frac{1}{\alpha_{\varphi_Y^1(x)}} D\varphi_Y^1(v^c)\right\|\right) d\mu(x)\\ &= \int_M \big(\log(|\alpha_x|) - \log|\alpha_{\varphi_Y^1(x)}|\big)\, d\mu(x)\\ &\quad + \int_M \log\|D\varphi_Y^1|_{\mathcal{E}_x^c}\|\, d\mu(x)\\ &= \int_M \log\|D\varphi_Y^1|_{\mathcal{E}_x^c}\|\, d\mu(x),\end{aligned}$$

where the last equality follows directly applying the change of variables theorem to the first integral and observing that $|\det(D\varphi_Y^1)| = 1$ because Y is divergence-free. This ends the proof of the lemma. □

We note that the one-dimensional assumption also implies that

$$\int_M \log\|D\varphi_Y^1|_{\mathscr{E}_x^c}\|\, d\mu(x) = \int_M \log|\det D\varphi_Y^1|_{\mathscr{E}_x^c}|\, d\mu(x).$$

5. Abundance of accessibility on $\mathcal{PH}_\mu^1(M)$

Although part of the results of this section holds in $\mathcal{PH}_\mu^1(M)$, here we assume that the central bundle is one-dimensional.

Let $X \in \mathcal{PH}_\mu^1(M)$ be a vector field and recall that both hyperbolic subbundles are nontrivial. For $x \in M$, the *accessibility class of* x, denoted by $AC(X, x)$, is the set of points y such that there exists a C^1 path from x to y whose tangent vectors belong to $E^s \cup E^u$ and vanishes at most finitely many times. This path is called an *us-path* and consists of a finite number of local stable and unstable manifolds, called *legs*. This notion defines an equivalence relation and we say that the vector field X has *the accessibility property* if $AC(X, x) = M$, for any $x \in M$. The formal definition of accessibility for flows, as far as we know, first appears in [16].

Following the ideas in [23] and adapting the notations to the setup of flows, let φ_X be a flow which preserves a (sectional) foliation $\mathscr{W}$ such that $T\mathscr{W} = E \subset N$. Let also $\mathscr{W}(x)$ be the leaf through x contained in $\mathcal{N}$ (where $T\mathcal{N} = N$) and

$$\mathscr{W}_\varepsilon(x) := \{y \in \gamma \subset \mathscr{W}(x) \mid x \in \gamma \text{ and } |\gamma| < \varepsilon\},$$

where γ is a curve and $|\gamma|$ denotes its arc-length.

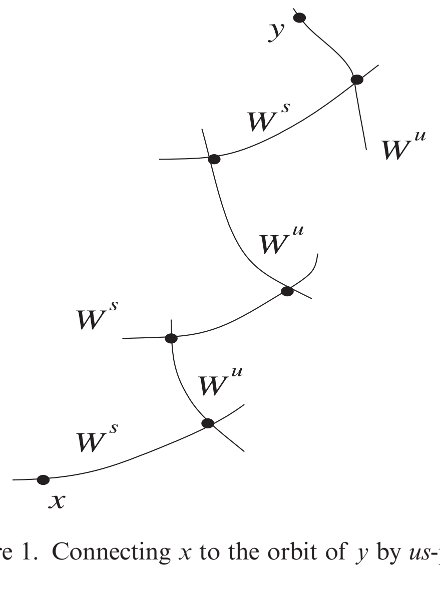

Figure 1. Connecting x to the orbit of y by *us*-paths

Let also V be a transversal disk to $\mathscr{W}$ in N such that $\dim(E) + \dim(V) = n - 1 = \dim(N)$, and

- the cylinder $B_\varepsilon(V) := \bigcup_{y \in V} \mathscr{W}_\varepsilon(y)$ and
- the cylinder with a hole $C_\varepsilon(V) := B_{5\varepsilon}(V) \backslash B_\varepsilon(V)$.

The proof of [23], Lemma A.4.2, can be directly adapted in order to obtain the following result.

Lemma 5.1 (Keepaway lemma for flows). *Assume that E is expanded by φ_X, say $\|D\varphi_X^{-1}|_E\| < \mu < 1$. Let k be such that $\mu^{-k} > 5$. Given a small disk transverse to $\mathscr{W}$ and $\varepsilon > 0$, if*

$$\varphi_X^t\big(C_\varepsilon(V)\big) \cap B_\varepsilon(V) = \emptyset \quad \textit{for all } t \in [0, k],$$

then given any $y \in V$, there exists $z \in \mathscr{W}_{5\varepsilon}(y) \backslash \mathscr{W}_\varepsilon(y)$ such that

$$\varphi_X^t(z) \notin B_\varepsilon(V) \quad \textit{for all } t \geq 0.$$

Assume that X is partially hyperbolic and let $\mathscr{W}_\varepsilon^u(x)$ (respectively, $\mathscr{W}_\varepsilon^s(x)$) denote the local unstable (respectively, stable) manifold of x of size ε. We define $\mathscr{W}_\varepsilon^s\big(\mathscr{W}_\varepsilon^u(x)\big) := \bigcup_{z \in \mathscr{W}_\varepsilon^u(x)} \mathscr{W}_\varepsilon^s(z)$.

Now we can obtain, as in [23], Corollary A.1, the next result.

Corollary 5.2. *Assume that φ_X leaves invariant an expanding foliation $\mathscr{W}$. Then:*

- *For every $x \in M$ the forward nonrecurrent points in $\mathscr{W}(x)$ are dense.*
- *If φ_X is a partially hyperbolic flow, then for any $x \in M$ and every $\varepsilon > 0$ there is $y \in \mathscr{W}_\varepsilon^s\big(\mathscr{W}_\varepsilon^u(x)\big)$ such that y is a forward and backward nonrecurrent point.*

Next result is a crucial step in the proof of Theorem 3.2, and borrows ([23], Lemma A.4.3) and its proof. Let $\Gamma(X)$ denote the closed and invariant set of points of M whose accessibility class is not open.

Lemma 5.3 (Unweaving lemma for flows). *For any $x \in \mathrm{Per}(X)$ there exists Y, C^1-close to X, such that $x \in \mathrm{Per}(Y)$ and $AC(Y, x)$ is open.*

Proof. We will give a sketch of the proof. Take $x \in \mathrm{Per}(X) \cap \Gamma(X)$, we will perturb X in order to obtain Y such that $x \in \mathrm{Per}(Y)$ and $x \notin \Gamma(Y)$, thus proving the lemma.

Since $x \in \Gamma(X)$ we have a *product structure* (cf. [23]), Remark 3.1, that is, for all $y \in \mathscr{W}_{\mathrm{loc}}^u(x)$ and for all $z \in \mathscr{W}_{\mathrm{loc}}^s(x)$ sufficiently close to x we have $\mathscr{W}_{\mathrm{loc}}^u(z) \cap \mathscr{W}_{\mathrm{loc}}^s(y) \neq \emptyset$.

Fixing $\varepsilon > 0$, we use Lemma 5.1 to obtain a "forward keepaway" point $y \in \mathscr{W}_{5\varepsilon}^u(x)$ and a "backward keepaway" point $z \in \mathscr{W}_{5\varepsilon}^s(x)$ (see Figure 2).

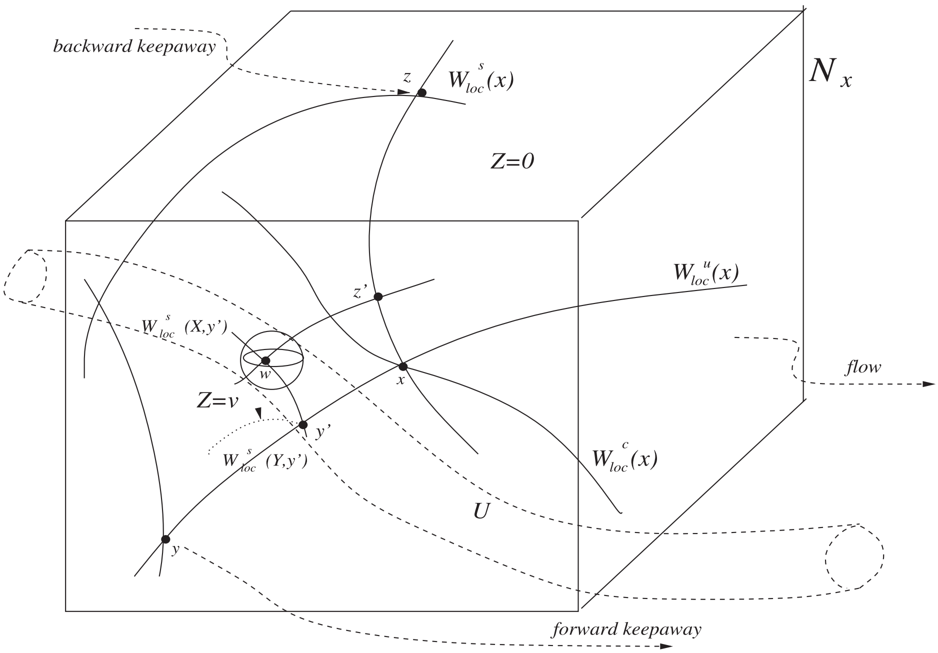

Figure 2. The perturbation scheme of Lemma 5.3

Then, as we need be close to x, in order to control the (invariance) constant δ associated to the size given by the central manifold theorem, we choose properly a backward iterate of y, y', and a forward iterate of z, z' (see Figure 2).

As $x \in \Gamma(X)$ we can take a three legs *su*-path connecting z' into x. Our main goal is to break this connection by a small conservative perturbation in a neighborhood U of the point $w = \mathcal{W}^u_{\mathrm{loc}}(z') \cap \mathcal{W}^s_{\mathrm{loc}}(y')$. The neighborhood U must be sufficiently small in order to have $\varphi^1_X(U) \cap U = \emptyset$.

The perturbation is defined in the following way; take some vector $\vec{v}$ transverse to the bundle $E^s_w \oplus E^u_w \oplus \mathbb{R}X(w)$, $\hat{U}$ a small neighborhood of w and contained in U. Applying the Arbieto and Matheus Pasting Lemma ([2], Theorem 3.2) we obtain a divergence-free vector field Z such that $Z(a) = \vec{v}$ if $a \in \hat{U}$, and $Z(a) = \vec{0}$ if $a \in M \backslash U$. Now we define the divergence-free vector field Y by $Y(a) := X(a) + Z(a)$; observe that if $\|\vec{v}\|$ is small then Y is close to X, $x \in \mathrm{Per}(Y)$, $Y \in \mathcal{PH}^1_\mu(M)$ and its central bundle is one-dimensional.

Moreover, as a consequence of the construction, we have that

- $\mathcal{W}^s_\varepsilon(x, X) = \mathcal{W}^s_\varepsilon(x, Y)$,
- $\mathcal{W}^u_\varepsilon(x, X) = \mathcal{W}^u_\varepsilon(x, Y)$,

- $\mathscr{W}_\delta^u(z', X) = \mathscr{W}_\delta^u(z', Y)$, and
- $\mathscr{W}_\delta^u(z', X) \cap \mathscr{W}_\delta^s(y', Y) = \emptyset$.

From this it follows that $x \notin \Gamma(Y)$, which ends the proof of the lemma. □

In order to end the proof of Theorem 3.2 let us define

$$\mathscr{D} := \{X \in \mathcal{PH}_\mu^1(M) \mid AC(X, x) \text{ is open for all } x \in \mathrm{Per}(X)\}.$$

For any $n \in \mathbb{N}$, let $\mathrm{Per}_n(X)$ denote the set of closed orbits of X with period less or equal to n, and $\mathcal{KS}_n$ the subset of $\mathcal{PH}_\mu^1(M)$ defined by the vector fields X such that all the closed orbits of X with period less or equal to n are hyperbolic. By Robinson's version of the Kupka–Smale theorem [20] $\mathcal{KS}_n$ is open and dense. Moreover, if $X \in \mathcal{KS}_n$, then it has only a finite number of closed orbits of period bounded by n. So, applying Lemma 5.3 a finite number of times, one gets that

$$\mathscr{D}_n := \{X \in \mathcal{PH}_\mu^1(M) \mid AC(X, x) \text{ is open for all } x \in \mathrm{Per}_n(X)\} \cap \mathcal{KS}_n$$

is open and dense in $\mathcal{PH}_\mu^1(M)$. Finally, as $\mathscr{D} \supset \bigcap_{n \in \mathbb{N}} \mathscr{D}_n$, it follows that $\mathscr{D}$ contains a residual subset of $\mathcal{PH}_\mu^1(M)$.

Considering the one-dimensional bundle $\mathscr{E}_x^c$, for all $x \in M$, the arguments used in ([23], Proposition A.5), can be adapted to get the following result.

Lemma 5.4. *If $\emptyset \neq \Gamma(X) \neq M$, then $\Gamma(X) \cap \mathrm{Per}(X) \neq \emptyset$.*

For the sake of completeness we will give a sketch of the proof of Lemma 5.4.

Recall that $\mathscr{W}_\delta^c(x)$ denotes a local central manifold associated to the one-dimensional central subbundle $\mathscr{E}_x^c$. The first step is the following classic result on invariant manifolds which is a reformulation of ([23], Lemma A.5.1), for the flow setting.

Lemma 5.5. *Given any $\varepsilon > 0$, there exists $\delta > 0$ such that if $d(y, z) < \delta$ and $z \in \mathscr{W}_\delta^c(x)$, then there exists a small $t \in \mathbb{R}$ such that $\mathscr{W}_{\mathrm{loc}}^c(X^t(y)) \cap \mathscr{W}_\varepsilon^s(\mathscr{W}_\varepsilon^u(x)) \neq \emptyset$ (see Figure 3).*

The next step is to fix a point $x \in \partial\Gamma(X)$ such that $I =]a_x, c_x[\subset \mathscr{W}_{\mathrm{loc}}^c(x) \backslash \Gamma(X)$ where $a_x = x$ and $c_x \notin \Gamma(X)$. Then one constructs an $(n-1)$-dimensional box $V = \mathscr{W}_\varepsilon^s(\mathscr{W}_\varepsilon^u(I))$ such that $\overline{V} \cap \Gamma(X) = \mathscr{W}_\varepsilon^s(\mathscr{W}_\varepsilon^u(x)) \subset \partial\Gamma(X)$. Notice that I can be taken sufficiently small in order to assure that for any $y \in V$ and $z \in I$, $\mathscr{W}_{\mathrm{loc}}^c(y) \cap \mathscr{W}_\varepsilon^s(\mathscr{W}_\varepsilon^u(z)) \neq \emptyset$. Given a small $\tau > 0$, define $W = \{X^t(V) \mid t \in [-\tau, \tau]\}$. As the nonwandering set is the whole manifold, there exist $w \in W$ and

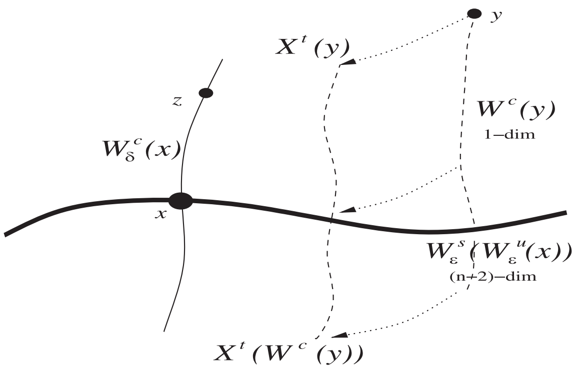

Figure 3. Illustration to Lemma 5.5

$t_w > 0$ such that $X^{t_w}(w) \in W$. Now, applying Lemma 5.5, there are t_0 and t_1 such that

$$\mathscr{W}^c_{\mathrm{loc}}\big(X^{t_0}(w)\big) \cap \mathscr{W}^s_\varepsilon\big(\mathscr{W}^u_\varepsilon(x)\big) = w_0$$

and

$$\mathscr{W}^c_{\mathrm{loc}}\big(X^{t_1+t_w}(w)\big) \cap \mathscr{W}^s_\varepsilon\big(\mathscr{W}^u_\varepsilon(x)\big) = w_1.$$

Finally, one can prove that $w_1 = X^{t_1+t_w-t_0}(w_0)$ and then, applying a suitable version of the Anosov closing lemma (cf. [23], Lemma A.5.2), conclude that there exists a closed orbit in $\Gamma(X)$.

Now let us return to the proof of Theorem 3.2. By Lemma 5.4, if $X \in \mathscr{D}$ then $\Gamma(X) = \emptyset$ or $\Gamma(X) = M$. If the first possibility holds then all the accessibility classes are open, thus, as M is connected, they are all equal, which implies that X has the accessibility property. If the second case holds then, by Lemma 5.4, $\mathrm{Per}(X) = \emptyset$.

Let $\mathscr{B}$ denote the set of vector field of $\mathscr{D}$ with $\mathrm{Per}(X) = \emptyset$. By the Pugh and Robinson version of the General Density Theorem [18], for C^1-generic X one has $\overline{\mathrm{Per}(X)} = M$, so it follows that $\mathscr{B}$ is a meager set.

Finally, as $\mathscr{D}$ contains a residual set, we conclude that accessibility is a generic property in $\mathcal{PH}^1_\mu(M)$, assuming, of course, that the central bundle is one-dimensional.

We are left to prove that accessibility is an open property in $\mathcal{PH}^1_\mu(M)$ (always assuming that the central bundle is one-dimensional). This follows, as in the discrete time case (see [22], Theorem 4.6) by the upper semicontinuity of the map $X \in \mathcal{PH}^1_\mu(M) \to \Gamma(X)$.

The proof of Theorem 3.2 is now complete.

We end this section by proving Corollary 1.

Proof. Let us first recall that a vector field X is *transitive* if there exists $x \in M$ whose forward orbit by the flow associated to X is dense in M.

Fix a $X \in \mathcal{PH}^1_\mu(M)$ and let $\mathcal{W}$ be an arbitrary neighborhood of X. Let $\mathcal{U}_\mathcal{W}$ be the C^1-open set given by Theorem 3.2. If $Y \in \mathcal{U}_\mathcal{W} \cap \mathcal{PH}^1_\mu(M)$ then Y has the accessibility property and, as it is divergence-free, its non-wandering set is equal to M. Therefore we can apply a result of Brin (a version for flows of [11], Theorem 1.2, see the remark immediately after this theorem) to conclude that Y is transitive. Finally, we take

$$\mathcal{T} = \bigcup_{X \in \mathcal{PH}^1_\mu(M)} \Big(\bigcup_{\mathcal{W} \in \mathcal{N}(X)} \mathcal{U}_\mathcal{W} \Big),$$

where $\mathcal{N}(X)$ denotes the set of all neighborhoods of X. □

6. From the mostly contracting and the accessibility conditions to ergodicity

Theorem 3.3 is just a weaker flow formulation of a theorem of Burns, Dolgopyat and Pesin ([12], Theorem 4). Also its proof is a direct adaptation of their arguments. Nevertheless, for the sake of completeness, we present an overline of the proof.

Let us fix $Z \in \mathcal{PH}^2_\mu(M)$ such that Z has the accessibility property and choose $\alpha < 0$ such that

$$\int_M \log\|D\varphi^1_Z|_{\mathscr{E}^c_x}\|\, d\mu(x) < \alpha. \tag{1}$$

Consider the set

$$\mathcal{A}_Z := \Big\{ x \in M \mid \lim_{t\to+\infty} \frac{1}{t}\int_0^t \log\|D\varphi^1_Z|_{\mathscr{E}^c_{\varphi^s_Z(x)}}\|\, ds < \alpha \Big\}.$$

This set is φ^t_Z-invariant and, as a consequence of the Birkhoff Ergodic theorem applied to the observable function $f(x) = \log\|D\varphi^1_Z|_{\mathscr{E}^c_x}\|$ and of inequality (1), it has positive Lebesgue measure.

As $\mathscr{E}^c$ is 1-dimensional, it is not difficult to obtain that

$$\lim_{t\to\pm\infty} \frac{1}{t} \log\|D\varphi^t_Z|_{\mathscr{E}^c_x}\| = \lim_{t\to+\infty} \frac{1}{t}\int_0^t \log\|D\varphi^1_Z|_{\mathscr{E}^c_{\varphi^s_Z(x)}}\|\, ds,$$

which implies that the Lyapunov exponent of Z in the direction $\mathscr{E}^c_x$ is negative, for any point $x \in \mathcal{A}_Z$.

From the last assertion we deduce that every ergodic component of $\mathcal{A}_Z$ is open (mod 0) and so is $\mathcal{A}_Z$.

In fact, let $\mathscr{B}$ be the full measure set of *Birkhoff's regular points*, i.e., points such the backwards and forwards time means coincide for any continuous observable function. Fix $x \in \mathcal{A}_Z \cap \mathscr{B}$; the set

$$\mathscr{N}(x) = \bigcup_{t \in [-\eta,\eta]} \varphi_Z^t\Big(\bigcup_{y \in V^{cs}(x)} V^u(y)\Big)$$

is a neighborhood of x, where $V^{cs}(x)$ is the central-stable disk associated to negative Lyapunov exponents, $V^u(y)$ is the local strong unstable manifold of y and $\eta > 0$.

Now define

$$\tilde{\mathscr{N}}(x) := \bigcup_{t \in [-\eta,\eta]} \varphi_Z^t\Big(\bigcup_{y \in V^{cs}(x)\cap\mathscr{B}} V^u(y)\Big).$$

The set $\tilde{\mathscr{N}}(x)$ is a full measure subset of $\mathscr{N}(x)$ and the Hopf argument allows us to conclude that, for any continuous observable function, the backwards Birkhoff mean is constant on $\tilde{\mathscr{N}}(x)$. Therefore $\tilde{\mathscr{N}}(x)$ is contained in the ergodic component of $\mathcal{A}_Z$ that contains the point x, which proves that this component is open (mod 0).

Therefore, by a result of Brin (see [17], Theorem 8.3, and also [16]), there exists a full measure set $\mathcal{R}$ such that if $x \in \mathcal{R}$ then $\{\varphi_Z^t(x) \mid t \in \mathbb{R}\}$ is dense in M. This result together with the fact that $\mathcal{A}_Z$ is φ_Z^t-invariant for all t, that it has positive measure, and that each ergodic component of $\varphi_Z^t|_{\mathcal{A}_Z}$ is open (mod 0) imply that $\mathcal{A}_Z$ has full Lebesgue measure and that Z is ergodic.

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