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Stabilization of the Schrödinger equation with a delay term in boundary feedback or internal feedback

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Abstract. In this paper, we investigate the effect of time delays in boundary or internal feedback stabilization of the Schrödinger equation. In both cases, under suitable assumptions, we establish sufficient conditions on the delay term that guarantee the exponential stability of the solution. These results are obtained by using suitable energy functionals and some observability estimates.

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1. Introduction

It is well known that certain infinite dimensional damped second order systems become unstable when arbitrary small time delays occur in the damping (see e.g. [4]). This lack of stability robustness was first shown to hold for the one-dimensional wave equation (see [3]). Later, further examples illustrating this phenomenon were given in [2]: the two-dimensional wave equation with damping introduced through Neumann-type boundary conditions on one edge of a square boundary and the Euler–Bernoulli beam equation in one dimension with damping introduced through a specific set of boundary conditions on the right end point.

More recently, Xu et al. [17] established sufficient conditions that guarantee the stability of the one-dimensional wave equation with a delay term in the boundary feedback. Nicaise and Pignotti [11] extended this result to the multidimensional wave equation with a delay term in the boundary or internal feedbacks; they further underline some instability phenomena. Similar results were obtained by Nicaise and Valein [12] for a class of second order evolution equations in one-dimensional networks with delay in unbounded feedbacks.

Motivated by the papers ([17], [11], [12]), we analyze in this paper the effect of time delays in internal feedback or boundary feedback stabilization of the Schrödinger equation in general domains of $\mathbb{R}^n$.

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with a boundary Γ of class C^2. Let (Γ_0, Γ_1) be a partition of Γ, i.e., $\Gamma = \Gamma_0 \cup \Gamma_1$ such that $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$, $\Gamma_0 \neq \emptyset$ and $\Gamma_1 \neq \emptyset$. In addition to these standard hypotheses, we assume the following.

(A) There exists a real-valued vector field $h \in \left(C^2(\overline{\Omega})\right)^n$ such that

(i) h is coercive in $\overline{\Omega}$, that is, there exists $\alpha > 0$ such that the Jacobian matrix J of h satisfies

$$\operatorname{Re}\left(J(x)\xi \cdot \bar{\xi}\right) \geq \alpha|\xi|^2 \qquad \text{for all } x \in \overline{\Omega},\ \xi \in \mathbb{C}^n,$$

(ii) $h(x) \cdot \nu(x) \leq 0$ for all $x \in \Gamma_0$,

where $\nu(x)$ is the unit normal to Γ at $x \in \Gamma$ pointing towards the exterior of Ω and $\operatorname{Re} z$ means the real part of the complex number z.

Remark 1.1. A particular example of a vector field h satisfying Assumption A is the radial vector field $h(x) = x - x_0$ for some $x_0 \in \mathbb{R}^n$. Another example is given by $h(x) = \nabla d(x)$ where d is a real strictly convex function in Ω. For further examples see [15] and the references therein.

In this paper, we are interested in the asymptotic behaviour of the solution of the initial boundary value problem

$$\begin{cases} y_t(x,t) - \mathbf{i}\Delta y(x,t) = 0 & \text{in } \Omega \times (0,+\infty), \\ y(x,0) = y_0(x) & \text{in } \Omega, \\ y(x,t) = 0 & \text{on } \Gamma_0 \times (0,+\infty), \\ \frac{\partial y}{\partial \nu}(x,t) = \mathbf{i}\mu_1 y(x,t) + \mathbf{i}\mu_2 y(x,t-\tau) & \text{on } \Gamma_1 \times (0,+\infty), \\ y(x,t-\tau) = f_0(x,t-\tau) & \text{on } \Gamma_1 \times (0,\tau), \end{cases} \tag{1.1}$$

where $\frac{\partial y}{\partial \nu}$ is the normal derivative, τ is the time delay, μ_1 and μ_2 are positive real numbers.

In the absence of delay, that is $\mu_2 = 0$, Lasiecka et al. [6] have shown that the solution of (1.1) decays exponentially to zero in the energy space $L^2(\Omega)$. If $\mu_2 > 0$, according to the results from ([4], [3], [2], [17], [11], [12]), we may expect to encounter either instability results or stability results according to the value of μ_2 with respect to μ_1. The main purpose of this work is to provide sufficient conditions on the coefficients μ_1 and μ_2 that guarantee that the system (1.1) remains exponentially stable. Indeed, we show as in ([11], [12]) that the exponential stability is preserved if

$$\mu_1 > \mu_2. \tag{1.2}$$

This is done, as in [11], by introducing the energy functional

$$E(t) = \frac{1}{2}\int_{\Omega} |y(x,t)|^2 \, dx + \frac{\xi}{2}\int_{\Gamma_1}\int_0^1 |y(x, t-\tau\rho)|^2 \, d\rho \, d\sigma(x), \tag{1.3}$$

where

$$\tau\mu_2 < \xi < \tau(2\mu_1 - \mu_2), \tag{1.4}$$

and using an energy estimate at the $L^2(\Omega)$ level for a Schrödinger equation with gradient and potentiel terms stated in [5], Theorem 2.6.1 and established in [6], Section 10. This result can be summarized as follows: Assume that the hypothesis (A) holds and let y be a smooth solution of the partial differential equation in (1.1) satisfying

$$y(x,t) = 0 \qquad \text{on } \Gamma_0 \times (0,T).$$

Then there exists a constant $c > 0$ depending on T such that

$$\int_{\Omega} |y(x,0)|^2 \, dx \le c\Bigg(\|y\|^2_{L^2(0,T,L^2(\Gamma_1))} + \int_0^T \int_{\Gamma_1} \left|\frac{\partial y(x,t)}{\partial \nu}\right| |y(x,t)| \, d\sigma(x) \, dt + \left\|\frac{\partial y}{\partial \nu}\right\|^2_{H_a^{-1}((0,T)\times\Gamma_1)} + \|y\|^2_{H^{-1}((0,T)\times\Omega)}\Bigg). \tag{1.5}$$

In (1.5), $H_a^{-1}\big((0,T)\times\Gamma_1\big)$ is the dual space of the space

$$H_a^1\big((0,T)\times\Gamma_1\big) = H^{1/2}\big(0,T; L^2(\Gamma_1)\big) \cap L^2\big(0,T; H^1(\Gamma_1)\big)$$

with respect to the pivot space $L^2\big((0,T)\times\Gamma_1\big)$.

On the other hand, if $\mu_2 \ge \mu_1$, we show that some instability results may appear, namely we show that there exists a sequence of delays for which the system (1.1) is not asymptotically stable.

To be more precise, our results concerning the system (1.1) are as follows.

Theorem 1.2. *Assume that there exists a vector field h satisfying* (A), *that* $\mu_1 > \mu_2$ (*see* (1.2)) *and that the energy* E *of the system* (1.1) *is given by* (1.3) *with* $\tau\mu_2 < \xi < \tau(2\mu_1 - \mu_2)$. *Then there exist constants* $M_b \ge 1$ *and* $\delta_b > 0$ *such that*

$$E(t) \le M_b e^{-\delta_b t} E(0).$$

Theorem 1.3. *If* $\mu_1 \le \mu_2$ (*i.e.*, (1.2) *is not satisfied*), *then there exists a sequence of delays for which the problem* (1.1) *is not asymptotically stable.*

Remark 1.4. Theorem 1.2 remains true if the Laplacian is replaced by a second order elliptic differential operator with space variable coefficients. To this end, one invokes the Riemannian geometric approach of [16] and [5], Remark 2.6.2.

In this paper, we also investigate the stability of the Schrödinger equation with a distributed delay term. More precisely, we consider the system described by

$$\begin{cases} y_t(x,t) = \mathbf{i}\Delta y(x,t) - a(x)\{\mu_1 y(x,t) \\ \qquad\qquad + \mu_2(x,t)y(x,t-\tau)\} & \text{in } \Omega \times (0,+\infty), \\ y(x,0) = y_0(x) & \text{in } \Omega, \\ y(x,t) = 0 & \text{on } \Gamma \times (0,+\infty), \\ y(x,t-\tau) = g_0(x,t-\tau) & \text{in } \Omega \times (0,\tau). \end{cases} \tag{1.6}$$

In (1.6) $a(\cdot)$ is an $L^\infty(\Omega)$-function which satisfies

$$a(x) \geq 0 \; a.e. \text{ in } \Omega \quad \text{and} \quad a(x) > a_0 > 0 \; a.e. \text{ in } \omega, \tag{1.7}$$

where $\omega \subset \Omega$ is an open neighborhood of Γ_0.

In [10], Machtyngier and Zuazua have shown in the case $\mu_2 = 0$ that the $L^2(\Omega)$-energy of the solution of (1.6) decays exponentially to zero. Their proof relies on an observability inequality established previously by the first author in [9]. We use this inequality together with (1.2) to establish the exponential decay of the energy of the solution of the system (1.6) defined by

$$F(t) = \frac{1}{2}\int_\Omega |y(x,t)|^2\,dx + \frac{\xi}{2}\int_\Omega a(x)\int_0^1 |y(x,t-\tau\rho)|^2\,d\rho\,dx. \tag{1.8}$$

As before if $\mu_2 \geq \mu_1$, we construct an explicit sequence of delays that destabilize the system.

The main results concerning the problem (1.6) can be summarized as follows.

Theorem 1.5. *Assume that there exists a vector field h satisfying (A), that $\mu_1 > \mu_2$ (see (1.2)) and that the energy F of the system (1.6) is given by (1.8) with $\tau\mu_2 < \xi < \tau(2\mu_1 - \mu_2)$. Then there exist constants $M_d \geq 1$ and $\delta_d > 0$ such that*

$$F(t) \leq M_d e^{-\delta_d t} F(0).$$

Theorem 1.6. *If $\mu_1 \leq \mu_2$ (i.e., (1.2) is not satisfied), then there exists a sequence of delays for which the problem (1.6) is not asymptotically stable.*

The paper is organized as follows. Theorem 1.2 and Theorem 1.3 are proved in Section 2 whereas Section 3 contains the proof of Theorem 1.5 and Theorem 1.6. Both sections start with the study of the well-posedness of the system under consideration.

2. Stability of the Schrödinger equation with a delay term in the boundary feedback

2.1. Well-posedness of system (1.1). In order to be able to manage the boundary condition with the delay term and inspired from ([17], [11]) we introduce the auxiliary variable $z(x, \rho, t) = y(x, t - \tau\rho)$. With this new unknown, problem (1.1) is equivalent to

$$\begin{cases} y_t(x,t) - \mathbf{i}\Delta y(x,t) = 0 & \text{in } \Omega \times (0,+\infty), \\ z_t(x,\rho,t) + \frac{1}{\tau} z_\rho(x,\rho,t) = 0 & \text{on } \Gamma_1 \times (0,1) \times (0,+\infty), \\ y(x,0) = y_0(x) & \text{in } \Omega, \\ z(x,\rho,0) = f_0(x,-\rho\tau) & \text{on } \Gamma_1 \times (0,1), \\ y(x,t) = 0 & \text{on } \Gamma_0 \times (0,+\infty), \\ \frac{\partial y}{\partial \nu}(x,t) = \mathbf{i}\mu_1 y(x,t) + \mathbf{i}\mu_2 z(x,1,t) & \text{on } \Gamma_1 \times (0,+\infty), \\ z(x,0,t) = y(x,t) & \text{on } \Gamma_1 \times (0,+\infty). \end{cases} \tag{2.1}$$

Let us define on the Hilbert space

$$H = L^2(\Omega) \times L^2\big(\Gamma_1 \times (0,1)\big),$$

the inner product

$$\left\langle \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}; \begin{pmatrix} y_2 \\ z_2 \end{pmatrix} \right\rangle = \operatorname{Re} \int_\Omega y_1(x)\overline{y_2(x)}\, dx + \xi \operatorname{Re} \int_{\Gamma_1} \int_0^1 z_1(x,\rho)\overline{z_2(x,\rho)}\, d\rho\, d\sigma(x).$$

Define further

$$H^1_{\Gamma_0}(\Omega) = \{v \in H^1(\Omega) \,|\, u = 0 \text{ on } \Gamma_0\}.$$

Setting $Y(t) = \begin{pmatrix} y(\cdot,t) \\ z(\cdot,\cdot,t) \end{pmatrix}$ (from now on the notation $y(\cdot,t)$ (resp. $z(\cdot,\cdot,t)$) means the function that maps x to $y(x,t)$ (resp. the function that maps (x,ρ) to $z(x,\rho,t)$)) we may rewrite problem (2.1) as follows

$$\begin{cases} \frac{d}{dt} Y(t) = \mathcal{A} Y(t), \\ Y(0) = \begin{pmatrix} y_0 \\ f_\tau^0 \end{pmatrix}, \end{cases} \tag{2.2}$$

where f_τ^0 means the function that maps (x,ρ) to $f_0(x,-\rho\tau)$ and the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} \mathbf{i}\Delta & 0 \\ 0 & -\tau^{-1}\frac{\partial}{\partial\rho} \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix}, \tag{2.3}$$

with domain $D(A)$ defined by

$$D(A) = \Big\{ (y,z) \in \big(H^{3/2}(\Omega) \cap H^1_{\Gamma_0}(\Omega)\big) \times L^2\big(\Gamma_1; H^1(0,1)\big) \mid$$
$$\Delta y \in L^2(\Omega), \frac{\partial y}{\partial \nu} = \mathbf{i}\mu_1 y + \mathbf{i}\mu_2 z(\cdot,1) \text{ on } \Gamma_1,\ y = z(\cdot,0) \text{ on } \Gamma_1 \Big\}.$$

Theorem 2.1. *For any initial data $Y_0 \in H$, there exists a unique (weak) solution $Y \in C\big([0,+\infty);H\big)$ of* (2.1). *If in addition we assume that $Y_0 \in D(A)$, then the solution $Y \in C\big(0,+\infty;D(A)\big) \cap C^1(0,+\infty;H)$ and is called a strong solution.*

Proof. The well-posedness of the problem (2.1) or its abstract version (2.2) follows via Lumer–Phillips Theorem (see for instance [13], Theorem I.4.3).

Let $Y = \begin{pmatrix} y \\ z \end{pmatrix} \in D(A)$. Then

$$\operatorname{Re}\langle AY, Y\rangle = \operatorname{Re}\left\langle \begin{pmatrix} \mathbf{i}\Delta y \\ -\tau^{-1} z_\rho \end{pmatrix}; \begin{pmatrix} y \\ z \end{pmatrix} \right\rangle$$
$$= \operatorname{Re}\int_\Omega \mathbf{i}\Delta y(x)\overline{y(x)}\,dx - \xi\tau^{-1}\operatorname{Re}\int_{\Gamma_1}\int_0^1 z_\rho(x,\rho)\overline{z(x,\rho)}\,d\rho\,d\sigma(x).$$

From Green's second theorem, we have

$$\operatorname{Re}\langle AY, Y\rangle = \operatorname{Re}\int_{\Gamma_1} \mathbf{i}\frac{\partial y}{\partial \nu}(x)\overline{y(x)}\,d\sigma(x) - \xi\tau^{-1}\operatorname{Re}\int_{\Gamma_1}\int_0^1 z_\rho(x,\rho)\overline{z(x,\rho)}\,d\rho\,d\sigma(x).$$

Integrating by parts in ρ, we obtain

$$\int_{\Gamma_1}\int_0^1 z_\rho(x,\rho)\overline{z(x,\rho)}\,d\rho\,d\sigma(x) = -\int_{\Gamma_1}\int_0^1 z(x,\rho)\overline{z_\rho(x,\rho)}\,d\rho\,d\sigma(x)$$
$$+\int_{\Gamma_1}\big(|z(x,1)|^2 - |z(x,0)|^2\big)\,d\sigma(x),$$

or equivalently

$$2\operatorname{Re}\int_{\Gamma_1}\int_0^1 z_\rho(x,\rho)\overline{z(x,\rho)}\,d\rho\,d\sigma(x) = \int_{\Gamma_1}\big(|z(x,1)|^2 - |z(x,0)|^2\big)\,d\sigma(x).$$

Therefore

$$\operatorname{Re}\langle AY, Y\rangle = \operatorname{Re}\int_{\Gamma_1} \mathbf{i}\frac{\partial y}{\partial \nu}(x)\overline{y(x)}\,d\sigma(x)$$
$$-\frac{\xi\tau^{-1}}{2}\int_{\Gamma_1}\big(|z(x,1)|^2 - |z(x,0)|^2\big)\,d\sigma(x). \qquad (2.4)$$

Insertion of the boundary conditions of (2.1) into (2.4) yields

$$\operatorname{Re}\langle AY, Y\rangle = -\mu_1 \int_{\Gamma_1} |y(x)|^2 \, d\sigma(x) - \mu_2 \operatorname{Re} \int_{\Gamma_1} z(x,1)\overline{y(x)} \, dx$$
$$- \frac{\xi\tau^{-1}}{2} \int_{\Gamma_1} \left(|z(x,1)|^2 - |z(x,0)|^2 \right) d\sigma(x),$$

from which follows, using the Cauchy–Schwarz inequality

$$\operatorname{Re}\langle AY, Y\rangle \leq -\left(\mu_1 - \frac{\mu_2}{2} - \frac{\xi\tau^{-1}}{2}\right) \int_{\Gamma_1} |y(x)|^2 \, d\sigma(x)$$
$$- \left(\frac{\xi\tau^{-1}}{2} - \frac{\mu_2}{2}\right) \int_{\Gamma_1} |z(x,1)|^2 \, d\sigma(x).$$

From (1.4), we conclude that

$$\operatorname{Re}\langle AY, Y\rangle \leq 0.$$

Thus A is dissipative.

Now we show that for a fixed $\lambda > 0$ and $(g,h) \in H$, there exists $Y = (y,z) \in D(A)$ such that

$$(\lambda I - A)\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} g \\ h \end{pmatrix},$$

or equivalently

$$\lambda y - \mathbf{i}\Delta y = g, \tag{2.5}$$
$$\lambda z + \tau^{-1} z_\rho = h. \tag{2.6}$$

Suppose that we have found y with the appropriate regularity, then we can determine z. Indeed, from (2.6) and the last line of (2.1) we have

$$z_\rho(x,\rho) = -\lambda\tau z(x,\rho) + \tau h(x,\rho), \qquad x \in \Gamma_1, \ \rho \in (0,1),$$
$$z(x,0) = y(x), \qquad x \in \Gamma_1.$$

The unique solution of the above initial value problem is given by

$$z(x,\rho) = y(x)e^{-\lambda\tau} + \tau e^{-\lambda\tau\rho} \int_0^\rho h(x,\sigma)e^{\lambda\tau\sigma} \, d\sigma(x), \qquad x \in \Gamma_1, \ \rho \in (0,1),$$

and in particular

$$z(x,1) = y(x)e^{-\lambda\tau} + \tau e^{-\lambda\tau}\int_0^1 h(x,\sigma)e^{\lambda\tau\sigma}\,d\sigma(x), \qquad x \in \Gamma_1.$$

The identity (2.5) can be reformulated as follows

$$\int_\Omega (\lambda y - \mathbf{i}\Delta y)\overline{w}\,dx = \int_\Omega g\overline{w}\,dx \quad \text{for all } w \in L^2(\Omega). \tag{2.7}$$

Integrating by parts, we get

$$\begin{aligned}\int_\Omega (\lambda y - \mathbf{i}\Delta y)\overline{w}\,dx &= \int_\Omega (\lambda y\overline{w} + \mathbf{i}\nabla y\nabla\overline{w})\,dx - \mathbf{i}\int_{\Gamma_1}\frac{\partial y}{\partial \nu}\overline{w}\,d\sigma(x)\\ &= \int_\Omega (\lambda y\overline{w} + \mathbf{i}\nabla y\nabla\overline{w})\,dx + \int_{\Gamma_1}\left(\mu_1 y\overline{w} + \mu_2 z(x,1)\overline{w}\right)d\sigma(x)\end{aligned}$$

for all $w \in H^1_{\Gamma_0}(\Omega)$. Therefore (2.7) can be rewritten as

$$\begin{aligned}&\int_\Omega (\lambda y\overline{w} + \mathbf{i}\nabla y\nabla\overline{w})\,dx + \int_{\Gamma_1}(\mu_1 + \mu_2 e^{-\lambda\tau})y\overline{w}\,d\sigma(x)\\ &\quad = \int_\Omega g\overline{w}\,dx - \int_{\Gamma_1}\left(\tau e^{-\lambda\tau}\int_0^1 h(x,\sigma)e^{\lambda\tau\sigma}d\sigma(x)\right)\overline{w}\,d\sigma(x)\end{aligned}$$

for all $w \in H^1_{\Gamma_0}(\Omega)$. Multiplying this equation by $1 - \mathbf{i}$, we obtain

$$\begin{aligned}&(1-\mathbf{i})\int_\Omega (\lambda y\overline{w} + \mathbf{i}\nabla y\nabla\overline{w})\,dx + (1-\mathbf{i})\int_{\Gamma_1}(\mu_1 + \mu_2 e^{-\lambda\tau})y\overline{w}\,d\sigma(x)\\ &\quad = (1-\mathbf{i})\int_\Omega g\overline{w}\,dx - (1-\mathbf{i})\int_{\Gamma_1}\left(\tau e^{-\lambda\tau}\int_0^1 h(x,\sigma)e^{\lambda\tau\sigma}d\sigma(x)\right)\overline{w}\,d\sigma(x)\end{aligned} \tag{2.8}$$

for all $w \in H^1_{\Gamma_0}(\Omega)$. Since the left-hand side of (2.8) is coercive on $H^1_{\Gamma_0}(\Omega)$ (in the sense that if we denote this left-hand side by $b(y,w)$, then $\operatorname{Re} b(y,y) \geq \min\{1,\lambda\}\|y\|^2_{H^1(\Omega)}$ for all $y \in H^1_{\Gamma_0}(\Omega)$), and since the right-hand side defines a continuous linear form on $H^1_{\Gamma_0}(\Omega)$ (since $(g,h) \in H$) the Lax–Milgram Theorem guarantees the existence and uniqueness of a solution $y \in H^1_{\Gamma_0}(\Omega)$ of (2.8).

If we consider $w \in \mathscr{D}(\Omega)$ in (2.8), then y has a solution in $\mathscr{D}'(\Omega)$,

$$\lambda y - \mathbf{i}\Delta y = g,$$

and thus $\Delta y \in L^2(\Omega)$. Using Green's formula in (2.8), we get

$$\int_{\Gamma_1} (\mu_1 + \mu_2 e^{-\lambda\tau}) y \overline{w} \, d\sigma(x)(x) + \mathbf{i} \int_{\Gamma_1} \frac{\partial y}{\partial \nu} \overline{w} \, d\sigma(x)$$
$$= \int_{\Gamma_1} \left(\tau e^{-\lambda\tau} \int_0^1 h(x,\eta) e^{\lambda\tau\eta} \, d\eta \right) \overline{w} \, d\sigma(x),$$

for all $w \in H^1_{\Gamma_0}(\Omega)$, from which it follows that

$$\mathbf{i} \frac{\partial y}{\partial \nu} + (\mu_1 + \mu_2 e^{-\lambda\tau}) y = \tau e^{-\lambda\tau} \int_0^1 h(x,\eta) e^{\lambda\tau\eta} \, d\eta \qquad \text{on } \Gamma_1.$$

Hence

$$\frac{\partial y}{\partial \nu} = \mathbf{i}\big(\mu_1 y + \mu_2 z(\cdot, 1)\big) \qquad \text{on } \Gamma_1.$$

As this right-hand side belongs to $L^2(\Gamma_1)$, we deduce that $\frac{\partial y}{\partial \nu} \in L^2(\Gamma_1)$ and by [8], Theorem 2.7.4 we deduce that $y \in H^{3/2}(\Omega)$ (reminding that Γ_0 and Γ_1 are disjoint, this theorem guarantees that if $y \in H^1_{\Gamma_0}(\Omega)$ is such that Δy belongs to $H^{1/2}(\Omega)'$ and $\frac{\partial y}{\partial \nu} \in L^2(\Gamma_1)$, then $y \in H^{3/2}(\Omega)$). So we have found $(y, z) \in D(A)$, which satisfies (2.5) and (2.6). By the Lumer–Phillips Theorem, A is the generator of a C_0-semigroup of contractions on H. □

2.2. Boundary feedback stabilization. Theorem 1.2 will be proved for smooth initial data. The general case follows by a standard density argument. We first show that the energy $E(t)$ of every solution of (1.1) is decreasing.

Proposition 2.2. *The energy corresponding to any strong solution of the problem* (1.1) *is decreasing and there exists* $C > 0$ *such that*

$$\frac{d}{dt} E(t) \le -C \int_{\Gamma_1} \left(|y(x,t)|^2 + |y(x,t-\tau)|^2 \right) d\sigma(x).$$

Proof. Differentiating $E(t)$ defined by (1.3) in time, we obtain

$$\frac{d}{dt} E(t) = \operatorname{Re} \int_\Omega y_t \overline{y} \, dx + \xi \operatorname{Re} \int_{\Gamma_1} \int_0^1 y_t(x, t-\tau\rho) \overline{y}(x, t-\tau\rho) \, d\rho \, d\sigma(x)$$
$$= \operatorname{Re} \int_\Omega (\mathbf{i}\Delta y) \overline{y} \, dx + \xi \operatorname{Re} \int_{\Gamma_1} \int_0^1 y_t(x, t-\tau\rho) \overline{y}(x, t-\tau\rho) \, d\rho \, d\sigma(x).$$

Applying Green's second Theorem and recalling the boundary conditions in (1.1), we obtain

$$\frac{d}{dt}E(t) = -\mu_1 \int_{\Gamma_1} |y(x,t)|^2 \, d\sigma(x) - \mu_2 \operatorname{Re} \int_{\Gamma_1} y(x,t-\tau)\,\bar{y}(x,t)\, d\sigma(x) \\ + \xi \operatorname{Re} \int_{\Gamma_1} \int_0^1 y_t(x,t-\tau\rho)\,\bar{y}(x,t-\tau\rho)\, d\rho\, d\sigma(x). \tag{2.9}$$

Now observe that

$$y_t(x,t-\tau\rho) = -\tau^{-1} y_\rho(x,t-\tau\rho),$$

and

$$\frac{d}{d\rho}|y(x,t-\tau\rho)|^2 = 2\operatorname{Re}\big(y_\rho(x,t-\tau\rho)\,\bar{y}(x,t-\tau\rho)\big). \tag{2.10}$$

Insertion of (2.10) into (2.9) yields

$$\begin{aligned} \frac{d}{dt}E(t) &= -\mu_1 \int_{\Gamma_1} |y(x,t)|^2 \, d\sigma(x) - \mu_2 \operatorname{Re} \int_{\Gamma_1} y(x,t-\tau)\,\bar{y}(x,t)\, d\sigma(x) \\ &\quad - \frac{\xi}{2\tau} \int_{\Gamma_1} \int_0^1 \frac{d}{d\rho} |y(x,t-\tau\rho)|^2 \, d\rho\, d\sigma(x) \\ &= -\mu_1 \int_{\Gamma_1} |y(x,t)|^2 \, d\sigma(x) - \mu_2 \operatorname{Re} \int_{\Gamma_1} y(x,t-\tau)\,\bar{y}(x,t)\, d\sigma(x) \\ &\quad - \frac{\xi}{2\tau} \int_{\Gamma_1} \big(|y(x,t-\tau)|^2 - |y(x,t)|^2\big)\, d\sigma(x). \end{aligned}$$

From Cauchy–Schwarz inequality, we have

$$\begin{aligned} \frac{d}{dt}E(t) \le &-\left(\mu_1 - \frac{\mu_2}{2} + \frac{\xi}{2\tau}\right) \int_{\Gamma_1} |y(x,t)|^2 \, d\sigma(x) \\ &- \left(\frac{\xi}{2\tau} - \frac{\mu_1}{2}\right) \int_{\Gamma_1} |y(x,t-\tau)|^2 \, d\sigma(x). \end{aligned}$$

This last inequality can be written

$$\frac{d}{dt}E(t) \le -C \int_{\Gamma_1} \big(|y(x,t)|^2 + |y(x,t-\tau)|^2\big)\, d\sigma(x),$$

where

$$C = \min\left\{\mu_1 - \frac{\mu_2}{2} + \frac{\xi}{2\tau}, \frac{\xi}{2\tau} - \frac{\mu_1}{2}\right\},$$

which is positive due to the assumption (1.4). □

We now establish an observability inequality which will be used to prove the exponential decay of the energy $E(t)$.

Proposition 2.3. *Let y be a strong solution of* (1.1). *Then there exists a positive constant C_0 depending on T such that for all $T > \tau$, the following inequality holds*

$$E(0) \leq C_0 \int_0^T \int_{\Gamma_1} \left(|y(x,t)|^2 + |y(x,t-\tau)|^2\right) d\sigma(x), \tag{2.11}$$

Proof. Set

$$E(t) = \mathscr{E}(t) + E_1(t),$$

where

$$\mathscr{E}(t) = \frac{1}{2}\int_\Omega |y(x,t)|^2\, dx \quad \text{ and } \quad E_1(t) = \frac{\xi}{2}\int_{\Gamma_1}\int_0^1 |y(x,t-\tau\rho)|^2\, d\rho\, d\sigma(x).$$

From [5], Theorem 2.6.1 (see (1.5)), we have the following estimate

$$\begin{aligned}\mathscr{E}(0) \leq c\Bigg(& \|y\|^2_{L^2(0,T,L^2(\Gamma_1))} + \int_0^T \int_{\Gamma_1} \left|\frac{\partial y}{\partial \nu}\right| |y|\, d\sigma(x)\, dt \\ &+ \left\|\frac{\partial y}{\partial \nu}\right\|^2_{H_a^{-1}(\Gamma_1\times(0,T))} + \|y\|^2_{H^{-1}(\Omega\times(0,T))}\Bigg)\end{aligned} \tag{2.12}$$

for $T > 0$ and for a suitable constant c depending on T.

We now impose the boundary conditions in (2.1). Then (2.12) becomes

$$\mathscr{E}(0) \leq c\left(\int_0^T \int_{\Gamma_1} \left(|y(x,t)|^2 + |y(x,t-\tau)|^2\right) d\sigma(x) + \|y\|^2_{H^{-1}(\Omega\times(0,T))}\right), \tag{2.13}$$

since the $H_a^{-1}(\Gamma_1 \times (0,T))$-norm is dominated by the $L^2(\Gamma_1 \times (0,T))$-norm.

$E_1(t)$ can be rewritten, via a change of variable, as

$$E_1(t) = \frac{\xi}{2}\int_{\Gamma_1}\int_{t-\tau}^t |y(x,s)|^2\, ds\, d\sigma(x).$$

Hence

$$E_1(0) \leq c \int_{\Gamma_1} \int_{-\tau}^{0} |y(x,s)|^2 \, ds \, d\sigma(x). \tag{2.14}$$

By another change of variable in (2.14), we have

$$E_1(0) \leq c \int_0^T \int_{\Gamma_1} |y(x,t-\tau)|^2 \, d\sigma(x) \, dt, \tag{2.15}$$

for $T \geq \tau$. Combining (2.13) and (2.15), for any $T \geq \tau$, we obtain

$$E(0) \leq c\Big(\int_0^T \int_{\Gamma_1} \big(|y(x,t)|^2 + |y(x,t-\tau)|^2\big) \, d\sigma(x) \, dt + \|y\|^2_{H^{-1}(\Omega\times(0,T))}\Big), \tag{2.16}$$

for a suitable constant c depending on T.

Naturally, (2.16) implies a fortiori

$$E(0) \leq c\Big(\int_0^T \int_{\Gamma_1} \big(|y(x,t)|^2 + |y(x,t-\tau)|^2\big) \, d\sigma(x) \, dt + \|y\|^2_{L^\infty(0,T;H^{-1}(\Omega))}\Big). \tag{2.17}$$

To get the requested inequality (2.11) from (2.17), we need to absorb the lower order term $\|y\|^2_{L^\infty(0,T,H^{-1}(\Omega))}$. To achieve this, we employ as in [11] and [15], a compactness/uniqueness contradiction argument.

Suppose that (2.11) does not hold. Then there exists a sequence y_n of solutions of problem (1.1) with $y_n(x,0) = y_{n,0}(x)$ and $y_n(x,t-\tau) = f_{n,0}(x,t-\tau)$ such that

$$E_n(0) > n \int_0^T \int_{\Gamma_1} \big(|y(x,t)|^2 + |y(x,t-\tau)|^2\big) \, d\sigma(x) \, dt. \tag{2.18}$$

Here $E_n(0)$ is the energy corresponding to y_n at time $t = 0$.

From (2.17), we have

$$\begin{aligned} E_n(0) \leq c\Big(&\int_0^T \int_{\Gamma_1} \big(|y_n(x,t)|^2 + |y_n(x,t-\tau)|^2\big) \, d\sigma(x) \, dt \\ &+ \|y_n\|^2_{L^\infty(0,T;H^{-1}(\Omega))}\Big). \end{aligned} \tag{2.19}$$

(2.19) together with (2.18) yield

$$\begin{aligned} n &\int_0^T \int_{\Gamma_1} \big(|y_n(x,t)|^2 + |y_n(x,t-\tau)|^2\big) \, d\sigma(x) \, dt \\ &\leq c\Big(\int_0^T \int_{\Gamma_1} \big(|y_n(x,t)|^2 + |y_n(x,t-\tau)|^2\big) \, d\sigma(x) \, dt + \|y_n\|^2_{L^\infty(0,T;H^{-1}(\Omega))}\Big), \end{aligned}$$

that is,

$$(n-c)\int_0^T\int_{\Gamma_1}\left(|y_n(x,t)|^2+|y_n(x,t-\tau)|^2\right)d\sigma(x)\,dt\le c\|y_n\|^2_{L^\infty(0,T;H^{-1}(\Omega))}. \quad (2.20)$$

Renormalizing, we obtain a sequence of solutions of problem (1.1) satisfying

$$\|y_n\|^2_{L^\infty(0,T;H^{-1}(\Omega))}=1 \qquad \text{for all } n>c, \quad (2.21)$$

and

$$\int_0^T\int_{\Gamma_1}\left(|y_n(x,t)|^2+|y_n(x,t-\tau)|^2\right)d\sigma(x)\,dt\le\frac{c}{n-c} \qquad \text{for all } n>c. \quad (2.22)$$

From (2.19), (2.21) and (2.22) we deduce that the sequence $Y_{n,0}=(y_{n,0},f_{n,0})$ is bounded in H. Thus there is a subsequence still denoted by $Y_{n,0}$ which converges weakly to some $Y_0=(y_0,f_0)\in H$. Let ψ be the solution of problem (1.1) with such initial condition Y_0. We have

$$\psi\in C\big(0,T;L^2(\Omega)\big)$$

from Theorem 2.1 and

$$\int_0^T\int_{\Gamma_1}|\psi(x,t)|^2\,d\sigma(x)\,dt+\int_0^T\int_{\Gamma_1}\left|\frac{\partial\psi(x,t)}{\partial\nu}\right|^2d\sigma(x)\,dt\le C$$

from Proposition 2.2 for some $C>0$. It then follows that

$$y_n\to\psi \qquad \text{in } L^\infty\big(0,T;L^2(\Omega)\big) \text{ weak star,}$$
$$(y_n)_t\to\psi_t \qquad \text{in } L^\infty\big(0,T;H^{-2}(\Omega)\big) \text{ weak star,}$$

and hence

$$\|y_n\|^2_{L^\infty(0,T;L^2(\Omega))}+\|(y_n)_t\|^2_{L^\infty(0,T;H^{-2}(\Omega))}\le C \qquad \text{for all } n\in\mathbb{N}. \quad (2.23)$$

Since the embedding $L^2(\Omega)\hookrightarrow H^{-1}(\Omega)$ is compact, (2.23) implies (see [1] and [14]) that for $0<T<+\infty$ the injection

$$Z\hookrightarrow L^\infty\big(0,T,H^{-1}(\Omega)\big)$$

is also compact, where Z is the Banach space equipped with the norm on the left-hand side of (2.23), is also compact. As a consequence there is a subsequence still denoted by y_n such that

$$y_n\to\psi \qquad \text{in } L^\infty\big(0,T,H^{-1}(\Omega)\big) \text{ strongly.}$$

Hence by (2.21) we obtain

$$\|\psi\|_{L^\infty(0,T;H^{-1}(\Omega))} = 1. \tag{2.24}$$

On the other hand, we have from (2.22)

$$\psi(x,t) = 0 \quad \text{on } \Gamma_1 \times (0,T).$$

Thus ψ satisfies

$$\begin{cases} \psi_t(x,t) - \mathbf{i}\Delta\psi(x,t) = 0, & \text{in } \Omega \times (0,T), \\ \psi(x,t) = 0, & \text{on } \Gamma \times (0,T), \\ \frac{\partial\psi}{\partial\nu}(x,t) = 0, & \text{on } \Gamma_1 \times (0,T). \end{cases}$$

From Holmgren's uniqueness theorem (see [7], Chapter 1, Theorem 8.2), we conclude that

$$\psi(x,t) = 0 \quad \text{in } \Omega \times (0,T),$$

which contradicts (2.24). This ends the proof of Proposition 2.3. □

We are now ready to finish the proof of Theorem 1.2.

Proof of Theorem 1.2. From Proposition 2.2, we have

$$E(T) - E(0) \leq -C \int_0^T \int_{\Gamma_1} \left(|y(x,t)|^2 + |y(x,t-\tau)|^2 \right) d\sigma(x).$$

The observability estimate (2.11) implies

$$\begin{aligned} E(T) \leq E(0) &\leq C_0 \int_0^T \int_{\Gamma_1} \left(|y(x,t)|^2 + |y(x,t-\tau)|^2 \right) d\sigma(x) \\ &\leq C_0 C^{-1} \big(E(0) - E(T) \big). \end{aligned}$$

Hence

$$E(T) \leq \frac{C_0 C^{-1}}{1 + C_0 C^{-1}} E(0).$$

Combining this estimate with the invariance by translation of the system (1.1), we obtain the desired conclusion.

2.3. A counter example. In this section we show through an example that the system (2.1) loses the property of exponential stability when $\mu_2 \geq \mu_1$.

We seek a solution of (2.1) in the form

$$y(x,t) = e^{\lambda t}\varphi(x),$$

where

$$\lambda = -\mathbf{i}\beta^2, \qquad \beta \in \mathbb{R}.$$

Then φ is a solution of the eigenvalue problem

$$\begin{cases} -\Delta\varphi = \mathbf{i}\lambda\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \Gamma_0, \\ \frac{\partial\varphi}{\partial\nu} = \mathbf{i}(\mu_1 + \mu_2 e^{-\lambda\tau})\varphi & \text{on } \Gamma_1. \end{cases} \tag{2.25}$$

Assume that

$$\cos(\beta^2\tau) = -\frac{\mu_1}{\mu_2}. \tag{2.26}$$

Then

$$\mu_2 \sin(\beta^2\tau) = \sqrt{\mu_2^2 - \mu_1^2}. \tag{2.27}$$

Inserting (2.26) and (2.27) into (2.25) yields

$$\begin{cases} -\Delta\varphi = \beta^2\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \Gamma_0, \\ \frac{\partial\varphi}{\partial\nu} + \sqrt{\mu_2^2 - \mu_1^2}\varphi = 0 & \text{on } \Gamma_1. \end{cases}$$

This is a classical eigenvalue problem for the Laplacian with Dirichlet–Robin boundary conditions.

Let $\{\beta_n^2; n \in \mathbb{N}\}$ be the set of these eigenvalues. It is well known that $\beta_n^2 \to +\infty$ as $n \to +\infty$. Taking $0 < \theta < 2\pi$ such that

$$\cos\theta = -\frac{\mu_1}{\mu_2} \qquad \text{and} \qquad \mu_2 \sin\theta = \sqrt{\mu_2^2 - \mu_1^2}, \tag{2.28}$$

we obtain a sequence of delays

$$\tau_{n,k} = \frac{1}{\beta_n^2}(\theta + 2k\pi), \qquad n, k \in \mathbb{N},$$

which become arbitrarily small or large for suitable choices of $n, k \in \mathbb{N}$, and for which the problem (2.1) is not asymtotically stable. Indeed, the energy of the solution $y(x,t) = e^{-\mathbf{i}\beta_n^2 t}\varphi(x)$ is constant. This proves Theorem 1.3.

3. Stability of the Schrödinger equation with a delay term in the internal feedback

3.1. Well-posedness of system (1.6). Proceeding as in the previous section, we can see that the system (1.6) is equivalent to

$$\begin{cases} y_t(x,t) = \mathbf{i}\Delta y(x,t) - a(x)\{\mu_1 y(x,t) \\ \qquad\qquad + \mu_2(x,t)z(x,1,t)\} & \text{in } \Omega \times (0,+\infty), \\ z_t(x,\rho,t) = -\tau^{-1} z_\rho(x,\rho,t) & \text{in } \Omega \times (0,1) \times (0,+\infty), \\ y(x,0) = y_0(x) & \text{in } \Omega, \\ z(x,\rho,0) = g_0(x,-\tau\rho) & \text{in } \Omega \times (0,\tau), \\ y(x,t) = 0 & \text{on } \Gamma \times (0,+\infty), \\ z(x,0,t) = y(x,t) & \text{in } \Omega \times (0,+\infty), \end{cases} \tag{3.1}$$

where we have set

$$z(x,\rho,t) = y(x,t-\tau\rho), \qquad x \in \Omega,\ \rho \in (0,1),\ t > 0.$$

Let us introduce the operator A^0 defined by

$$A^0\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} \mathbf{i}\Delta y - a\mu_1 y - a\mu_2 z(\cdot,1) \\ -\tau^{-1} z_\rho \end{pmatrix},$$

and

$$D(A^0) = \{(y,z) \in (H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)) \times L^2(\Omega, H^1(0,1)) \mid y = z(\cdot,0) \text{ in } \Omega\}.$$

Then we rewrite the system (3.1) as

$$\begin{cases} U'(t) = A^0 U(t), \\ U(0) = U_0, \end{cases}$$

where

$$U(t) = \begin{pmatrix} y(t) \\ z(t) \end{pmatrix}.$$

Denote by $\mathscr{H}^0$ the Hilbert space

$$\mathscr{H}^0 = L^2(\Omega) \times L^2\big(\Omega \times (0,1)\big)$$

equipped with the inner product

$$\left\langle \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} y_2 \\ z_2 \end{pmatrix} \right\rangle_{\mathscr{H}^0} = \operatorname{Re} \int_\Omega y_1(x)\overline{y_2(x)}\, dx + \xi \operatorname{Re} \int_\Omega \int_0^1 z(x,\rho)\overline{z(x,\rho)}\, d\rho\, dx.$$

Repeating the argument used in the proof of Theorem 2.1, we obtain the following well-posedness result for the problem (3.1).

Theorem 3.1. *For any $U_0 \in \mathscr{H}^0$, there exists a unique (weak) solution*

$$U \in C(0, +\infty; \mathscr{H}^0)$$

of the problem (3.1). *Moreover if $U_0 \in D(A^0)$, then the solution U is more regular, namely*

$$U \in C\big(0, +\infty; D(A^0)\big) \cap C^1(0, +\infty; \mathscr{H}^0)$$

and is called a strong solution.

3.2. Internal feedback stabilization

Proposition 3.2. *The energy corresponding to any strong solution of the problem* (3.1) *is decreasing and there exists a positive constant C such that*

$$\frac{d}{dt}F(t) \le -C \int_\Omega a(x)\{|y(x,t)|^2 + |y(x,t-\tau)|^2\}\, dx. \tag{3.2}$$

Proof. We differentiate $F(t)$ in (1.8) and use (3.1) to obtain

$$\begin{aligned} \frac{d}{dt}F(t) = \operatorname{Re} \int_\Omega &\big(\mathbf{i}\Delta y(x,t)\big)\overline{y(x,t)}\, dx \\ &- \operatorname{Re} \int_\Omega a(x)\{\mu_1 y(x,t) + \mu_2(x,t) y(x,t-\tau)\}\overline{y(x,t)}\, dx \\ &+ \xi \operatorname{Re} \int_\Omega a(x) \int_0^1 y_t(x, t-\tau\rho)\overline{y(x,t-\tau\rho)}\, d\rho\, dx. \end{aligned}$$

Applying Green's second theorem and recalling the boundary condition in (3.1), we get

$$\frac{d}{dt}F(t) = -\mu_1 \operatorname{Re}\int_\Omega a(x)|y(x,t)|^2\,dx - \mu_2 \operatorname{Re}\int_\Omega a(x)\,y(x,t)\overline{y(x,t)}\,dx$$
$$+\,\xi \operatorname{Re}\int_\Omega a(x)\int_0^1 y_t(x,t-\tau\rho)\overline{y(x,t-\tau\rho)}\,d\rho\,dx. \tag{3.3}$$

As in the proof of Proposition 2.2, we have

$$\begin{aligned}\operatorname{Re}&\int_\Omega a(x)\int_0^1 y_t(x,t-\tau\rho)\overline{y(x,t-\tau\rho)}\,d\rho\,dx\\ &= -\tau^{-1}\operatorname{Re}\int_\Omega a(x)\int_0^1 y_\rho(x,t-\tau\rho)\overline{y(x,t-\tau\rho)}\,d\rho\,dx\\ &= -\frac{\tau^{-1}}{2}\int_\Omega a(x)\{|y(x,t-\tau)|^2 - |y(x,t)|^2\}\,dx.\end{aligned} \tag{3.4}$$

Insertion of (3.4) into (3.3) yields

$$\frac{d}{dt}F(t) = -\mu_1 \operatorname{Re}\int_\Omega a(x)|y(x,t)|^2\,dx - \mu_2 \operatorname{Re}\int_\Omega a(x)\,y(x,t)\overline{y(x,t)}\,dx$$
$$-\,\frac{\xi\tau^{-1}}{2}\int_\Omega a(x)|y(x,t-\tau)|^2 + \frac{\xi\tau^{-1}}{2}\int_\Omega |y(x,t)|^2\,dx. \tag{3.5}$$

The desired estimate (3.2) follows from (3.5) via the Cauchy–Schwarz inequality. □

The key step in the proof of Theorem 1.5 is the following observability inequality.

Proposition 3.3. *Let y be a strong solution of* (3.1). *Then there exists a positive constant C_0 depending on T such that for all $T > \tau$, the following estimate holds true*

$$F(0) \le C_0 \int_0^T \int_\Omega a(x)\{|y(x,t)|^2 + |y(x,t-\tau)|^2\}\,dx\,dt. \tag{3.6}$$

Proof. Following [10] and [11], we write the solution y of (3.1) as $y = u + v$ where u solves

$$\begin{cases} u_t(x,t) = \mathbf{i}\Delta u(x,t) & \text{in } \Omega\times(0,+\infty),\\ u(x,0) = y_0(x) & \text{in } \Omega,\\ u(x,t) = 0 & \text{on } \Gamma\times(0,+\infty), \end{cases} \tag{3.7}$$

and v satisfies

$$\begin{cases} v_t(x,t) = \mathbf{i}\Delta v(x,t) - a(x)\{\mu_1 y(x,t) \\ \qquad\qquad + \mu_2(x,t)y(x,t-\tau)\} & \text{in } \Omega \times (0,+\infty), \\ v(x,0) = 0 & \text{in } \Omega, \\ v(x,t) = 0 & \text{on } \Gamma \times (0,+\infty). \end{cases} \tag{3.8}$$

Let us denote by

$$\mathscr{E}_u(t) = \int_\Omega |u(x,t)|^2\,dx$$

the energy corresponding to the solution of (3.7). Then it follows from [10], Proposition 3.1 that for all $T > 0$, there exists a positive constant c depending on T such that

$$\mathscr{E}_u(0) \le c\int_0^T \int_\omega |u(t,x)|^2\,dx\,dt.$$

Using (1.7) we get

$$\mathscr{E}_u(0) \le \frac{c}{a_0}\int_0^T \int_\Omega a(x)|u(t,x)|^2\,dx\,dt.$$

On the other hand we have, for $T > \tau$,

$$\frac{\xi}{2}\int_\Omega a(x)\int_0^1 |y(x,-\tau\rho)|^2\,d\rho\,dx \le c\int_0^T \int_\Omega a(x)|y(x,t-\tau)|^2\,dx\,dt.$$

Hence, for $T > \tau$,

$$\begin{aligned} F(0) &= \mathscr{E}_u(0) + \frac{\xi}{2}\int_\Omega a(x)\int_0^1 |y(x,-\tau\rho)|^2\,d\rho\,dx \\ &\le c\int_0^T \int_\Omega a(x)\{|u(t,x)|^2 + |y(x,t-\tau)|^2\}\,dx\,dt \\ &\le c\int_0^T \int_\Omega a(x)\{|y(t,x)|^2 + |v(t,x)|^2 + |y(x,t-\tau)|^2\}\,dx\,dt. \end{aligned}$$

By classical energy estimates on Schrödinger equation we deduce that

$$F(0) \le C_0\int_0^T \int_\Omega a(x)\{|y(t,x)|^2 + |y(x,t-\tau)|^2\}\,dx\,dt. \qquad \square$$

Combining the estimates (3.2) and (3.6), as in the case of a boundary feedback, we obtain the exponential stability result of Theorem 1.5.

3.3. A counter example. We proceed as in the case of boundary delay. We assume (2.26) and we look for a solution of the problem (3.1) in the form

$$y(x,t) = e^{\lambda t}\varphi(x) \qquad \text{with } \lambda = -\mathbf{i}\beta^2,\ \beta \in \mathbb{R}.$$

Then φ is a solution of the boundary value problem

$$\begin{cases} (-\Delta + a(x)\sqrt{\mu_2^2 - \mu_1^2})\varphi = \beta^2\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \Gamma. \end{cases}$$

The operator $-\Delta + a(x)\sqrt{\mu_2^2 - \mu_1^2}$ with Dirichlet boundary condition is positive self-adjoint in $L^2(\Omega)$ with a compact resolvent. Let $\{\beta_n^2 \mid n \in \mathbb{N}\}$ be the set of its eigenvalues. Then $\beta_n^2 \to +\infty$ as $n \to +\infty$. For $0 < \theta < 2\pi$ given by (2.28), we obtain a sequence of delays

$$\tau_{n,k} = \frac{1}{\beta_n^2}(\theta + 2k\pi), \qquad n,k \in \mathbb{N},$$

for which the problem (3.1) loses its asymptotic stability. The proof of Theorem 1.6 is complete.

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