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Bi-Lipschitz equivalent metrics on groups, and a problem in additive number theory

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To José Dias da Silva on the occasion of his retirement

Abstract. There is a standard "word length" metric canonically associated to any set of generators for a group. In particular, for any integers a and b greater than 1, the additive group $\mathbb{Z}$ has generating sets $\{a^i\}_{i=0}^{\infty}$ and $\{b^j\}_{j=0}^{\infty}$ with associated metrics d_A and d_B, respectively. It is proved that these metrics are bi-Lipschitz equivalent if and only if there exist positive integers m and n such that $a^m = b^n$.

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1. Groups as metric spaces

A beautiful unsolved problem in metric geometry and geometric group theory is to determine if the additive group of integers with the word metric determined by the generating set of powers of 2 is quasi-isometric to the additive group of integers with the word metric determined by the generating set of powers of 3. A related question is to determine if these two arithmetically defined metric spaces are bi-Lipschitz equivalent or if their metrics are bi-Lipschitz equivalent. This paper gives a simple necessary and sufficient condition for word metrics on $\mathbb{Z}$ defined by geometric series to be bi-Lipschitz equivalent. In particular, it follows that the word metrics defined by the sets $\{2^i\}_{i=1}^{\infty}$ and $\{3^j\}_{j=1}^{\infty}$ are not bi-Lipschitz equivalent.

This work is related to recent work on phase transitions in infinitely generated groups (Alpert [1], Jin [2], Nathanson [6], [7]).

In order to make this paper accessible to number theorists as well as geometers, and to fix terminology, I begin with a brief introduction to bi-Lipschitz equivalence and word metrics on groups. For more information on these topics, see Burago, Burago, and Ivanov [3], de la Harpe [4], and Gromov [5].

Two metrics d_A and d_B on the same set X are called *bi-Lipschitz equivalent metrics* if there exists a number $K \geq 1$ such that

$$\frac{1}{K} d_A(x_1, x_2) \leq d_B(x_1, x_2) \leq K d_A(x_1, x_2) \tag{1}$$

for all $x_1, x_2 \in X$. In this paper we study metrics on the additive group $\mathbb{Z}$ of integers that are defined in a natural way by geometric progressions, and give a necessary and sufficient condition for two such metrics to be bi-Lipschitz equivalent.

More generally, two metric spaces (X, d_X) and (Y, d_Y) are called *bi-Lipschitz equivalent* if there exists a function f from X onto Y such that

$$\frac{1}{K} d_X(x_1, x_2) \leq d_Y\bigl(f(x_1), f(x_2)\bigr) \leq K d_X(x_1, x_2)$$

for some number $K \geq 1$ and all $x_1, x_2 \in X$. The function f is called a *bi-Lipschitz equivalence*; it is necessarily a homeomorphism. In particular, the metrics d_A and d_B on a set X are bi-Lipschitz equivalent metrics if and only if the identity map from (X, d_A) to (X, d_B) is a bi-Lipschitz equivalence.

Let G be a group with identity e, and let A be a set of generators for G. The set A may be finite or infinite. Let $A^{-1} = \{a^{-1} : a \in A\}$. A *word with respect to A* is a finite product of the form $a_1 a_2 \ldots a_n$, where $a_i \in A \cup A^{-1}$ for $i = 1, \ldots, n$. We call n the *length* of this word. An element $x \in G \backslash \{e\}$ has *length* $\ell_A(x) = n$ with respect to the generating set A if n is the least positive integer such that x can be written as a word with respect to A of length n. We define $\ell_A(e) = 0$. Every element of G has finite length because A generates G.

The length function has the following properties. First, $\ell_A(x) = 0$ if and only if $x = e$. Second, $\ell_A(x) = \ell_A(x^{-1})$ for all $x \in G$. Third, there is the subadditivity condition

$$\ell_A(xy) \leq \ell_A(x) + \ell_A(y) \tag{2}$$

for all $x, y \in G$.

We use the length function associated with the generating set A to construct a distance function d_A on the group G. For all $x, y \in G$, let $d_A(x, y)$ be the length of the group element $x^{-1}y$, that is,

$$d_A(x, y) = \ell_A(x^{-1}y)$$

for all $x, y \in X$. (If G is an abelian group, written additively, we define $d_A(x, y) = \ell_A(y - x)$.) In particular, $d_A(x, e) = \ell_A(x)$ for all $x \in G$. It follows that $d_A(x, y) = 0$ if and only if $\ell_A(x^{-1}y) = 0$, that is, if and only if $x^{-1}y = e$ or $x = y$. Similarly,

$$d_A(x, y) = \ell_A(x^{-1}y) = \ell\big((x^{-1}y)^{-1}\big) = \ell(y^{-1}x) = d_A(y, x).$$

By inequality (2), we have

$$d_A(x, z) = \ell(x^{-1}z) = \ell(x^{-1}yy^{-1}z) \le \ell(x^{-1}y) + \ell(y^{-1}z) = d_A(x, y) + d_A(y, z)$$

for all $x, y, z \in G$, and so d satisfies the triangle inequality. Thus, (G, d_A) is a metric space. We call d_A the *metric associated to the generating set* A.

Lemma 1.1. *Let* $A = \{a_i\}_{i \in I}$ *and* $B = \{b_j\}_{j \in J}$ *be generating sets for the group* G. *Let* ℓ_A *and* ℓ_B *be the corresponding length functions, and* d_A *and* d_B *the associated metrics on* G. *The following conditions are equivalent:*

(i) $\sup\{\ell_A(b_j) : j \in J\} < \infty$ *and* $\sup\{\ell_B(a_i) : i \in I\} < \infty$.

(ii) *The metrics* d_A *and* d_B *are bi-Lipschitz equivalent.*

Proof. If $K_B = \sup\{\ell_A(b_j) : j \in J\} < \infty$, then every generator $b_j \in B$ can be represented as product of at most K_B elements of $A \cup A^{-1}$. Taking the inverse of this representation, we see that the inverse generator b_j^{-1} can also be represented as a product of at most K_B elements of $A \cup A^{-1}$. Let $x, y \in G$. If $d_B(x, y) = n$, then $x^{-1}y$ is a product of n elements of $B \cup B^{-1}$. Writing each of these as a product of at most K_B elements of $A \cup A^{-1}$, we obtain a representation of $x^{-1}y$ as the product of at most $K_B n$ elements of $A \cup A^{-1}$, and so $d_A(x, y) \le K_B n = K_B d_B(x, y)$. Similarly, if $K_A = \sup\{\ell_B(a_i) : i \in I\} < \infty$, then every element in $A \cup A^{-1}$ can be represented as a product of at most K_A elements of $B \cup B^{-1}$, and $d_B(x, y) \le K_A d_A(x, y)$. Thus, if $K_A < \infty$ and $K_B < \infty$, then inequality (1) holds with $K = \max(K_A, K_B)$, and the metrics d_A and d_B are equivalent. This proves that (i) implies (ii).

Conversely, if the metrics are d_A and d_B are bi-Lipschitz equivalent, then there exists a number $K \ge 1$ such that inequality (1) holds. For every generator $a_i \in A$ we have

$$\ell_B(a_i) = d_B(a_i, e) \le K d_A(a_i, e) = K\ell_A(a_i) = K$$

and so $\sup\{\ell_B(a_i) : i \in I\} \le K < \infty$. Similarly, $\sup\{\ell_A(b_j) : j \in J\} \le K < \infty$. Therefore, (ii) implies (i). This completes the proof. □

Corollary 1.2. *Let $A = \{a_i\}_{i \in I}$ and $B = \{b_j\}_{j \in J}$ be generating sets for the group G. If A and B are finite sets, then the associated metrics d_A and d_B are bi-Lipschitz equivalent.*

Proof. It suffices to observe that the finite sets of numbers $\{\ell_B(a_i) : i \in I\}$ and $\{\ell_A(b_j) : j \in J\}$ have finite upper bounds. □

Corollary 1.3. *Let $A = \{a_i\}_{i \in I}$ and $B = \{b_j\}_{j \in J}$ be generating sets for the group G. If A is a finite set and B is an infinite set, then the associated metrics d_A and d_B are not bi-Lipschitz equivalent.*

Proof. If $|A| = r < \infty$, then $|A \cup A^{-1}| \le 2r$, and for every positive integer s there are less than $(2r)^{s+1}$ words with respect to $A \cup A^{-1}$ of length at most s. Since B is infinite, it follows that there are infinitely many generators $b_j \in B$ with $\ell_A(b_j) > s$, and so $\sup\{\ell_A(b_j) : j \in J\} = \infty$. Therefore, the metrics d_A and d_B are not bi-Lipschitz equivalent. □

It remains to determine when the metrics associated with different infinite generating sets for a group are bi-Lipschitz equivalent. This is an open problem even for $\mathbb{Z}$, the additive group of integers, for which the generating sets are the sets (finite or infinite) of relatively prime integers. We shall prove the following theorem, which determines when the metrics associated with infinite geometric sequences of integers are bi-Lipschitz equivalent.

Main Theorem. *Let a and b be integers greater than 1, and consider the additive group $\mathbb{Z}$ with generating sets $A = \{a^i\}_{i=0}^{\infty}$ and $B = \{b^j\}_{j=0}^{\infty}$. Let d_A and d_B be the metrics on $\mathbb{Z}$ associated with the generating sets A and B, respectively. These metrics are bi-Lipschitz equivalent if and only if there exist positive integers m and n such that $a^m = b^n$.*

2. Representations of integer powers to various integer bases

Let a be an integer greater than 1. Every nonnegative integer n has a unique a-adic representation

$$n = \sum_{i=0}^{\infty} \delta_i a^i \tag{3}$$

where $\delta_i \in \{0, 1, 2, \dots, a-1\}$ for all nonnegative integers i, and $\delta_i = 0$ for all sufficiently large i. For integers $u < v$ we denote by $[u, v)$ the interval of integers $\{u, u+1, \dots, v-1\}$. An interval $[u, v)$ is called an *a-adic block* for the positive integer n if, in the a-adic expansion (3), we have $\delta_i \neq 0$ for all $i \in [u, v)$. The a-adic block $[u, v)$ for the positive integer n is called a *maximal a-adic block* if either $u = 0$ and $\delta_v = 0$, or $u \geq 1$ and $\delta_{u-1} = \delta_v = 0$. We define the *maximal a-adic block function* $M_A(n)$ as the number of maximal a-adic blocks in the a-adic expansion of n. For example, if I is a set of k nonnegative integers, if $\delta_i \in \{1, 2, \ldots, a-1\}$ for $i \in I$, and if $n = \sum_{i \in I} \delta_i a^i$, then $M_A(n) \leq k$. Moreover, $M_A(n) = k$ if and only if no two elements of I are consecutive.

Lemma 2.1. *Let $a \geq 2$ and $r \geq 1$. Let $J = \{j_i\}_{i=0}^r$ be a strictly decreasing sequence of nonnegative integers and let $\delta_{j_i} \in \{1, 2, \ldots, a-1\}$ for $i = 0, 1, \ldots, r$. If*

$$n_J = \delta_{j_0} a^{j_0} - \sum_{i=1}^{r} \delta_{j_i} a^{j_i}$$

then

$$M_A(n_J) \leq r$$

and

$$n_J = (\delta_{j_0} - 1) a^{j_0} + \sum_{i=j_r+1}^{j_0-1} \delta'_i a^i + \delta'_{j_r} a^{j_r},$$

where $\delta'_i \in \{0, 1, 2, \ldots, a-1\}$ for $i = j_r, \ldots, j_0 - 1$ and $\delta'_{j_r} \neq 0$.

Proof. The proof is by induction on r. For $r = 1$ we have

$$\begin{aligned} n_J &= \delta_{j_0} a^{j_0} - \delta_{j_1} a^{j_1} \\ &= (\delta_{j_0} - 1) a^{j_0} + (a^{j_0} - a^{j_1}) - (\delta_{j_1} - 1) a^{j_1} \\ &= (\delta_{j_0} - 1) a^{j_0} + \sum_{i=j_1}^{j_0-1} (a-1) a^i - (\delta_{j_1} - 1) a^{j_1} \\ &= (\delta_{j_0} - 1) a^{j_0} + \sum_{i=j_1+1}^{j_0-1} (a-1) a^i + (a - \delta_{j_1}) a^{j_1} \end{aligned}$$

and

$$1 \leq a - \delta_{j_1} \leq a - 1.$$

The nonzero digits of n_J form a single a-adic block, and so $M_A(n_J) = 1$.

Let $r \geq 1$ and suppose that the Lemma is true for r. Let $J = \{j_i\}_{i=0}^{r+1}$ be a strictly decreasing sequence of nonnegative integers and let $\delta_{j_i} \in \{1, 2, \ldots, a-1\}$ for $i = 0, 1, \ldots, r, r+1$. We consider the integer

$$n_J = \delta_{j_0} a^{j_0} - \sum_{i=1}^{r+1} \delta_{j_i} a^{j_i}.$$

Let

$$m_J = \delta_{j_0} a^{j_0} - \sum_{i=1}^{r} \delta_{j_i} a^{j_i}.$$

By the induction hypothesis, $M_A(m_J) \leq r$, and a^{j_r} is the smallest power of a that appears in the a-adic expansion of m_J with a nonzero digit. It follows that

$$M_A(m_J) - 1 \leq M_A(m_J - a^{j_r}) \leq M_A(m_J) \leq r.$$

We write

$$n_J = m_J - \delta_{j_{r+1}} a^{j_{r+1}} = (m_J - a^{j_r}) + (a^{j_r} - \delta_{j_{r+1}} a^{j_{r+1}}).$$

Again applying the induction hypothesis, we see that the positive integer $a^{j_r} - \delta_{j_{r+1}} a^{j_{r+1}}$ has exactly one maximal a-adic block, and that the largest power of a that appears in its a-adic expansion with a nonzero digit is less than a^{j_r}. It follows that

$$M_A(m_J) \leq M_A(n_J) \leq M_A(m_J) + 1 \leq r + 1.$$

Moreover, $a^{j_{r+1}}$ is the smallest power of a that appears in the a-adic expansion of $a^{j_r} - \delta_{j_{r+1}} a^{j_{r+1}}$ with a nonzero digit, and so $a^{j_{r+1}}$ is the smallest power of a that appears in the a-adic expansion of n_J with a nonzero digit. This completes the proof. □

Lemma 2.2. *Let I and W be disjoint finite sets of nonnegative integers. Let $a \geq 2$, and let $\delta_i \in \{1, 2, \ldots, a-1\}$ for $i \in I$ and $\delta_w \in \{1, 2, \ldots, a-1\}$ for $w \in W$. Then*

$$M_A\Big(\sum_{i \in I} \delta_i a^i\Big) - |W| \leq M_A\Big(\sum_{i \in I} \delta_i a^i + \sum_{w \in W} \delta_w a^w\Big) \leq M_A\Big(\sum_{i \in I} \delta_i a^i\Big) + |W|.$$

Proof. It suffices to prove the Lemma for $|W| = 1$. Adding a "new" power of a to an a-adic representation changes a zero digit to a nonzero digit. If the former zero digit was adjacent to two nonzero digits, then the number of maximal a-adic blocks decreases by 1. If the former zero digit was adjacent to one zero digit and to one nonzero digit, then the number of maximal a-adic blocks does not change. If the former zero digit was adjacent to two zero digits, then the number of maximal a-adic blocks increases by 1. This completes the proof. $\square$

Lemma 2.3. *Let $a \geq 2$ and $k \geq 1$. If n is a positive integer such that*

$$n = \sum_{t \in T} \varepsilon_t \delta_t a^t,$$

where T is a set of k nonnegative integers, $\delta_t \in \{1, 2, \dots, a-1\}$ and $\varepsilon_t \in \{1, -1\}$ for all $t \in T$, then $M_A(n) \leq k$.

Proof. Since n is positive, it follows that $\varepsilon_{t^*} = 1$ for $t^* = \max(T)$. If $\varepsilon_t = 1$ for all $t \in T$, then $n = \sum_{t \in T} \delta_t a^t$ is the a-adic representation, which has exactly k nonzero digits, and so $M_A(n) \leq k$.

Suppose that $\varepsilon_t = -1$ for some $t \in T$. Arrange T in strictly increasing order $t_1 < t_2 < \cdots < t_k$. Let U be the set of all t_i with $i \geq 2$ such that $\varepsilon_{t_i} = 1$ and $\varepsilon_{t_{i-1}} = -1$. Let $\ell = \operatorname{card}(U)$. We observe that $\varepsilon_{t^*} = 1$ implies that $\ell \geq 1$. Arrange the elements of U in strictly increasing order

$$u_1 < \cdots < u_\ell.$$

Define $u_0 = -1$. For $j = 1, 2, \dots, \ell$, let

$$V_j = \{t \in T : u_{j-1} < t < u_j \text{ and } \varepsilon_t = -1\}.$$

The set V_j is nonempty for all $j = 1, 2, \dots, \ell$. If

$$v_j = \min(V_j)$$

then

$$u_{j-1} < v_j < u_j \qquad \text{for } j = 1, \dots, \ell.$$

Moreover, $\varepsilon_{t^*} = 1$ implies that if $t \in T$ and $\varepsilon_t = -1$, then $t \in V_j$ for some j. Let $V = \bigcup_{j=1}^{\ell} V_j$. We define

$$n_{V_j} = \varepsilon_{u_j} \delta_{u_j} a^{u_j} + \sum_{v \in V_j} \varepsilon_v \delta_v a^v = \delta_{u_j} a^{u_j} - \sum_{v \in V_j} \delta_v a^v.$$

By Lemma 2.1,

$$a^{u_{j-1}+1} \leq a^{v_j} \leq n_{V_j} < a^{u_j+1} \tag{4}$$

and a^{v_j} is the smallest power of a that appears in the a-adic expansion of n_{V_j} with a nonzero digit. Also,

$$M_A(n_{V_j}) \leq \text{card}(V_j).$$

We define

$$n' = \sum_{t \in U \cup V} \varepsilon_t \delta_t a^t = \sum_{j=1}^{\ell} n_{V_j}.$$

Then

$$M_A(n') \leq \sum_{j=1}^{\ell} M_A(n_{V_j}) \leq \sum_{j=1}^{\ell} \text{card}(V_j) = \text{card}(V).$$

Let $W = T \backslash (U \cup V)$. Then $\varepsilon_w = 1$ for all $w \in W$, and

$$n = n' + \sum_{w \in W} \delta_w a^w.$$

Let I be the set of all nonnegative integers i such that a^i occurs with a nonzero digit in the a-adic representation of n'. If $i \in I$, then $v_j \leq i \leq u_j$ for some $j \in \{1, 2, \ldots, \ell\}$. On the other hand, if $w \in W$, then $w > u_\ell$ or $u_{j-1} < w < v_j$ for some $j \in \{1, 2, \ldots, \ell\}$. Therefore, $I \cap W = \emptyset$. An application of Lemma 2.2 gives

$$M_A(n) \leq M_A(n') + |W| \leq |V| + |W| = k - |U| \leq k - 1.$$

This completes the proof. $\square$

Let $n = \sum_{i=0}^{\infty} \delta_i a^i$ be the a-adic expansion of the positive integer n. We introduce the function

$$\text{ord}_A(n) = \max\{i : \delta_i \neq 0\}.$$

Let $r = \text{ord}_A(n)$. For every positive integer $k \leq r + 1$, we call the k-tuple

$$(\delta_{r-k+1}, \delta_{r-k+2}, \ldots, \delta_{r-1}, \delta_r) \in \{0, 1, 2, \ldots, a-1\}^{k-1} \times \{1, 2, \ldots, a-1\}$$

the *leading k-digit string of n with respect to a*.

The following result is presumably well known.

Lemma 2.4. *Let a and b be integers greater than 1 such that $a^m \neq b^n$ for all positive integers m and n. Let $(\gamma_0, \gamma_1, \ldots, \gamma_{k-2}, \gamma_{k-1})$ be a k-tuple in $\{0, 1, 2, \ldots, a-1\}^{k-1} \times \{1, 2, \ldots, a-1\}$. There exist infinitely many positive integers n such that b^n has leading k-digit string $(\gamma_0, \gamma_1, \ldots, \gamma_{k-2}, \gamma_{k-1})$ with respect to a.*

Proof. We claim that the positive real number $\log b/(k \log a)$ is irrational for all positive integers k. If not, then there exist positive integers r and s such that $\log b/(k \log a) = r/s$, or equivalently, $a^{kr} = b^s$, which is absurd.

Let $t = \sum_{i=0}^{k-1} \gamma_i a^i$. Since $\gamma_{k-1} \in \{1, 2, \ldots, a-1\}$, we have $a^{k-1} \leq t < a^k$, and so

$$0 \leq \frac{k-1}{k} \leq \frac{\log t}{k \log a} < \frac{\log(t+1)}{k \log a} \leq 1.$$

Let $\{x\}$ denote the fractional part of the real number x. Since the sequence of fractional parts of the positive integral multiples of an irrational number is uniformly distributed in the unit interval $[0, 1)$, it follows that there exists a set $\mathcal{N}$ of positive integers of positive asymptotic density $\log\big((t+1)/t\big)/(k \log a)$ such that, for every $n \in \mathcal{N}$, we have

$$\frac{\log t}{k \log a} \leq \left\{\frac{n \log b}{k \log a}\right\} < \frac{\log(t+1)}{k \log a}.$$

Thus, for every $n \in \mathcal{N}$ there is a positive integer m such that

$$\frac{\log t}{k \log a} \leq \frac{n \log b}{k \log a} - m < \frac{\log(t+1)}{k \log a}.$$

This implies that

$$ta^{km} \leq b^n < (t+1)a^{km}$$

and so $\operatorname{ord}_A(b^n - ta^{km}) \leq km - 1$. It follows that there exist $\delta_i \in \{0, 1, \ldots, a-1\}$ for $i = 0, 1, \ldots, km-1$ such that

$$b^n - ta^{km} = \sum_{i=0}^{km-1} \delta_i a^i.$$

Therefore,

$$b^n = \sum_{i=0}^{km-1} \delta_i a^i + ta^{km} = \sum_{i=0}^{km-1} \delta_i a^i + \Big(\sum_{i=0}^{k-1} \gamma_i a^i\Big) a^{km} = \sum_{i=0}^{km-1} \delta_i a^i + \sum_{i=0}^{k-1} \gamma_i a^{km+i}.$$

Thus, b^n has leading k-digit string $(\gamma_0, \gamma_1, \ldots, \gamma_{k-2}, \gamma_{k-1})$ with respect to a. □

Lemma 2.5. *Let a and b be integers greater than 1 such that $a^m \neq b^n$ for all positive integers m and n. For every positive integer ℓ there exist infinitely many positive integers n such that $M_A(b^n) \geq \ell$.*

Proof. By Lemma 2.4, there exist infinitely many positive integers n such that b^n has leading 2ℓ-digit string $(0, 1, 0, 1, \ldots, 0, 1, 0, 1)$ with respect to a. For each such n we have $M_A(b^n) \geq \ell$. □

3. Proof of the Main Theorem

Theorem 3.1. *Let a and b be integers greater than 1, and consider the additive group $\mathbb{Z}$ with generating sets $A = \{a^i\}_{i=0}^{\infty}$ and $B = \{b^j\}_{j=0}^{\infty}$. Let d_A and d_B be the metrics on $\mathbb{Z}$ associated with the generating sets A and B, respectively. If there exist positive integers m and n such that $a^m = b^n$, then these metrics are bi-Lipschitz equivalent.*

Proof. If $a^m = b^n$, then for all nonnegative integers q we have $a^{qm} = b^{qn}$, and so $\ell_B(a^{qm}) = 1$.

Let i be a nonnegative integer. By the division algorithm, there exist nonnegative integers q and r such that $i = qm + r$ and $0 \leq r \leq m - 1$. Then

$$a^i = a^{qm} a^r = \underbrace{a^{qm} + \cdots + a^{qm}}_{a^r \text{ summands}}$$

By inequality (2),

$$\ell_B(a^i) \leq a^r \ell_B(a^{qm}) \leq a^r \leq a^{m-1}$$

and so $\sup\{\ell_B(a^i) : a^i \in A\} < \infty$. Similarly, $\sup\{\ell_A(b^j) : b^j \in B\} < \infty$. Lemma 1.1 implies that the metrics d_A and d_B are bi-Lipschitz equivalent. This completes the proof. □

Theorem 3.2. *Let a and b be integers greater than 1, and consider the additive group $\mathbb{Z}$ with generating sets $A = \{a^i\}_{i=0}^{\infty}$ and $B = \{b^j\}_{j=0}^{\infty}$. Let d_A and d_B be the metrics on $\mathbb{Z}$ associated with the generating sets A and B, respectively. If these metrics are bi-Lipschitz equivalent, then there exist positive integers m and n such that $a^m = b^n$.*

Proof. Since d_A and d_B are bi-Lipschitz equivalent metrics on the group G, Lemma 1.1 implies that

$$L = \sup\{\ell_A(b^j) : b^j \in B\} < \infty$$

and so every generator $b^j \in B$ can be represented as a word with respect to A of length at most L. By Lemma 2.3, $M_A(b^j) \le L$. If $a^m \ne b^n$ for all positive integers m and n, then Lemma 2.5 implies that there exist infinitely many n such that b^n has leading $2(L+1)$-digit string $(0,1,0,1,\ldots,0,1)$, and for these numbers b^n we have $M_A(b^n) \ge L+1$. This is a contradiction, and so $a^m = b^n$ for some m and n. □

It is worthwhile to record the following elementary number theoretic observation.

Lemma 3.3. *Let a and b be positive integers. There exist positive integers m and n such that $a^m = b^n$ if and only if there exist relatively prime positive integers m' and n' such that $a^{m'} = b^{n'}$. There exist relatively prime positive integers m and n such that $a^m = b^n$ if and only if there exists a positive integer c such that $a = c^n$ and $b = c^m$.*

Proof. Let m and n be positive integers such that $a^m = b^n$. If d is the greatest common divisor of m and n, then $m = m'd$ and $n = n'd$, where m' and n' are relatively prime positive integers. We have

$$(a^{m'})^d = a^m = b^n = (b^{n'})^d$$

and so $a^{m'} = b^{n'}$.

Let m and n be relatively prime positive integers such that $a^m = b^n$. Let $\mathbf{P}$ be the set of prime numbers. By the fundamental theorem of arithmetic, for every prime p there exist unique nonnegative integers α_p and β_p such that $\alpha_p = \beta_p = 0$ for all sufficiently large p, and

$$a = \prod_{p \in \mathbf{P}} p^{\alpha_p} \quad \text{and} \quad b = \prod_{p \in \mathbf{P}} p^{\beta_p}.$$

Then

$$\prod_{p \in \mathbf{P}} p^{m\alpha_p} = a^m = b^n = \prod_{p \in \mathbf{P}} p^{n\beta_p}$$

and so $m\alpha_p = n\beta_p$ for all $p \in \mathbf{P}$. Because m and n are relatively prime, it follows that, for all p, there is a nonnegative integer γ_p with $\alpha_p = n\gamma_p$, hence $\beta_p = m\gamma_p$. Let $c = \prod_{p \in \mathbf{P}} p^{\gamma_p}$. We have

$$a = \prod_{p \in \mathbf{P}} p^{\alpha_p} = \prod_{p \in \mathbf{P}} p^{n\gamma_p} = c^n$$

and

$$b = \prod_{p \in \mathbf{P}} p^{\beta_p} = \prod_{p \in \mathbf{P}} p^{m\gamma_p} = c^m.$$

Conversely, if $a = c^n$ and $b = c^m$, then $a^m = c^{mn} = b^n$. This completes the proof. □

4. Quasi-isometry

Problem 1. *Let a and b be integers greater than* 1*, and let d_A and d_B be the metrics on $\mathbb{Z}$ associated with the generating sets $A = \{a^i\}_{i=0}^{\infty}$ and $B = \{b^j\}_{j=0}^{\infty}$, respectively. By the Main Theorem, the identity map from $\mathbb{Z}$ to $\mathbb{Z}$ is a bi-Lipschitz equivalence if and only if there exist positive integers m and n such that $a^m = b^n$. It is an open problem to determine if there exists some map $f : \mathbb{Z} \to \mathbb{Z}$ that is a bi-Lipschitz equivalence with respect to these metrics, that is, to determine if the metric spaces $(\mathbb{Z}, d_A)$ and $(\mathbb{Z}, d_B)$ are bi-Lipschitz equivalent.*

Metric spaces (X, d_X) and (Y, d_Y) are called *quasi-isometric* if there are subspaces $X' \subseteq X$ and $Y' \subseteq Y$ and a number $C > 0$ such that (i) the metric spaces (X', d_X) and (Y', d_Y) are bi-Lipschitz equivalent, and (ii) for every $x \in X$ there exists $x' \in X'$ such that $d_X(x, x') < C$, and for every $y \in Y$ there exists $y' \in Y'$ such that $d_Y(y, y') < C$.

Problem 2. *Let a and b be integers greater than* 1*, and let d_A and d_B be the metrics on $\mathbb{Z}$ associated with the generating sets $A = \{a^i\}_{i=0}^{\infty}$ and $B = \{b^j\}_{j=0}^{\infty}$, respectively. Are the metric spaces $(\mathbb{Z}, d_A)$ and $(\mathbb{Z}, d_B)$ quasi-isometric? This problem was first posed by Richard E. Schwartz for the generating sets $A = \{2^i\}_{i=0}^{\infty}$ and $B = \{3^j\}_{j=0}^{\infty}$.*

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