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On the endomorphism monoid of a profinite semigroup

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Abstract. Necessary and sufficient conditions are given for the endomorphism monoid of a profinite semigroup to be profinite. A similar result is established for the automorphism group.

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1. Introduction

A classical result in profinite group theory says that if G is a profinite group with a fundamental system of neighborhoods of the identity consisting of open characteristic subgroups, then the group $\operatorname{Aut} G$ of continuous automorphisms of G is profinite with respect to the compact-open topology [8]. This applies in particular to finitely generated profinite groups. Hunter proved that the monoid of continuous endomorphisms $\operatorname{End} S$ of a finitely generated profinite semigroup S is profinite in the compact-open topology [6]. This result was later rediscovered by Almeida [4], who was unaware of Hunter's result. Almeida was the first to use to good effect that $\operatorname{End} S$ is profinite [2], [5], [3], in particular with respect to the study of maximal subgroups of finitely generated free profinite semigroups. In this note I give necessary and sufficient conditions for $\operatorname{Aut} S$ and $\operatorname{End} S$ to be profinite for a profinite semigroup S. This came out of trying to find an easier proof than Almeida's for the finitely generated case. This led me unawares to exactly Hunter's proof, which I afterwards discovered via a Google search. Like Hunter [6] and Ribes and Zalesskii [8], I give an explicit description of $\operatorname{End} S$ and $\operatorname{Aut} S$ as inverse limits in the case they are profinite. I also deduce the Hopfian property for S in this case (in [6], [8] this is done only in the finitely generated case). Recall that a topological

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semigroup S is *Hopfian* if each surjective continuous endomorphism of S is an automorphism.

2. The main result

My approach, like that of Hunter [6], but unlike that of Almeida [4] and Ribes and Zalesskii [8], relies on the uniform structure on a profinite semigroup and Ascoli's theorem. Recall that a congruence ρ on a profinite semigroup S is called *open* if it is an open subset of $S \times S$. It is easy to see that open congruences are precisely the kernels of continuous surjections from S to finite semigroups [7], Chapter 3. A congruence ρ on S is called *fully invariant* if, for all continuous endomorphisms $f : S \to S$, one has $(x, y) \in \rho$ implies $\big(f(x), f(y)\big) \in \rho$. Equivalently, ρ is fully invariant if and only if $\rho \subseteq (f \times f)^{-1}(\rho)$ for all $f \in \operatorname{End} S$. In group theory, it is common to call a subgroup invariant under all automorphisms 'characteristic.' As I do not know of any terminology in vogue for the corresponding notion for congruences, it seems reasonable to define a congruence ρ on S to be *characteristic* if $(x, y) \in \rho$ implies $\big(f(x), f(y)\big) \in \rho$ for all continuous automorphisms f of S. Again, this amounts to $\rho \subseteq (f \times f)^{-1}(\rho)$ for all $f \in \operatorname{Aut} S$. Clearly, any fully invariant congruence is characteristic.

Let $(X, \mathscr{U})$ and $(Y, \mathscr{V})$ be uniform spaces. Recall that a family $\mathscr{F}$ of functions from X to Y is said to be *uniformly equicontinuous* if, for all entourages $R \in \mathscr{V}$, one has $\bigcap_{f \in \mathscr{F}} (f \times f)^{-1}(R) \in \mathscr{U}$. Of course, a uniformly equicontinuous family consists of uniformly continuous functions. It is clearly enough to have this condition satisfied for all R running over a fundamental system of entourages for the uniformity $\mathscr{V}$.

Every compact Hausdorff space X has a unique uniformity compatible with its topology, namely the collection of all neighborhoods (not necessarily open) of the diagonal in $X \times X$. The following theorem is a special case of the Ascoli theorem for uniform spaces.

Theorem 1 (Ascoli). *Let X, Y be compact Hausdorff spaces equipped with their unique uniform structures and let $\mathscr{C}(X, Y)$ be the space of continuous map from X to Y equipped with the compact-open topology. Then, for a family $\mathscr{F} \subseteq \mathscr{C}(X, Y)$, the following are equivalent:*

(1) *$\mathscr{F}$ is compact in the compact-open topology;*

(2) *$\mathscr{F}$ is closed and (uniformly) equicontinuous.*

In the case of a profinite semigroup S the uniform structure is given by taking the open congruences as a fundamental system of entourages. The multiplication on S is uniformly continuous and all continuous endomorphisms of S are uniformly continuous. The following result gives a sufficient condition for a profinite semigroup to have a fundamental system of entourages consisting of open fully invariant congruences. The *index* of an open congruence ρ on a profinite semigroup S is the cardinality of S/ρ.

Proposition 2. *Let S be a profinite semigroup admitting only finitely many open congruences of index n for each $n \geq 1$. Then S has a fundamental system of open fully invariant congruences. This applies in particular if S is finitely generated.*

Proof. Let $\mathscr{F}_n$ be the set of open congruences on S of index at most n and let $\rho_n = \bigcap \mathscr{F}_n$; it is open because $\mathscr{F}_n$ is finite. Clearly, the family $\{\rho_n \mid n \geq 1\}$ is a fundamental system of entourages for the uniformity. I claim ρ_n is fully invariant. Indeed, let $f \in \operatorname{End} S$ and $\sigma \in \mathscr{F}_n$. Then if $p : S \to S/\sigma$ is the quotient map, one has $(f \times f)^{-1}(\sigma) = \ker pf$ and hence is of index at most n. Thus

$$(f \times f)^{-1}(\rho_n) = (f \times f)^{-1}\left(\bigcap \mathscr{F}_n\right) = \bigcap_{\sigma \in \mathscr{F}_n} (f \times f)^{-1}(\sigma) \supseteq \bigcap \mathscr{F}_n = \rho_n$$

as required.

To prove the final statement, suppose that S has a finite generating set X. Let T_{n+1} be the monoid of all maps on an $(n+1)$-element set. As T_{n+1} contains an isomorphic copy of each semigroup of order n, it follows that every open congruence of index n on S can be realized as the kernel of a continuous homomorphism from S to T_{n+1}. Such a homomorphism is determined by its restriction to X. Since there are only finitely many maps from X to T_{n+1}, it follows that S has only finitely many open congruences of index n. □

Remark 3. There are non-finitely generated profinite groups that satisfy the hypothesis of Proposition 2. For example, one can take the direct product of all finite simple groups (one copy per isomorphism class); see [8], Exercise 4.4.5.

It is well known that, for any locally compact Hausdorff space X, the compact-open topology turns $\mathscr{C}(X, X)$ into a topological monoid with respect to the operation of composition. The main result of this note is:

Theorem 4. *Let S be a profinite semigroup. Then* $\operatorname{End} S$ (*respectively,* $\operatorname{Aut} S$) *is compact in the compact-open topology if and only if S admits a fundamental system of open fully invariant* (*respectively, characteristic*) *congruences. Moreover, if* $\operatorname{End} S$ (*respectively,* $\operatorname{Aut} S$) *is compact, then it is profinite and the compact-open topology coincides with the topology of pointwise convergence.*

Proof. I just handle the case of $\operatorname{End} S$ as the corresponding result for $\operatorname{Aut} S$ is obtained by simply replacing the words ‘fully invariant’ by ‘characteristic’ and ‘endomorphism’ by ‘automorphism’.

First observe that $\operatorname{End} S$ is closed in $\mathscr{C}(S,S)$. Indeed, suppose that $f : S \to S$ is a continuous map that is not a homomorphism. Then there are elements $s, t \in S$ such that $f(st) \neq f(s)f(t)$. Choose disjoint open neighborhoods U, V of $f(st)$ and $f(s)f(t)$ respectively. By continuity of multiplication one can find open neighborhoods W, W' of $f(s)$ and $f(t)$ so that $W \cdot W' \subseteq V$. Then let N be the set of all continuous functions $g : S \to S$ such that $g(st) \in U$, $g(s) \in W$ and $g(t) \in W'$. Then $f \in N$ and N is open in the compact-open topology. Clearly, if $g \in N$, then $g(s)g(t) \in W \cdot W' \subseteq V$ and $g(st) \in U$, whence $g(st) \neq g(s)g(t)$. Thus $\operatorname{End} S$ is closed.

Assume that $\operatorname{End} S$ is compact. By Ascoli's theorem, it is uniformly equicontinuous. Let ρ be an open congruence on S. Then uniform equicontinuity implies that

$$\sigma = \bigcap_{f \in \operatorname{End} S} (f \times f)^{-1}(\rho)$$

is an entourage of the uniformity on S. Evidentally, σ is a congruence. It must contain an open congruence by definition of the uniformity on S and so σ is an open congruence (the open congruences being a filter in the lattice of congruences on S). Since the identity belongs to $\operatorname{End} S$, trivially $\sigma \subseteq \rho$. It remains to observe that σ is fully invariant. Indeed, if $g \in \operatorname{End} S$, then

$$(g \times g)^{-1}(\sigma) = \bigcap_{f \in \operatorname{End} S} (fg \times fg)^{-1}(\rho) \supseteq \bigcap_{h \in \operatorname{End} S} (h \times h)^{-1}(\rho) = \sigma$$

establishing that σ is fully invariant. Thus S has a fundamental system of open fully invariant congruences.

Conversely, suppose that S has a fundamental system of open fully invariant congruences. Uniform equicontinuity follows because if ρ is an open fully invariant congruence, then for any $f \in \operatorname{End} S$, one has $(f \times f)^{-1}(\rho) \supseteq \rho$ and hence $\bigcap_{f \in \operatorname{End} S}(f \times f)^{-1}(\rho) \supseteq \rho$. Since the set of entourages is a filter, it follows that $\bigcap_{f \in \operatorname{End} S}(f \times f)^{-1}(\rho)$ is an entourage. Because the open fully invariant congruences form a fundamental system of entourages for the uniformity on S, this shows that $\operatorname{End} S$ is uniformly equicontinuous.

Compactness of $\operatorname{End} S$ is now direct from Ascoli's theorem. Let us equip S^S with the topology of pointwise convergence. Since the compact-open topology is finer than the topology of pointwise convergence, the natural inclusion $i : \operatorname{End} S \to S^S$ is continuous. As $\operatorname{End} S$ and S^S are compact Hausdorff, it follows that i is a topological embedding and hence the compact-open topology on $\operatorname{End} S$ coincides with the topology of pointwise convergence. Also $\operatorname{End} S$ is totally disconnected being a subspace of S^S. Thus $\operatorname{End} S$ is profinite. □

In light of Proposition 2, Hunter's result for finitely generated profinite semigroups (and the corresponding well-known result for automorphism groups of finitely generated profinite groups) is immediate.

Corollary 5. *If S is a finitely generated profinite semigroup, then* $\operatorname{End} S$ *is a profinite monoid and* $\operatorname{Aut} S$ *is a profinite group in the compact-open topology, which coincides with the topology of pointwise convergence.*

Theorem 4 also implies the converse of [8], Proposition 4.4.3: a profinite group G has profinite automorphism group if and only if it has a fundamental system of neighborhoods of the identity consisting of open characteristic subgroups.

Remark 6. If S is a profinite semigroup generated by a finite set X, then we have the composition of continuous maps $\operatorname{End} S \to S^S \to S^X$ where the last map is induced by restriction. Moreover, this composition is injective. Since $\operatorname{End} S$ is compact, it follows that $\operatorname{End} S$ is homeomorphic to the closed space of all maps $X \to S$ that extend to an endomorphism of S equipped with the topology of pointwise convergence. In the case S is a relatively free profinite semigroup on X, we in fact have $\operatorname{End} S$ is homeomorphic to S^X. Under this assumption, if T is the abstract subsemigroup generated by X (which is relatively free in some variety of semigroups), then it easily follows that T^X is dense in S^X and so $\operatorname{End} T$ is dense in $\operatorname{End} S$.

A corollary is the well-known fact that finitely generated profinite semigroups are Hopfian. In fact, there is the following stronger result.

Corollary 7. *Let S be a profinite semigroup admitting a fundamental system of open fully invariant congruences, e.g., if S is finitely generated. Then S is Hopfian.*

Proof. Suppose that $f : S \to S$ is a surjective continuous endomorphism that is not an automorphism and let $f(x) = f(y)$ with $x \neq y \in S$. Then there is an open fully invariant congruence ρ so that $(x, y) \notin \rho$. Since ρ is fully invariant, there is an induced endomorphism $f' : S/\rho \to S/\rho$, which evidentally is surjective. Thus f' is an automorphism by finiteness. But if $[x]$, $[y]$ are the classes of x, y respectively, then $f'([x]) = f'([y])$ but $[x] \neq [y]$. This contradiction shows that S is Hopfian. □

Remark 8. In fact a more general result is true. Let X be a compact Hausdorff space and let M be a compact monoid of continuous maps on X with respect to the compact-open topology. Then every surjective element of M is invertible cf. [1]. The proof goes like this. First one shows that the surjective elements of M form a closed subsemigroup S (its complement is the union over all points $x \in X$

of the open sets $\mathcal{N}(X, X\setminus\{x\})$ of maps f with $f(X) \subseteq X\setminus\{x\})$. Clearly, the identity is the only idempotent of S. But a compact Hausdorff monoid with a unique idempotent is a compact group so every element of S is invertible. Consequently, any compact Hausdorff semigroup whose endomorphism monoid is compact must be Hopfian.

Not all profinite semigroups have a fundamental system of open fully invariant congruences. For instance, if S is any infinite profinite space equipped with the left zero multiplication, then S is a profinite semigroup, every continuous map on S is an endomorphism and every equivalence relation on S is a congruence. We claim that the only open congruence on S which is fully invariant is the universal equivalence relation. Indeed, if ρ is an open equivalence relation that is not universal, then since S is infinite, some class of ρ contains two distinct elements x, y. Because S is a profinite space, there is continuous map $f : S \to \{0, 1\}$ with $f(x) \neq f(y)$. As ρ contains at least two classes, there is a map $g : \{0, 1\} \to S$ with $g(0)$ and $g(1)$ in distinct ρ-classes. The composition $gf : S \to S$ is a continuous endomorpism of S with $\big(gf(x), gf(y)\big) \notin \rho$. We conclude ρ is not fully invariant.

As another example, let F be a free profinite group on a countable set of generators $X = \{x_1, x_2, \ldots\}$ converging to 1 [8]. Let $\sigma : F \to F$ be the continuous endomorphism induced by the shift $x_1 \mapsto 1$ and $x_i \mapsto x_{i-1}$ for $i \geq 2$. Then σ is surjective but not injective and so $\operatorname{End} F$ is not profinite.

As is the case for automorphism groups of profinite groups [8], Proposition 4.4.3, $\operatorname{End} S$ can be explicitly realized as a projective limit of finite monoids given a fundamental system of open fully invariant congruences on S. For finitely generated profinite semigroups, this was observed by Hunter [6].

Theorem 9. *Let S be a profinite semigroup and suppose that $\mathscr{F}$ is a fundamental system of entourages for S consisting of open fully invariant congruences. If $\rho \in \mathscr{F}$, then there is a natural continuous projection $r_\rho : \operatorname{End} S \to \operatorname{End} S/\rho$. Let $\hat{\rho}$ be the corresponding open congruence on* $\operatorname{End} S$. *Let $\hat{\mathscr{F}} = \{\hat{\rho} \mid \rho \in \mathscr{F}\}$. Then*

$$\operatorname{End} S \cong \varprojlim_{\hat{\mathscr{F}}} \operatorname{End} S/\hat{\rho}. \tag{1}$$

The analogous result holds for $\operatorname{Aut} S$ *if there exists a fundamental system of open characteristic congruences for S.*

Proof. First we must show that r_ρ is continuous so that $\hat{\rho}$ is indeed an open congruence. Indeed, if $f \in \operatorname{End} S$, then $r_\rho^{-1} r_\rho(f)$ consists of those endomorphisms $g \in \operatorname{End} S$ that take each block B of ρ into the block of ρ containing $f(B)$. But since each block of ρ is compact and open, and there are only finitely many blocks, it follows that $r_\rho^{-1} r_\rho(f)$ is an open set in the compact-open topology on $\operatorname{End} S$. Thus $\hat{\rho}$ is an open congruence.

Since the open fully invariant congruences on S are closed under intersection, the set $\hat{\mathscr{F}}$ is closed under intersection and so it makes sense to form the projective limit in (1). Since the canonical homomorphism from $\operatorname{End} S$ to the inverse limit on the right hand side of (1) is surjective, to prove that it is an isomorphism it suffices to show that $\hat{\mathscr{F}}$ separates points. If f, g are distinct endomorphisms of S, we can find $s \in S$ so that $f(s) \neq g(s)$. Then since $\mathscr{F}$ is a fundamental system of entourages, there exists $\rho \in \mathscr{F}$ such that $\big(f(s), g(s)\big) \notin \rho$. It follows that $r_\rho(f) \neq r_\rho(g)$. □

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