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Uniform decay rates of coupled anisotropic elastodynamic/Maxwell equations with nonlinear damping

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Abstract. This work is devoted to study the asymptotic behavior of the total energy associated with a coupled system of anisotropic hyperbolic models: the elastodynamic equations and Maxwell's system in the exterior of a bounded body in $\mathbb{R}^3$. Our main result says that in the presence of nonlinear damping, a unique solution of small initial data exists globally in time and the total energy as well as higher order energies decay at a uniform rate as $t \to +\infty$.

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1. Introduction

The propagation of electromagnetic waves in very special materials (like crystals) is quite different and interesting: The energy in general does not propagate along the normals to the fronts, but along rays which may be distinct from the normals (see [18]). Thus, in these kind of mediums the so-called permittivity and permeability are no more scalar-valued functions, but 3×3 symmetric matrices. Very seldom both will be diagonal matrices (see [18]). As a consequence, in this case the Maxwell equations cannot be reduced (in general) to a second order vector-wave equation for which a large amount of results are available.

Maxwell's equations provide a natural mathematical framework to understand the propagation of electromagnetic waves through bodies like the above special materials. These are the so-called anisotropic Maxwell equations. Due to recent Industrial applications specially with the so-called "smart materials" (see [1]) engineers needed to consider the interaction of anisotropic Maxwell equations with anisotropic elastic waves. In the mathematical literature we find very few articles giving exact properties of such coupled systems (see [13], [16] and references therein).

Motivated by the above discussion this paper is devoted to study the asymptotic behavior of a coupled system of equations: The elastodynamic system and Maxwell equations both anisotropic. The phenomenon happens in the exterior of a bounded body. Let us describe the model: Let $\Omega \subset \mathbb{R}^3$ be an exterior domain (unbounded with bounded complement) with boundary $\partial\Omega$ of class $\mathscr{C}^2$. We denote points in the space/time cylinder $\Omega \times (0, +\infty)$ by (x, t) where $x \in \Omega$ is the spatial variable and t denotes time. Let $u = u(x, t)$, $E = E(x, t)$ and $H = H(x, t)$ be vector valued functions each of them with three components denoting the displacement vector, the electric field intensity and the magnetic field intensity, respectively. We consider the coupled system

$$u_{tt} - \sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\left(A_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \gamma \operatorname{curl} E + F(u_t) = 0, \tag{1.1}$$

$$\varepsilon(x)E_t - \operatorname{curl} H + \sigma E - \gamma \operatorname{curl} u_t = 0, \tag{1.2}$$

$$\mu(x)H_t + \operatorname{curl} E = 0, \tag{1.3}$$

$$\operatorname{div}\big(\mu(x)H\big) = 0 \tag{1.4}$$

in $\Omega \times (0, +\infty)$. Here, $\varepsilon = \varepsilon(x)$ and $\mu = \mu(x)$ denote the electric permittivity and magnetic permeability respectively. They are 3×3 symmetric matrices which are uniformly positive definite almost everywhere for x in Ω. The parameter $\sigma > 0$ is called the electric conductivity, γ is the coupling constant and $F(u_t) = \big(F_1(u_t), F_2(u_t), F_3(u_t)\big)$ is a nonlinear damping term which will satisfy suitable growth assumptions.

We complement system (1.1)–(1.4) with initial conditions

$$(u, u_t, E, H)|_{t=0} = (u_0, u_1, E_0, H_0) \quad \text{in } \Omega \tag{1.5}$$

and boundary conditions

$$u = 0, \qquad \eta \times E = 0 \qquad \text{on } \partial\Omega \times (0, +\infty), \tag{1.6}$$

where $\eta = \eta(x)$ denotes the unit normal vector at $x \in \partial\Omega$ pointing the exterior of Ω. Here $\times$ is the usual vector product in $\mathbb{R}^3$.

The total energy associated with system (1.1)–(1.6) is given by

$$\mathscr{L}_1(t) = \frac{1}{2}\int_\Omega [|u_t(t)|^2 + Ju(t) + \varepsilon(x)E(t)\cdot E(t) + \mu(x)H(t)\cdot H(t)]\,dx \tag{1.7}$$

where

$$Ju(t) = \sum_{i,j=1}^{3} A_{ij}(x) \frac{\partial u}{\partial x_j}(t) \cdot \frac{\partial u}{\partial x_i}(t) \quad \text{and} \quad |u_t(t)|^2 = \sum_{j=1}^{3} \left(\frac{\partial u_j}{\partial t}(t) \right)^2. \tag{1.8}$$

Here the dot $\cdot$ means the usual inner product in $\mathbb{R}^3$.

Let us mention briefly a reason for choosing the coupling $\gamma \operatorname{curl} E$ and $-\gamma \operatorname{curl} u_t$ between the Lame system and the Maxwell system: It is known that several materials of the family of crystals, polymers or ceramics have the property that an electric field acting on the material creates stress and as a response to deformations is "produced" a polarization vector. These are the so called piezoelectric materials. The theory of linear piezoelectricity can be found in references [11] or [14]. The coupling for these electromechanical interaction is given by the terms $\sum_{i=1}^{3} \frac{\partial}{\partial x_i}(A_i^* E)$ and $\sum_{i=1}^{3} A_i \frac{\partial u_t}{\partial x_i}$ (see the above references or [17]) where A_i are 3×3 symmetric matrices. In the simplest case, if the medium is isotropic, then the matrices A_i are such that $\sum_{i=1}^{3} \frac{\partial}{\partial x_i}(A_i^* E) = \gamma \operatorname{curl} E$ and $\sum_{i=1}^{3} A_i \frac{\partial u_t}{\partial x_i} = -\gamma \operatorname{curl} u_t$.

There is a large literature concerning the decay of semilinear hyperbolic problems in exterior domains. E. Zuazua [23] considered the semilinear wave equation with localized damping in unbounded domains. M. Nakao [19] considered the semilinear scalar wave equation with a "localized" dissipation on a neighborhood of a part of the boundary and "near" infinity. He proved the existence of global solutions for small initial data and found polynomial decay rates in time for the total energy (see also R. Ikehata [15] and C. R. da Luz and R. C. Charão [4]). R. C. Charão and R. Ikehata [3] proved that the solutions of a nonlinearly damped of elastic waves with a localized damping near infinity decay in an algebraic rate to zero (see also [12]). Recently, the coupled model of elastodynamic with the isotropic Maxwell equation was treated in exterior domains by M. V. Ferreira and G. Perla Menzala [13] assuming that $F(u_t) = u_t - f(u_t)$ with suitable assumptions on f. The main result in [13] was the uniform decay as $t \to +\infty$ of the "second level" energy

$$\frac{1}{2} \int_{\Omega} [|u_{tt}(t)|^2 + Ju_t(t) + \varepsilon |E_t(t)|^2 + \mu |H_t(t)|^2] \, dx.$$

The final results given in [13] (see Theorems 3.1 and 3.2) did not give any information about the decay of the quantities $\int_{\Omega} \mu |H|^2 \, dx$ and $\int_{\Omega} Ju \, dx$. The results presented in this work improve the ones in [13] considering model (1.1)–(1.6) in appropriate function spaces and treating the full anisotropic case.

In [7] we have found polynomial decay rates for the total energy of the linear coupled system of anisotropic elastic waves and by the anisotropic Maxwell system in exterior domains. In the present work we use some ideas given in [7] in order to treat the coupled system (1.1)–(1.4). The discussion given in [7] for the linear problem is not enough to obtain the existence of solutions for the nonlinearly damped problem. We will need more regular solutions to conclude our results. In Theorem 4.1 we obtain several uniform rates of decay as $t \to +\infty$ using an iterative procedure using ideas in [7].

Let us mention recent related results: For the anisotropic Maxwell equations in bounded regions Ω, M. Eller [10] established an observability inequality also known as an inverse inequality. By a duality argument this observability inequality implies exact controllability of an electromagnetic field in Ω by a current flux on the boundary $\partial\Omega$. C. R. Luz and G. P. Menzala [5] studied the asymptotic behavior of the anisotropic Maxwell equations with internal dissipation in exterior domains. In [6], the problem with boundary dissipation of Silver-Muller's type in bounded domains was treated. B. V. Kapitonov and G. P. Menzala [16] studied a transmission problem for a system of isotropic electromagneto-elasticity in a bounded domain. Under suitable geometric conditions imposed on the domain they proved results of stabilization and exact controllability for the model. S. Nicaise [20] studied the stabilization problem for the electromagneto-elastic system. Higher order energy decay for damped wave equations was recently studied by P. Radu et.al. in [22].

The paper is organized as follows. Well posedness of the linear problem is analyzed in Section 3 using semigroup theory. In Section 4 we study the asymptotic behavior for the linear problem using multiplier methods and properties of an auxiliary evolution coupled system of first order. In Section 5 we study global existence and decay properties for the nonlinearly damped system. Here we use some ideas due to M. Nakao [19] and R. Ikehata [15] where they studied the wave equation in exterior domains in the presence of dissipations. We adapted their techniques to our more complicated situation.

2. Notations and assumptions

Let Ω be an exterior domain in $\mathbb{R}^3$, that is, $\Omega = \mathbb{R}^3 \backslash \bar{\mathcal{O}}$, being $\mathcal{O}$ open and bounded with boundary $\partial\mathcal{O}$ of class $\mathscr{C}^2$. We consider the set $\mathscr{M}$ of all 3×3 matrices $\alpha = \alpha(x) = [\alpha_{ij}(x)]_{3\times 3}$ which are symmetric and uniformly positive definite ones for almost every x in Ω, that is, there exist $\alpha_0 > 0$ in such a way that

$$\xi\alpha(x)\xi^t \geq \alpha_0|\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^3 \text{ almost everywhere in } \Omega. \tag{2.1}$$

The entries α_{ij} are real-valued functions and belong to $L^\infty(\Omega)$. In (2.1) we denote by $\xi^t = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$ whenever $\xi = (\xi_1\ \xi_2\ \xi_3)$ with $\xi_j \in \mathbb{R}$, $j = 1, 2, 3$. Also $|\xi|^2 = \sum_{j=1}^3 \xi_j^2$.

Let $\alpha \in \mathcal{M}$, we define the space

$$L^2(\Omega;\alpha) = \Big\{ v(x) = \big(v_1(x), v_2(x), v_3(x)\big) \text{ in such a way that}$$
$$\int_\Omega v(x)\alpha(x)v^t(x)\,dx = \sum_{i,j=1}^{3} \int_\Omega \alpha_{i,j}(x)v_i(x)v_j(x)\,dx < +\infty \Big\}$$

with the norm

$$\|v\|^2_{L^2(\Omega;\alpha)} = \sum_{i,j=1}^{3} \int_\Omega \alpha_{i,j}(x)v_i(x)v_j(x)\,dx.$$

Clearly $L^2(\Omega;\alpha) = [L^2(\Omega)]^3$ where $[L^2(\Omega)]^3 = L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ and the norms in $L^2(\Omega;\alpha)$ and $[L^2(\Omega)]^3$ are equivalent in the space $[L^2(\Omega)]^3$. Besides that, $L^2(\Omega;\alpha)$ is a Hilbert space with the following inner product:

$$(v, w)_{L^2(\Omega;\alpha)} = \sum_{i,j=1}^{3} \int_\Omega \alpha_{i,j}(x)v_i(x)w_j(x)\,dx$$

where $v(x) = \big(v_1(x), v_2(x), v_3(x)\big)$ and $w(x) = \big(w_1(x), w_2(x), w_3(x)\big)$.

From now on, we will always assume the following conditions:

(H1): The matrices ε and μ belong to $\mathcal{M}$.

(H2): Each A_{ij} is a 3×3 matrix whose entries belong to $W^{1,\infty}(\Omega)$ and there exists a positive constant a_0 such that

$$\sum_{i,j=1}^{3} [A_{ij}(x)\xi_j] \cdot \xi_i \ge a_0 \sum_{i=1}^{3} |\xi_i|^2 \tag{2.2}$$

for any vectors $\xi_i \in \mathbb{R}^3$, $i = 1, 2, 3$.

(H3): The entries $C^{ij}_{kl}(x)$ of the matrix $A_{ij}(x)$ are of the form

$$C^{ij}_{kl}(x) = (1 - \delta_{il}\delta_{jk})a_{ikjl}(x) + \delta_{ik}\delta_{jl}a_{iljk}(x)$$

where $\delta_{lk} = \begin{cases} 1 & \text{if } l = k, \\ 0 & \text{if } l \ne k, \end{cases}$ and $a_{ikjl}(x)$ are the Cartesian components of the elastic tensor with the symmetric properties

$$a_{ijkl}(x) = a_{jikl}(x) = a_{klij}(x) \quad \text{almost everywhere in } \Omega. \tag{2.3}$$

The symmetric assumptions (2.3) imply that the transpose of $A_{ij}(x)$ is $A_{ji}(x)$.

Observe that (H1) implies that ε and μ are invertible almost everywhere in Ω. In fact, since they belong to $\mathscr{M}$ then their eigenvalues are positive. Consequently, the determinant of each one of them is also positive. Hence, $\varepsilon(x)$ and $\mu(x)$ are invertible. We can easily prove that the entries of ε^{-1} and μ^{-1} belong to $L^\infty(\Omega)$.

Without loss of generalization we can assume that $F(0) = 0$ thus we will consider F of the form $F(\xi) = \kappa\xi + f(\xi)$ where $\kappa > 0$, $\xi \in \mathbb{R}^3$ and $f(\xi) = \big(f_1(\xi), f_2(\xi), f_3(\xi)\big)$ will satisfy suitable growth conditions given in the Section 5.

In order to simplify notations we will denote by $\|v\|$ the norm of v in $[L^2(\Omega)]^3$. All notations we use in this article follow the ones given in [8]. From now on we will denote by C a positive constant which may be of different values from line to line.

3. Linear system: existence and uniqueness

In this section we recall a result proved in [7] (Theorem 3.1) where we considered the linear coupled system:

$$u_{tt} - \sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\left(A_{ij}\frac{\partial u}{\partial x_j}\right) + \kappa u_t + \gamma \operatorname{curl} E = 0 \qquad \text{in } \Omega \times (0,\infty), \tag{3.1}$$

$$\varepsilon E_t - \operatorname{curl} H + \sigma E - \gamma \operatorname{curl} u_t = 0 \qquad \text{in } \Omega \times (0,\infty), \tag{3.2}$$

$$\mu H_t + \operatorname{curl} E = 0 \qquad \text{in } \Omega \times (0,\infty), \tag{3.3}$$

$$\operatorname{div}(\mu H) = 0 \qquad \text{in } \Omega \times (0,\infty), \tag{3.4}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x) \qquad \text{in } \Omega, \tag{3.5}$$

$$E(x,0) = E_0(x), \qquad H(x,0) = H_0(x) \qquad \text{in } \Omega, \tag{3.6}$$

$$u = 0, \qquad E \times \eta = 0 \qquad \text{on } \partial\Omega \times (0,\infty). \tag{3.7}$$

We used semigroup theory in the Hilbert space $X = [H_0^1(\Omega)]^3 \times [L^2(\Omega)]^3 \times L^2(\Omega;\varepsilon) \times L^2(\Omega;\mu)$ with the inner product:

$$\langle v, w\rangle_X = \int_\Omega \left\{ \sum_{i,j=1}^{3} \left(A_{ij}(x)\frac{\partial v_1}{\partial x_j}(x)\right) \cdot \frac{\partial w_1}{\partial x_i}(x) + v_1(x)\cdot w_1(x) \right\} dx$$
$$+ \int_\Omega v_2(x)\cdot w_2(x)\, dx + (v_3, w_3)_{L^2(\Omega;\varepsilon)} + (v_4, w_4)_{L^2(\Omega;\mu)}$$

for any $v = (v_1, v_2, v_3, v_4)$, $w = (w_1, w_2, w_3, w_4)$ in X.

Next, we consider the unbounded linear operator $A\colon D(A) \subset X \to X$, with domain

$$D(A) = [H^2(\Omega) \cap H_0^1(\Omega)]^3 \times [H_0^1(\Omega)]^3 \times H_0(\text{curl};\Omega) \times H(\text{curl};\Omega) \tag{3.8}$$

given by

$$Aw = (w_2, Lw_1 - w_1 - \gamma \operatorname{curl} w_3, \gamma\varepsilon^{-1} \operatorname{curl} w_2 + \varepsilon^{-1} \operatorname{curl} w_4, -\mu^{-1} \operatorname{curl} w_3)$$

for any $w = (w_1, w_2, w_3, w_4) \in D(A)$, where L is the operator defined by

$$L = \sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\left(A_{ij}(x)\frac{\partial}{\partial x_j}\right).$$

In (3.8) we denote by

$$H(\operatorname{curl};\Omega) = \{v \text{ in } [L^2(\Omega)]^3 \text{ such that } \operatorname{curl} v \text{ belong to } [L^2(\Omega)]^3\}$$

with inner product

$$\langle v, w\rangle_{H(\operatorname{curl};\Omega)} = \int_\Omega [v(x)\cdot w(x) + \operatorname{curl} v(x)\cdot \operatorname{curl} w(x)]\, dx$$

and

$$H_0(\operatorname{curl};\Omega) = \{v \text{ in } H(\operatorname{curl};\Omega) \text{ such that } \eta \times v|_{\partial\Omega} = 0\}$$

where $\eta = \eta(x)$ is the unit normal vector at $x \in \partial\Omega$ pointing the exterior of Ω. It can be verified that $H_0(\operatorname{curl};\Omega)$ is a closed subspace of $H(\operatorname{curl};\Omega)$ (see [9]) and the property

$$\int_\Omega v(x)\cdot \operatorname{curl} w(x)\, dx = \int_\Omega \operatorname{curl} v(x)\cdot w(x)\, dx \tag{3.9}$$

holds for any $v \in H_0(\operatorname{curl};\Omega)$ and $w \in H(\operatorname{curl};\Omega)$.

We consider now the bounded linear operator $B : X \to X$ given by

$$Bw = (0, w_1 - \kappa w_2, -\sigma\varepsilon^{-1} w_3, 0)$$

for any $w = (w_1, w_2, w_3, w_4)$ in X.

The infinitesimal generator of problem (3.1)–(3.3), (3.5)–(3.7) is given by $\mathscr{A} = A + B$ with domain $D(\mathscr{A}) = D(A)$. Clearly (3.4) will be satisfy for any t if we choose initial data H_0 such that $\operatorname{div}(\mu H_0) = 0$. Since we are interested in decay properties of the solutions of problem (1.1)–(1.6) using the techniques we will describe in the following sections, we will need more regular solutions. Therefore by standard procedure we can obtain:

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^3$ be as in Section 2 and assume that* (H1), (H2) *and* (H3) *hold. If (u_0, u_1, E_0, H_0) belongs to $D(\mathscr{A}^2) \cap Y$, then, system* (3.1)–(3.7) *has a unique* (*strong*) *solution*

$$(u, u_t, E, H) \in \mathscr{C}\big([0, +\infty); D(\mathscr{A}^2) \cap Y\big) \\ \cap \mathscr{C}^1\big([0, +\infty); D(\mathscr{A}) \cap Y\big) \cap \mathscr{C}^2\big([0, +\infty); Y\big)$$

where $Y = \{(w_1, w_2, w_3, w_4)$ in X such that $\operatorname{div}(\mu w_4) = 0$ *in* $\Omega\}$.

By definition, $D(\mathscr{A}^2) = \{w \in D(\mathscr{A})$ such that $\mathscr{A}w \in D(\mathscr{A})\}$. We can easily verify that

$$\begin{aligned} D(\mathscr{A}^2) = \{w = (w_1, w_2, w_3, w_4) &\text{ such that } w_1, w_2 \in [H^2(\Omega) \cap H_0^1(\Omega)]^3; \\ &w_3 \in H_0(\mathrm{curl}; \Omega); w_4 \in H(\mathrm{curl}; \Omega); Lw_1 - \gamma \,\mathrm{curl}\, w_3 \in [H_0^1(\Omega)]^3; \\ &-\sigma\varepsilon^{-1} w_3 + \gamma\varepsilon^{-1} \,\mathrm{curl}\, w_2 + \varepsilon^{-1} \,\mathrm{curl}\, w_4 \in H_0(\mathrm{curl}; \Omega); \\ &\mu^{-1} \,\mathrm{curl}\, w_3 \in H(\mathrm{curl}; \Omega)\} \end{aligned}$$

and the norms

$$\|w\|^2_{D(\mathscr{A}^2)} = \|w\|^2_{D(\mathscr{A})} + \|\mathscr{A}w\|^2_{D(\mathscr{A})}$$

and

$$\begin{aligned} \|w\|^2 = \|w_1\|^2_{[H^2(\Omega)]^3} &+ \|w_2\|^2_{[H^2(\Omega)]^3} + \|w_3\|^2_{H(\mathrm{curl}; \Omega)} + \|w_4\|^2_{H(\mathrm{curl}; \Omega)} \\ &+ \|-\sigma\varepsilon^{-1} w_3 + \gamma\varepsilon^{-1} \,\mathrm{curl}\, w_2 + \varepsilon^{-1} \,\mathrm{curl}\, w_4\|^2_{H(\mathrm{curl}; \Omega)} \\ &+ \|Lw_1 - \gamma \,\mathrm{curl}\, w_3\|^2_{[H^1(\Omega)]^3} + \|\mu^{-1} \,\mathrm{curl}\, w_3\|^2_{H(\mathrm{curl}; \Omega)} \end{aligned} \tag{3.10}$$

are equivalent.

4. Linear system: asymptotic behavior

In this section we study the asymptotic behavior of the solutions of the linear coupled system described in Theorem 3.1. The information we have from our previous work [7] are not enough to obtain de decay of the nonlinearly damped problem (1.1)–(1.6). We have:

Theorem 4.1. *Let us assume that Ω, ε, μ and the matrices A_{ij}, $1 \le i, j \le 3$, have the same assumptions as in Theorem* 3.1. *Let $(u_0, u_1, E_0, H_0) \in D(\mathscr{A}^2) \cap Y$ such* *that* $\mu H_0 = \operatorname{curl}\psi_0$, *for some* $\psi_0 \in H_0(\operatorname{curl};\Omega)$ *and* $(u_0+u_1) \in [L^{6/5}(\Omega)]^3$. *Then, the corresponding solution* (u, u_t, E, H) *of system* (3.1)–(3.7) *satisfies the decay properties*

i) $\|u(t)\|^2 + \|H(t)\|^2 \le CI_0(1+t)^{-1}$

ii) $\int_\Omega Ju(x,t)\,dx + \|E(t)\|^2 + \|\operatorname{curl} H(t)\|^2 \le CI_0(1+t)^{-2}$

iii) $\|u_t(t)\|^2 + \|Lu(t)\|^2 + \|H_t(t)\|^2 + \|\operatorname{curl} E(t)\|^2 \le CI_0(1+t)^{-3}$

iv) $\|E_t(t)\|^2 + \|\operatorname{curl} H_t(t)\|^2 + \int_\Omega Ju_t(x,t)\,dx \le CI_0(1+t)^{-4}$

v) $\|u_{ttt}(t)\|^2 + \int_\Omega Ju_{tt}(x,t)\,dx + \|E_{tt}(t)\|^2 + \|H_{tt}(t)\|^2 + \|\operatorname{curl} E_t(t)\|^2$
$+ \|u_{tt}(t)\|^2 + \|Lu_t(t)\|^2 \le CI_0(1+t)^{-5}$

where $C > 0$ *is a constant independent of the initial data,*

$$I_0 = \|(u_0, u_1, E_0, H_0)\|^2_{D(\mathscr{A}^2)} + \|u_0+u_1\|^2_{[L^{6/5}(\Omega)]^3} + \|\psi_0\|^2,$$

J *is given by* (1.8) *and* i)–v) *hold for any* $t > 0$.

In order to provide a more transparent proof, we divided the discussion into some Lemmas. In all Lemmas below we will assume all hypothesis of Theorem 4.1, C will denote a positive constant which may be of different values from line to line.

Lemma 4.2. *The estimate*

$$(1+t)\mathscr{L}_1(t) + \kappa\int_0^t (1+s)\|u_s(s)\|^2\,ds + \sigma\int_0^t (1+s)\|E(s)\|^2\,ds \le CI_0$$

holds for any $t \ge 0$.

Proof. Let us take the inner product in $[L^2(\Omega)]^3$ of (3.1) with $u_t(t)$, (3.2) with $E(t)$ and (3.3) with $H(t)$. By adding the corresponding identities we find

$$\frac{d\mathscr{L}_1}{dt}(t) + \kappa\|u_t(t)\|^2 + \sigma\|E(t)\|^2 = 0 \tag{4.1}$$

where $\mathscr{L}_1(t)$ is given by (1.7). Multiplying (4.1) by $(1+t)$ and integrating by parts over $[0,t]$ we obtain

$$(1+t)\mathscr{L}_1(t) + \kappa \int_0^t (1+s)\|u_s(s)\|^2\, ds + \sigma \int_0^t (1+s)\|E(s)\|^2\, ds$$
$$= \mathscr{L}_1(0) + \int_0^t \mathscr{L}_1(s)\, ds. \tag{4.2}$$

We take the inner product in $[L^2(\Omega)]^3$ of (3.1) with $u(t)$ and integrate it over the interval $[0, t]$ to obtain

$$\frac{\kappa}{2}\|u(t)\|^2 + \int_0^t \int_\Omega Ju(s)\, dx\, ds = \frac{\kappa}{2}\|u_0\|^2 + \int_0^t \|u_s(s)\|^2\, ds$$
$$- \int_\Omega u_t(t)\cdot u(t)\, dx + \int_\Omega u_1 \cdot u_0\, dx - \gamma \int_0^t \int_\Omega \operatorname{curl} E(s)\cdot u(s)\, dx\, ds. \tag{4.3}$$

Clearly if $v \in [H^1(\Omega)]^3$ then by condition (2.2) we have

$$\int_\Omega |\operatorname{curl} v(x)|^2\, dx \le 2\sum_{i=1}^3 \int_\Omega \left|\frac{\partial v}{\partial x_i}(x)\right|^2 dx \le \frac{2}{a_0}\int_\Omega Jv(x)\, dx. \tag{4.4}$$

Using (3.9) and (4.4) we obtain from (4.3) for any $\delta > 0$ the estimate

$$\frac{\kappa}{2}\|u(t)\|^2 + \int_0^t \int_\Omega Ju(s)\, dx\, ds \le CI_0 + \int_0^t \|u_s(s)\|^2\, ds + \frac{1}{\kappa}\|u_t(t)\|^2$$
$$+ \frac{\kappa}{4}\|u(t)\|^2 + \frac{\gamma}{\delta}\int_0^t \|E(s)\|^2\, ds + C\delta \int_0^t \int_\Omega Ju(s)\, dx\, ds. \tag{4.5}$$

Choosing $\delta > 0$ sufficiently small in (4.5) it follows that

$$\int_0^t \int_\Omega Ju(s)\, dx\, ds \le CI_0 \tag{4.6}$$

for some positive constant C.

We also know (see [7]) that

$$\int_0^t \|H(s)\|^2_{L^2(\Omega;\mu)}\, ds \le CI_0 + C\int_0^t \int_\Omega Ju(s)\, dx\, ds. \tag{4.7}$$

Using (4.6), (4.7) and (4.1) we deduce from (4.2) the estimate the conclusion of Lemma 4.2. □

Lemma 4.3. *The estimate*

$$(1+t)^{\beta}\mathscr{L}_2(t) + \frac{\sigma\beta}{4}(1+t)^{\beta-1}\|E(t)\|^2 + 2\kappa\int_0^t (1+s)^{\beta}\|u_{ss}(s)\|^2\,ds$$
$$+\,2\sigma\int_0^t (1+s)^{\beta}\|E_s(s)\|^2\,ds \le CI_0$$

holds for $\beta = 1, 2$ *or* 3 *and any* $t \ge 0$.

Proof. By taking the derivative with respect to t of problem (3.1)–(3.7) we obtain

$$u_{ttt} - \sum_{i,j=1}^{3}\frac{\partial}{\partial x_i}\left(A_{ij}\frac{\partial u_t}{\partial x_j}\right) + \kappa u_{tt} + \gamma\,\mathrm{curl}\,E_t = 0 \qquad \text{in } \Omega\times(0,\infty), \tag{4.8}$$

$$\varepsilon E_{tt} - \mathrm{curl}\,H_t + \sigma E_t - \gamma\,\mathrm{curl}\,u_{tt} = 0 \qquad \text{in } \Omega\times(0,\infty), \tag{4.9}$$

$$\mu H_{tt} + \mathrm{curl}\,E_t = 0 \qquad \text{in } \Omega\times(0,\infty), \tag{4.10}$$

$$\mathrm{div}(\mu H_t) = 0 \qquad \text{in } \Omega\times(0,\infty), \tag{4.11}$$

$$u_{tt}(0) = u_2 = Lu_0 - \kappa u_1 - \gamma\,\mathrm{curl}\,E_0 \qquad \text{in } \Omega, \tag{4.12}$$

$$E_t(0) = E_1 = \varepsilon^{-1}\,\mathrm{curl}\,H_0 - \sigma\varepsilon^{-1}E_0 + \gamma\varepsilon^{-1}\,\mathrm{curl}\,u_1 \qquad \text{in } \Omega, \tag{4.13}$$

$$H_t(0) = H_1 = -\mu^{-1}\,\mathrm{curl}\,E_0 \qquad \text{in } \Omega, \tag{4.14}$$

$$u_t = 0, \qquad E_t\times\eta = 0 \qquad \text{on } \partial\Omega\times(0,\infty). \tag{4.15}$$

We take the inner product in $[L^2(\Omega)]^3$ of (4.8), (4.9) and (4.10) with $u_{tt}(t)$, $E_t(t)$ and $H_t(t)$ respectively. By adding the corresponding identities we find

$$\frac{d\mathscr{L}_2}{dt}(t) + \kappa\|u_{tt}(t)\|^2 + \sigma\|E_t(t)\|^2 = 0 \qquad \text{for any } t \ge 0 \tag{4.16}$$

where $\mathscr{L}_2(t)$ is the second order analogous of (1.7)

$$\mathscr{L}_2(t) = \frac{1}{2}\left\{\|u_{tt}(t)\|^2 + \int_\Omega Ju_t(t)\,dx + \|E_t(t)\|^2_{L^2(\Omega;\varepsilon)} + \|H_t(t)\|^2_{L^2(\Omega;\mu)}\right\}.$$

Integration of identity (4.16) on $[0, t]$ give us

$$\mathscr{L}_2(t) + \kappa\int_0^t \|u_{ss}(s)\|^2\,ds + \sigma\int_0^t \|E_s(s)\|^2\,ds = \mathscr{L}_2(0) \le CI_0. \tag{4.17}$$

Multiplying (4.16) by $(1+t)^\beta$ where $\beta = 1, 2$ or 3 and integrating the result by parts over $[0, t]$ we obtain

$$(1+t)^\beta \mathscr{L}_2(t) + \kappa \int_0^t (1+s)^\beta \|u_{ss}(s)\|^2\, ds + \sigma \int_0^t (1+s)^\beta \|E_s(s)\|^2\, ds$$
$$= \mathscr{L}_2(0) + \beta \int_0^t (1+s)^{\beta-1} \mathscr{L}_2(s)\, ds. \tag{4.18}$$

Clearly our next step will be to estimate the term $\beta \int_0^t (1+s)^{\beta-1} \mathscr{L}_2(s)\, ds$ and $(1+t)^{\beta-1}\|E(t)\|^2$ by CI_0.

Next, we take the inner product in $[L^2(\Omega)]^3$ of (3.1), (4.9) and (3.3) with $u_{tt}(t)$, $E(t)$ and $H_t(t)$ respectively and add the corresponding identities to obtain

$$\|u_{tt}(t)\|^2 + \|H_t(t)\|^2_{L^2(\Omega;\mu)} - \|E_t(t)\|^2_{L^2(\Omega;\varepsilon)} - \int_\Omega J u_t(t)\, dx$$
$$+ \frac{d}{dt}\left\{ \frac{\kappa}{2}\|u_t(t)\|^2 + \frac{\sigma}{2}\|E(t)\|^2 + \big(E_t(t), E(t)\big)_{L^2(\Omega;\varepsilon)} \right.$$
$$\left. + \int_\Omega \sum_{i,j=1}^3 \left[A_{ij} \frac{\partial u}{\partial x_j}(t) \right] \cdot \frac{\partial u_t}{\partial x_i}(t)\, dx \right\} = 0. \tag{4.19}$$

Multiplying (4.19) by $(1+t)^{\beta-1}$ and integrating the result by parts over $[0, t]$ give us

$$\int_0^t (1+s)^{\beta-1}\|u_{ss}(s)\|^2\, ds + \int_0^t (1+s)^{\beta-1}\|H_s(s)\|^2_{L^2(\Omega;\mu)}\, ds$$
$$+ \frac{\kappa}{2}(1+t)^{\beta-1}\|u_t(t)\|^2 + \frac{\sigma}{2}(1+t)^{\beta-1}\|E(t)\|^2 = \frac{\kappa}{2}\|u_1\|^2 + \frac{\sigma}{2}\|E_0\|^2$$
$$+ \frac{\kappa}{2}(\beta-1)\int_0^t (1+s)^{\beta-2}\|u_s(s)\|^2\, ds + \frac{\sigma}{2}(\beta-1)\int_0^t (1+s)^{\beta-2}\|E(s)\|^2\, ds$$
$$+ \int_0^t (1+s)^{\beta-1}\|E_s(s)\|^2_{L^2(\Omega;\varepsilon)}\, ds + \int_0^t \int_\Omega (1+s)^{\beta-1} J u_s(s)\, dx\, ds$$
$$- (1+t)^{\beta-1}\big(E_t(t), E(t)\big)_{L^2(\Omega;\varepsilon)} + (E_1, E_0)_{L^2(\Omega;\varepsilon)}$$
$$+ (\beta-1)\int_0^t (1+s)^{\beta-2}\big(E_s(s), E(s)\big)_{L^2(\Omega;\varepsilon)}\, ds$$
$$+ \int_\Omega \sum_{i,j=1}^3 \left[A_{ij}\frac{\partial u_0}{\partial x_j}\right] \cdot \frac{\partial u_1}{\partial x_i}\, dx - (1+t)^{\beta-1}\int_\Omega \sum_{i,j=1}^3 \left[A_{ij}\frac{\partial u}{\partial x_j}(t)\right] \cdot \frac{\partial u_t}{\partial x_i}(t)\, dx$$
$$+ (\beta-1)\int_0^t \int_\Omega (1+s)^{\beta-2} \sum_{i,j=1}^3 \left[A_{ij}\frac{\partial u}{\partial x_j}(s)\right] \cdot \frac{\partial u_s}{\partial x_i}(s)\, dx\, ds. \tag{4.20}$$

Using condition (2.2) we obtain from (4.20) for any $\delta > 0$ the estimate

$$\begin{aligned}
&\int_0^t (1+s)^{\beta-1}\|H_s(s)\|^2_{L^2(\Omega;\mu)}\,ds + \frac{\sigma}{2}(1+t)^{\beta-1}\|E(t)\|^2 \le CI_0 \\
&\quad + C\int_0^t (1+s)^{\beta-2}\|u_s(s)\|^2\,ds + C\int_0^t (1+s)^{\beta-2}\|E(s)\|^2\,ds \\
&\quad + C\int_0^t (1+s)^{\beta-1}\|E_s(s)\|^2\,ds + C\int_0^t\int_\Omega (1+s)^{\beta-1}Ju_s(s)\,dx\,ds \\
&\quad + \frac{1}{\delta}(1+t)^{\beta-1}\|E_t(t)\|^2_{L^2(\Omega;\varepsilon)} + C\delta(1+t)^{\beta-1}\|E(t)\|^2 \\
&\quad + \frac{C}{\delta}(1+t)^{\beta-2}\int_\Omega Ju(t)\,dx + C\delta(1+t)^{\beta}\int_\Omega Ju_t(t)\,dx \\
&\quad + C\int_0^t\int_\Omega (1+s)^{\beta-3}Ju(s)\,dx\,ds
\end{aligned} \tag{4.21}$$

for some positive constant C.

Using (4.6) and Lemma 4.2 we deduce from (4.21), $\delta > 0$ sufficiently small and $\beta = 1, 2$ or 3

$$\begin{aligned}
&\int_0^t (1+s)^{\beta-1}\|H_s(s)\|^2_{L^2(\Omega;\mu)}\,ds + \frac{\sigma}{4}(1+t)^{\beta-1}\|E(t)\|^2 \le CI_0 \\
&\quad + C\int_0^t (1+s)^{\beta-1}\|E_s(s)\|^2\,ds + C\int_0^t\int_\Omega (1+s)^{\beta-1}Ju_s(s)\,dx\,ds \\
&\quad + C(1+t)^{\beta-1}\|E_t(t)\|^2_{L^2(\Omega;\varepsilon)} + \frac{1}{2\beta}(1+t)^{\beta}\int_\Omega Ju_t(t)\,dx.
\end{aligned} \tag{4.22}$$

Next, we take the inner product in $[L^2(\Omega)]^3$ of (4.8) with $u_t(t)$. The resulting identity we multiply by $(1+t)^{\beta-1}$ and integrate by parts over $[0, t]$ to obtain

$$\begin{aligned}
&\frac{\kappa}{2}(1+t)^{\beta-1}\|u_t(t)\|^2 + \int_0^t\int_\Omega (1+s)^{\beta-1}Ju_s(s)\,dx\,ds = \frac{\kappa}{2}\|u_1\|^2 \\
&\quad + \frac{\kappa}{2}(\beta-1)\int_0^t (1+s)^{\beta-2}\|u_s(s)\|^2\,ds - (1+t)^{\beta-1}\int_\Omega u_{tt}(t)\cdot u_t(t)\,dx \\
&\quad + (\beta-1)\int_0^t\int_\Omega (1+s)^{\beta-2}u_{ss}(s)\cdot u_s(s)\,dx\,ds \\
&\quad + \int_\Omega u_2\cdot u_1\,dx + \int_0^t (1+s)^{\beta-1}\|u_{ss}(s)\|^2\,ds \\
&\quad - \gamma\int_0^t\int_\Omega (1+s)^{\beta-1}E_s(s)\cdot \operatorname{curl} u_s(s)\,dx\,ds
\end{aligned}$$

$$\le CI_0 + C\int_0^t (1+s)^{\beta-2}\|u_s(s)\|^2\,ds + \frac{1}{\kappa}(1+t)^{\beta-1}\|u_{tt}(t)\|^2$$
$$+\frac{\kappa}{4}(1+t)^{\beta-1}\|u_t(t)\|^2 + C\int_0^t (1+s)^{\beta-1}\|u_{ss}(s)\|^2\,ds$$
$$+\frac{\gamma}{2\delta}\int_0^t (1+s)^{\beta-1}\|E_s(s)\|^2\,ds + \frac{\gamma\delta}{2}\int_0^t (1+s)^{\beta-1}\|\operatorname{curl} u_s(s)\|^2\,ds \tag{4.23}$$

for any $\delta > 0$ and $\beta = 1, 2$ or 3. Using (4.23), Lemma 4.2 and choosing $\delta > 0$ sufficiently small we deduce

$$\frac{1}{2}\int_0^t\int_\Omega (1+s)^{\beta-1} Ju_s(s)\,dx\,ds \le CI_0 + C(1+t)^{\beta-1}\|u_{tt}(t)\|^2$$
$$+ C\int_0^t (1+s)^{\beta-1}\|u_{ss}(s)\|^2\,ds + C\int_0^t (1+s)^{\beta-1}\|E_s(s)\|^2\,ds. \tag{4.24}$$

Now, we use estimate (4.22) into (4.18) and use (4.24) to obtain

$$\frac{(1+t)^\beta}{2}\mathscr{L}_2(t) + \frac{\sigma\beta}{8}(1+t)^{\beta-1}\|E(t)\|^2 + \kappa\int_0^t (1+s)^\beta\|u_{ss}(s)\|^2\,ds$$
$$+\sigma\int_0^t (1+s)^\beta\|E_s(s)\|^2\,ds \le CI_0 + C\beta\int_0^t (1+s)^{\beta-1}\|u_{ss}(s)\|^2\,ds$$
$$+ C\beta\int_0^t (1+s)^{\beta-1}\|E_s(s)\|^2\,ds + C\beta(1+t)^{\beta-1}\|E_t(t)\|^2_{L^2(\Omega;\varepsilon)}$$
$$+ C\beta(1+t)^{\beta-1}\|u_{tt}(t)\|^2 \tag{4.25}$$

for $\beta = 1, 2$ or 3 and any $t \ge 0$.

Setting $\beta = 1$ in (4.25) and using (4.17) we get

$$(1+t)\mathscr{L}_2(t) + \frac{\sigma}{4}\|E(t)\|^2 + 2\kappa\int_0^t (1+s)\|u_{ss}(s)\|^2\,ds$$
$$+ 2\sigma\int_0^t (1+s)\|E_s(s)\|^2\,ds \le CI_0. \tag{4.26}$$

Now, letting $\beta = 2$ in (4.25) and using (4.26) it follows

$$(1+t)^2\mathscr{L}_2(t) + \frac{\sigma}{2}(1+t)\|E(t)\|^2 + 2\kappa\int_0^t (1+s)^2\|u_{ss}(s)\|^2\,ds$$
$$+ 2\sigma\int_0^t (1+s)^2\|E_s(s)\|^2\,ds \le CI_0.$$

Similarly if $\beta = 3$ we get

$$(1+t)^3 \mathscr{L}_2(t) + \frac{3\sigma}{4}(1+t)^2 \|E(t)\|^2 + 2\kappa \int_0^t (1+s)^3 \|u_{ss}(s)\|^2 \, ds$$
$$+ 2\sigma \int_0^t (1+s)^3 \|E_s(s)\|^2 \, ds \le CI_0 \tag{4.27}$$

which concludes the proof of Lemma 4.3. □

Lemma 4.4. *The estimates*

a) $(1+t)^2 \int_\Omega Ju(t)\,dx + \kappa \int_0^t (1+s)^2 \|u_s(s)\|^2 \, ds \le CI_0$

and

b) $\kappa(1+t)^3 \|u_t(t)\|^2 + 2 \int_0^t \int_\Omega (1+s)^3 Ju_s(s)\,dx\,ds \le CI_0$

hold for any $t \ge 0$.

Proof. Taking inner product in $[L^2(\Omega)]^3$ of (3.1) with $u_t(t)$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_t(t)\|^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega Ju(t)\,dx + \kappa \|u_t(t)\|^2 + \gamma \int_\Omega \operatorname{curl} E(t) \cdot u_t(t)\,dx = 0. \tag{4.28}$$

Multiplying identity (4.28) by $(1+t)^2$ and integrating by parts over $[0, t]$ we obtain

$$\frac{1}{2}(1+t)^2 \|u_t(t)\|^2 + \frac{1}{2}(1+t)^2 \int_\Omega Ju(t)\,dx + \kappa \int_0^t (1+s)^2 \|u_s(s)\|^2 \, ds$$
$$= \frac{1}{2}\|u_1\|^2 + \frac{1}{2}\int_\Omega Ju_0\,dx + \int_0^t (1+s)\|u_s(s)\|^2 \, ds$$
$$+ \int_0^t \int_\Omega (1+s) Ju(s)\,dx\,ds - \gamma \int_0^t \int_\Omega (1+s)^2 \operatorname{curl} E(s) \cdot u_s(s)\,dx\,ds. \tag{4.29}$$

Now, we use Lemma 4.1, (3.3) and the following result obtained in [7] (Theorem 4.2)

$$(1+t)\|u(t)\|^2 + \int_0^t \int_\Omega (1+s) Ju(s)\,dx\,ds \le CI_0 \tag{4.30}$$

to deduce from (4.29) for any $\delta > 0$ the estimate

$$
\begin{aligned}
&\frac{1}{2}(1+t)^2\int_\Omega Ju(t)\,dx+\kappa\int_0^t(1+s)^2\|u_s(s)\|^2\,ds\\
&\quad\le CI_0+\gamma\int_0^t(1+s)^2\big(H_s(s),u_s(s)\big)_{L^2(\Omega;\mu)}\,ds\\
&\quad\le C_1I_0+\frac{\gamma}{\delta}\int_0^t(1+s)^2\|H_s(s)\|^2_{L^2(\Omega;\mu)}\,ds+C_1\delta\int_0^t(1+s)^2\|u_s(s)\|^2\,ds. \qquad (4.31)
\end{aligned}
$$

Choosing $\delta>0$ sufficiently small, using (4.22), (4.24) (with $\beta=3$) and (4.27) together with (4.31) we obtain the conclusion of part a) of Lemma 4.4.

Observe that (4.23) remains valid for $\beta=4,5,\ldots$. Finally, we use (4.23) with $\beta=4$ to obtain

$$
\begin{aligned}
&\frac{\kappa}{4}(1+t)^3\|u_t(t)\|^2+\int_0^t\int_\Omega(1+s)^3Ju_s(s)\,dx\,ds\\
&\quad\le CI_0+C\int_0^t(1+s)^2\|u_s(s)\|^2\,ds+\frac{1}{\kappa}(1+t)^3\|u_{tt}(t)\|^2\\
&\qquad+C\int_0^t(1+s)^3\|u_{ss}(s)\|^2\,ds+\frac{\gamma}{2\delta}\int_0^t(1+s)^3\|E_s(s)\|^2\,ds\\
&\qquad+\frac{\gamma\delta}{2}\int_0^t(1+s)^3\|\operatorname{curl}u_s(s)\|^2\,ds.
\end{aligned}
$$

Using (4.27), part a) of lemma and choosing $\delta>0$ sufficiently small we conclude part b) of Lemma 4.4. □

Lemma 4.5. *The estimate*

$$
\begin{aligned}
&(1+t)^5\|u_{ttt}(t)\|^2+(1+t)^5\int_\Omega Ju_{tt}(t)\,dx+(1+t)^5\|E_{tt}(t)\|^2_{L^2(\Omega;\varepsilon)}\\
&\quad+(1+t)^5\|H_{tt}(t)\|^2_{L^2(\Omega;\mu)}+\frac{5\sigma}{2}(1+t)^4\|E_t(t)\|^2\\
&\quad+4\sigma\int_0^t(1+s)^5\|E_{ss}(s)\|^2\,ds\le CI_0
\end{aligned}
$$

hold for any $t\ge0$.

Proof. We differentiate system (4.8)–(4.15) with respect to t to get

$$
u_{tttt}-\sum_{i,j=1}^3\frac{\partial}{\partial x_i}\left(A_{ij}\frac{\partial u_{tt}}{\partial x_j}\right)+\kappa u_{ttt}+\gamma\operatorname{curl}E_{tt}=0\quad\text{in }\Omega\times(0,\infty), \qquad (4.32)
$$

$$
\varepsilon E_{ttt}-\operatorname{curl}H_{tt}+\sigma E_{tt}-\gamma\operatorname{curl}u_{ttt}=0\quad\text{in }\Omega\times(0,\infty), \qquad (4.33)
$$

$$\mu H_{ttt} + \operatorname{curl} E_{tt} = 0 \quad \text{in } \Omega \times (0, \infty), \tag{4.34}$$

$$\operatorname{div}(\mu H_{tt}) = 0 \quad \text{in } \Omega \times (0, \infty), \tag{4.35}$$

$$U_{tt}(0) = (u_2, u_3, E_2, H_2) = \mathscr{A}^2 U_0 \quad \text{in } \Omega, \tag{4.36}$$

$$u_{tt} = 0, \qquad E_{tt} \times \eta = 0 \quad \text{on } \partial\Omega \times (0, \infty) \tag{4.37}$$

where $\mathscr{A}^2$ is given as in Section 3. We take the inner product in $[L^2(\Omega)]^3$ of (4.32) with $u_{ttt}(t)$, (4.33) with $E_{tt}(t)$ and (4.34) with $H_{tt}(t)$. By adding the corresponding identities we obtain

$$\frac{d\mathscr{L}_3}{dt}(t) + \kappa\|u_{ttt}(t)\|^2 + \sigma\|E_{tt}(t)\|^2 = 0, \qquad \text{for any } t \geq 0 \tag{4.38}$$

where

$$\mathscr{L}_3(t) = \frac{1}{2}\left\{\|u_{ttt}(t)\|^2 + \int_\Omega Ju_{tt}(t)\,dx + \|E_{tt}(t)\|^2_{L^2(\Omega;\varepsilon)} + \|H_{tt}(t)\|^2_{L^2(\Omega;\mu)}\right\}.$$

Multiplying (4.38) by $(1+t)^\lambda$ where $\lambda = 1, 2, 3, 4$ or 5 and integrate the resulting identity by parts over $[0, t]$ give us

$$\begin{aligned}(1+t)^\lambda \mathscr{L}_3(t) &+ \kappa\int_0^t (1+s)^\lambda \|u_{sss}(s)\|^2\,ds + \sigma\int_0^t (1+s)^\lambda\|E_{ss}(s)\|^2\,ds \\ &= \mathscr{L}_3(0) + \lambda\int_0^t (1+s)^{\lambda-1}\mathscr{L}_3(s).\end{aligned} \tag{4.39}$$

Now, we take the inner product in $[L^2(\Omega)]^3$ of identities (4.8), (4.33) and (4.10) with $u_{ttt}(t)$, $E_t(t)$ and $H_{tt}(t)$ respectively. Adding the corresponding results we obtain

$$\begin{aligned}&\|u_{ttt}(t)\|^2 + \|H_{tt}(t)\|^2_{L^2(\Omega;\mu)} - \|E_{tt}(t)\|^2_{L^2(\Omega;\varepsilon)} - \int_\Omega Ju_{tt}(t)\,dx \\ &\quad + \frac{d}{dt}\left\{\frac{\kappa}{2}\|u_{tt}(t)\|^2 + \frac{\sigma}{2}\|E_t(t)\|^2 + \big(E_{tt}(t), E_t(t)\big)_{L^2(\Omega;\varepsilon)}\right. \\ &\quad \left. + \int_\Omega \sum_{i,j=1}^3 \left[A_{ij}\frac{\partial u_t}{\partial x_j}(t)\right]\cdot\frac{\partial u_{tt}}{\partial x_i}(t)\,dx\right\} = 0.\end{aligned} \tag{4.40}$$

Now, we can use similar calculations we did to get (4.22) in order to obtain the estimate

$$\int_0^t (1+s)^{\lambda-1}\|H_{ss}(s)\|^2_{L^2(\Omega;\mu)}\,ds + \frac{\sigma}{4}(1+t)^{\lambda-1}\|E_t(t)\|^2$$
$$\leq CI_0 + C\int_0^t (1+s)^{\lambda-1}\|E_{ss}(s)\|^2\,ds + C(1+t)^{\lambda-1}\|E_{tt}(t)\|^2_{L^2(\Omega;\varepsilon)}$$
$$+ C\int_0^t\int_\Omega (1+s)^{\lambda-1}Ju_{ss}(s)\,dx\,ds + \frac{1}{2\lambda}(1+t)^\lambda\int_\Omega Ju_{tt}(t)\,dx \tag{4.41}$$

for $\lambda = 1, 2, 3, 4$ or 5. Let us take the inner product in $[L^2(\Omega)]^3$ of (4.32) with $u_{tt}(t)$ we obtain the equality

$$\frac{\kappa}{2}\frac{d}{dt}\|u_{tt}(t)\|^2 + \int_\Omega Ju_{tt}(t)\,dx = -\frac{d}{dt}\int_\Omega u_{ttt}(t)\cdot u_{tt}(t)\,dx$$
$$+ \|u_{ttt}(t)\|^2 - \gamma\int_\Omega E_{tt}(t)\cdot \operatorname{curl} u_{tt}(t)\,dx. \tag{4.42}$$

We multiply (4.42) by $(1+t)^{\lambda-1}$ and integrate the result by parts over $[0, t]$ to obtain for any $\delta > 0$

$$\frac{\kappa}{2}(1+t)^{\lambda-1}\|u_{tt}(t)\|^2 + \int_0^t\int_\Omega (1+s)^{\lambda-1}Ju_{ss}(s)\,dx\,ds$$
$$= \frac{\kappa}{2}\|u_2\|^2 + \frac{\kappa}{2}(\lambda-1)\int_0^t (1+s)^{\lambda-2}\|u_{ss}(s)\|^2\,ds - (1+t)^{\lambda-1}\int_\Omega u_{ttt}(t)\cdot u_{tt}(t)\,dx$$
$$+ \int_\Omega u_3\cdot u_2\,dx + (\lambda-1)\int_0^t\int_\Omega (1+s)^{\lambda-2}u_{sss}(s)\cdot u_{ss}(s)\,dx\,ds$$
$$+ \int_0^t (1+s)^{\lambda-1}\|u_{sss}(s)\|^2\,ds - \gamma\int_0^t\int_\Omega (1+s)^{\lambda-1}E_{ss}(s)\cdot \operatorname{curl} u_{ss}(s)\,dx\,ds$$
$$\leq CI_0 + C\int_0^t (1+s)^{\lambda-2}\|u_{ss}(s)\|^2\,ds + \frac{1}{\kappa}(1+t)^{\lambda-1}\|u_{ttt}(t)\|^2$$
$$+ \frac{\kappa}{4}(1+t)^{\lambda-1}\|u_{tt}(t)\|^2 + C\int_0^t (1+s)^{\lambda-1}\|u_{sss}(s)\|^2\,ds$$
$$+ \frac{\gamma}{2\delta}\int_0^t (1+s)^{\lambda-1}\|E_{ss}(s)\|^2\,ds + \frac{\gamma\delta}{2}\int_0^t (1+s)^{\lambda-1}\|\operatorname{curl} u_{ss}(s)\|^2\,ds. \tag{4.43}$$

Now we use (4.27) to get from (4.43), $\lambda = 1, 2, 3, 4$ or 5 and $\delta > 0$ sufficiently small the estimate

$$\frac{1}{2}\int_0^t\int_\Omega (1+s)^{\lambda-1}Ju_{ss}(s)\,dx\,ds \leq CI_0 + C(1+t)^{\lambda-1}\|u_{ttt}(t)\|^2$$
$$+ C\int_0^t (1+s)^{\lambda-1}\|u_{sss}(s)\|^2\,ds + C\int_0^t (1+s)^{\lambda-1}\|E_{ss}(s)\|^2\,ds.$$

Using the previous estimate and (4.41) in (4.39) we obtain

$$\frac{(1+t)^{\lambda}}{2}\mathscr{L}_3(t)+\frac{\sigma\lambda}{8}(1+t)^{\lambda-1}\|E_t(t)\|^2+\kappa\int_0^t(1+s)^{\lambda}\|u_{sss}(s)\|^2\,ds$$
$$+\sigma\int_0^t(1+s)^{\lambda}\|E_{ss}(s)\|^2\,ds\le CI_0+C\lambda\int_0^t(1+s)^{\lambda-1}\|u_{sss}(s)\|^2\,ds$$
$$+C\lambda\int_0^t(1+s)^{\lambda-1}\|E_{ss}(s)\|^2\,ds+C\lambda(1+t)^{\lambda-1}\|E_{tt}(t)\|^2_{L^2(\Omega;\varepsilon)}$$
$$+C\lambda(1+t)^{\lambda-1}\|u_{ttt}(t)\|^2. \tag{4.44}$$

Choosing $\lambda=1$ in (4.44) we obtain

$$(1+t)\mathscr{L}_3(t)+2\kappa\int_0^t(1+s)\|u_{sss}(s)\|^2\,ds+2\sigma\int_0^t(1+s)\|E_{ss}(s)\|^2\,ds\le CI_0 \tag{4.45}$$

due to (4.38). Next, choosing $\lambda=2$ in (4.44) and using (4.45) we have

$$(1+t)^2\mathscr{L}_3(t)+2\kappa\int_0^t(1+s)^2\|u_{sss}(s)\|^2\,ds+2\sigma\int_0^t(1+s)^2\|E_{ss}(s)\|^2\,ds\le CI_0.$$

Similarly, choosing $\lambda=3,4$ and 5 using the same idea we obtain the estimates

$$(1+t)^3\mathscr{L}_3(t)+2\kappa\int_0^t(1+s)^3\|u_{sss}(s)\|^2\,ds+2\sigma\int_0^t(1+s)^3\|E_{ss}(s)\|^2\,ds\le CI_0,$$
$$(1+t)^4\mathscr{L}_3(t)+2\kappa\int_0^t(1+s)^4\|u_{sss}(s)\|^2\,ds+2\sigma\int_0^t(1+s)^4\|E_{ss}(s)\|^2\,ds\le CI_0$$

and

$$(1+t)^5\mathscr{L}_3(t)+\frac{5\sigma}{4}(1+t)^4\|E_t(t)\|^2+2\kappa\int_0^t(1+s)^5\|u_{sss}(s)\|^2\,ds$$
$$+2\sigma\int_0^t(1+s)^5\|E_{ss}(s)\|^2\,ds\le CI_0 \tag{4.46}$$

which proves Lemma 4.5. □

Lemma 4.6. *The estimates*

a) $(1+t)^4\int_\Omega Ju_t(t)\,dx\le CI_0$

and

b) $(1+t)^5\|u_{tt}(t)\|^2\le CI_0$

hold for any $t\ge 0$.

Proof. We take the inner product in $[L^2(\Omega)]^3$ of (4.8) with $u_{tt}(t)$. Afterwards, we multiply the identity by $(1+t)^4$, integrate by parts over $[0,t]$ and use estimates obtained in Lemmas 4.3 and 4.4 to obtain

$$\begin{aligned}
&\frac{1}{2}(1+t)^4\|u_{tt}(t)\|^2 + \frac{1}{2}(1+t)^4\int_\Omega Ju_t(t)\,dx + \kappa\int_0^t (1+s)^4\|u_{ss}(s)\|^2\,ds \\
&\quad = \frac{1}{2}\|u_2\|^2 + \frac{1}{2}\int_\Omega Ju_1\,dx + 2\int_0^t (1+s)^3\|u_{ss}(s)\|^2\,ds \\
&\qquad + 2\int_0^t\int_\Omega (1+s)^3 Ju_s(s)\,dx\,ds - \gamma\int_0^t\int_\Omega (1+s)^4 \operatorname{curl} E_s(s)\cdot u_{ss}(s)\,dx\,ds \\
&\quad \le CI_0 - \gamma\int_0^t\int_\Omega (1+s)^4 \frac{d}{ds}[\operatorname{curl} E_s(s)\cdot u_s(s)]\,dx\,ds \\
&\qquad + \gamma\int_0^t\int_\Omega (1+s)^4 \operatorname{curl} E_{ss}(s)\cdot u_s(s)\,dx\,ds \\
&\quad = CI_0 - \gamma(1+t)^4\int_\Omega \operatorname{curl} E_t(t)\cdot u_t(t)\,dx + \gamma\int_\Omega \operatorname{curl} E_1\cdot u_1\,dx \\
&\qquad + 4\gamma\int_0^t\int_\Omega (1+s)^3 \operatorname{curl} E_s(s)\cdot u_s(s)\,dx\,ds \\
&\qquad + \gamma\int_0^t\int_\Omega (1+s)^4 \operatorname{curl} E_{ss}(s)\cdot u_s(s)\,dx\,ds.
\end{aligned}$$

Using equation (4.10) it follows

$$\begin{aligned}
&\frac{1}{2}(1+t)^4\int_\Omega Ju_t(t)\,dx + \kappa\int_0^t (1+s)^4\|u_{ss}(s)\|^2\,ds \le CI_0 \\
&\quad + C(1+t)^5\|H_{tt}(t)\|^2_{L^2(\Omega;\mu)} + C(1+t)^3\|u_t(t)\|^2 + C\int_0^t (1+s)^3\|E_s(s)\|^2\,ds \\
&\quad + C\int_0^t (1+s)^3\|\operatorname{curl} u_s(s)\|^2\,ds + C\int_0^t (1+s)^5\|E_{ss}(s)\|^2\,ds. \qquad (4.47)
\end{aligned}$$

Finally, using our estimates obtained in Lemmas 4.3, 4.4 and 4.5 we conclude that all terms on the right hand side of (4.47) are bounded by CI_0, consequently

$$\frac{1}{2}(1+t)^4\int_\Omega Ju_t(t)\,dx + \kappa\int_0^t (1+s)^4\|u_{ss}(s)\|^2\,ds \le CI_0. \qquad (4.48)$$

Next, we consider $\lambda = 6$ in (4.43) to obtain

$$\frac{\kappa}{4}(1+t)^5\|u_{tt}(t)\|^2 + \int_0^t\int_\Omega (1+s)^5 Ju_{ss}(s)\,dx\,ds \le CI_0$$
$$+ C\int_0^t (1+s)^4\|u_{ss}(s)\|^2\,ds + \frac{1}{\kappa}(1+t)^5\|u_{ttt}(t)\|^2 + C\int_0^t (1+s)^5\|u_{sss}(s)\|^2\,ds$$
$$+ \frac{\gamma}{2\delta}\int_0^t (1+s)^5\|E_{ss}(s)\|^2\,ds + \frac{\gamma\delta}{2}\int_0^t (1+s)^5\|\operatorname{curl} u_{ss}(s)\|^2\,ds.$$

Choosing $\delta > 0$ sufficiently small and using (4.46) and (4.48) we deduce the part b) of Lemma 4.6. □

Proof of Theorem 4.1. i) follows using Lemma 4.2 and (4.30). Part ii) follows using Lemmas 4.3, 4.4 together with (3.2). To prove iii) we can use Lemmas 4.3, 4.4 and (3.3) to prove the decay rate for the terms $\|u_t(t)\|$, $\|H_t(t)\|$ and $\|\operatorname{curl} E(t)\|$. The term $\|Lu(t)\|$ decays at that rate using Lemmas 4.3 and 4.4 together with (3.1). Next, iv) follows using Lemmas 4.5 and 4.6 together with (4.9). Finally, v) follows for the terms $\|u_{ttt}(t)\|$, $\int_\Omega Ju_{tt}(t)\,dx$, $\|E_{tt}(t)\|$ and $\|H_{tt}(t)\|$ due to Lemma 4.5. The term $\|\operatorname{curl} E_t(t)\|$ decay to the required rate due to Lemma 4.5 and (4.10). The terms $\|u_{tt}(t)\|$ and $\|Lu_t(t)\|$ decay at rate $(1+t)^{-5}$ using Lemma 4.6 and Lemmas 4.5 and 4.6 together with (4.8) respectively.

Corollary 4.7. *Under the assumptions of Theorem* 4.1, *the solution* (u, u_t, E, H) *of system* (3.1)–(3.7) *satisfies the estimates:*

i) $\|Lu(t) - \gamma \operatorname{curl} E(t)\|^2_{[H^1(\Omega)]^3} \le CI_0(1+t)^{-3}$

ii) $\|-\sigma\varepsilon^{-1}E(t) + \gamma\varepsilon^{-1}\operatorname{curl} u_t(t) + \varepsilon^{-1}\operatorname{curl} H(t)\|^2_{H(\mathrm{curl};\Omega)} \le CI_0(1+t)^{-2}$

iii) $\|\mu^{-1}\operatorname{curl} E(t)\|^2_{H(\mathrm{curl};\Omega)} \le CI_0(1+t)^{-2}$

for any $t \ge 0$ *where* C *is positive constant independent of the initial data.*

Proof. By having $U_0 \in D(\mathscr{A}^2)$ and $U_t(t) = \mathscr{A}U(t)$ then $U_{tt}(t) = \mathscr{A}U_t(t)$ and $\mathscr{A}U_t(t) = \mathscr{A}^2U(t)$. Thus, $U_{tt}(t) = \mathscr{A}^2U(t)$, that is,

$$u_{tt}(t) = Lu(t) - \kappa u_t(t) - \gamma \operatorname{curl} E(t), \tag{4.49}$$

$$u_{ttt}(t) = Lu_t(t) - \kappa Lu(t) + \kappa^2 u_t(t) + \kappa\gamma \operatorname{curl} E(t) - \gamma \operatorname{curl}\big(-\sigma\varepsilon^{-1}E(t) + \gamma\varepsilon^{-1}\operatorname{curl} u_t(t) + \varepsilon^{-1}\operatorname{curl} H(t)\big), \tag{4.50}$$

$$E_{tt}(t) = \sigma^2\varepsilon^{-1}\varepsilon^{-1}E(t) - \sigma\gamma\varepsilon^{-1}\varepsilon^{-1}\operatorname{curl} u_t(t) - \sigma\varepsilon^{-1}\varepsilon^{-1}\operatorname{curl} H(t) + \gamma\varepsilon^{-1}\operatorname{curl}\big(Lu(t) - \kappa u_t(t) - \gamma\operatorname{curl} E(t)\big) - \varepsilon^{-1}\operatorname{curl}\big(\mu^{-1}\operatorname{curl} E(t)\big), \tag{4.51}$$

$$H_{tt}(t) = -\mu^{-1}\operatorname{curl}\big(-\sigma\varepsilon^{-1}E(t) + \gamma\varepsilon^{-1}\operatorname{curl} u_t(t) + \varepsilon^{-1}\operatorname{curl} H(t)\big). \tag{4.52}$$

Using (4.49) and Theorem 4.1 we obtain

$$\|Lu(t) - \gamma \operatorname{curl} E(t)\|^2_{[H^1(\Omega)]^3} = \|u_{tt}(t) + \kappa u_t(t)\|^2_{[H^1(\Omega)]^3} \le CI_0(1+t)^{-3}.$$

By (4.50) we have

$$\begin{aligned} &\operatorname{curl}\big(-\sigma\varepsilon^{-1}E(t) + \gamma\varepsilon^{-1}\operatorname{curl} u_t(t) + \varepsilon^{-1}\operatorname{curl} H(t)\big) \\ &\quad = -\frac{1}{\gamma}u_{ttt}(t) + \frac{1}{\gamma}Lu_t(t) - \frac{\kappa}{\gamma}Lu(t) + \frac{\kappa^2}{\gamma}u_t(t) + \kappa \operatorname{curl} E(t). \qquad (4.53) \end{aligned}$$

It follows from (4.53) and Theorem 4.1 the estimates

$$\begin{aligned} &\|-\sigma\varepsilon^{-1}E(t) + \gamma\varepsilon^{-1}\operatorname{curl} u_t(t) + \varepsilon^{-1}\operatorname{curl} H(t)\|^2_{H(\mathrm{curl};\Omega)} \\ &\quad = \|-\sigma\varepsilon^{-1}E(t) + \gamma\varepsilon^{-1}\operatorname{curl} u_t(t) + \varepsilon^{-1}\operatorname{curl} H(t)\|^2 \\ &\qquad + \frac{1}{\gamma^2}\|-u_{ttt}(t) + Lu_t(t) - \kappa Lu(t) + \kappa^2 u_t(t) + \gamma\kappa \operatorname{curl} E(t)\|^2 \le CI_0(1+t)^{-2}. \end{aligned}$$

By (4.51) and (4.49) we have

$$\begin{aligned} \operatorname{curl}\big(\mu^{-1}\operatorname{curl} E(t)\big) &= -\varepsilon E_{tt}(t) + \sigma^2\varepsilon^{-1}E(t) \\ &\quad - \sigma\gamma\varepsilon^{-1}\operatorname{curl} u_t(t) - \sigma\varepsilon^{-1}\operatorname{curl} H(t) + \gamma \operatorname{curl} u_{tt}(t). \qquad (4.54) \end{aligned}$$

Using (4.54) and Theorem 4.1 we obtain

$$\begin{aligned} \|\mu^{-1}\operatorname{curl} E(t)\|^2_{H(\mathrm{curl};\Omega)} &= \|\mu^{-1}\operatorname{curl} E(t)\|^2 \\ &\quad + \|-\varepsilon E_{tt}(t) + \sigma^2\varepsilon^{-1}E(t) - \sigma\gamma\varepsilon^{-1}\operatorname{curl} u_t(t) \\ &\quad - \sigma\varepsilon^{-1}\operatorname{curl} H(t) + \gamma \operatorname{curl} u_{tt}(t)\|^2 \\ &\le CI_0(1+t)^{-2} \end{aligned}$$

where we used the estimates

$$\|\operatorname{curl} u_t(t)\|^2 \le C\int_\Omega Ju_t(t)\,dx \quad \text{and} \quad \|\operatorname{curl} u_{tt}(t)\|^2 \le C\int_\Omega Ju_{tt}(t)\,dx.$$

Thus, the Corollary 4.7 is proved. □

5. The nonlinearly damped system: well posedness and asymptotic behavior

Using the results obtained in Section 4 for the linear system we now we will prove the global well posedness of system (1.1)–(1.6) and the asymptotic behavior of the

solutions. Let us write $F(\xi) = \kappa\xi + f(\xi)$ with $\kappa > 0$ and $\xi \in \mathbb{R}^3$. We will assume:
(H4): Let $f = (f_1, f_2, f_3)$ with $f_i : \mathbb{R}^3 \to \mathbb{R}$ such that $f_i \in \mathscr{C}^2(\mathbb{R}^3)$. There exist positive constants k_j, $j = 1, 2, 3$ such that for some $p \geq 2$ we have the growth conditions

$$\begin{aligned}
&|f(\xi)| \leq k_1|\xi|^p \\
&|\nabla f_i(\xi)| \leq k_2|\xi|^{p-1}, \qquad i = 1, 2, 3 \\
&\left|\nabla \frac{\partial f_i}{\partial x_j}(\xi)\right| \leq k_3|\xi|^{p-2}, \qquad i, j = 1, 2, 3
\end{aligned}$$

for any $\xi \in \mathbb{R}^3$. Here $|\cdot|$ denotes the norm in $\mathbb{R}^3$.

Problem (1.1)–(1.3) with $F(\xi) = \kappa\xi + f(\xi)$ and initial condition (1.5) is equivalent to

$$\begin{cases} \dfrac{dU}{dt}(t) = \mathscr{A}U(t) + G\big(U(t)\big) \\ U(0) = U_0 \end{cases} \tag{5.1}$$

where $U(t) = \big(u(t), u_t(t), E(t), H(t)\big)$, $U_0 = (u_0, u_1, E_0, H_0)$, $\mathscr{A}$ is the operator defined in Section 3 and $G\big(U(t)\big) = \big(0, -f\big(u_t(t)\big), 0, 0\big)$.

Lemma 5.1. *Assume that the entries of ε and μ belong to $W^{1,\infty}(\Omega)$. The map $G : D(\mathscr{A}^2) \to D(\mathscr{A}^2)$ satisfies*

i) *Given any positive constant M, there exists L_M such that*

$$\|G(v) - G(w)\|_{D(\mathscr{A})} \leq L_M\|v - w\|_{D(\mathscr{A})}$$

for any $w, v \in D(\mathscr{A}^2)$ such that $\|w\|_{D(\mathscr{A}^2)} \leq M$ and $\|v\|_{D(\mathscr{A}^2)} \leq M$.

ii) *The map G takes bounded sets of $D(\mathscr{A}^2)$ into bounded sets of $D(\mathscr{A}^2)$.*

Proof. With the same notations as in Section 3, let $w, v \in D(\mathscr{A}^2)$ such that $\|w\|_{D(\mathscr{A}^2)} \leq M$ and $\|v\|_{D(\mathscr{A}^2)} \leq M$. Let us denote by $w = (w_1, w_2, w_3, w_4)$ and $v = (v_1, v_2, v_3, v_4)$. Clearly

$$\|G(v) - G(w)\|_X^2 = \|f(v_2) - f(w_2)\|^2 = \sum_{i=1}^{3} \|f_i(v_2) - f_i(w_2)\|_{L^2(\Omega)}^2. \tag{5.2}$$

Using the mean value theorem in (5.2) follows the existence of a positive constant $C = C(M)$ such that

$$\|G(v) - G(w)\|_X^2 \leq C(M)\|v_2 - w_2\|^2 \leq C(M)\|v - w\|_{D(\mathscr{A})}^2.$$

By definition of the operator $\mathscr{A}$ we have

$$\begin{aligned}\|\mathscr{A}(G(v)-G(w))\|_X^2 &= \|\mathscr{A}(0,f(v_2)-f(w_2),0,0)\|_X^2 \\ &= \|(f(v_2)-f(w_2),-\kappa f(v_2)+\kappa f(w_2),\gamma\varepsilon^{-1}\operatorname{curl} f(v_2)-\gamma\varepsilon^{-1}\operatorname{curl} f(w_2),0)\|_X^2 \\ &\le C\|f(v_2)-f(w_2)\|_{[H^1(\Omega)]^3}^2 = C\sum_{i=1}^{3}\|f_i(v_2)-f_i(w_2)\|_{H^1(\Omega)}^2.\end{aligned}$$

Remains to prove that there exists a positive constant $C = C(M)$ such that

$$\left\|\frac{\partial}{\partial x_j}(f_i(v_2)-f_i(w_2))\right\|_{L^2(\Omega)}^2 \le C(M)\|v-w\|_{D(\mathscr{A})}^2, \qquad \text{holds for } i,j=1,2,3.$$

Let $v_2 = (v_{2,1}, v_{2,2}, v_{2,3})$ and $w_2 = (w_{2,1}, w_{2,2}, w_{2,3})$. Using the chain rule we obtain for each $i,j = 1,2,3$.

$$\begin{aligned}&\left\|\frac{\partial}{\partial x_j}(f_i(v_2)-f_i(w_2))\right\|_{L^2(\Omega)}^2 \\ &= \int_\Omega\left|\nabla f_i(v_2)\cdot\left(\frac{\partial v_{2,1}}{\partial x_j},\frac{\partial v_{2,2}}{\partial x_j},\frac{\partial v_{2,3}}{\partial x_j}\right)-\nabla f_i(w_2)\cdot\left(\frac{\partial w_{2,1}}{\partial x_j},\frac{\partial w_{2,2}}{\partial x_j},\frac{\partial w_{2,3}}{\partial x_j}\right)\right|^2 dx \\ &\le 2\int_\Omega\left|\nabla f_i(v_2)\cdot\left(\frac{\partial v_{2,1}}{\partial x_j},\frac{\partial v_{2,2}}{\partial x_j},\frac{\partial v_{2,3}}{\partial x_j}\right)-\nabla f_i(v_2)\cdot\left(\frac{\partial w_{2,1}}{\partial x_j},\frac{\partial w_{2,2}}{\partial x_j},\frac{\partial w_{2,3}}{\partial x_j}\right)\right|^2 dx \\ &\quad + 2\int_\Omega\left|\nabla f_i(v_2)\cdot\left(\frac{\partial w_{2,1}}{\partial x_j},\frac{\partial w_{2,2}}{\partial x_j},\frac{\partial w_{2,3}}{\partial x_j}\right)-\nabla f_i(w_2)\cdot\left(\frac{\partial w_{2,1}}{\partial x_j},\frac{\partial w_{2,2}}{\partial x_j},\frac{\partial w_{2,3}}{\partial x_j}\right)\right|^2 dx \\ &\le C\|\nabla f_i(v_2)\|_{[L^\infty(\Omega)]^3}^2\|v_2-w_2\|_{[H^1(\Omega)]^3}^2 \\ &\quad + C\|\nabla f_i(v_2)-\nabla f_i(w_2)\|_{[L^4(\Omega)]^3}^2\left\|\left(\frac{\partial w_{2,1}}{\partial x_j},\frac{\partial w_{2,2}}{\partial x_j},\frac{\partial w_{2,3}}{\partial x_j}\right)\right\|_{[L^4(\Omega)]^3}^2 \\ &\le C_1(M)\|v_2-w_2\|_{[H^1(\Omega)]^3}^2 + C_1(M)\|v_2-w_2\|_{[L^4(\Omega)]^3}^2 \le C(M)\|v-w\|_{D(\mathscr{A})}^2\end{aligned}$$

which proves i).

Next, let $w \in D(\mathscr{A}^2)$, $w = (w_1, w_2, w_3, w_4)$ such that $\|w\|_{D(\mathscr{A}^2)} \le M$. Due to item i) we know

$$\|G(w)\|_X^2 + \|\mathscr{A}(G(w))\|_X^2 \le C(M).$$

Using the definition of the operator $\mathscr{A}$ (see Section 3), we have

$$\|\mathscr{A}^2(G(w))\|_X^2 = \|\mathscr{A}^2(0,f(w_2),0,0)\|_X^2 \le C\|f(w_2)\|_{[H^2(\Omega)]^3}^2.$$

To conclude the prove of ii) remains only to verify the existence of a positive constant $C = C(M)$ such that

$$\left\| \frac{\partial^2 f_i}{\partial x_k \partial x_j}(w_2) \right\|^2_{L^2(\Omega)} \leq C(M), \qquad \text{for any } i, j, k = 1, 2, 3.$$

Let $w_2 = (w_{2,1}, w_{2,2}, w_{2,3})$ then an straightforward calculation

$$\frac{\partial^2 f_i}{\partial x_k \partial x_j}(w_2) = \sum_{m,l=1}^{3} \left\{ \frac{\partial^2 f_i}{\partial \xi_m \partial \xi_l}(w_2) \frac{\partial w_{2,m}}{\partial x_k} \frac{\partial w_{2,l}}{\partial x_j} \right\} + \sum_{m=1}^{3} \frac{\partial f_i}{\partial \xi_m}(w_2) \frac{\partial^2 w_{2,m}}{\partial x_k \partial x_j}$$

where $f_i(\xi) = f_i(\xi_1, \xi_2, \xi_3)$.

Therefore, we obtain the estimate

$$\left\| \frac{\partial^2 f_i}{\partial x_k \partial x_j}(w_2) \right\|^2_{L^2(\Omega)} \leq C \sum_{m,l=1}^{3} \left\| \frac{\partial^2 f_i}{\partial \xi_m \partial \xi_l}(w_2) \right\|^2_{L^\infty(\Omega)} \left\| \frac{\partial w_{2,m}}{\partial x_k} \right\|^2_{L^4(\Omega)} \left\| \frac{\partial w_{2,l}}{\partial x_j} \right\|^2_{L^4(\Omega)}$$

$$+ C \|\nabla f_i(w_2)\|^2_{[L^\infty(\Omega)]^3} \sum_{m=1}^{3} \left\| \frac{\partial^2 w_{2,m}}{\partial x_k \partial x_j} \right\|^2_{L^2(\Omega)} \leq C(M)$$

which proves item ii) of Lemma 5.1. □

We will use a well known result for evolution systems, see for instance the book of H. Brezis and T. Cazenave [2].

Theorem 5.2. *Let X_1 be a reflexive Banach space and $G : D(\mathscr{A}_1) \to D(\mathscr{A}_1)$ where $\mathscr{A}_1$ is the infinitesimal generator of a $\mathscr{C}_0$ semigroup in X_1. Assume*

i) *G maps bounded sets of $D(\mathscr{A}_1)$ into bounded sets of $D(\mathscr{A}_1)$.*

ii) *For every $M > 0$ there exists a positive constant $C = C(M)$ such that*

$$\|G(v) - G(w)\|_{X_1} \leq C(M)\|v - w\|_{X_1},$$

for all $w, v \in D(\mathscr{A}_1)$ such that $\|w\|_{D(\mathscr{A}_1)} \leq M$, $\|v\|_{D(\mathscr{A}_1)} \leq M$.

Then, for every $v_0 \in D(\mathscr{A}_1)$ there exists a unique strong solution of the problem

$$\begin{cases} \dfrac{dv}{dt}(t) = \mathscr{A}_1 v(t) + G\big(v(t)\big) \\ v(0) = v_0 \end{cases}$$

defined on the maximal interval of existence $[0, T_m)$. Furthermore, either $T_m = +\infty$ or $T_m < \infty$ In the later case

$$\lim_{t \to T_m} (\|v(t)\|_{X_1} + \|\mathscr{A}_1 v(t)\|_{X_1}) = +\infty.$$

We can now use Theorem 5.2 to prove local existence for system (5.1).

Theorem 5.3. *Let $\Omega \subset \mathbb{R}^3$ be an exterior domain* (*unbounded with bounded complement*) *with boundary $\partial\Omega$ of class $\mathscr{C}^2$. Assume conditions* (H1), (H2), (H3) *and* (H4). *Furthermore, let us suppose that the entries of ε and μ belong to $W^{1,\infty}(\Omega)$. If (u_0, u_1, E_0, H_0) belongs to $D(\mathscr{A}^2) \cap Y$, then there exist $T_m > 0$ such that problem* (1.1)–(1.6) (*with $F(u_t) = \kappa u_t + f(u_t)$*) *has a unique solution*

$$(u, u_t, E, H) \in \mathscr{C}\big([0, T_m); D(\mathscr{A}^2) \cap Y\big) \cap \mathscr{C}^1\big([0, T_m); D(\mathscr{A}) \cap Y\big).$$

Furthermore, $T_m = +\infty$ or $T_m < +\infty$ in this case

$$\lim_{t \to T_m} \big\| \big(u(t), u_t(t), E(t), H(t)\big) \big\|_{D(\mathscr{A}^2)} = +\infty$$

where $\| \cdot \|_{D(\mathscr{A}^2)}$ is the norm defined in (3.10).

Proof. Let $X_1 = D(\mathscr{A})$. We define the operator $\mathscr{A}_1$ with domain $D(\mathscr{A}_1) = \{w \in X_1; \mathscr{A}w \in X_1\}$ and given by

$$\mathscr{A}_1 w = \mathscr{A} w, \qquad \text{for any } w \in D(\mathscr{A}_1).$$

Clearly, $\mathscr{A}_1$ is the infinitesimal generator of a semigroup $\mathscr{C}_0$ in X_1 and if we denote by $\{S_1(t)\}_{t \geq 0}$ the semigroup generator by $\mathscr{A}_1$ and $\{S(t)\}_{t \geq 0}$ the one generated by $\mathscr{A}$ we have

$$S_1(t)w = S(t)w, \qquad \text{for any } w \in X_1 \text{ and } t \geq 0.$$

Observe that $D(\mathscr{A}_1) = D(\mathscr{A}^2)$, therefore using Lemma 5.1 the map $G : D(\mathscr{A}_1) \to D(\mathscr{A}_1)$ satisfies the assumptions of Theorem 5.2. In a similar way as done in [7] we can show that whenever $\operatorname{div} \mu H_0 = 0$ then $\operatorname{div} \mu H(t) = 0$ for any $t > 0$. The proof of Theorem 5.3 is now complete. □

Next, we prove global existence of the nonlinearly damped system using the decay estimates for the associated linear system. We recall that $\| \cdot \|$ means the norm in $[L^2(\Omega)]^3$.

Let $w = (w_1, w_2, w_3, w_4) \in D(\mathscr{A}^2) \cap Y$ and define the quantities

$$\|w\|_1 = \|w_1\| + \|w_4\|;$$

$$\|w\|_2 = \|w_3\| + \|\operatorname{curl} w_4\| + \left(\int_\Omega J w_1(x)\, dx \right)^{1/2} + \|-\sigma\varepsilon^{-1} w_3 + \gamma\varepsilon^{-1} \operatorname{curl} w_2 + \varepsilon^{-1} \operatorname{curl} w_4\|_{H(\mathrm{curl};\Omega)} + \|\mu^{-1} \operatorname{curl} w_3\|_{H(\mathrm{curl};\Omega)};$$

$$\|w\|_3 = \|w_2\| + \|Lw_1\| + \|\operatorname{curl} w_3\| + \|Lw_1 - \gamma \operatorname{curl} w_3\|_{[H^1(\Omega)]^3};$$

$$\|w\|_4 = \left(\int_\Omega Jw_2(x)\,dx\right)^{1/2};$$

$$\|w\|_5 = \|Lw_2\|. \tag{5.3}$$

Our aim in this section is to prove the following.

Theorem 5.4. *Under the assumptions of Theorem* 5.3, *let* $(u_0, u_1, E_0, H_0) \in D(\mathscr{A}^2) \cap Y$ *such that* $\mu H_0 = \operatorname{curl} \psi_0$ *for some* $\psi_0 \in H_0(\operatorname{curl}; \Omega)$ *and* $(u_0 + u_1) \in [L^{6/5}(\Omega)]^3$. *Then, there exists* $\delta_0 > 0$ *sufficiently small such that if* $I_0 < \delta_0$, *system* (1.1)–(1.6) (*with* $F(u_t) = \kappa u_t + f(u_t)$) *has a unique strong solution*

$$(u, u_t, E, H) \in \mathscr{C}\big([0, +\infty); D(\mathscr{A}^2) \cap Y\big) \cap \mathscr{C}^1\big([0, +\infty); D(\mathscr{A}) \cap Y\big)$$

and has the following decay rates

i) $\|u(t)\|^2 + \|H(t)\|^2 \le CI_0(1+t)^{-1}$

ii) $\|E(t)\|^2 + \|\operatorname{curl} H(t)\|^2 + \int_\Omega Ju(x,t)\,dx \le CI_0(1+t)^{-2}$

iii) $\|u_t(t)\|^2 + \|Lu(t)\|^2 + \|\operatorname{curl} E(t)\|^2 \le CI_0(1+t)^{-3}$

iv) $\int_\Omega Ju_t(x,t)\,dx \le CI_0(1+t)^{-4}$

v) $\|Lu_t(t)\|^2 \le CI_0(1+t)^{-5}$

for any $t \ge 0$ *where* C *is a positive constant* (*independent of the initial data*) *and* I_0 *is given as in Theorem* 4.1.

Proof. We recall that $\{S(t)\}_{t\ge 0}$ is the semigroup generated by $\mathscr{A}$. With our notations in (5.3) and the results of Theorem 4.1 we have that

$$\begin{aligned}
\|S(t)[u_0, u_1, E_0, H_0]\|_1 &\le CI_0^{1/2}(1+t)^{-1/2}\\
\|S(t)[u_0, u_1, E_0, H_0]\|_2 &\le CI_0^{1/2}(1+t)^{-1}\\
\|S(t)[u_0, u_1, E_0, H_0]\|_3 &\le CI_0^{1/2}(1+t)^{-3/2}\\
\|S(t)[u_0, u_1, E_0, H_0]\|_4 &\le CI_0^{1/2}(1+t)^{-2}\\
\|S(t)[u_0, u_1, E_0, H_0]\|_5 &\le CI_0^{1/2}(1+t)^{-5/2}
\end{aligned} \tag{5.4}$$

holds for some positive constant C and for any $t \ge 0$. Let us define

$$I_0(s) = \|f(u_t(s))\|^2_{[H^2(\Omega)]^3} + \|f(u_t(s))\|^2_{[L^{6/5}(\Omega)]^3}$$

for $0 \leq s < T_m$ and u is the solution given in Theorem 5.3. We claim that

$$I_0(s) \leq \tilde{C} \|u_t(s)\|^{2p}_{[H^2(\Omega)]^3} \qquad \text{for } 0 \leq s < T_m. \tag{5.5}$$

In fact, using (H4) and denoting by $u_t(s) = \big(u_t^1(s), u_t^2(s), u_t^3(s)\big)$ we have

$$\begin{aligned}
I_0(s) &= \sum_{i=1}^{3} \big\| f_i\big(u_t(s)\big)\big\|^2_{H^2(\Omega)} + \big\| f\big(u_t(s)\big)\big\|^2_{[L^{6/5}(\Omega)]^3} \\
&\leq C \sum_{i=1}^{3} \Big(\big\| f_i\big(u_t(s)\big)\big\|^2_{L^2(\Omega)} + \big\| \Delta f_i\big(u_t(s)\big)\big\|^2_{L^2(\Omega)} \Big) + C \|u_t(s)\|^{2p}_{[L^{6p/5}(\Omega)]^3} \\
&\leq C_1 \|u_t(s)\|^{2p}_{[L^{2p}(\Omega)]^3} + C_1 \|u_t(s)\|^{2p}_{[L^{6p/5}(\Omega)]^3} \\
&\quad + C_1 \sum_{i=1}^{3} \Bigg\| \sum_{j,m=1}^{3} \frac{\partial f_i}{\partial \xi_m}\big(u_t(s)\big) \frac{\partial^2 u_t^m}{\partial x_j^2}(s) \\
&\qquad\qquad + \sum_{j,k,m=1}^{3} \frac{\partial^2 f_i}{\partial \xi_m \partial \xi_k}\big(u_t(s)\big) \frac{\partial u_t^m}{\partial x_j}(s) \frac{\partial u_t^k}{\partial x_j}(s) \Bigg\|^2_{L^2(\Omega)} \leq C_2 \|u_t(s)\|^{2p}_{[H^2(\Omega)]^3} \\
&\quad + C_2 \sum_{j=1}^{3} \int_\Omega |u_t(x,s)|^{2(p-1)} \left| \left(\frac{\partial^2 u_t^1}{\partial x_j^2}(x,s), \frac{\partial^2 u_t^2}{\partial x_j^2}(x,s), \frac{\partial^2 u_t^3}{\partial x_j^2}(x,s) \right) \right|^2 dx \\
&\quad + C_2 \sum_{j=1}^{3} \int_\Omega |u_t(x,s)|^{2(p-2)} \left| \left(\frac{\partial u_t^1}{\partial x_j}(x,s), \frac{\partial u_t^2}{\partial x_j}(x,s), \frac{\partial u_t^3}{\partial x_j}(x,s) \right) \right|^4 dx \\
&\leq \tilde{C} \|u_t(s)\|^{2p}_{[H^2(\Omega)]^3}
\end{aligned}$$

which prove our claim (5.5). As a consequence of (5.4) we deduce

$$\begin{aligned}
\big\| S(t-s) G\big(U(s)\big)\big\|_1 &\leq C I_0^{1/2}(s) (1+t-s)^{-1/2} \\
\big\| S(t-s) G\big(U(s)\big)\big\|_2 &\leq C I_0^{1/2}(s) (1+t-s)^{-1} \\
\big\| S(t-s) G\big(U(s)\big)\big\|_3 &\leq C I_0^{1/2}(s) (1+t-s)^{-3/2} \\
\big\| S(t-s) G\big(U(s)\big)\big\|_4 &\leq C I_0^{1/2}(s) (1+t-s)^{-2} \\
\big\| S(t-s) G\big(U(s)\big)\big\|_5 &\leq C I_0^{1/2}(s) (1+t-s)^{-5/2}
\end{aligned} \tag{5.6}$$

for any $0 \leq s \leq t$ and $0 \leq t < T_m$. Let K a positive constant such that $K > C$ where C is the constant which appears in (5.4) and (5.6).

We will prove Theorem 5.4 by contradiction: Assume that at least one of the following inequalities is untrue for $0 \le t < T_m$

$$\begin{aligned}
&(1+t)^{1/2}\|U(t)\|_1 \le KI_0^{1/2}\\
&(1+t)\|U(t)\|_2 \le KI_0^{1/2}\\
&(1+t)^{3/2}\|U(t)\|_3 \le KI_0^{1/2}\\
&(1+t)^{2}\|U(t)\|_4 \le KI_0^{1/2}\\
&(1+t)^{5/2}\|U(t)\|_5 \le KI_0^{1/2}.
\end{aligned} \tag{5.7}$$

Suppose (for example) that the first inequality of (5.7) is untrue. Then, by continuity we should have some T_1 with $0 < T_1 < T_m$ such that

$$\begin{aligned}
&(1+t)^{1/2}\|U(t)\|_1 < KI_0^{1/2}, &&\text{for any } 0 \le t < T_1\\
&(1+T_1)^{1/2}\|U(T_1)\|_1 = KI_0^{1/2}\\
&(1+t)^{1/2}\|U(t)\|_1 > KI_0^{1/2}, &&\text{for any } T_1 < t < T_1+\varepsilon_1
\end{aligned}$$

for some $\varepsilon_1 > 0$. It may happen that other inequality in (5.7) be also untrue. Suppose (for example) the second inequality in (5.7) is untrue. Then, by continuity we will have some T_2 with $0 < T_2 < T_m$ such that

$$\begin{aligned}
&(1+t)\|U(t)\|_2 < KI_0^{1/2}, &&\text{for any } 0 \le t < T_2\\
&(1+T_2)\|U(T_2)\|_2 = KI_0^{1/2}\\
&(1+t)\|U(t)\|_2 > KI_0^{1/2}, &&\text{for any } T_2 < t < T_2+\varepsilon_2
\end{aligned}$$

for some $\varepsilon_2 > 0$. Taking $\widetilde{T_0} = \min\{T_1, T_2\}$ we will have

$$\begin{aligned}
&(1+t)^{1/2}\|U(t)\|_1 < KI_0^{1/2}, &&\text{for any } 0 \le t < \widetilde{T_0}\\
&(1+t)\|U(t)\|_2 < KI_0^{1/2}, &&\text{for any } 0 \le t < \widetilde{T_0}
\end{aligned}$$

and at least one of the following identities be valid

$$\begin{aligned}
&(1+\widetilde{T_0})^{1/2}\|U(\widetilde{T_0})\|_1 = KI_0^{1/2}\\
&(1+\widetilde{T_0})\|U(\widetilde{T_0})\|_2 = KI_0^{1/2}.
\end{aligned}$$

Then, by continuity we should have some T_0, $0 < T_0 < T_m$ such that

$$\begin{aligned}
&(1+t)^{1/2}\|U(t)\|_1 < KI_0^{1/2}, &&\text{for any } 0 \le t < T_0\\
&(1+t)\|U(t)\|_2 < KI_0^{1/2}, &&\text{for any } 0 \le t < T_0\\
&(1+t)^{3/2}\|U(t)\|_3 < KI_0^{1/2}, &&\text{for any } 0 \le t < T_0\\
&(1+t)^{2}\|U(t)\|_4 < KI_0^{1/2}, &&\text{for any } 0 \le t < T_0\\
&(1+t)^{5/2}\|U(t)\|_5 < KI_0^{1/2}, &&\text{for any } 0 \le t < T_0
\end{aligned}$$

and at least one of the following identities be valid

$$\begin{aligned}
&(1+T_0)^{1/2}\|U(T_0)\|_1 = KI_0^{1/2}\\
&(1+T_0)\|U(T_0)\|_2 = KI_0^{1/2}\\
&(1+T_0)^{3/2}\|U(T_0)\|_3 = KI_0^{1/2}\\
&(1+T_0)^{2}\|U(T_0)\|_4 = KI_0^{1/2}\\
&(1+T_0)^{5/2}\|U(T_0)\|_5 = KI_0^{1/2}.
\end{aligned} \tag{5.8}$$

Now, we will reach the contradiction using the estimates below. We obtain the existence of some $\delta_i > 0$, $i = 1, \dots, 5$ such that if $I_0 < \min\{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$ then

$$\begin{aligned}
&(1+t)^{1/2}\|U(t)\|_1 < KI_0^{1/2}, &&\text{for any } 0 \le t \le T_0\\
&(1+t)\|U(t)\|_2 < KI_0^{1/2}, &&\text{for any } 0 \le t \le T_0\\
&(1+t)^{3/2}\|U(t)\|_3 < KI_0^{1/2}, &&\text{for any } 0 \le t \le T_0\\
&(1+t)^{2}\|U(t)\|_4 < KI_0^{1/2}, &&\text{for any } 0 \le t \le T_0\\
&(1+t)^{5/2}\|U(t)\|_5 < KI_0^{1/2}, &&\text{for any } 0 \le t \le T_0
\end{aligned}$$

which is in contradiction with (5.8). The estimates are the following:

Since $g = -Lu_t(t) + u_t(t)$ belongs to $[L^2(\Omega)]^3$ in our region Ω (for $0 \le t < T_m$) we can use elliptic regularity to obtain from $-Lu_t(t) + u_t(t) = g$ that $u_t(t) \in [H^2(\Omega)]^3$ and

$$\|u_t(t)\|_{[H^2(\Omega)]^3} \le C_0\{\|Lu_t(t)\| + \|u_t(t)\|\}$$

for some positive constant C_0. Therefore, for any $p \ge 2$ we have

$$\|u_t(t)\|^p_{[H^2(\Omega)]^3} \le C_0^p\{\|U(t)\|_5 + \|U(t)\|_3\}^p$$

(see (5.3)), where $U(t) = \big(u(t), u_t(t), E(t), H(t)\big)$. Using (5.8) it follows that

$$\begin{aligned}\|u_t(t)\|^p_{[H^2(\Omega)]^3} &\le C_0^p K^p I_0^{p/2}\{(1+t)^{-5/2} + (1+t)^{-3/2}\}^p \\ &\le 2C_0^p K^p I_0^{p/2}(1+t)^{-3p/2} \end{aligned} \tag{5.9}$$

for any $0 \le t < T_0$.

Now, we use (5.4), (5.6) and the variation of parameter's formula

$$U(t) = S(t)U_0 + \int_0^t S(t-s)G\big(U(s)\big)\,ds$$

to estimate in the norma $\|\cdot\|_1$ for any $0 \le t \le T_0$

$$\begin{aligned}\|U(t)\|_1 &\le CI_0^{1/2}(1+t)^{-1/2} + C\int_0^t (1+t-s)^{-1/2} I_0^{1/2}(s)\,ds \\ &\le CI_0^{1/2}(1+t)^{-1/2} + C\tilde{C}^{1/2}\int_0^t (1+t-s)^{-1/2}\|u_s(s)\|^p_{[H^2(\Omega)]^3}\,ds, \\ &\le CI_0^{1/2}(1+t)^{-1/2} \\ &\quad + 2C_0^p C\tilde{C}^{1/2}\int_0^t (1+t-s)^{-1/2}K^p I_0^{p/2}(1+s)^{-3p/2}\,ds \end{aligned} \tag{5.10}$$

where we used (5.5) and (5.9). In the last term on the right hand side of (5.10) we can use a calculus lemma (see R. Racke [21] or R. Ikehata [15]) which says that the estimate

$$\int_0^t (1+t-s)^{-1/2}(1+s)^{-\beta}\,ds \le C_\beta(1+t)^{-1/2} \qquad \text{for any } t > 0$$

holds as long as $\beta > 1$. Thus, from (5.10) we obtain

$$\|U(t)\|_1 \le CI_0^{1/2}(1+t)^{-1/2} + 2C_0^p C\tilde{C}^{1/2}C_\beta K^p I_0^{p/2}(1+t)^{-1/2}$$

for any $0 \le t \le T_0$. Let $\delta_1 > 0$ such that

$$\delta_1 \le \left(\frac{K - C}{2C_0^p C\tilde{C}^{1/2}C_\beta K^p}\right)^{2/(p-1)}.$$

Therefore, if we take $I_0 < \delta_1$ it follows that

$$\|U(t)\|_1 < KI_0^{1/2}(1+t)^{-1/2}, \qquad \text{for any } 0 \le t \le T_0. \tag{5.11}$$

Similarly, if we use the calculus estimates (see [21])

$$\int_0^t (1+t-s)^{-1}(1+s)^{-\beta}\,ds \le C_\beta (1+t)^{-1}, \qquad \text{whenever } \beta > 1$$

and

$$\int_0^t (1+t-s)^{-m}(1+s)^{-\beta}\,ds \le C(\beta,m)(1+t)^{-m}, \qquad \text{whenever } 1 < m \le \beta$$

we obtain

$$\begin{aligned} \|U(t)\|_2 &< K I_0^{1/2}(1+t)^{-1} \\ \|U(t)\|_3 &< K I_0^{1/2}(1+t)^{-3/2} \\ \|U(t)\|_4 &< K I_0^{1/2}(1+t)^{-2} \\ \|U(t)\|_5 &< K I_0^{1/2}(1+t)^{-5/2} \end{aligned} \tag{5.12}$$

for any $0 \le t \le T_0$.

Observe that the norms $\sum_{i=1}^5 \|\cdot\|_i$ and $\|\cdot\|_{D(\mathscr{A}^2)}$ are equivalent in $D(\mathscr{A}^2)$. Thus, we conclude the existence of $\delta > 0$ such that if $I_0 < \delta$ then the solution of problem (5.1) satisfies $\|U(t)\|_{D(\mathscr{A}^2)} \le K_2$ for any $t \in [0, T_m)$ and some positive constant K_2. Therefore, Theorem 5.3 implies that $T_m = +\infty$. □

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