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Fixed point theorems for 1-set weakly contractive and pseudocontractive operators on an unbounded domain

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Abstract. We establish new fixed point results for some nonlinear weakly condensing, 1-set weakly contractive, pseudo-contractive and nonexpansive operators defined on unbounded domains under different boundary conditions as well as other additional assumptions. An existence result of positive eigenvalues and eigenvectors for nonlinear weakly condensing operators and an application to generalized Hammerstein integral equations are given.

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1. Introduction

For more than forty years, the study of condensing, 1-set contractive, nonexpansive and pseudo-contractive operators, has been the main object of research in Nonlinear Functional Analysis, especially, the existence of fixed points for a closed convex subset into itself for one of this class of operators and was started by Browder [5], [6], [7], Sadovskii [34], Petryshyn [31], [32], Nussbaum [28], Kirk [4], [20], [21], Morales [26] and others. These studies were mainly based upon the potential tool of degree theory, geometry of the ambient Banach space (reflexivity, uniform convexity, normal structure, etc.), properties of operators (semi-closed, demi-continuous, demi-closed, etc.) and boundary conditions in particular the famous Leray–Schauder condition. Since then, whether the mentioned operators defined on the closure of bounded subset of a Banach space has a fixed point has become an interesting problem.

Recently, some existence results for fixed points, positive eigenvalues, and eigenvectors for 1-set contractive, pseudo-contractive, nonexpansive operators

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under several boundary conditions (Leray-Schauder, weak inwardness and others) as well as others additional assumptions and without exigence of the boundedness of the domain have been considered by some authors [15], [18], [24], [27], [29]. Due to the lack of compactness in L_1-spaces which equipped with their weak topology, are the convenient and natural setting to investigate the existence problems of fixed point and eigenvectors for operators and solutions of various kinds of nonlinear differential equations and nonlinear integral equations in Banach spaces, the above mentioned results cannot be easily applied. These equations can be transformed into fixed point problems and nonlinear equations involving a broader class of nonlinear operators which are (ws)-compact (See Definition 2.3) and among them have the property that the image of any set is in a certain sense more weakly compact than the original set itself (see [3], [12], [14], [23], [22]).

In this paper, we introduce the concept of demi-weakly compact at the origin (see Definition 3.5) and examine weakly condensing, 1-set weakly contractive, pseudo-contractive and nonexpansive operators mainly in the case when the domain is an unbounded subset of a Banach space. We are able to find fixed points for such operators under several boundary conditions as well as some additional assumptions (Leray-schauder, weak inwardness and others) and, if useful, by imposing a boundedness condition on the operators. Our results generalize and extend relevant and recent ones (see [1], [23]). In addition, our arguments and methods are elementary in the sense that they do not need any recourse to degree theory or theory of homotopy-extensions.

2. Preliminaries

Before we state and prove our fixed point theorems, we first collect some notation and preliminary facts from the theory of operators defined on Banach spaces.

Throughout this paper we assume that $(E, \|\cdot\|)$ is a real Banach space, E^* is its topological dual and θ means the zero vector of the space E. As usual, we will denote $B_r(z)$, and $S_r(z)$, the closed ball, and the sphere, with radius $r > 0$ and center $z \in E$, respectively. Here $\overset{w}{\to}$ denotes weak convergence and $\to$ denotes strong convergence in E.

If $x \in E$, we will denote as $J(x)$ the normalized duality mapping at x defined by $J(x) := \{j \in E^* : j(x) = \|x\|^2, \|j\| = \|x\|\}$. We will use the mapping $\langle \cdot, \cdot \rangle_+ : E \times E \to \mathbb{R}$ defined by $\langle y, x \rangle_+ := \max\{j(y) : j \in J(x)\}$.

Let C be a nonempty subset of E. Recall that a mapping $T : C \to E$ is said to be nonexpansive whenever $\|T(x) - T(y)\| \leq \|x - y\|$ for every $x, y \in C$. Recall that a sequence (x_n) of elements of C is said to be an a.f.p. sequence for T whenever $\lim_{n\to\infty} \|x_n - T(x_n)\| = 0$. It is well known that if T is a nonexpansive mapping which maps a closed convex bounded subset C of E into itself, then such mapping always has an a.f.p. sequence in C.

A mapping $A : D(A) \subseteq E \to E$ will be called an accretive operator on E if and only if $\langle A(x) - A(y), x - y \rangle_+ \geq 0$ for all $x, y \in D(A)$.

If, in addition, $R(I + \lambda A)$ is for one, hence for all, $\lambda > 0$, precisely E, then A is called m-accretive. We say that A satisfies the range condition if $\overline{D(A)} \subset R(I + \lambda A)$ for all $\lambda > 0$.

We now recall some important facts regarding accretive operators which will be used in our paper (see for example [8]).

Proposition 2.1. *Let $A : D(A) \to E$ be a mapping. The following conditions are equivalent:*

- *A is an accretive operator.*
- *The inequality $\|x - y\| \leq \|x - y + \lambda(A(x) - A(y))\|$ holds for all $\lambda \geq 0$, and for every $x, y \in D(A)$.*
- *For each $\lambda > 0$ the resolvent $J_\lambda := (I + \lambda A)^{-1} : R(I + \lambda A) \to D(A)$ is a single-valued nonexpansive mapping.*

A mapping $T : C \to E$ is said to be pseudo-contractive if for every $x, y \in C$, and for all positive r, $\|x - y\| \leq \|(1 + r)(x - y) - r(T(x) - T(y))\|$. Pseudo-contractive mappings are easily seen to be more general than nonexpansive mappings ones. The interest in these mappings also stems from the fact that they are firmly connected to the well known class of accretive mappings. Specifically T is pseudo-contractive if and only if $I - T$ is accretive where I is the identity mapping.

We say that the mapping $T : C \to E$ is weakly inward on C if

$$\lim_{\lambda \to 0^+} d((1 - \lambda)x + \lambda T(x), C) = 0$$

for all $x \in C$. Such condition is always weaker than the assumption of T mapping the boundary of C into C. Recall that if $A : D(A) \to X$ is a continuous accretive mapping, $D(A)$ is convex and closed and $I - A$ is weakly inward on $D(A)$, then A has the range condition (see [25]).

In [15] the authors considered several fixed points results for continuous pseudo-contractive mappings with unbounded domains satisfying additional conditions in terms of a function $G : E \times E \to \mathbb{R}$ under the following assumptions:

(g1) $G(\lambda x, y) \leq \lambda G(x, y)$ for any $x, y \in E$ and $\lambda > 0$,

(g2) there exists $S > 0$ such that $0 < G(x, x)$ for any $x \in E$ with $\|x\| \geq S$,

(g3) $G(x + y, z) \leq G(x, z) + G(y, z)$ for any $x, y, z \in E$,

(g4) for each $y \in E$, there exist $t > 0$ (depending on y) such that if $\|x\| \geq t$, then $|G(y, x)| < G(x, x)$.

Definition 2.2. Let E be a Banach space. An operator $F : E \to E$ is said to be weakly compact if $F(\Omega)$ is relatively weakly compact for every bounded subset $\Omega \subseteq E$.

Definition 2.3 [36]. Let E be a Banach space. An operator (not necessary linear) $F : E \to E$ is said to be strongly continuous on E if for every sequence $(x_n)_n$ with $x_n \xrightarrow{w} x$, we have $F(x_n) \to F(x)$.

Next we introduce the notion of (ws)-compact operators.

Definition 2.4 [17]. Let D a subset of a Banach space $(E, \|\cdot\|)$. An operator (not necessarily linear) $F : D \to E$ is said to be (ws)-compact if F is $\|\cdot\|$-continuous and, for every weakly convergent sequence $(x_n)_{n\in\mathbb{N}}$ elements of D, the sequence $\big(F(x_n)\big)_{n\in\mathbb{N}}$ admits a strongly convergent subsequence.

Remark 2.5. (i) As examples of (ws)-compact operators we have compact operators and strongly continuous operators.

(ii) A map F is (ws)-compact if and only if it maps relatively weakly compact sets into relatively compact ones.

(iii) F is (ws)-compact does not imply that F is sequentially weakly continuous (i.e., $x_n \xrightarrow{w} x$ implies $F(x_n) \xrightarrow{w} F(x)$); see [23], [22].

The following fixed point result stated in [23], will be used in the next section. The proof follows from Schauder's fixed point theorem.

Theorem 2.6. *Let Ω be a nonempty closed convex subset of a Banach space E. Assume that $F : \Omega \to \Omega$ is (ws)-compact. If $F(\Omega)$ is relatively weakly compact, then there exists $x \in \Omega$ such that $F(x) = x$.*

Definition 2.7. Let E be a Banach space and C a lattice with a least element, which is denoted by 0. By a measure of weak non-compactness $(MNWC)$ on E, we mean a function Φ defined on the set of all bounded subsets of E with values in C satisfying:

(1) $\Phi\big(\overline{\mathrm{conv}}(\Omega)\big) = \Phi(\Omega)$, for all bounded subsets $\Omega \subseteq E$, where $\overline{\mathrm{conv}}$ denotes the closed convex hull of Ω.

(2) For any bounded subsets Ω_1, Ω_2 of E we have

$$\Omega_1 \subseteq \Omega_2 \implies \Phi(\Omega_1) \leq \Phi(\Omega_2).$$

(3) $\Phi(\Omega \cup \{a\}) = \Phi(\Omega)$ for all $a \in E$, Ω bounded set of E.

(4) $\Phi(\Omega) = 0$ if and only if Ω is relatively weakly compact in E.

The *MNWC* Φ is said positive homogenous provided $\Phi(\lambda\Omega) = \lambda\Phi(\Omega)$ for all $\lambda > 0$ and Ω is a bounded set in E.

The above notion is a generalization of the important well known De Blasi measure of weak non-compactness β (see [9]) defined on each bounded set Ω of E by

$$\beta(\Omega) = \inf\{\varepsilon > 0 : \text{there exists a weakly compact set } D \text{ such that } \Omega \subseteq D + B_\varepsilon(\theta)\}.$$

It is well known that β enjoys these properties for all bounded subsets Ω, Ω_1, Ω_2 of E:

(5) $\beta(\Omega_1 \cup \Omega_2) = \max\{\beta(\Omega_1), \beta(\Omega_2)\}$.

(6) $\beta(\lambda\Omega) = \lambda\beta(\Omega)$ for all $\lambda > 0$.

(7) $\beta(\Omega_1 + \Omega_2) \leq \beta(\Omega_1) + \beta(\Omega_2)$.

Definition 2.8. Let Ω be a nonempty subset of Banach space E and Φ a MNWC on E. If F maps Ω into E, we say that

(a) F is Φ-condensing if $\Phi\big(F(D)\big) < \Phi(D)$ for all bounded sets $D \subseteq \Omega$ with $\Phi(D) \neq 0$, and

(b) F is Φ-nonexpansive map if $\Phi\big(F(D)\big) \leq \Phi(D)$ for all bounded sets $D \subseteq \Omega$.

3. Fixed point theorems

First, we state and prove an analogue of Sadovskii's fixed point theorem for *ws*-compact, weakly condensing mapping defined on unbounded closed convex set.

Theorem 3.1. *Let Ω be a non-empty unbounded closed convex set in a Banach space E. Assume Φ is a MNWC on E and $F : \Omega \to \Omega$ is a Φ-condensing (ws)-compact mapping. In addition, suppose that $F(\Omega)$ is bounded. Then the set of fixed points of F in Ω is nonempty and compact.*

Proof. Let $x_0 \in \Omega$ and $D = \{F^n(x_0), n = 0, 1, 2, \ldots\}$ where $F^0(x_0) = x_0$. Then $D = F(D) \cup \{x_0\}$ and so $\Phi\big(F(D)\big) = \Phi(D)$ which means that $\Phi(D) = 0$ and D is relatively weakly compact. By Remark 2.5 (ii), $F(D)$ is relatively compact. Also, $F\big(F(D)\big) \subseteq F(D)$ so by [16], Lemma 1, one may choose a compact set $D_0 \subseteq F(D)$ with $D_0 \subseteq \overline{\mathrm{conv}}\big(F(D_0)\big)$. Let $\mathscr{T} = \{Q \mid D_0 \subseteq Q, Q = \overline{\mathrm{conv}}\, Q, F(Q) \subseteq Q\}$. It is obvious that $\mathscr{T} \neq \emptyset$, since $\Omega \in \mathscr{T}$. If ξ is a chain in the ordered set $(E, \subseteq)$ then $\bigcap_{Q \in \xi} Q$ is a lower bound of ξ, which can be easily verified. Hence by Zorn's lemma $\mathscr{T}$ has a minimal element K. From $F(K) \subseteq K$, since K is closed and convex, it follows that the set $\overline{\mathrm{conv}}(F(K))$ is a subset of K. So we have

$$F\left(\overline{\mathrm{conv}}(F(K))\right) \subseteq F(K) \subseteq \overline{\mathrm{conv}}(F(K)).$$

From $D_0 \subseteq \overline{\mathrm{conv}}(F(K))$, it follows that the set $\overline{\mathrm{conv}}(F(K))$ is in $\mathscr{T}$. Since K is a minimal element of $\mathscr{T}$ it follows that $K = \overline{\mathrm{conv}}(F(K))$. Hence, $\Phi(K) = \Phi\left(\overline{\mathrm{conv}}(F(K))\right) = \Phi(F(K))$. Since F is Φ-condensing, we obtain $\Phi(K) = \Phi(F(K)) = 0$, and $F(K)$ is relatively weakly compact. Now, F is a (ws)-compact map from the closed convex set K into itself. From Theorem 2.6, F has a fixed point in $K \subseteq \Omega$. Let $\mathscr{S} = \{x \in \Omega : F(x) = x\}$, be the fixed point set of F. Since F is continuous, $\mathscr{S}$ is obviously a closed subset of Ω such that $F(\mathscr{S}) = \mathscr{S}$. Since $\mathscr{S} \subseteq F(\Omega)$, $F(\mathscr{S}) = \mathscr{S}$ and F is Φ-condensing, we have $\Phi(\mathscr{S}) = 0$ and so $\mathscr{S}$ is a relatively weakly compact subset of Ω. Besides, F is (ws)-compact so $F(\mathscr{S}) = \mathscr{S}$ is relatively compact. Since $\mathscr{S}$ is closed, we obtain that $\mathscr{S}$ is compact. This proof is complete. □

Remark 3.2. Theorem 3.1 extends and improves Theorem 2.2 in [23].

Corollary 3.3. *Let Ω be a nonempty unbounded closed convex set in a Banach space E. Assume that $F : \Omega \to \Omega$ is a (ws)-compact mapping which satisfies that $F(\Omega)$ is bounded and $F(D)$ is relatively weakly compact whenever D is a bounded set of Ω. Then the set of fixed points of F in Ω is nonempty and compact.*

Proof. This is an immediate consequence of Theorem 3.1 since F is clearly Φ-condensing where Φ is any $MNWC$ on E. □

Remark 3.4. Corollary 3.3 is a sharpening of Theorem 3.1.

Definition 3.5. A mapping $F : \Omega \to E$ is said to be demi-weakly compact at θ ((dwc) for short) if for every bounded a.f.p. sequence $(x_n)_n$ in Ω (i.e., $x_n - F(x_n) \to \theta$) then $(x_n)_n$ has a weakly convergent subsequence.

Now, let us recall the following well known concept of mapping due to Petryshyn [30]:

Definition 3.6. A mapping $F : \Omega \to E$ is said to be *demi-compact* at $\theta \in E$((dc) for short) if, for every bounded a.f.p. sequence (x_n) in Ω, there exists a strongly convergent subsequence of (x_n).

Remark 3.7. If Ω is a closed subset of E and $F : \Omega \to E$ is a continuous mapping, demi-compact at θ and it admits a bounded a.f.p. sequence (x_n), then it has a fixed point. Indeed, suppose that (x_n) is a bounded sequence in Ω such that

$x_n - F(x_n) \to \theta$. It follows from the demi-compactness of F, that there exists a subsequence of (x_n) which converges strongly to some $x \in \Omega$. Without loss of generality, we may assume that (x_n) converges strongly to $x \in \Omega$. Hence, taking into account that $x_n - F(x_n) \to \theta$ and the continuity of F, we derive, $F(x) = x$.

Clearly if $F : \Omega \to E$ is demi-compact at θ then it is demi-weakly compact at θ, but to be demi-weakly compact at θ does not mean that this mapping becomes demi-compact at θ (for instance see Example 3.12 below). Nevertheless, we have the following result.

Lemma 3.8. *Let E be a Banach space and let Ω be a nonempty closed subset of E. Assume $F : \Omega \to E$ is a (ws)-compact and (dwc) mapping. Then F is a continuous (dc)-mapping.*

Proof. Suppose that (x_n) is a bounded sequence in Ω such that $x_n - F(x_n) \to \theta$. Since F is (dwc) we know that there exist a subsequence (x_{n_k}) of (x_n) and an element $x \in E$ such that $x_{n_k} \xrightarrow{w} x$.

We claim that there exists a subsequence $(x_{n_{k_s}})$ of (x_n) such that $x_{n_{k_s}} \to x$.

Indeed, by definition of (wc)-mapping, we know that there exist a subsequence $(x_{n_{k_s}})$ of (x_{n_k}) and an element $y \in E$ such that $F(x_{n_{k_s}}) \to y$. Hence,

$$\|x_{n_{k_s}} - y\| \le \|x_{n_{k_s}} - F(x_{n_{k_s}})\| + \|F(x_{n_{k_s}}) - y\| \to 0.$$

This means that $x_{n_{k_s}} \to y$ and since Ω is closed, $x = y \in \Omega$. This completes the proof. □

Theorem 3.9. *Let Ω be a nonempty unbounded closed convex subset of a Banach space E. Assume Φ is a positive homogenous MNWC on E satisfying condition* (7) *and $F : \Omega \to \Omega$ is a (ws)-compact (dwc) and Φ-nonexpansive mapping with $F(\Omega)$ is bounded. Then F has a fixed point in Ω.*

Proof. Let z be a fixed element of Ω. Define $F_n = t_n F + (1 - t_n)z$, $n = 1, 2, \ldots,$ where $(t_n)_n$ is a sequence of $(0, 1)$ such that $t_n \to 1$. Since $z \in \Omega$ and Ω is convex, it follows that F_n maps Ω into itself. Let D be an arbitrary bounded subset of Ω. Then we have

$$\Phi\big(F_n(D)\big) \le \Phi\big(\{t_n F(D)\} + \{(1 - t_n)z\}\big) \le t_n \Phi\big(F(D)\big) \le t_n \Phi(D).$$

So, if $\Phi(D) \neq 0$ we have

$$\Phi\big(F_n(D)\big) < \Phi(D).$$

Therefore F_n is Φ-condensing on Ω. Clearly F_n is (ws)-compact mapping, so by Theorem 3.1, F_n has a fixed point, say, x_n in Ω. Consequently, $\|x_n - F(x_n)\| =$

$\|(t_n - 1)(F(x_n) - z)\| \to 0$ as $n \to \infty$, since $t_n \to 1$ as $n \to \infty$ and $F(\Omega)$ is bounded.

Finally, by Lemma 3.8 we have that F is a continuous (dc)-mapping. Since F admits a bounded sequence (x_n) satisfying that $x_n - F(x_n) \to \theta$, by Remark 3.7 we conclude that F has a fixed point. $\square$

Remark 3.10. If in Theorem 3.9 we add the hypothesis $\theta \in \Omega$, then we obtain the same conclusion without assuming that Φ satisfies condition (7).

Next, we state and prove new fixed point results for pseudo-contractive mappings defined on unbounded closed convex subsets of E using the notion of demi-compactness and tools provided in [15].

Proposition 3.11. *Let Ω be a nonempty unbounded closed convex subset of a Banach space E. Assume that $F : \Omega \to \Omega$ is a continuous (dc) and pseudo-contractive mapping such that $F(\Omega)$ is bounded. Then F has a fixed point.*

Proof. Since Ω is a closed convex subset and $F(\Omega)$ is a bounded subset of Ω, clearly $K := \overline{\text{conv}}(F(\Omega))$ is a bounded closed convex and F-invariant subset of Ω. Thus, since F is a continuous pseudo-contractive mapping from K into itself, it is well known that $A := I - F : K \to E$ is an accretive operator with the range condition.

Consequently, the resolvent $J_1 := (I + A)^{-1} : K \to K$ is single valued and non-expansive mapping. Moreover, since K is bounded closed and convex there exists a sequence (w_n) in K such that $w_n - J_1(w_n) \to \theta$.

If we let $x_n = J_1(w_n)$, then $x_n + x_n - F(x_n) = w_n$ and therefore

$$x_n - F(x_n) = w_n - J_1(w_n),$$

which implies that (x_n) is a bounded sequence in K such that $x_n - F(x_n) \to \theta$ which implies that (x_n) is a bounded a.f.p. sequence in Ω and since F is a continuous (dc)-mapping by Remark 3.7 we have that F has a fixed point. $\square$

The next example shows that in Theorem 3.1 and in Proposition 3.11 we cannot remove the condition F is a (ws)-compact mapping.

Example 3.12. Let E be the Banach space $(L^1[0,1], \|\cdot\|)$ and consider the Alspach mapping, i.e., first let

$$\Omega := \left\{ f \in L^1[0,1] : 0 \le f \le 1, \|f\|_1 = \frac{1}{2} \right\}.$$

It is well known that Ω is a weakly compact convex subset of E.

Now consider $F : \Omega \to \Omega$ such that for each $f \in \Omega$, $F(f)$ is defined by

$$F\big(f(t)\big) = \begin{cases} \min\{2f(2t), 1\}, & t \in \big[0, \frac{1}{2}\big], \\ \max\{2f(2t-1) - 1, 0\}, & t \in \big(\frac{1}{2}, 1\big]. \end{cases}$$

Since Ω is a weakly compact set, then it is clear that F is β-condensing (dwc)-mapping with $F(\Omega)$ bounded. Nevertheless, in [2] it is proved that F is a fixed point free nonexpansive mapping.

In Proposition 3.11 we assume that $F(\Omega)$ is a bounded set, which implies that we can find a bounded closed convex F-invariant set. If we omit such hypothesis we still obtain the following result:

Proposition 3.13. *Let E be a real Banach space. Suppose that $G : E \times E \to \mathbb{R}$ is a mapping satisfying conditions* (g1)–(g4). *Let Ω be a nonempty closed convex subset of E. Assume that $F : \Omega \to \Omega$ is a continuous (dc) and pseudo-contractive mapping. If the following condition is satisfied:*

(a) *There exists $R > 0$ such that for every $x \in \Omega \backslash B_R(\theta)$ the inequality $G\big(F(x), x\big) \le G(x, x)$ holds.*

Then F has a fixed point.

Proof. In order to obtain the conclusion, from Remark 3.7, it will be enough to see that there exists a bounded a.f.p. sequence (x_n) in Ω. To see this, we argue as follows:

It is clear that the operator $A = I - F : \Omega \to E$, where I is the identity operator, is an accretive operator with the range condition. Then, by the proof of Theorem 3.7 of [15], we know that there exists a bounded sequence (w_n) in Ω such that $w_n - (I + A)^{-1}(w_n) \to \theta$. Therefore, following the proof of Proposition 3.11, we achieve the result. □

Corollary 3.14. *Let Ω be a nonempty unbounded closed convex subset of a Banach space E. Assume that $F : \Omega \to \Omega$ is a continuous (dc) and pseudo-contractive mapping. If there exist $x_0 \in E$ and $R > 0$ such that for all $x \in \Omega$ with $\|x\| \ge R$ the inequality*

$$\|F(x) - x_0\| \le \|x - x_0\|$$

holds. Then F has a fixed point.

Proof. If we define $G(x, y) = \langle x, y - x_0 \rangle_+$, by Corollary 4.8 of [15], it is clear that G satisfies conditions (g1)–(g4) and moreover there exists $R > 0$ such that $G\big(T(x), x\big) \le G(x, x)$ for all $x \in \Omega \backslash B_R(\theta)$, then we can apply Proposition 3.13. □

Remark 3.15. All the previous results give a new fixed point existence for nonexpansive operators defined on unbounded closed convex subsets of E.

In the rest of this section we shall discuss a fixed point result for sequentially weakly continuous and Φ-nonexpansive mappings. In order to obtain this result let us recall Theorem 12 of [13]:

Theorem 12 ([13]). *Let M be a nonempty closed convex and bounded subset of a Banach space E and consider Φ a MNWC on E. Assume that $F : M \to M$ is a sequentially weakly continuous and Φ-condensing map, then F has a fixed point.*

Proposition 3.16. *Let M be a nonempty unbounded closed convex subset of a Banach space E and consider Φ a positive homogenous MNWC on E satisfying condition* (7). *Assume that $F : M \to M$ is a sequentially weakly continuous (dwc) and Φ-nonexpansive map with $F(M)$ bounded. Then F has a fixed point.*

Proof. Since M is a closed convex subset and $F(M)$ is a bounded subset of M, clearly $K := \overline{\text{conv}}(F(M))$ is a bounded closed convex and F-invariant subset of Ω. Now, arguing as the proof of Theorem 3.9 and applying Theorem 12 we can obtain a sequence (x_n) in K such that $\|x_n - F(x_n)\| \to 0$. Since F is (dwc) and sequentially weakly continuous we achieve the result. $\square$

The next example shows that in Proposition 3.16 the condition F is a (dwc)-mapping cannot be omitted.

Example 3.17. Let E be the Banach space $(C[0,1], \|\cdot\|_\infty)$ and consider the mapping $F : E \to E$ defined by $F(u(t)) = tu(t)$. It is easy to see that F is a nonexpansive mapping on E and therefore it is β-nonexpansive (see Corollary 4.13).

On the other hand, if we consider the set $M := \{u \in E : 0 = u(0) \leq u(t) \leq u(1) = 1\}$. It is not difficult to see that M is a bounded closed convex and F-invariant subset of E, moreover F is fixed point free sequentially weakly continuous mapping on M.

4. Leray–Schauder type fixed point theorems for 1-set weakly contractive operators

First, we prove some Leray–Schauder type theorems for a broader class of nonlinear operators, in which the operators have the property that the image of any set is in a certain sense more weakly compact than the original set itself.

Theorem 4.1. *Let E be a Banach space, Ω a nonempty unbounded closed convex subset of E and $U \subseteq \Omega$ an open set (with respect to the topology of Ω) and let z be* *an element of U. Assume Φ is a MNWC on E and $F : \overline{U} \to \Omega$ a Φ-condensing (ws)-compact mapping with $F(\overline{U})$ is bounded. Then either*

($\mathscr{A}_1$) *F has a fixed point, or*

($\mathscr{A}_2$) *there is a point $u \in \partial_\Omega U$ (the boundary of U in Ω) and $\lambda \in (0,1)$ with $u = \lambda F(u) + (1-\lambda)z$.*

Remark 4.2. $\overline{U}$ and $\partial_\Omega U$ denote the closure and boundary of U in Ω, respectively.

Proof. Suppose ($\mathscr{A}_2$) does not hold and F does not have a fixed point in $\partial_\Omega U$ (otherwise, we are finished, i.e. ($\mathscr{A}_1$) occurs). Let D be the set defined by

$$D = \{x \in \overline{U} : x = \lambda F(x) + (1-\lambda)z, \text{ for some } \lambda \in [0,1]\}.$$

D is nonempty and bounded, because $z \in D$ and $F(\overline{U})$ is bounded. We have $D \subseteq \operatorname{conv}(\{z\} \cup F(D))$. So, $\Phi(D) \neq 0$ implies

$$\Phi(D) \leq \Phi\big(\operatorname{conv}(\{z\} \cup F(D))\big) \leq \Phi\big(F(D)\big) < \Phi(D),$$

which is a contradiction. Hence, $\Phi(D) = 0$ and D is relatively weakly compact. We will show that D is compact. The continuity of F implies that D is closed. For that, let $(x_n)_n$ be a sequence of D such that $x_n \to x$, $x \in \overline{U}$. For all $n \in \mathbb{N}$, there exists a $\lambda_n \in [0,1]$ such that $x_n = \lambda_n F(x_n) + (1-\lambda_n)z$. Since $\lambda_n \in [0,1]$, we can extract a subsequence $(\lambda_{n_j})_j$ such that $\lambda_{n_j} \to \lambda \in [0,1]$. So, $\lambda_{n_j} F(x_{n_j}) \to \lambda F(x)$. Hence $x = \lambda F(x) + (1-\lambda)z$ and $x \in D$. Now, we prove that D is sequentially compact. To see this, let $(x_n)_n$ be a sequence of D. For all $n \in \mathbb{N}$, there exists a $\lambda_n \in [0,1]$ such that $x_n = \lambda_n F(x_n) + (1-\lambda_n)z$. $\lambda_n \in [0,1]$, we can extract a subsequence $(\lambda_{n_j})_j$ such that $\lambda_{n_j} \to \lambda \in [0,1]$. We have that the set $\{x_n, n \in \mathbb{N}\}$ is contained in D, so it is relatively weakly compact and consequently by the Eberlein–Šmulian theorem [11], Theorem 8.12.4, p. 549, it is weakly sequentially compact. Hence, without loss of generality, the sequence $(x_n)_n$ has a weakly convergent subsequence $(x_{n_j})_j$. Since F is (ws)-compact, then the sequence $\big(F(x_{n_j})\big)_j$ has a strongly convergent subsequence, say $\big(F(x_{n_{j_k}})\big)_k$. Hence, the sequence $\big(\lambda_{n_{j_k}} F(x_{n_{j_k}})\big)_k$ is strongly convergent which means that the sequence $(x_{n_{j_k}})_k$ is also strongly convergent. Accordingly, D is compact. Because E is a Hausdorff locally convex space, we have that E is completely regular [35], p. 16. Since $D \cap (\Omega \backslash U) = \emptyset$, then by [19], p. 146, there is a continuous function $\varphi : \Omega \to [0,1]$, such that $\varphi(x) = 1$ for $x \in D$ and $\varphi(x) = 0$ for $x \in \Omega \backslash U$. Let $F^* : \Omega \to \Omega$ be the mapping defined by

$$F^*(x) = \varphi(x)F(x) + \big(1 - \varphi(x)\big)z.$$

Clearly, $F^*(\Omega)$ is bounded. Because $\partial_\Omega U = \partial_\Omega \overline{U}$, φ is continuous, $[0,1]$ is compact and F is (ws)-compact, we have that F^* is (ws)-compact. Let $X \subseteq \Omega$ be bounded. Then, since

$$F^*(X) \subseteq \operatorname{conv}\bigl(\{z\} \cup F(X \cap U)\bigr),$$

we have

$$\Phi\bigl(F^*(X)\bigr) \leq \Phi\bigl(F(X \cap U)\bigr) \leq \Phi\bigl(F(X)\bigr),$$

and $\Phi\bigl(F^*(X)\bigr) < \Phi(X)$ if $\Phi(X) \neq 0$. So, F^* is Φ-condensing . Therefore Theorem 3.1, implies that F^* has a fixed point $x_0 \in \Omega$. If $x_0 \notin U$, $\varphi(x_0) = 0$ and $x_0 = z$, which contradicts the hypothesis $z \in U$. Then $x_0 \in U$ and $x_0 = \varphi(x_0)F(x_0) + \bigl(1 - \varphi(x_0)\bigr)z$ which implies that $x_0 \in D$, and so $\varphi(x_0) = 1$. □

Corollary 4.3. *Let E be a Banach space, Ω a nonempty unbounded closed convex subset of E and $U \subseteq \Omega$ an open set (with respect to the topology of Ω) and let z be an element of U. Assume Φ a MNWC on E and $F : \overline{U} \to \Omega$ a Φ-condensing (ws)-compact mapping with $F(\overline{U})$ bounded. Suppose that F satisfies the Leray–Schauder boundary condition*

$$u - z \neq \lambda\bigl(F(u) - z\bigr), \qquad \lambda \in (0,1),\ u \in \partial_\Omega U,$$

then the set of fixed points of F in $\overline{U}$ is nonempty and compact.

Proof. By Theorem 4.1, F has a fixed point. Let $\mathscr{S} = \{x \in \overline{U} : F(x) = x\}$ be the fixed point set of F. Since F is continuous, $\mathscr{S}$ is obviously a closed subset of $\overline{U}$ such that $F(\mathscr{S}) = \mathscr{S}$. Now, arguing as the proof of Theorem 3.1 concerning the subset D, we have that $\mathscr{S}$ is sequentially compact and hence it is compact. □

As a special case, we obtain a fixed point theorem of the Rothe type [33] for Φ-condensing (ws)-compact mapping.

Corollary 4.4. *Let E be a Banach space, Ω a closed convex subset of E and $U \subseteq \Omega$ an open set (with respect to the topology of Ω) such that $\theta \in U$. Assume Φ a MNWC on E and $F : \overline{U} \to \Omega$ a Φ-condensing (ws)-compact mapping with $F(\overline{U})$ bounded. In addition, assume that $\overline{U}$ is starshaped with respect to θ and $F(\partial_\Omega U) \subseteq \overline{U}$. Then the set of fixed points of F in $\overline{U}$ is nonempty and compact.*

Proof. Since $\overline{U}$ is starshaped with respect to θ and $F(\partial_\Omega U) \subseteq \overline{U}$, then $x \neq \lambda F(x)$ for every $x \in \partial_\Omega U$ and $\lambda \in (0,1)$. Applying Theorem 3.1, then the set of fixed points of F in $\overline{U}$ is nonempty and compact. □

Corollary 4.5. *Let E be a Banach space, $\Omega \subset E$ a nonempty unbounded closed convex subset, $U \subset \Omega$ an open set (with respect to the topology of Ω) and let z be an element of U. Assume that $F : \overline{U} \to \Omega$ is a (ws)-compact mapping which satisfies $F(\overline{U})$ is bounded and $F(D)$ is relatively weakly compact whenever D is a bounded set of $\overline{U}$. Then either*

($\mathscr{A}_1$) *F has a fixed point, or*

($\mathscr{A}_2$) *there is a point $u \in \partial_\Omega U$ (the boundary of U in Ω) and $\lambda \in (0,1)$ with $u = \lambda F(u) + (1-\lambda)z$.*

Proof. This is an immediate consequence of Theorem 4.1 since F is Φ-condensing and $F(\overline{U})$ is bounded. □

Remark 4.6. Corollary 4.5 extends and improves Theorem 2.12 in [1].

Next, we show the existence of positive eigenvalues and eigenvectors of (ws)-compact, weakly compact and weakly condensing mappings defined on unbounded domains.

Corollary 4.7. *Let E be a Banach space, $\Omega \subset E$ a nonempty unbounded closed convex subset, $U \subset \Omega$ an open set (with respect to the topology of Ω) and such that $\theta \in U$. Assume that $F : \overline{U} \to \Omega$ is a (ws)-compact mapping which satisfies $F(\overline{U})$ is bounded and $F(D)$ is relatively weakly compact whenever D is a bounded set of $\overline{U}$. In addition suppose F has no fixed point in $\overline{U}$. Then there exist an $x \in \partial_\Omega U$ and $\lambda \in (0,1)$ such that $x = \lambda F(x)$.*

Corollary 4.8. *Let E be a Banach space, $\Omega \subset E$ a nonempty unbounded closed convex subset, $U \subset \Omega$ an open set (with respect to the topology of Ω) and such that $\theta \in U$. In addition let Φ be a positive homogenous MNWC on E, $k \geq 1$ and $F : \overline{U} \to \Omega$ a Φ-nonexpansive (ws)-compact mapping, with $F(\overline{U})$ bounded. Suppose that there is a real number $c > k$ such that*

$$F(\overline{U}) \cap (cU) = \emptyset.$$

Then there exists an $x \in \partial_\Omega U$ and $\lambda \geq c$ such that $F(x) = \lambda x$.

Proof. We suppose that for all $x \in \partial_\Omega U$ and $\lambda \geq c$, $F(x) \neq \lambda x$. Let $F_1 = \frac{1}{c}F$ and

$$D = \{x \in \overline{U} : x = \lambda F_1(x) \text{ for some } \lambda \in [0,1]\}.$$

D is nonempty and bounded. Because $\theta \in D$ and $F(\overline{U})$ is bounded. We have $D \subseteq \operatorname{conv}(\{\theta\} \cup F_1(D))$. So, since $\Phi(D) \neq 0$, F is Φ-nonexpansive and $c > 1$ we have

$$\Phi(D) \leq \Phi\big(\operatorname{conv}(\{\theta\} \cup F_1(D))\big) \leq \frac{1}{c}\Phi\big(F(D)\big) < \Phi(D),$$

which is a contradiction. Hence, $\Phi(D) = 0$ and D is relatively weakly compact. Clearly F_1 is a (ws)-compact mapping and so D is compact. We claim that $D \cap (\Omega \backslash U) = \emptyset$. We suppose to the contrary that $D \cap (\Omega \backslash U) \neq \emptyset$. Then there exists an $x \in \Omega \backslash U$ and $\alpha \in [0, 1]$ such that $\alpha F_1(x) = x$. If $\alpha = 0$, then $x = \theta$, which contradicts $\theta \in U$. If $\alpha \neq 0$, then $F(x) = \frac{c}{\alpha} x$ $\left(\frac{c}{\alpha} \geq c\right)$, which contradicts the hypothesis. Thus, $D \cap (\Omega \backslash U) = \emptyset$. Let $F_1^* : \Omega \to \Omega$ be the mapping defined by:

$$F_1^*(x) = \varphi(x) F_1(x),$$

where $\varphi : \Omega \to [0, 1]$, is a continuous function such that $\varphi(x) = 1$ for $x \in D$ and $\varphi(x) = 0$ for $x \in \Omega \backslash U$. Arguing as in the proof of Theorem 4.1, we prove that F_1 is a Φ-condensing (ws)-compact mapping with $F_1(\Omega)$ bounded. Therefore, Theorem 3.1 implies that F_1^* has a fixed point $x_1 \in \Omega$. If $x_1 \notin U$, $\varphi(x_1) = 0$ and $x_1 = \theta$, which contradicts the hypothesis $\theta \in U$. Then $x_1 \in U$ and $x_1 = \varphi(x_1) F_1(x_1)$, which implies that $x_1 \in D$, and so $\varphi(x_1) = 1$ and $F(x_1) = cx_1$. Hence, $F(\overline{U}) \cap (cU) \neq \emptyset$, another contradiction. Accordingly, there exist an $x \in \partial_\Omega U$ and $\lambda \geq c$ such that $F(x) = \lambda x$. □

Next, we obtain the following applicable form of Corollary 4.5.

Proposition 4.9. *Let E be a Banach space, $k \geq 1$ and $F : B_1(\theta) \to E$ a β-nonexpansive (ws)-compact mapping. Suppose that there is a real number $c > k$ such that $\|F(x)\| \geq c$ for all $x \in B_1(\theta)$. Then there exist an $x \in S_1(\theta)$ and $\lambda \geq c$ such that $F(x) = \lambda x$.*

Proof. It suffices to note that De Blasi's measure β of weak noncompactness is positive and homogenous. □

As an application of Proposition 4.9, we work with a nonlinear eigenvalue problem in concrete situation, see [12].

Theorem 4.10. *Let X, Y be finite dimensional Banach spaces, D a compact subset of $\mathbb{R}^n$, $\lambda \in \mathbb{R}$ and $E = L^1(D, X)$. Assume that*

(a) *$G : B_1(\theta) \to E$ is a ws-compact, weakly compact operator,*

(b) *$f : D \times X \to Y$ verifies Carathéodory hypotheses i.e., f is strongly measurable with respect to $t \in D$, for all $x \in X$, and continuous with respect to $x \in X$, for almost all $t \in D$,*

(c) *there are $a \in L^1(D)$ and $b \geq 0$ such that*

$$\|f(t, x)\| \leq a(t) + b\|x\|, \qquad t \in D,\ x \in X,$$

(d) $k : D \times D \to L(Y, X)$ (*the space of bounded linear operators from* Y *into* X) *is strongly measurable and the linear operator* K *defined by*

$$\big(K(z)\big)(t) = \int_D k(t,s)z(s)\,ds,$$

maps $L^1(D, Y)$ *into* $L^1(D, X)$ *continuously,*

(e) *the functions* $s \to k(t,s)$ *are in* $L^\infty\big(D, L(Y, X)\big)$ *for almost all* $t \in D$,

(f) $|\lambda|b\|K\| \le 1$ ($\|K\|$ denotes the operator norm of K).

Consider the nonlinear operator $F : B_1(\theta) \to E$ *given by*

$$F(y) = G(y) + L(y) = G(y) + \lambda \int_D k(t,s) f\big(s, y(s)\big)\,ds.$$

Set

$$\alpha = |\lambda|(\|a\| + b)\|K\|, \qquad \gamma = \inf_{y \in B_1(\theta)} \|G(y)\|.$$

If $\gamma > \alpha + 1$, *then* F *has a positive eigenvalue whose corresponding vector lies in* $S_1(\theta)$.

Proof. We shall use some ideas of [12]. First, we prove that L is (ws)-compact, β-nonexpansive operator. By the assumption (b), we obtain that the Nemytskii operator generated by f and defined by

$$N_f(y)(t) := f\big(t, y(t)\big), \qquad y \in L^1(D, X)$$

maps continuously $L^1(D, X)$ into $L^1(D, Y)$. So, by the assumption (d) the operator $L = \lambda K N_f$ is continuous. Using the assumptions (b), (c) and (f) and arguing as in [12], we prove that the operator L is β-nonexpansive. Now, let $(y_n)_n$ be a weakly convergent sequence of $L^1(D, X)$. Then $(y_n)_n$ is uniformly bounded and by the assumption (b) we obtain

$$\|f(t, y_n)\| \le a(t) + b\|y_n\|. \tag{1}$$

Since $(y_n)_n$ is weakly compact in $L^1(D, X)$, by the Dunford–Pettis criterion it turns out to be equi-absolutely integrable on D, that is

$$\forall \varepsilon > 0,\ \exists \delta > 0, \qquad |D_0| < \delta \implies \int_{D_0} \|y_n(t)\|\,dt < \varepsilon \qquad \forall n \in \mathbb{N}.$$

Therefore by (1), also $\big(N_f(y_n)\big)_n$ is equi-absolutely integrable on D, which implies the weak compactness in $L^1(D, Y)$ of $\big(N_f(y_n)\big)_n$, and hence by the Eberlein-Šmulian Theorem [11], theorem 8.12.4, p. 549, $N_f(y_n)_n$ has a weakly convergent subsequence, say $\big(N_f(y_{n_j})\big)_j$.

On the other hand, the continuity of the linear operator K, implies its weak continuity on $L^1(D, Y)$ for almost all $t \in D$. Consequently, we obtain that $\big(KN_f(y_{n_j})\big)_j$ and so $\big(L(y_{n_j})\big)_j$ is pointwise converging, for almost all $t \in D$. Using again the weak continuity of the linear operator K, we infer that $\big(L(y_{n_j})\big)_j$ is equi-absolutely integrable on D. Hence, by Vitali's convergence theorem ([10]), $\big(L(y_{n_j})\big)_j$ is strongly convergent in $L^1(D, X)$. Accordingly, the operator L is (ws)-compact. For all $y \in B_1(\theta)$, we have

$$\|F(y)\| \geq \|G(y)\| - \|L(y)\| \geq \gamma - \alpha > 1.$$

Since G is weakly compact, (ws)-compact operator and hence F is (ws)-compact and β-nonexpansive operator, Proposition 4.9, implies that F has an eigenvalue $\eta > 1$ with corresponding eigenvector $y \in S_1(\theta)$. □

Theorem 4.11. *Let E be a Banach space, Ω be a nonempty unbounded closed convex of E and $U \subseteq \Omega$ an open set (with respect to the topology of Ω). In addition, let Φ be a positive homogenous MNWC on E satisfying condition (7) and $F : \overline{U} \to \Omega$ a Φ-nonexpansive (ws)-compact mapping, with $F(\overline{U})$ bounded. Assume that*

(a) *There exists $z \in U$ such that $u - z \neq \lambda\big(F(u) - z\big)$, $\lambda \in (0,1)$, $u \in \partial_\Omega U$,*

(b) *F is (dwc).*

Then F has a fixed point in $\overline{U}$.

Proof. Let $F_n = t_nF + (1 - t_n)z$, $n = 1, 2, \ldots$, where $(t_n)_n$ is a sequence of $(0,1)$ such that $t_n \to 1$. Since $z \in \Omega$ and Ω is convex, it follows that F_n maps $\overline{U}$ into Ω. Suppose that $\lambda_n\big(F_n(y_n) - z\big) = y_n - z$ for some $y_n \in \partial_\Omega U$ and for some $\lambda_n \in (0,1)$. Then we have

$$y_n - z = \lambda_n\big(F_n(y_n) - z\big) = \lambda_n t_n F(y_n) + \lambda_n(1 - t_n)z - \lambda_n z = \lambda_n t_n\big(F(y_n) - z\big),$$

which contradicts the hypothesis (a) since $\lambda_n t_n \in (0,1)$. Let X be an arbitrary bounded subset of $\overline{U}$. Then we have

$$\Phi\big(F_n(X)\big) = \Phi\big(\{t_nF(X)\} + \{(1 - t_n)z\}\big) \leq t_n\Phi\big(F(X)\big) \leq t_n\Phi(X).$$

So, if $\Phi(X) \neq 0$ we have

$$\Phi\big(F_n(X)\big) < \Phi(X).$$

Therefore, F_n is Φ-condensing on $\overline{U}$. From Theorem 4.1, F_n has a fixed point, say, x_n in $\overline{U}$. Now arguing as in the proof of Theorem 3.9, we prove that F has a fixed point in $\overline{U}$. □

Remark 4.12. If in Theorem 4.11 we add the hypothesis $\theta \in U$ and replace condition (a) by

(a′) $u \neq \lambda F(u)$, $\lambda \in (0,1)$, $u \in \partial_\Omega U$,

then we obtain the same conclusion without assuming that Φ satisfies condition (7).

Corollary 4.13. *Let $(E, \|\cdot\|$ be a Banach space, $\Omega \subset E$ a nonempty unbounded closed convex subset, $U \subset \Omega$ an open set (with respect to the topology of Ω) and let z be an element of U. Assume $F : E \to E$ is nonexpansive and $F : \overline{U} \to \Omega$ is a (ws)-compact such that $F(\overline{U})$ is bounded. In addition suppose that*

(a) $u - z \neq \lambda\big(F(u) - z\big)$, $\lambda \in (0,1)$, $u \in \partial_\Omega U$,

(b) *F is (dwc).*

Then F has a fixed point in $\overline{U}$.

Proof. The proof follows immediately from Theorem 4.11, once we show that F is β-nonexpansive. To see this, let D be a bounded set of Ω and $d = \beta(D)$. Let $\varepsilon > 0$. Then there exists a weakly compact set K of E with $D \subseteq K + B_{d+\varepsilon}(\theta)$. So for $x \in D$ there exist $y \in K$ and $z \in B_{d+\varepsilon}(\theta)$ such that $x = y + z$ and so

$$\|F(x) - F(y)\| \leq \|x - y\| \leq d + \varepsilon.$$

It follows immediately, that

$$F(D) \subseteq F(K) + B_{d+\varepsilon}(\theta) \subseteq \overline{F(K)} + B_{d+\varepsilon}(\theta).$$

Since F is a (ws)-compact mapping and K is weakly compact then $\overline{F(K)}$ is compact and hence weakly compact. Thus, $\beta\big(F(D)\big) \leq (d + \varepsilon)$. Since $\varepsilon > 0$ is arbitrary, then $\beta\big(F(D)\big) \leq \beta(D)$. Accordingly, F is β-nonexpansive. □

Proposition 4.14. *Let E be a Banach space, Ω be a nonempty unbounded closed convex subset of E such that $\theta \in \Omega$. Assume $F : \Omega \to E$ is a continuous (dc) and pseudo-contractive mapping. In addition suppose that*

(a) *F is weakly inward on Ω,*

(b) *there exists $r > 0$ such that for every $u \in \Omega \backslash B_r(\theta)$, $u \neq \lambda F(u)$, $\lambda \in (0,1)$.*

Then F has a fixed point.

Proof. It is a consequence of Proposition 3.11, since following the same argument as in Corollary 4.6 and Theorem 4.1 of [15], we obtain that there exists K a nonempty bounded closed convex and F-invariant subset of Ω. $\square$

Proposition 4.15. *Let U be a bounded open subset of a Banach space E, and let $F : \overline{U} \to E$ be a continuous (dc) pseudo-contractive mapping. Suppose that F satisfies the following conditions:*

(i) *There exists $z \in U$ satisfying $F(x) - z \neq \lambda(x - z)$ for $x \in \partial U$ and $\lambda \geq 1$.*

Then F has a fixed point in U.

Proof. Under these hypotheses we can invoke the argument given in the proof of Theorem 1 of [27] and thus we can guarantee that

$$\inf\{\|x - T(x)\| : x \in U\} = 0.$$

Consequently, there exists a bounded a.f.p. sequence (x_n) in U and since F is a continuous (dc)-mapping by Remark 3.7 we derive the result. $\square$

Remark 4.16. We remark that for all previous results, no convexity of the domain of F is required.

Theorem 4.17. *Let M be a nonempty closed convex subset of a Banach space E and consider Φ a positive homogenous MNWC on E satisfying condition (7). Assume that $F : M \to M$ is a mapping with the following properties:*

(i) *F is Φ-nonexpansive.*

(ii) *F is a (ws)-compact mapping.*

(iii) *F is (dwc).*

(iv) *There exists $x_0 \in M$ and $R > 0$ such that $F(x) - x_0 \neq \lambda(x - x_0)$ for every $\lambda > 1$ and for every $x \in M \cap S_R(x_0)$.*

Then F has a fixed point.

Proof. Define $F_n = t_n F + (1 - t_n)x_0$, $n = 1, 2, \ldots,$ where $(t_n)_n$ is a sequence of $(0, 1)$ such that $t_n \to 1$. Since $x_0 \in M$ and M is convex, it follows that F_n maps M into itself. Moreover, F_n is a Φ-condensing and (ws)-compact mapping, for instance see the proof of Theorem 3.9.

By assumption (iv), we have that $F_n(x) - x_0 \neq \lambda(x - x_0)$ for all $\lambda > 1$ and for every $x \in M \cap S_R(x_0)$. Otherwise, we can find $z \in M \cap S_R(x_0)$ and $\lambda > 1$ such that $F_n(x) - x_0 = \lambda(x - x_0)$, but if this holds, then

$$\lambda(z - x_0) = F_n(z) - x_0 = t_n\big(F(z) - x_0\big),$$

consequently $F(z) - x_0 = \frac{\lambda}{t_n}(z - x_0)$ which is a contradiction.

These properties allow us to invoke Theorem 9 of [13] and hence we have that there exists a bounded sequence (x_n) such that $x_n = F_n(x_n)$. Now, following the steps of the proof of Theorem 3.9 we achieve the conclusion. $\square$

In the rest of this section we shall discuss a nonlinear Leray–Schauder alternative for positive operators. Let E_1 and E_2 be two Banach lattices, with positive cônes E_1^+ and E_2^+, respectively. An operator T from E_1 into E_2 is said to be positive, if it carries the positive cône E_1^+ into E_2^+ (i.e., $T(E_1^+) \subseteq E_2^+$).

Theorem 4.18. *Let Ω be a nonempty unbounded closed convex subset of a Banach lattice E such that $\Omega^+ := \Omega \cap E^+ \neq \emptyset$. Assume $F : \Omega \to \Omega$ is a positive (ws)-compact operator. If $F(\Omega)$ is relatively weakly compact, then F has at least a positive fixed point in Ω.*

Proof. Since Ω^+ is a closed convex subset of E^+ and $F(\Omega^+) \subseteq \Omega^+$. Also, $F(\Omega^+) \subseteq F(\Omega)$, so $F(\Omega^+)$ is relatively weakly compact. Now, it suffices to apply Theorem 2.6 to prove that F has fixed point in $\Omega^+ \subseteq \Omega$. $\square$

Theorem 4.19. *Let Ω be a nonempty unbounded closed convex subset of a Banach lattice E such that $\Omega^+ \neq \emptyset$. Assume Φ is a MNWC on E and $F : \Omega \to \Omega$ is a positive Φ-condensing (ws)-compact mapping with $F(\Omega)$ bounded. Then the set of positive fixed points of F in Ω is nonempty and compact.*

Proof. Let $x_0 \in \Omega^+$ and $D = \{F^n(x_0), n \in \mathbb{N}\}$ where $F^0(x_0) = x_0$. Then $D = F(D) \cup \{x_0\}$ and $D \subseteq \Omega^+$. Arguing as in the proof of Theorem 3.1, there exists a closed convex subset K such that $K \cap E^+ \neq \emptyset$, $F(K) \subseteq K$ and $F(K)$ is relatively weakly compact. So, by Theorem 4.18, F has a positive fixed point in Ω. $\square$

Theorem 4.20. *Let Ω be a nonempty unbounded closed convex subset of a Banach lattice space E. In addition, let $U \subseteq \Omega$ be an open set (with respect to the topology of Ω) and let z be an element of $U \cap E^+$. Assume Φ is a MNWC on E and $F : \overline{U} \to \Omega$ is a positive Φ-condensing (ws)-compact mapping with $F(\overline{U})$ bounded. Then either*

$(\mathscr{A}_1)$ *F has a positive fixed point, or*

$(\mathscr{A}_2)$ *there is a point $u \in \partial_\Omega U \cap E^+$ (the positive boundary of U in Ω) and $\lambda \in (0,1)$ with $u = \lambda F(u) + (1-\lambda)z$.*

Proof. Suppose $(\mathscr{A}_2)$ does not hold and F does not have a positive fixed point in $\partial_\Omega U$ (otherwise, we are finished, i.e. $\mathscr{A}_1$ occurs). Let D be the set defined by

$$D = \{x \in \overline{U} \cap E^+ : x = \lambda F(x) + (1-\lambda)z \text{ for some } \lambda \in [0,1]\}.$$

Since E is a normed lattice, E^+ is closed, and so, $\overline{U} \cap E^+$ is a closed subset of Ω. Arguing as in the proof of Theorem 4.1, we prove that D is compact and that there is a continuous function $\varphi : \Omega \to [0, 1]$, such that $\varphi(x) = 1$ for $x \in D$ and $\varphi(x) = 0$ for $x \in \Omega \backslash U$. Let $F^* : \Omega \to \Omega$ be the mapping defined by

$$F^*(x) = \varphi(x)F(x) + \big(1 - \varphi(x)\big)z.$$

Clearly $F^*(\Omega)$ is bounded. Because $\partial_\Omega U = \partial_\Omega \overline{U}$, φ is continuous and F is a positive (ws)-compact and Φ-condensing operator, we have that F^* is a positive (ws)-compact and Φ-condensing operator. Therefore, following again the proof of Theorem 4.1 we achieve the result. □

Corollary 4.21. *Let Ω be a nonempty unbounded closed convex subset of a Banach space E. In addition, let $U \subseteq \Omega$ be an open set (with respect to the topology of Ω) such that $\theta \in U$, let $F : \overline{U} \to \Omega$ be a positive (ws)-compact map, Φ-condensing and let $F(\overline{U})$ be bounded. We suppose that $y \notin \{\lambda F(y), \lambda \in (0, 1)\}$ for all $y \in \partial_\Omega U \cap E^+$. Then the set of positive fixed points of F in $\overline{U}$ is nonempty and compact.*

Proposition 4.22. *Let Ω be a nonempty closed convex subset of a Banach lattice E such that $\Omega^+ \neq \emptyset$. Assume that $F : \Omega \to \Omega$ is a positive continuous (dc) pseudo-contractive mapping such that $F(\Omega)$ is bounded. Then F has a positive fixed point.*

Proof. Since Ω^+ is a nonempty closed convex subset of E^+ and $F(\Omega^+) \subseteq F(\Omega)$, clearly $F(\Omega^+)$ is a bounded set. On the other hand, since F is a positive mapping, we have that $F(\Omega^+) \subseteq \Omega^+$.

These facts allow us to define $K = \overline{\mathrm{conv}}\big(F(\Omega^+)\big) \subseteq \Omega^+$ and then arguing as in the proof of Proposition 3.11 we obtain the result. □

Proposition 4.23. *Let E be a Banach lattice. Suppose that $G : E \times E \to \mathbb{R}$ is a mapping satisfying conditions (g1)–(g2). Let Ω be a nonempty unbounded closed convex subset of E such that $\theta \in \Omega^+$. Assume $F : \Omega \to \Omega$ is a positive continuous (dc)-pseudocontractive mapping. In addition suppose that there exists $r > 0$ such that for every $u \in \Omega^+ \backslash B_r(\theta)$, the inequality $G\big(F(u), u\big) \leq G(u, u)$ holds. Then F has a positive fixed point.*

Proof. Since F is a positive mapping and $\theta \in \Omega^+$, we have that Ω^+ is a nonempty closed convex F-invariant set. Thus, we can consider that $F : \Omega^+ \to \Omega^+$ is a continuous pseudocontractive mapping and then F is weakly inward on Ω^+. Now, arguing as in Theorem 4.1 of [15] we derive that there exists a bounded sequence (x_n) in Ω^+ such that $x_n - F(x_n) \to \theta$ and since F is a continuous (dc)-mapping we achieve the result. □

Corollary 4.24. *Let E be a Banach lattice, let Ω be a nonempty unbounded closed convex subset of E such that $\theta \in \Omega^+$. Assume $F : \Omega \to \Omega$ is a positive continuous (dc)-pseudocontractive mapping. In addition suppose that there exists $r > 0$ such that $u \neq \lambda F(u)$, $\lambda \in (0,1)$, for every $u \in \Omega^+ \backslash B_r(\theta)$. Then F has a positive fixed point.*

In order to prove this corollary it is enough to consider, in Proposition 4.23, the mapping

$$G(x,y) = \begin{cases} \lambda, & \text{if } x = \lambda y,\ \lambda > 0,\ x \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

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