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On the porosity of the free boundary in the $p(x)$-obstacle problem

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Abstract. In this paper we consider the obstacle problem for the $p(x)$-Laplace operator. Assuming that p is locally Lipschitz continuous, we establish the growth rate of the solution near the free boundary from which we deduce its porosity.

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1. Introduction

Let Ω be a bounded open connected subset of $\mathbb{R}^n$, $n \geq 2$, $f \in L^\infty(\Omega)$ and $g \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$, $g \geq 0$. We consider the $p(x)$-obstacle problem with a zero obstacle, i.e. the obstacle problem for the $p(x)$-Laplacian

$$\begin{cases} \Delta_{p(x)} u = \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u) = f & \text{in } [u > 0], \\ u \geq 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

The weak formulation of this problem is given by the following variational inequality:

$$(\mathrm{P}) \qquad \begin{cases} \text{Find } u \in K_g \text{ such that:} \\ \displaystyle\int_\Omega \big(|\nabla u|^{p(x)-2} \nabla u \cdot \nabla(v - u) + f.(v - u)\big)\, dx \geq 0 \quad \text{for all } v \in K_g, \end{cases}$$

where $K_g = \{v \in W^{1,p(x)}(\Omega) : v - g \in W_0^{1,p(x)}(\Omega), v \geq 0 \text{ a.e. in } \Omega\}$, p is a measurable real valued function defined in Ω and satisfying

$$1 < p_- \leq p(x) \leq p_+ \qquad \text{a.e. } x \in \Omega \tag{1.1}$$

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for some positive numbers p_- and p_+. The space $W_0^{1,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$, where $W^{1,p(x)}(\Omega)$ is the variable exponent Sobolev space

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : \nabla u \in \left(L^{p(x)}(\Omega)\right)^n\right\}$$

and $L^{p(x)}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable} : \rho(u) = \int_\Omega |u(x)|^{p(x)} < \infty\}$ is equipped with the Luxembourg norm

$$\|u\|_{p(x)} = \inf\{\lambda > 0 : \rho(u/\lambda) \le 1\}.$$

$W^{1,p(x)}(\Omega)$ is equipped with the norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}, \qquad \text{where } \|\nabla u\|_{p(x)} = \sum_{i=1}^n \left\|\frac{\partial u}{\partial x_i}\right\|_{p(x)}.$$

If p is also log-Hölder continuous, i.e. satisfies for some $L > 0$

$$-|p(x) - p(y)| \log|x - y| \le L \qquad \text{for all } x, y \in \overline{\Omega}, \tag{1.2}$$

then we have [5] $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega) = W^{1,p(x)}(\Omega) \cap W_0^{1,1}(\Omega)$.

By $B_r(x)$ we shall denote the open ball in $\mathbb{R}^n$ with center x and radius r. The conjugate of $p(x)$ defined by $\frac{p(x)}{p(x)-1}$ will be denoted by $q(x)$.

We first give some classical properties of the solution in Section 1. In Section 2, we establish the growth rate of a class of functions. In Section 3, we obtain the exact growth rate of the solution of the problem (P) near the free boundary, from which we deduce its porosity. Our result on the porosity of the free boundary extends similar results for the Laplacian [1], for the p-Laplacian [8], and for the A-Laplacian [2]. As it was observed in [8], the free boundary has therefore Hausdorff dimension less than n and hence it is of Lebesgue measure zero.

First, we recall the following existence and uniqueness result established in [7]. We refer also to [11] for a much more general framework of entropy solution.

Proposition 1.1. *Assume that $f \in L^{q(x)}(\Omega)$ and $g \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$. Then there exists a unique solution u to the problem* (P).

In the following proposition, we generalize some classical properties of the obstacle problem.

Proposition 1.2. *Let u be the solution of* (P).

(i) *If* $f \geq 0$ *in* Ω, *then* $0 \leq u \leq \|g\|_\infty$ *in* Ω.

(ii) $f\chi([u > 0]) \leq \Delta_{p(x)} u \leq f$ *a.e. in* Ω.

(iii) *If* $u \in C^0(\Omega)$, *then* $\Delta_{p(x)} u = f$ *a.e. in* $[u > 0]$.

Proof. (i) Note that $u \geq 0$ since $u \in K_g$. To get the upper bound of u, we take $\min(u, \|g\|_\infty) = u - (u - \|g\|_\infty)^+$ as a test function in (P). We get

$$\int_\Omega |\nabla(u - \|g\|_\infty)^+|^{p(x)} \leq -\int_\Omega f(u - \|g\|_\infty)^+ \leq 0.$$

Then $\nabla(u - \|g\|_\infty)^+ = 0$ a.e. in Ω. Since $u = g$ on $\partial\Omega$, we deduce that $(u - \|g\|_\infty)^+ = 0$ a.e. in Ω.

(ii) Let $\zeta \in \mathscr{D}(\Omega)$, $\zeta \geq 0$.

First taking $u + \zeta \in K_g$ as a test function for (P), we obtain

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \zeta + f\zeta \geq 0,$$

which leads to $\Delta_{p(x)} u \leq f$ in $\mathscr{D}'(\Omega)$.

Now without loss of generality, we can assume that $\zeta \neq 0$. For $\varepsilon > 0$, let $H_\varepsilon(s) = \min\left(1, \frac{s^+}{\varepsilon}\right)$. Taking $u - \frac{\varepsilon}{|\zeta|_\infty} H_\varepsilon(u - \varepsilon)\zeta \in K_g$ as a test function for (P), we get

$$\int_\Omega H_\varepsilon(u - \varepsilon)|\nabla u|^{p(x)-2} \nabla u \nabla \zeta + f H_\varepsilon(u - \varepsilon)\zeta \leq 0.$$

Letting $\varepsilon \to 0$, we obtain

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \nabla \zeta + f\chi([u > 0])\zeta \leq 0,$$

which leads to $\Delta_{p(x)} u \geq f\chi([u > 0])$ in $\mathscr{D}'(\Omega)$.

(iii) Assume that $u \in C^0(\Omega)$, and let $\zeta \in \mathscr{D}([u > 0])$, $\zeta \geq 0$. Setting $\delta = \min_{\mathrm{supp}\,\zeta} u$ and taking $u \pm \delta \frac{\zeta}{\|\zeta\|_\infty}$ as test functions for (P), we get

$$\int_{[u>0]} |\nabla u|^{p(x)-2} \nabla u \nabla \zeta + f\zeta = 0.$$

We obtain $\Delta_{p(x)} u = f$ in $\mathscr{D}'([u > 0])$. □

Remark 1.3. Inequalities (ii) and equation (iii) of Proposition 1.2 were established in [11] in the framework of entropy solution, under the condition: $\operatorname{ess\,inf}_{x\in\Omega}\big(q_1(x) - \big(p(x)-1\big)\big) > 0$, where $q_1(x) = \frac{q_0(x)p(x)}{q_0(x)+1}$ and $q_0(x) = \frac{np(x)}{n-p(x)}\,\frac{p_- - 1}{p_-}$.

Remark 1.4. If $f \geq 0$ in Ω and $f \in L^\infty_{\mathrm{loc}}(\Omega)$, we know from Proposition 1.2 that u is bounded and $\Delta_{p(x)}u$ is locally bounded in Ω. If moreover $p \in C^{0,\beta}_{\mathrm{loc}}(\Omega)$, then we have (see [4]) $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$, for some $\alpha \in (0,1)$.

2. A class of functions on the unit ball

In all what follows, we assume that p is Lipschitz continuous, i.e. there exists a positive constant L such that

$$|p(x) - p(y)| \leq L|x-y| \qquad \text{for all } x, y \in \Omega. \tag{2.1}$$

In this section, we study a family $\mathscr{F}_{p(x)}$ of problems defined on the unit ball $B_1 = B_1(0)$. More precisely, $u \in \mathscr{F}_{p(x)}$ if it satisfies:

$$\begin{cases} u \in W^{1,p(x)}(B_1), & u(0) = 0, \\ 0 \leq u \leq 1 \text{ in } B_1, & \|\Delta_{p(x)}u\|_{L^\infty(B_1)} \leq 1. \end{cases}$$

We know (see [4]) that $u \in C^{1,\alpha}_{\mathrm{loc}}(B_1)$ for some $\alpha \in (0,1)$. In particular there exist two positive constants $\alpha = \alpha(n, p_-, p_+, L)$ and $C = C(n, p_-, p_+, L)$ such that

$$|u|_{1,\alpha,\bar{B}_{3/4}} \leq C \qquad \text{for all } u \in \mathscr{F}_{p(x)}. \tag{2.2}$$

The following theorem gives a growth rate of the elements in the class $\mathscr{F}_{p(x)}$.

Theorem 2.1. *There exists a positive constant $C_0 = C_0(n, p_-, p_+, L)$ such that for every $u \in \mathscr{F}_{p(x)}$, we have*

$$0 \leq u(x) \leq C_0|x|^{q_0} \qquad \textit{for all } x \in B_1,$$

where $q_0 = \dfrac{p_0}{p_0 - 1}$ *is the conjugate of* $p_0 = p(0)$.

Let us first introduce some notations. For a nonnegative bounded function u, we define the quantity $S(r,u) = \sup_{x\in B_r} u(x)$. We also define for each $u \in \mathscr{F}_{p(x)}$ the set

$$\mathbb{M}(u) = \{j \in \mathbb{N}/2^{q_0}S(2^{-j-1}, u) \geq S(2^{-j}, u)\}.$$

Then we have

Lemma 2.2. *If $\mathbb{M}(u) \neq \emptyset$, then there exists a constant $c_0 = c_0(n, p_-, p_+, L)$ such that*

$$S(2^{-j-1}, u) \leq c_0(2^{-j})^{q_0} \quad \textit{for all } u \in \mathscr{F}_{p(x)} \textit{ and } j \in \mathbb{M}(u).$$

Proof. Arguing by contradiction, we assume that

$$\forall k \in \mathbb{N}\ \exists u_k \in \mathscr{F}_{p(x)}\ \exists j_k \in \mathbb{M}(u_k) \quad \text{such that } S(2^{-j_k-1}, u_k) \geq k(2^{-j_k})^{q_0}. \tag{2.3}$$

Consider the function $v_k(x) = \dfrac{u_k(2^{-j_k}x)}{S(2^{-j_k-1}, u_k)}$ defined in B_1. By definition of v_k and $\mathbb{M}(u_k)$, we have

$$\begin{cases} 0 \leq v_k \leq \dfrac{S(2^{-j_k}, u_k)}{S(2^{-j_k-1}, u_k)} \leq 2^{q_0} \quad \text{in } B_1, \\ \sup_{x \in \bar{B}_{1/2}} v_k(x) = 1, \quad v_k(0) = 0. \end{cases}$$

Now let $p_k(x) = p(2^{-j_k}x)$. We claim that there exists $k_0 \in \mathbb{N}$ and a positive constant C independent of k such that

$$|\Delta_{p_k(x)} v_k(x)| \leq C\left(\frac{1}{k^{p_0-1}} + \frac{1}{k^{\alpha(p_0-1)^2}}\right) \leq 1 \quad \text{for all } k \geq k_0. \tag{2.4}$$

Indeed let $s_k = \dfrac{2^{-j_k}}{S(2^{-j_k-1}, u_k)}$. Then one can easily verify that

$$\begin{aligned} \Delta_{p_k(x)} v_k(x) &= 2^{-j_k} s_k^{p_k(x)-1} \Delta_{p(x)} u_k(2^{-j_k}x) \\ &\quad + 2^{-j_k}(\ln(s_k)) s_k^{p_k(x)-1} |\nabla u_k(2^{-j_k}x)|^{p_k(x)-2} \nabla u_k(2^{-j_k}x) \nabla p(2^{-j_k}x). \end{aligned}$$

Using the fact that $u_k \in \mathscr{F}_{p(x)}$ and $|\nabla p|_{L^\infty(\Omega)} \leq L$ (by (2.1)), this leads to

$$|\Delta_{p_k(x)} v_k(x)| \leq 2^{-j_k} s_k^{p_k(x)-1} + L 2^{-j_k} |\ln(s_k)| s_k^{p_k(x)-1} |\nabla u_k(2^{-j_k}x)|^{p_k(x)-1}.$$

Since $u_k \geq 0$ in B_1, $u_k(0) = 0$ and $u_k \in C^1(\bar{B}_{3/4})$, we have $\nabla u_k(0) = 0$. Combining this result and (2.2), we get for all $k \in \mathbb{N}$ and for all $x \in B_1$:

$$|\nabla u_k(2^{-j_k}x)| \leq C(2^{-j_k})^{\alpha}.$$

It follows that

$$|\Delta_{p_k(x)} v_k(x)| \leq 2^{-j_k} s_k^{p_k(x)-1} \left(1 + L(C)^{p_k(x)-1} |\ln(s_k)| (2^{-j_k})^{\alpha(p_k(x)-1)}\right). \tag{2.5}$$

Note that $S(2^{-j_k-1}, u_k) = u_k(z_k)$, for some $z_k \in \bar{B}_{2^{-j_k-1}}$. Since $u_k(0) = 0$ and $u_k \in C^1(\bar{B}_{3/4})$, we deduce that

$$S(2^{-j_k-1}, u_k) \le C|z_k| \le C2^{-j_k-1}.$$

Consequently, we obtain

$$s_k = \frac{2^{-j_k}}{S(2^{-j_k-1}, u_k)} \ge \frac{2^{-j_k}}{C2^{-j_k-1}} = \frac{2}{C} = \mu.$$

Now to estimate the righthand side of (2.5), we proceed as follows:

Estimate of $|\ln(s_k)|(2^{-j_k})^{\alpha(p_k(x)-1)}$: Since ln is a continuous function in $[\mu, \infty)$ and satisfies $\lim_{t\to\infty} t^{-\alpha(p_0-1)^2} \ln(t) = 0$, we deduce that there exists a positive constant $c_1 = c_1(\alpha, p_0, \mu)$ such that

$$|\ln(t)| \le c_1 t^{\alpha(p_0-1)^2} \qquad \text{for all } t \ge \mu.$$

In particular we have

$$|\ln(s_k)| \le c_1 s_k^{\alpha(p_0-1)^2} \qquad \text{for all } k \in \mathbb{N}.$$

We infer from (2.3) that

$$s_k^{p_0-1} 2^{-j_k} \le \frac{1}{k^{p_0-1}} \qquad \text{for all } k \in \mathbb{N}. \tag{2.6}$$

It follows that

$$\begin{aligned} |\ln(s_k)|(2^{-j_k})^{\alpha(p_0-1)} &\le c_1 s_k^{\alpha(p_0-1)^2} (2^{-j_k})^{\alpha(p_0-1)} = c_1 (s_k^{p_0-1} 2^{-j_k})^{\alpha(p_0-1)} \\ &\le \frac{c_1}{k^{\alpha(p_0-1)^2}} \qquad \forall k \in \mathbb{N}. \end{aligned}$$

Now we write

$$|\ln(s_k)|(2^{-j_k})^{\alpha(p_k(x)-1)} = |\ln(s_k)|(2^{-j_k})^{\alpha(p_0-1)} (2^{-j_k})^{\alpha(p_k(x)-p_0)}.$$

Using the Lipschitz continuity of p, we get

$$(2^{-j_k})^{\alpha(p_k(x)-p_0)} = e^{\alpha(p(0)-p(2^{-j_k}x))\ln(2^{j_k})} \le e^{\alpha L 2^{-j_k} |\ln(2^{j_k})|} \le c_2 = c_2(\alpha, L).$$

We deduce that we have for $c_3 = c_1 c_2$

$$|\ln(s_k)|(2^{-j_k})^{\alpha(p_k(x)-1)} \le \frac{c_3}{k^{\alpha(p_0-1)^2}} \qquad \text{for all } k \in \mathbb{N}. \tag{2.7}$$

Estimate of $2^{-j_k}s_k^{p_k(x)-1}$: We first write

$$2^{-j_k}s_k^{p_k(x)-1} = 2^{-j_k}s_k^{p_0-1}s_k^{p_k(x)-p_0}. \tag{2.8}$$

As we did above, we can find a positive constant $c_4 = c_4(p_0, \mu)$ such that

$$|\ln(t)| \leq c_4 t^{p_0-1} \qquad \text{for all } t \geq \mu.$$

In particular, we have

$$|\ln(s_k)| \leq c_4 s_k^{p_0-1} \qquad \text{for all } k \in \mathbb{N}.$$

Then we have by using (2.6)

$$\begin{aligned} s_k^{p_k(x)-p_0} &= e^{\alpha(p(2^{-j_k}x)-p(0))\ln(s_k)} \leq e^{\alpha L c_4 2^{-j_k}s_k^{p_0-1}} \\ &\leq e^{\alpha L c_4/k^{p_0-1}} \leq e^{\alpha L c_4} = c_5(\alpha, L, p_0, \mu). \end{aligned}$$

We deduce then from (2.6) and (2.8) that we have

$$2^{-j_k}s_k^{p_k(x)-1} \leq \frac{c_5}{k^{p_0-1}} \qquad \text{for all } k \in \mathbb{N}. \tag{2.9}$$

We conclude from (2.5), (2.7) and (2.9) that (2.4) is true.

Conclusion: Taking into account the uniform bound of v_k, (2.4) and the fact that p_k satisfies (1.1) and (2.1) with the same constants, we deduce (see [4]) that there exist two positive constants $\delta = \delta(n, p_-, p_+, L)$ and $C = C(n, p_-, p_+, L)$ such that $v_k \in C^{1,\delta}(\bar{B}_{3/4})$ and $|v_k|_{1,\delta,\bar{B}_{3/4}} \leq C$, for all $k \geq k_0$.

It follows then from Ascoli–Arzela's theorem that there exists a subsequence, still denoted by v_k and a function $v \in C^{1,\delta}(\bar{B}_{3/4})$ such that $v_k \to v$ in $C^1(\bar{B}_{3/4})$. Moreover, it is clear that v satisfies

$$\begin{cases} \Delta_{p_0} v = 0 \text{ in } B_{3/4}, \quad v \geq 0 \text{ in } B_{3/4}, \\ \sup_{x \in B_{1/2}} v(x) = 1, \quad v(0) = 0. \end{cases}$$

By the maximum principle we have necessarily $v \equiv 0$ in $B_{3/4}$, which is in contradiction with $\sup_{x \in B_{1/2}} v(x) = 1$. □

Proof of Theorem 2.1. Let $x \in B_1 \backslash \{0\}$. There exists $j \in \mathbb{N} \cup \{0\}$ such that $2^{-j-1} \leq |x| \leq 2^{-j}$. Then we have

$$u(x) \leq S(2^{-j}, u). \tag{2.10}$$

We shall prove by induction that we have

$$S(2^{-j}, u) \leq c_0'(2^{-j})^{q_0} \qquad \text{for all } j \in \mathbb{N} \cup \{0\}. \tag{2.11}$$

for some constant $c_0' = \max(c_0 2^{q_0}, 1)$

For $j = 0$, we have $S(2^{-0}, u) = S(1, u) \leq 1 = (2^{-0})^{q_0} \leq c_0'(2^{-0})^{q_0}$.

Let $j \geq 1$. Assume that $S(2^{-j}, u) \leq c_0'(2^{-j})^{q_0}$. We distinguish two cases:

– If $j \in \mathbb{M}(u)$, we have by Lemma 2.2,

$$S(2^{-(j+1)}, u) = S(2^{-j-1}, u) \leq c_0(2^{-j})^{q_0} = c_0 2^{q_0}(2^{-(j+1)})^{q_0} \leq c_0'(2^{-(j+1)})^{q_0}.$$

– If $j \notin \mathbb{M}(u)$, we have $S(2^{-(j+1)}, u) = S(2^{-j-1}, u) < 2^{-q_0} S(2^{-j}, u)$. Using the induction assumption, we get

$$S(2^{-(j+1)}, u) \leq 2^{-q_0} c_0'(2^{-j})^{q_0} = c_0'(2^{-(j+1)})^{q_0}.$$

We conclude from (2.10)–(2.11) that

$$u(x) \leq S(2^{-j}, u) \leq c_0'(2^{-j})^{q_0} \leq c_0'(2|x|)^{q_0} = C_0|x|^{q_0}. \qquad \square$$

3. Porosity of the free boundary

In all what follows, we assume that there exist positive constants λ_0, Λ_0 such that

$$0 < \lambda_0 \leq f \leq \Lambda_0 \qquad \text{a.e. in } \Omega. \tag{3.1}$$

The following lemma and Theorem 2.1 give the exact growth rate of the solution of the problem (P) near the free boundary. This extends a result established in [1] for the Laplacian, and generalized in [8] for the p-Laplacian (see also [2] for the A-Laplacian).

Lemma 3.1. *Suppose that $u \in W^{1,p(x)}(\Omega)$ is a nonnegative continuous function satisfying*

$$\Delta_{p(x)} u = f \qquad \text{in } \mathscr{D}'([u > 0]).$$

Then there exists $r_* > 0$ such that for each $y \in \overline{[u > 0]}$ and $r \in (0, r_*)$ satisfying $B_r(y) \subset \Omega$, we have

$$\sup_{\partial B_r(y)} u \geq C(y) r^{p(y)/(p(y)-1)} + u(y),$$

where $C(y) = \left(1 - \frac{1}{p(y)}\right)\left(\frac{\lambda_0}{2n}\right)^{1/(p(y)-1)}$.

Proof. It is enough to prove the result for $y \in [u > 0]$. We consider the function defined by

$$v(x, y) = C(y)|x - y|^{p(y)/(p(y)-1)}.$$

Then it is not difficult to verify that

$$\Delta_{p(y)} v = \frac{\lambda_0}{2} \quad \text{and} \quad \Delta_{p(x)} v = \left(\frac{\lambda_0}{2n}\right)^{(p(x)-1)/(p(y)-1)} |x - y|^{(p(x)-p(y))/(p(y)-1)} \theta(x, y),$$

where

$$\begin{aligned} \theta(x, y) = n + \frac{p(x) - p(y)}{p(y) - 1} + \big(q(y) - 1\big) \ln\left(\frac{\lambda_0}{2n}\right) \nabla p(x) \cdot (x - y) \\ + \big(q(y) - 1\big) \cdot (x - y) \cdot \nabla p(x) \cdot \ln(|x - y|). \end{aligned}$$

We claim that there exists $r_* > 0$ such that

$$\forall r \in (0, r_*)\ \forall y \in \Omega\ \forall x \in B_r(y) \subset \Omega : 0 \le \Delta_{p(x)} v \le \lambda_0. \tag{3.2}$$

To prove (3.2), we first write $\Delta_{p(x)} v - \dfrac{\lambda_0}{2}$ in the form

$$\begin{aligned} \Delta_{p(x)} v - \frac{\lambda_0}{2} &= \frac{\lambda_0}{2} \left[\left(\frac{\lambda_0}{2n}\right)^{(p(x)-p(y))/(p(y)-1)} |x - y|^{(p(x)-p(y))/(p(y)-1)} - 1 \right] \\ &\quad + \frac{1}{p(y) - 1} \frac{\lambda_0}{2n} \left(\frac{\lambda_0}{2n}\right)^{(p(x)-p(y))/(p(y)-1)} |x - y|^{(p(x)-p(y))/(p(y)-1)} \\ &\quad \times \left[p(x) - p(y) + \ln\left(\frac{\lambda_0}{2n}\right) \nabla p(x).(x - y) \right. \\ &\qquad \left. + (x - y).\nabla p(x).\ln(|x - y|) \right]. \end{aligned}$$

For $|x - y| < r < \frac{1}{e}$, we have

$$\begin{aligned} |x - y|^{(p(x)-p(y))/(p(y)-1)} &= e^{(p(x)-p(y))/(p(y)-1)\ln(|x-y|)} \\ &\le e^{L/(p_- - 1)|x-y|\,|\ln(|x-y|)|} \le e^{L/(p_- - 1)r|\ln(r)|}. \end{aligned}$$

Similarly, we have for $|x - y| < r$ and $\Lambda_1 = \left|\ln\left(\dfrac{\lambda_0}{2n}\right)\right|$

$$\left(\frac{\lambda_0}{2n}\right)^{(p(x)-p(y))/(p(y)-1)} = e^{(p(x)-p(y))/(p(y)-1)\ln(\lambda_0/2n)}$$
$$\leq e^{L/(p_- -1)|x-y|\,|\ln(\lambda_0/2n)|} \leq e^{\Lambda_1 Lr/(p_- -1)}.$$

We deduce that for $|x-y| < r < \frac{1}{e}$, we have

$$\left|\Delta_{p(x)}v - \frac{\lambda_0}{2}\right| \leq \frac{\lambda_0}{2}[e^{(L\Lambda_1/(p_- -1))r}e^{(L/(p_- -1))r|\ln(r)|} - 1]$$
$$+ \frac{1}{p_- -1}\frac{\lambda_0}{2n}e^{(L\Lambda_1/(p_- -1))r}e^{(L/(p_- -1))r|\ln(r)|}[Lr + L\Lambda_1 r + Lr|\ln(r)|].$$

It is clear now that there exists $r_* > 0$ such that for all $r \in (0, r_*)$, the right-hand side of the above inequality is less than $\lambda_0/2$. Hence (3.2) holds.

Now let $\varepsilon > 0$ and consider the following function $u_\varepsilon(x) = u(x) - (1-\varepsilon)u(y)$.

We have from (3.1)–(3.2)

$$\Delta_{p(x)}u_\varepsilon = \Delta_{p(x)}u = f \geq \lambda_0 \geq \Delta_{p(x)}v \quad \text{in } B_r(y) \cap [u > 0].$$

Moreover

$$u_\varepsilon = -(1-\varepsilon)u(y) \leq 0 \leq v \quad \text{on } (\partial[u > 0]) \cap B_r(y).$$

If we also have

$$u_\varepsilon \leq v \quad \text{on } \big(\partial B_r(y)\big) \cap [u > 0],$$

then we get by the weak maximum principle

$$u_\varepsilon \leq v \quad \text{in } B_r(y) \cap [u > 0].$$

But $u_\varepsilon(y) = \varepsilon u(y) > 0 = v(y)$ which constitutes a contradiction.

So there exists $z \in \big(\partial B_r(y)\big) \cap [u > 0]$ such that $u_\varepsilon(z) > v(z)$. Since v is radial, we get

$$\sup_{\partial B_r(y)}\big(u - (1-\varepsilon)u(y)\big) = \sup_{\partial B_r(y)} u_\varepsilon \geq \sup_{\partial B_r(y)\cap[u>0]} u_\varepsilon \geq u_\varepsilon(z)$$
$$> v(z) = C(y)r^{p(y)/(p(y)-1)}.$$

Letting $\varepsilon \to 0$, we get

$$\sup_{\bar{B}_r(y)} u \geq \sup_{\partial B_r(y)} u \geq C(y)r^{p(y)/(p(y)-1)} + u(y). \qquad \square$$

We shall denote by u the solution of the problem (P). The main result of this section is the porosity of the free boundary $(\partial[u>0])\cap\Omega$.

We recall that a set $E\subset\mathbb{R}^n$ is called porous with porosity δ, if there is an $r_0>0$ such that

$$\forall x\in E\ \forall r\in(0,r_0)\ \exists y\in\mathbb{R}^n \quad \text{such that } B_{\delta r}(y)\subset B_r(x)\backslash E.$$

A porous set has Hausdorff dimension not exceeding $n-c\delta^n$, where $c=c(n)>0$ is a constant depending only on n. In particular, a porous set has Lebesgue measure zero.

Theorem 3.2. *Let r_* be as in Lemma 3.1, $R\in(0,r_*)$ and $x_0\in\Omega$ such that $\overline{B_{4R}(x_0)}\subset\Omega$. Then $(\partial[u>0])\cap\overline{B_R(x_0)}$ is porous with porosity constant depending only on $\|g\|_\infty$, λ_0, Λ_0, R, $\operatorname{dist}(\overline{B_R(x_0)},\partial\Omega)$, p_-, p_+, L and n. As an immediate consequence, we have*

$$\Delta_{p(x)}u=f\chi([u>0]) \quad \text{a.e. in } \Omega.$$

We need a lemma.

Lemma 3.3. *Let $R>0$ and $x_0\in\Omega$ such that $\overline{B_{4R}(x_0)}\subset\Omega$. We consider, for $y_0\in\overline{B_{2R}(x_0)}\cap[u=0]$ and $M\geq1$, the functions defined in $\bar{B}_1$ by*

$$\tilde{p}(z)=p(y_0+Rz), \qquad \tilde{u}(z)=\frac{u(y_0+Rz)}{MR}. \tag{3.3}$$

Then there exists M_0 depending on $\|g\|_\infty$, Λ_0, R, $\operatorname{dist}(\overline{B_{3R}(y_0)},\partial\Omega)$, p_-, p_+, L and n such that for any $M\geq M_0$, we have $\tilde{u}\in\mathscr{F}_{\tilde{p}(z)}$.

Proof. First, note that $\tilde{p}$ and $\tilde{u}$ are well defined, since we have $\overline{B_R(y_0)}\subset\overline{B_{3R}(x_0)}\subset\Omega$. Moreover we have $\tilde{u}(0)=\dfrac{u(y_0)}{MR}=0$, and for $M\geq\dfrac{\|g\|_\infty}{R}$, we have $0\leq\tilde{u}\leq1$ in B_1.

Next, one can easily verify that $\tilde{u}$ satisfies

$$\begin{aligned}\Delta_{\tilde{p}(z)}\tilde{u}&=\frac{R}{M^{\tilde{p}(z)-1}}(\Delta_{p(.)}u)(y_0+Rz)\\&\quad-\frac{R\ln(M)}{M^{\tilde{p}(z)-1}}|\nabla u(y_0+Rz)|^{\tilde{p}(z)-2}\nabla u(y_0+Rz).\nabla p(x_0+Rz).\end{aligned}$$

It follows that $\tilde{u}$ satisfies

$$\begin{aligned}
&\|\Delta_{\tilde{p}(z)}\tilde{u}\|_{\infty, B_1} \\
&\quad \le \frac{R}{M^{p_- -1}}\left(\Lambda_0 + L|\ln(M)|\max(|\nabla u|^{p_- -1}_{\infty, \overline{B_{3R}(x_0)}}, |\nabla u|^{p_+ -1}_{\infty, \overline{B_{3R}(x_0)}})\right) \\
&\quad \le \frac{R}{M^{p_- -1}}\left(\Lambda_0 + L|\ln(M)|C\big(n, p_-, p_+, L, \|g\|_\infty, \Lambda_0, \operatorname{dist}\big(\overline{B_{3R}(x_0)}, \partial\Omega\big)\big)\right).
\end{aligned}$$

Hence there exists M_0 depending on $\|g\|_\infty$, Λ_0, R, $\operatorname{dist}\big(\overline{B_{3R}(y_0)}, \partial\Omega\big)$, p_-, p_+, L and n such that for all $M \ge M_0$, we have $\|\Delta_{\tilde{p}(z)}\tilde{u}\|_{\infty, B_1} \le 1$. We conclude that $\tilde{u} \in \mathscr{F}_{\tilde{p}(z)}$ for all $M \ge M_0$. $\square$

Proof of Theorem 3.2. Let $R \in (0, r_*)$ such that $\overline{B_{4R}(x_0)} \subset \Omega$, and let $x \in E = \partial[u > 0] \cap \overline{B_R(x_0)}$. For each $0 < r < R$, we have $\overline{B_r(x)} \subset B_{2R}(x_0) \subset \Omega$. Let $y \in \partial B_r(x)$ such that $u(y) = \sup_{\partial B_r(x)} u$. Then we have by Lemma 3.1

$$u(y) \ge C_0 r^{p(x)/(p(x)-1)} + u(x) = C_0 r^{p(x)/(p(x)-1)}, \tag{3.4}$$

with $C_0 = \left(1 - \frac{1}{p_-}\right)\min\left(\left(\frac{\lambda_0}{2n}\right)^{1/(p_- -1)}, \left(\frac{\lambda_0}{2n}\right)^{1/(p_+ -1)}\right)$.

Hence $y \in B_{2R}(x_0) \cap [u > 0]$. We denote by $d(y) = \operatorname{dist}\big(y, \overline{B_{2R}(x_0)} \cap [u = 0]\big)$ the distance from y to the set $\overline{B_{2R}(x_0)} \cap [u = 0]$. By continuity of d, there exists $y_0 \in \overline{B_{2R}(x_0)} \cap [u = 0]$ such that $d(y) = |y - y_0|$.

Now we claim that there exists a constant C_1 such that

$$u(y) \le C_1\big(d(y)\big)^{p(y_0)/(p(y_0)-1)}. \tag{3.5}$$

To prove (3.5), we will apply Theorem 2.1 to the functions defined in $\bar{B}_1$ by (3.3).

First note that $\overline{B_R(y_0)} \subset \overline{B_{3R}(x_0)} \subset \Omega$. Indeed let $z \in \overline{B_R(y_0)}$. We have

$$|z - x_0| \le |z - y_0| + |y_0 - x_0| \le R + 2R = 3R, \qquad \text{which means that } z \in \overline{B_{3R}(x_0)}.$$

Next, it is easy to see that $\tilde{p}$ satisfies (1.1) and (2.1) with the constants p_-, p_+ and LR respectively. Moreover by Lemma 3.3, there exits M_0 such that for all $M \ge M_0$, we have $\tilde{u} \in \mathscr{F}_{\tilde{p}(z)}$. Applying Theorem 2.1, we obtain for a positive constant C depending only on n, p_-, p_+ and LR, that

$$\tilde{u}(z) \le C|z|^{\tilde{p}(0)/(\tilde{p}(0)-1)} \qquad \text{for all } z \in B_1.$$

Since $x \in \overline{B_R(x_0)} \cap [u = 0]$, we have $d(y) \le |y - x| = r < R$. Therefore we have $|y - y_0| < R$, and we can apply the previous inequality to $z = \dfrac{y - y_0}{R} \in B_1$. We obtain

$$\frac{1}{MR}u(y) \le C\left(\frac{|y-y_0|}{R}\right)^{p(y_0)/(p(y_0)-1)} \qquad \text{or}$$

$$u(y) \le CMR^{-1/(p(y_0)-1)}\big(d(y)\big)^{p(y_0)/(p(y_0)-1)},$$

which is (3.5). We deduce from (3.4)–(3.5) that

$$C_0 r^{p(x)/(p(x)-1)} \le u(y) \le C_1\big(d(y)\big)^{p(y_0)/(p(y_0)-1)}. \tag{3.6}$$

Using (1.1) and (2.1), we get

$$\begin{aligned} r^{p(x)/(p(x)-1)} &= r^{(p(y)-p(x))/(p(x)-1)(p(y)-1)} r^{p(y)/(p(y)-1)} \\ &= e^{(p(y)-p(x))/(p(x)-1)(p(y)-1)\ln(r)} r^{p(y)/(p(y)-1)} \\ &\ge e^{-Lr|\ln(r)|/(p_- -1)^2} r^{p(y)/(p(y)-1)} \\ &\ge m_0 r^{p(y)/(p(y)-1)}, \end{aligned} \tag{3.7}$$

where

$$m_0 = \min_{t\in[0,D(\Omega)/2]} e^{-Lt|\ln(t)|/(p_- -1)^2}, \qquad D(\Omega) = \sup_{x,y\in\bar{\Omega}} |x-y|.$$

Similarly, we have

$$\begin{aligned} \big(d(y)\big)^{p(y_0)/(p(y_0)-1)} &= \big(d(y)\big)^{(p(y)-p(y_0))/(p(y_0)-1)(p(y)-1)}\big(d(y)\big)^{p(y)/(p(y)-1)} \\ &= e^{(p(y)-p(y_0))/(p(y_0)-1)(p(y)-1)\ln(d(y))}\big(d(y)\big)^{p(y)/(p(y)-1)} \\ &\le e^{Ld(y)|\ln(d(y))|/(p_- -1)^2}\big(d(y)\big)^{p(y)/(p(y)-1)} \\ &\le m_1\big(d(y)\big)^{p(y)/(p(y)-1)}, \end{aligned} \tag{3.8}$$

where

$$m_1 = \max_{t\in[0,D(\Omega)]} e^{Lt|\ln(t)|/(p_- -1)^2}.$$

Hence we obtain from (3.6)–(3.8)

$$C_0 m_0 r^{p(y)/(p(y)-1)} \le u(y) \le C_1 m_1\big(d(y)\big)^{p(y)/(p(y)-1)},$$

which leads to

$$d(y) \ge \left(\frac{m_0C_0}{m_1C_1}\right)^{1/q(y)} r \ge \delta r, \qquad \text{where}$$

$$\delta = \min\left(\frac{1}{2}, \left(\frac{m_0C_0}{m_1C_1}\right)^{1-1/p_-}, \left(\frac{m_0C_0}{m_1C_1}\right)^{1-1/p_+}\right) < 1.$$

Let now $y^* \in [x, y]$ such that $|y - y^*| = \delta r/2$. Then we have

$$B_{(\delta/2)r}(y^*) \subset B_{\delta r}(y) \cap B_r(x).$$

Indeed, we have for each $m \in B_{\delta r/2}(y^*)$

$$|m - y| \leq |m - y^*| + |y^* - y| < \frac{\delta r}{2} + \frac{\delta r}{2} = \delta r$$
$$|m - x| \leq |m - y^*| + (|x - y| - |y^* - y|) < \frac{\delta r}{2} + \left(r - \frac{\delta r}{2}\right) = r.$$

Moreover, we have

$$B_{\delta r}(y) \cap B_r(x) \subset [u > 0]$$

since $B_{\delta r}(y) \subset B_{d(y)}(y) \subset [u > 0]$ and $d(y) \geq \delta r$.

Hence we have

$$B_{(\delta/2)r}(y^*) \subset B_{\delta r}(y) \cap B_r(x) \subset B_r(x) \backslash \partial[u > 0] \subset B_r(x) \backslash E. \qquad \square$$

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