Portugal. Math. (N.S.)
Vol. 68, Fasc. 3, 2011, 239–257
DOI 10.4171/PM/1890

Portugaliae Mathematica

Agreeable solutions of variational problems

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(Communicated by Irene Fonseca)

Abstract. In this paper we study solutions of infinite horizon variational problems associated with a certain class of integrands. We consider c-optimal solutions, which were introduced and used for models of solid-state physics and in the theory of thermodynamical equilibrium for materials and agreeable solutions introduced for models of economic dynamics. We show that if an integrand possesses an asymptotic turnpike property, then these two optimality notions are equivalent.

Mathematics Subject Classification (2010). 49J99.

Keywords. Agreeable function, c-optimal function, good function, infinite horizon problem, integrand.

1. Introduction

In this paper we analyze solutions of infinite horizon variational problems associated with the functional

$$\int_{T_1}^{T_2} f\big(v(t), v'(t)\big)\,dt,$$

where $T_1 \geq 0$, $T_2 > T_1$, $v : [T_1, T_2] \to \mathbb{R}^n$ is an absolutely continuous (a.c.) function and $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^1$ belongs to a space of integrands described below. It should be mentioned that the study of properties of solutions of optimal control problems and variational problems defined on infinite domains and on sufficiently large domains has recently been a rapidly growing area of research. See, for example, [2], [3], [5], [10], [15], [17], [24], [26] and the references mentioned therein.

In this paper we study solutions of infinite horizon variational problems associated with a certain class of integrands. We consider c-optimal solutions which were introduced and used for models of solid-state physics [1], [20] and in the theory of thermodynamical equilibrium for materials [4], [11]–[14] and agreeable solutions which were introduced for models of economic dynamics [7]–[9]. We show that if an integrand possesses an asymptotic turnpike property, then these two optimality notions are equivalent.

Denote by $|\cdot|$ the Euclidean norm in $\mathbb{R}^n$. Let a be a positive constant and let $\psi : [0, \infty) \to [0, \infty)$ be an increasing function such that $\psi(t) \to \infty$ as $t \to \infty$. Denote by $\mathscr{A}$ the set of all continuous functions $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^1$ which satisfy the following assumptions:

A(i) for each $x \in \mathbb{R}^n$ the function $f(x, \cdot) : \mathbb{R}^n \to \mathbb{R}^1$ is convex;

A(ii) $f(x, y) \geq \max\{\psi(|x|), \psi(|y|)|y|\} - a$ for each $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$;

A(iii) for each $M, \varepsilon > 0$ there exist $\Gamma, \delta > 0$ such that

$$|f(x_1, y_1) - f(x_2, y_2)| \leq \varepsilon \max\{f(x_1, y_1), f(x_2, y_2)\}$$

for each $y_1, y_2, x_1, x_2 \in \mathbb{R}^n$ which satisfy

$$|x_i| \leq M, \quad |y_i| \geq \Gamma, \qquad i = 1, 2, \qquad |x_1 - x_2|, |y_1 - y_2| \leq \delta.$$

The set $\mathscr{A}$ contains many integrands. Examples of functions $f \in \mathscr{A}$ can be found in [21]–[24].

It is easy to show that an integrand $f = f(x, y) \in C^1(\mathbb{R}^{2n})$ belongs to $\mathscr{A}$ if f satisfies assumptions A(i), A(ii) and if there exists an increasing function $\psi_0 : [0, \infty) \to [0, \infty)$ such that

$$\max\{|\partial f/\partial x(x, y)|, |\partial f/\partial y(x, y)|\} \leq \psi_0(|x|)\big(1 + \psi(|y|)|y|\big)$$

for each $x, y \in \mathbb{R}^n$.

Example 1.1. It is not difficult to see that if $\psi(t) = t$ for all $t \geq 0$, $n = 1$, if functions $h_1, h_2 \in C^1(\mathbb{R}^1)$ satisfy

$$h_1(x) \geq |x| + 1, \qquad x \in \mathbb{R}^1$$

and if the function $h_2 : \mathbb{R}^1 \to \mathbb{R}^1$ is convex and

$$y^2 + 1 \leq h_2(y) \leq c_0(y^2 + 1), \qquad |h_2'(y)| \leq c_0(y^2 + 1)$$

for all $y \in \mathbb{R}^1$, where c_0 is a positive constant, then the function

$$f(x, y) = h_1(x)h_2(y), \qquad (x, y) \in \times\mathbb{R}^1 \times \mathbb{R}^1,$$

belongs to $\mathscr{A}$.

For the set $\mathcal{A}$ we consider the uniformity which is determined by the following base:

$$E(N,\varepsilon,\lambda) = \{(f,g) \in \mathcal{A} \times \mathcal{A} : |f(x,y) - g(x,y)| \le \varepsilon \text{ for all } x, y \in \mathbb{R}^n \text{ satisfying } |x|, |y| \le N\}$$
$$\cap \{(f,g) \in \mathcal{A} \times \mathcal{A} \mid (|f(x,y)| + 1)(|g(x,y)| + 1)^{-1} \in [\lambda^{-1}, \lambda] \text{ for all } x, y \in \mathbb{R}^n \text{ satisfying } |x| \le N\},$$

where $N, \varepsilon > 0$ and $\lambda > 1$. In this paper we consider the space $\mathcal{A}$ equipped with the topology induced by this uniformity. It was shown in [21], [24] that the uniform space $\mathcal{A}$ is metrizable and complete.

We consider functionals of the form

$$I^f(T_1, T_2, v) = \int_{T_1}^{T_2} f(v(t), v'(t))\, dt, \tag{1.1}$$

where $f \in \mathcal{A}$, $0 \le T_1 < T_2 < \infty$ and $v : [T_1, T_2] \to \mathbb{R}^n$ is an absolutely continuous (a.c.) function.

For $f \in \mathcal{A}$, $y, z \in \mathbb{R}^n$ and real numbers T_1, T_2 satisfying $0 \le T_1 < T_2$ we set

$$U^f(T_1, T_2, y, z) = \inf\{I^f(T_1, T_2, v) \mid v : [T_1, T_2] \to \mathbb{R}^n \text{ is an a.c. function satisfying } v(T_1) = y, v(T_2) = z\} \tag{1.2}$$

and

$$\sigma^f(T_1, T_2, y) = \inf\{U^f(T_1, T_2, y, z) \mid z \in \mathbb{R}^n\}. \tag{1.3}$$

It is easy to see that $-\infty < U^f(T_1, T_2, y, z) < \infty$ for each $f \in \mathcal{A}$, each $y, z \in \mathbb{R}^n$ and all numbers T_1, T_2 satisfying $0 \le T_1 < T_2$.

Let $f \in \mathcal{A}$. For any a.c. function $v : [0, \infty) \to \mathbb{R}^n$ we set

$$J(v) = \liminf_{T \to \infty} T^{-1} I^f(0, T, v). \tag{1.4}$$

Of special interest is the minimal long-run average cost growth rate

$$\mu(f) = \inf\{J(v) \mid v : [0, \infty) \to \mathbb{R}^n \text{ is an a.c. function}\}. \tag{1.5}$$

Clearly $-\infty < \mu(f) < \infty$.

Here we follow [5], [10], [24] in defining good functions for variational problems.

Let $f \in \mathscr{A}$. An a.c. function $v : [0, \infty) \to \mathbb{R}^n$ is called an (f)-good function if the function

$$T \to I^f(0, T, v) - \mu(f)T, \qquad T \in (0, \infty),$$

is bounded.

In [21] we showed that for each $f \in \mathscr{A}$ and each $z \in \mathbb{R}^n$ there exists an (f)-good function $v : [0, \infty) \to \mathbb{R}^n$ satisfying $v(0) = z$.

We follow [12] in defining c-optimal functions.

An a.c. function $v : [0, \infty) \to \mathbb{R}^n$ is called c-optimal with respect to f (or just c-optimal if the function f is understood) if $\sup\{|v(t)| \mid t \in [0, \infty)\} < \infty$ and if for each $T > 0$ the equality

$$I^f(0, T, v) = U^f\big(0, T, v(0), v(T)\big)$$

holds.

Note that any c-optimal with respect to f function is (f)-good (see Proposition 5.2 of [21]).

For the proof of the following result see Theorem 1.1 of [21] and Theorem 1.1 of [22].

Proposition 1.2. *For each $f \in \mathscr{A}$ and any $z \in \mathbb{R}^n$ there exists a c-optimal with respect to f function $v : [0, \infty) \to \mathbb{R}^n$ such that $v(0) = z$.*

The notion of c-optimality is a slight modification of the notion of minimality introduced in [6] and discussed in [16], [18], [19]. The difference is that in our paper c-optimal solutions are bounded and defined on the interval $[0, \infty)$ while in [16], [18], [19] minimal solutions are defined on the whole space $\mathbb{R}^n$ and the boundedness is not assumed. Note that an analogous notion of minimality was used in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals [1], [20].

In the sequel we use the following result (see Proposition 1.1 of [23]).

Proposition 1.3. *Let $f \in \mathscr{A}$. Then for any a.c. function $v : [0, \infty) \to \mathbb{R}^n$ either $I^f(0, T, v) - T\mu(f) \to \infty$ as $T \to \infty$ or*

$$\sup\{|I^f(0, T, v) - T\mu(f)| \mid T \in (0, \infty)\} < \infty.$$

Moreover any (f)-good function $v : [0, \infty) \to \mathbb{R}^n$ is bounded.

An a.c. function $v : [0, \infty) \to \mathbb{R}^n$ is called (f)-agreeable if for each $T_0 > 0$ and each $\varepsilon > 0$ there exists $T_\varepsilon > T_0$ such that for each $T > T_\varepsilon$ there exists an a.c. function $w : [0, T] \to \mathbb{R}^n$ such that

$$w(t) = v(t), \qquad t \in [0, T_0],$$

and

$$I^f(0, T, w) \le \sigma^f\big(0, T, v(0)\big) + \varepsilon.$$

The notion of agreeable functions (programs) is well-known in the economic literature [7]–[9]. In the present paper we introduce its following strong version.

An a.c. function $v : [0, \infty) \to \mathbb{R}^n$ is called strongly (f)-agreeable if for each $T_0 > 0$ and each $\varepsilon > 0$ there exist $T_\varepsilon > T_0$ and a neighborhood $\mathscr{U}$ of f in $\mathscr{A}$ such that for each $g \in \mathscr{U}$ and each $T > T_\varepsilon$ there exists an a.c. function $w : [0, T] \to \mathbb{R}^n$ such that

$$w(t) = v(t), \qquad t \in [0, T_0]$$

and

$$I^g(0, T, w) \le \sigma^g\big(0, T, v(0)\big) + \varepsilon.$$

Results known in the literature which establish existence of agreeable functions (solutions) were obtained under strong assumptions on an objective function which determines an optimality criterion [7]–[9]. In particular, it was assumed that the objective function is convex (concave) as a function of all its variables. In the present paper we show that for many integrands c-optimality and agreeability are equivalent.

In the definition of c-optimal functions we assume that they are bounded while in the definition of agreeable functions there is no boundedness requirement. On the other hand in view of Proposition 3.1 any agreeable function is bounded.

We denote $d(x, B) = \inf\{|x - y| \mid y \in B\}$ for $x \in \mathbb{R}^n$, $B \subset \mathbb{R}^n$ and denote by $\mathrm{dist}(A, B)$ the Hausdorff metric for two sets $A, B \subset \mathbb{R}^n$. For every bounded a.c. function $v : [0, \infty) \to \mathbb{R}^n$ define

$$\begin{aligned} \Omega(v) = \{y \in \mathbb{R}^n \mid &\text{there exists a sequence } \{t_i\}_{i=0}^\infty \subset (0, \infty) \\ &\text{for which } t_i \to \infty, v(t_i) \to y \text{ as } i \to \infty\} \end{aligned} \tag{1.6}$$

which is called a limiting set of v.

We say that an integrand $f \in \mathscr{A}$ has the *asymptotic turnpike property*, or briefly ATP, if $\Omega(v_2) = \Omega(v_1)$ for all (f)-good functions $v_i : [0, \infty) \to \mathbb{R}^n$, $i = 1, 2$ (see [12], [21], [24]). In other words $\Omega(v)$ is the same for all (f)-good functions v.

In [21] we established the existence of a set $\mathscr{F} \subset \mathscr{A}$ which is a countable intersection of open everywhere dense subsets of $\mathscr{A}$ such that each integrand $f \in \mathscr{F}$ possesses ATP. Thus a typical integrand possesses ATP.

Denote by $\mathscr{M}$ the set of all functions $f = f(x, y) \in C^1(\mathbb{R}^{2n})$ which satisfy the following assumptions:

$$\partial f/\partial y_i \in C^1(\mathbb{R}^{2n}) \qquad \text{for } i = 1, \dots, n;$$

the matrix $(\partial^2 f/\partial y_i \partial y_j)(x, y)$, $i, j = 1, \dots, n$ is positive definite for all $(x, y) \in \mathbb{R}^{2n}$;

$$f(x, y) \geq \max\{\psi(|x|), \psi(|y|)|y|\} - a \qquad \text{for all } (x, y) \in \mathbb{R}^n \times \mathbb{R}^n;$$

there exist a number $c_0 > 1$ and monotone increasing functions $\phi_i : [0, \infty) \to [0, \infty)$, $i = 0, 1, 2$ such that

$$\begin{aligned} &\phi_0(t)/t \to \infty \qquad \text{as } t \to \infty, \\ &f(x, y) \geq \phi_0(c_0|y|) - \phi_1(|x|), \qquad x, y \in \mathbb{R}^n, \\ &\max\{|\partial f/\partial x_i(x, y)|, |\partial f/\partial y_i(x, y)|\} \\ &\quad \leq \phi_2(|x|)\big(1 + \phi_0(|y|)\big), \qquad x, y \in \mathbb{R}^n,\ i = 1, \dots, n. \end{aligned}$$

It is easy to see that $\mathscr{M} \subset \mathscr{A}$.

The following theorem is our main result.

Theorem 1.4. *Let $f \in \mathscr{M}$ possess ATP and $v : [0, \infty) \to \mathbb{R}^n$ be an a.c. function. Then the following assertions are equivalent:*

1. *v is strongly (f)-agreeable;*
2. *v is (f)-agreeable;*
3. *v is c-optimal with respect f.*

It is clear that assertion 1 of Theorem 1.4 implies assertion 2. In the proof of Theorem 1.4 we show that assertion 2 implies assertion 3 and that assertion 3 implies assertion 1. Note that assertion 2 implies assertion 3 for any $f \in \mathscr{A}$. In order to show that assertion 3 implies assertion 1 we need to assume that $f \in \mathscr{M}$ and that f possesses ATP.

Note that in the literature there are no examples of agreeable functions which are not strongly agreeable. It is interesting to construct such an example but this problem is not simple because most integrands possess ATP and in this case by Theorem 1.4 our two notions are equivalent.

The paper is organized as follows. In Section 2 we consider perfect solutions and state a result (Theorem 2.1) which shows that if $f \in \mathscr{M}$ possesses ATP, then a function v is c-optimal if and only if it is perfect. Section 3 contains auxiliary results. Theorem 2.1 is proved in Section 4 while Theorem 1.4 is proved in Section 5.

2. Perfect functions

Let $f \in \mathscr{A}$. By a simple modification of the proof of Proposition 4.4 of [11] (see also Theorems 8.1 and 8.2 of [21]) we obtain the following representation formula

$$U^f(0, T, x, y) = T\mu(f) + \pi^f(x) - \pi^f(y) + \theta_T^f(x, y), \qquad x, y \in \mathbb{R}^n,\ T > 0, \quad (2.1)$$

where $\pi^f : \mathbb{R}^n \to \mathbb{R}^1$ is a continuous function defined by

$$\pi^f(x) = \inf\{\liminf_{T\to\infty}[I^f(0, T, v) - \mu(f)T] \mid v : [0, \infty) \to \mathbb{R}^n \text{ is an a.c. function satisfying } v(0) = x\}, \qquad x \in \mathbb{R}^n, \quad (2.2)$$

and $(T, x, y) \to \theta_T^f(x, y) \in \mathbb{R}^1$, $(T, x, y) \in (0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n$, is a continuous non-negative function which satisfies the following condition: for every $T > 0$ and every $x \in \mathbb{R}^n$ there is $y \in \mathbb{R}^n$ for which $\theta_T^f(x, y) = 0$.

For each $\tau_1 \ge 0$, $\tau_2 > \tau_1$, each $r_1, r_2 \in [\tau_1, \tau_2]$ satisfying $r_1 < r_2$ and each a.c. function $v : [\tau_1, \tau_2] \to \mathbb{R}^n$ set

$$\Gamma^f(r_1, r_2, v) = I^f(r_1, r_2, v) - \pi^f\big(v(r_1)\big) + \pi^f\big(v(r_2)\big) - (r_2 - r_1)\mu(f). \quad (2.3)$$

In view of (2.1), (2.3) and nonnegativity of θ_T^f,

$$\Gamma^f(r_1, r_2, v) \ge 0 \qquad \text{for each } \tau_1 \ge 0, \tau_2 > \tau_1, \text{ each } r_1, r_2 \in [\tau_1, \tau_2] \text{ satisfying } r_1 < r_2 \text{ and each a.c. function } v : [\tau_1, \tau_2] \to \mathbb{R}^n. \quad (2.4)$$

We follow [14] in defining perfect functions.

An a.c. function $v : [0, \infty) \to \mathbb{R}^n$ is called (f)-perfect if for all $T > 0$,

$$\Gamma^f(0, T, v) = 0.$$

We will prove the following result.

Theorem 2.1. *Let $f \in \mathscr{M}$ possess ATP and $v : [0, \infty) \to \mathbb{R}^n$ be an a.c. function. Then the following assertions are equivalent:*

1. *v is c-optimal with respect f;*
2. *v is f-perfect.*

A prototype of Theorem 2.1 was obtained in [14] for one-dimensional second order variational problems with real valued functions arising in continuum mechanics. Here the result of [14] is extended for the variational problems with vector valued functions considered in this paper.

Note that in [26] we constructed examples of c-optimal functions which are not perfect.

3. Auxiliary results

In order to prove Theorem 1.1 we need the following results.

Proposition 3.1 ([22], Theorem 1.3). *Let $f \in \mathcal{A}$ and M_1, M_2, c be positive numbers. Then there exist a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$ and a number $S > 0$ such that for each $g \in \mathcal{U}$, each $T_1 \in [0, \infty)$ and each $T_2 \in [T_1 + c, \infty)$ the following properties hold:*

(i) *if $x, y \in \mathbb{R}^n$ satisfy $|x|, |y| \le M_1$ and if an a.c. function $v : [T_1, T_2] \to \mathbb{R}^n$ satisfies*

$$v(T_1) = x,\ v(T_2) = y, \qquad I^g(T_1, T_2, v) \le U^g(T_1, T_2, x, y) + M_2,$$

then

$$|v(t)| \le S, \qquad t \in [T_1, T_2]; \tag{3.1}$$

(ii) *if $x \in \mathbb{R}^n$ satisfies $|x| \le M_1$ and if an a.c. function $v : [T_1, T_2] \to \mathbb{R}^n$ satisfies*

$$v(T_1) = x, \qquad I^g(T_1, T_2, v) \le \sigma^g(T_1, T_2, x) + M_2,$$

then the inequality (3.1) *is valid.*

Proposition 3.2 ([25], Lemma 4.2). *Let $f \in \mathcal{M}$ possess ATP and let $H(f) \subset \mathbb{R}^n$ be such that $\Omega(v) = H(f)$ for each (f)-good function v.*

Assume that $\varepsilon \in (0, 1)$. Then there exist numbers $q, \delta > 0$ such that for each $h_1, h_2 \in \mathbb{R}^n$ satisfying $d(h_i, H(f)) \le \delta$, $i = 1, 2$, and each $T \ge q$ there exists an a.c. function $v : [0, T] \to \mathbb{R}^n$ which satisfies

$$v(0) = h_1, \qquad v(T) = h_2, \qquad \Gamma^f(0, T, v) \le \varepsilon.$$

Proposition 3.3 ([21], Theorem 8.3). *Let $f \in \mathcal{A}$ and $x \in \mathbb{R}^n$. Then there exists an (f)-good function $v : [0, \infty) \to \mathbb{R}^n$ such that*

$$v(0) = x \quad \text{and} \quad \Gamma^f(0, T, v) = 0 \quad \text{for all } T > 0.$$

Proposition 3.4 ([22], Corollary 2.1). *For each $f \in \mathcal{A}$, each pair of numbers T_1, T_2 satisfying $0 \le T_1 < T_2$ and each $z_1, z_2 \in \mathbb{R}^n$ there exists an a.c. function $v : [T_1, T_2] \to \mathbb{R}^n$ such that $v(T_i) = z_i$, $i = 1, 2$, $I^f(T_1, T_2, v) = U^f(T_1, T_2, z_1, z_2)$.*

Proposition 3.5 ([22], Corollary 2.2). *For each $f \in \mathcal{A}$, each pair of numbers T_1, T_2 satisfying $0 \le T_1 < T_2$ and each $z \in \mathbb{R}^n$ there exists an a.c. function $v : [T_1, T_2] \to \mathbb{R}^n$ such that $v(T_1) = z$, $I^f(T_1, T_2, v) = \sigma^f(T_1, T_2, z)$.*

Proposition 3.6 ([23], Theorem 1.2). *Assume that $f \in \mathcal{M}$ and that there exists a nonempty compact set $H(f) \subset \mathbb{R}^n$ such that $\Omega(v) = H(f)$ for each (f)-good function v.*

Let $\varepsilon, K > 0$. Then there exist a neighborhood $\mathcal{U}$ of f in $\mathcal{A}$ and positive numbers l_0, δ such that the following assertion holds.

For each $g \in \mathcal{U}$, each $T \ge 2l_0$ and each a.c. function $v : [0, T] \to \mathbb{R}^n$ which satisfies

$$|v(0)|, |v(T)| \le K, \qquad I^g(0, T, v) \le U^g\big(0, T, v(0), v(T)\big) + \delta$$

the following inequality holds:

$$d\big(v(t), H(f)\big) \le \varepsilon, \qquad t \in [l_0, T - l_0].$$

Proposition 3.7 ([22], Proposition 2.9). *Assume that $f \in \mathcal{A}$, $0 < c_1 < c_2 < \infty$ and $c_3, \varepsilon > 0$. Then there exists a neighborhood V of f in $\mathcal{A}$ such that for each $g \in V$, each pair of numbers $T_1, T_2 \ge 0$ satisfying $T_2 - T_1 \in [c_1, c_2]$ and each $y, z \in \mathbb{R}^n$ satisfying $|y|, |z| \le c_3$ the inequality*

$$|U^f(T_1, T_2, y, z) - U^g(T_1, T_2, y, z)| \le \varepsilon.$$

holds.

Proposition 3.8 ([22], Proposition 2.8). *Let $f \in \mathcal{A}$, $0 < c_1 < c_2 < \infty$, $D, \varepsilon > 0$. Then there exists a neighborhood V of f in $\mathcal{A}$ such that for each $g \in V$, each pair of numbers $T_1, T_2 \ge 0$ satisfying $T_2 - T_1 \in [c_1, c_2]$ and each an a.c. function $v : [T_1, T_2] \to \mathbb{R}^n$ satisfying*

$$\min\{I^f(T_1, T_2, v), I^g(T_1, T_2, v)\} \le D,$$

the inequality

$$|I^f(T_1, T_2, v) - I^g(T_1, T_2, v)| \le \varepsilon$$

holds.

The following useful result was obtained in [23].

Proposition 3.9. *Let $f \in \mathcal{A}$. Then $\pi^f(x) \to \infty$ as $|x| \to \infty$.*

4. Proof of Theorem 2.1

Assume that for all $T > 0$

$$\Gamma^f(0, T, v) = 0. \tag{4.1}$$

By Proposition 1.3 there is $c_0 > 0$ such that for all $T > 0$

$$I^f(0, T, v) - T\mu(f) \geq -c_0. \tag{4.2}$$

It follows from (4.1), (4.2) and (2.3) that

$$-c_0 \leq \Gamma^f(0, T, v) + \pi^f\big(v(0)\big) - \pi^f\big(v(T)\big) = \pi^f\big(v(0)\big) - \pi^f\big(v(T)\big) \tag{4.3}$$

and

$$\pi^f\big(v(T)\big) \leq c_0 + \pi^f\big(v(0)\big). \tag{4.4}$$

Together with Proposition 3.9 this implies that

$$\sup\{|v(t)| \mid t \in [0, \infty)\} < \infty. \tag{4.5}$$

Let $T > 0$. By the representation formula (2.1), the nonnegativity of the function θ_T^f, (4.1) and (2.3)

$$\begin{aligned} U^f\big(0, T, v(0), v(T)\big) &= T\mu(f) + \pi^f\big(v(0)\big) - \pi^f\big(v(T)\big) + \theta_T^f\big(v(0), v(T)\big) \\ &\geq T\mu(f) + \pi^f\big(v(0)\big) - \pi^f\big(v(T)\big) \\ &= T\mu(f) + \pi^f\big(v(0)\big) - \pi^f\big(v(T)\big) + \Gamma^f(0, T, v) = I^f(0, T, v). \end{aligned}$$

This implies that

$$U^f\big(0, T, v(0), v(T)\big) = I^f(0, T, v)$$

for all $T > 0$. Combined with (4.5) this implies that v is c-optimal with respect to f.

Assume now that the function v is c-optimal with respect to f. By Proposition 3.3 there exists an (f)-good function $u : [0, \infty) \to \mathbb{R}^n$ such that

$$u(0) = v(0) \quad \text{and} \quad \Gamma^f(0, T, u) = 0 \quad \text{for all } T > 0. \tag{4.6}$$

Assume that there is $T_0 > 0$ such that

$$\Delta := \Gamma^f(0, T_0, v) > 0. \tag{4.7}$$

There is a nonempty compact set $H(f) \subset \mathbb{R}^n$ such that $\Omega(w) = H(f)$ for each (f)-good function w.

By Proposition 3.2 there exist numbers $q, \delta > 0$ such that the following property holds:

(P1) for each $h_1, h_2 \in \mathbb{R}^n$ satisfying $d\big(h_i, H(f)\big) \le \delta$, $i = 1, 2$, and each $T \ge q$ there exists an a.c. function $w : [0, T] \to \mathbb{R}^n$ which satisfies

$$w(0) = h_1, \qquad w(T) = h_2, \qquad \Gamma^f(0, T, w) \le \Delta/4.$$

Since the functions v and u are (f)-good we have

$$\Omega(v) = \Omega(u) = H(f) \tag{4.8}$$

and there exists $T_1 > T_0$ such that

$$d\big(v(t), H(f)\big) \le \delta/2, \quad d\big(u(t), H(f)\big) \le \delta/2 \qquad \text{for all } t \ge T_1. \tag{4.9}$$

In view of (4.9)

$$d\big(u(T_1), H(f)\big) \le \delta/2, \qquad d\big(v(T_1 + q), H(f)\big) \le \delta/2. \tag{4.10}$$

By (4.10) and (P1) there exists an a.c. function $w : [T_1, T_1 + q] \to \mathbb{R}^n$ such that

$$\begin{gathered} w(T_1) = u(T_1), \qquad w(T_1 + q) = v(T_1 + q), \\ \Gamma^f(T_1, T_1 + q, w) \le \Delta/4. \end{gathered} \tag{4.11}$$

Put

$$\begin{gathered} v_1(t) = u(t), \quad t \in [0, T_1], \qquad v_1(t) = w(t), \quad t \in (T_1, T_1 + q], \\ v_1(t) = v(t), \quad t \in (T_1 + q, \infty). \end{gathered} \tag{4.12}$$

Clearly, the a.c. function v_1 is well defined. By (4.6), (4.11) and (4.12)

$$v_1(0) = v(0), \qquad v_1(T_1 + q) = v(T_1 + q). \tag{4.13}$$

It follows from (2.3), (4.7), the inequality $T_1 > T_0$, (2.4), (4.12), (4.6) and (4.11) that

$$\begin{aligned} I^f(0, T_1 + q, v) - I^f(0, T_1 + q, v_1) &= \Gamma^f(0, T_1 + q, v) - \Gamma^f(0, T_1 + q, v_1) \\ &\ge \Delta - \Gamma^f(0, T_1, v_1) - \Gamma^f(T_1, T_1 + q, v_1) \\ &= \Delta - \Gamma^f(0, T_1, u) - \Gamma^f(T_1, T_1 + q, w) \\ &\ge \Delta - \Delta/4. \end{aligned}$$

Combined with (4.13) this contradicts c-optimality of v. The contradiction we have reached proves that $\Gamma^f(0, T, v) = 0$ for all $T > 0$. Theorem 2.1 is proved.

5. Proof of Theorem 1.4

Clearly, assertion 1 implies assertion 2. We show that assertion 2 implies assertion 3. Assume that the function v is (f)-agreeable and show that it is c-optimal with respect to f. First we show that

$$\sup\{|v(t)| \mid t \in [0, \infty)\} < \infty. \tag{5.1}$$

By Proposition 3.1 there exists a number $S_0 > 0$ such that the following property holds:

(P2) for each $T \geq 1$, each $x \in \mathbb{R}^n$ satisfying $|x| \leq |v(0)| + 1$ and each a.c. function $u : [0, T] \to \mathbb{R}^n$ satisfying

$$u(0) = x, \qquad I^f(0, T, u) \leq \sigma^f(0, T, x) + 1$$

we have

$$|u(t)| \leq S_0, \qquad t \in [0, T].$$

Fix $T_0 > 1$. Since v is (f)-agreeable there exists $T_1 > T_0$ such that the following property holds:

(P3) for each $T > T_1$ there exists an a.c. function $u : [0, T] \to \mathbb{R}^n$ such that

$$u(t) = v(t), \qquad t \in [0, T_0] \tag{5.2}$$

and

$$I^f(0, T, u) \leq \sigma^f\big(0, T, v(0)\big) + 4^{-1}. \tag{5.3}$$

Let $T > T_1$ and let an a.c. function $u : [0, T] \to \mathbb{R}^n$ be as guaranteed by property (P3). Thus (5.2) and (5.3) holds. It follows from (5.2), (5.3) and (P2) that for all $t \in [0, T_0]$

$$|v(t)| = |u(t)| \leq S_0. \tag{5.4}$$

Since T_0 is an arbitrary number larger than 1 we conclude that

$$|v(t)| \leq S_0, \qquad t \in [0, \infty). \tag{5.5}$$

Let $T_0 > 0$. We show that

$$I^f(0, T_0, v) = U^f\big(0, T_0, v(0), v(T_0)\big). \tag{5.6}$$

Assume the contrary and put

$$\varepsilon = 8^{-1}\big[I^f(0, T_0, v) - U^f\big(0, T_0, v(0), v(T_0)\big)\big]. \tag{5.7}$$

Clearly, $\varepsilon > 0$ and there exists an a.c. function $v_1 : [0, T_0] \to \mathbb{R}^n$ such that

$$v_1(0) = v(0), \qquad v_1(T_0) = v(T_0), \tag{5.8}$$

$$I^f(0, T_0, v_1) < I^f(0, T_0, v) - 7\varepsilon. \tag{5.9}$$

Since the function v is (f)-agreeable there exists $T_1 > T_0$ such that the following property holds:

(P4) for each $T > T_1$ there exists an a.c. function $u : [0, T] \to \mathbb{R}^n$ such that

$$u(t) = v(t), \qquad t \in [0, T_0] \tag{5.10}$$

and

$$I^f(0, T, u) \le \sigma^f\big(0, T, v(0)\big) + \varepsilon. \tag{5.11}$$

Fix $T > T_1$ and let an a.c. function $u : [0, T] \to \mathbb{R}^n$ be as guaranteed by property (P4). Thus (5.10) and (5.11) hold. Define a function $u_1 : [0, T] \to \mathbb{R}^n$ by

$$u_1(t) = v_1(t), \quad t \in [0, T_0], \qquad u_1(t) = u(t), \quad t \in (T_0, T]. \tag{5.12}$$

In view of (5.8), (5.10) and (5.12) the a.c. function u_1 is well-defined and

$$u_1(0) = u(0) = v(0). \tag{5.13}$$

It follows from (5.12), (5.10) and (5.9) that

$$\begin{aligned} I^f(0, T, u) - I^f(0, T, u_1) &= I^f(0, T_0, u) - I^f(0, T_0, u_1) \\ &= I^f(0, T_0, v) - I^f(0, T_0, v_1) > 7\varepsilon. \end{aligned}$$

Combined with (5.13) this implies that

$$I^f(0, T, u) > 7\varepsilon + \sigma^f\big(0, T, v(0)\big).$$

This contradicts (5.11). The contradiction we have reached proves that (5.6) holds. Since T_0 is any positive number (5.5) and (5.6) imply that the function v is c-optimal with respect to f.

We show that assertion 3 implies assertion 1. Assume that the function v is c-optimal with respect to f. We show that the function v is strongly (f)-agreeable.

There is a nonempty compact set $H(f) \subset \mathbb{R}^n$ such that $\Omega(u) = H(f)$ for each (f)-good function u.

Choose a number $M_0 > 0$ such that

$$M_0 > 4 + \sup\{|z| \mid z \in H(f)\} + \sup\{|v(t)| \mid t \in [0, \infty)\}. \tag{5.14}$$

By Proposition 3.1 there exist a neighborhood $\mathscr{U}_1$ of f in $\mathscr{A}$ and a number $M_1 > M_0$ such that the following properties hold:

(P5) for each $g \in \mathscr{U}_1$, each $T \geq 1$, each $x, y \in \mathbb{R}^n$ satisfying $|x|, |y| \leq M_0$ and each a.c. function $w : [0, T] \to \mathbb{R}^n$ satisfying

$$w(0) = x, \qquad w(T) = y, \qquad I^g(0, T, w) \leq U^g(0, T, x, y) + 4$$

the following inequality holds:

$$|w(t)| \leq M_1, \qquad t \in [0, T]; \tag{5.15}$$

(P6) for each $g \in \mathscr{U}_1$, each $T \geq 1$, each $x \in \mathbb{R}^n$ satisfying $|x| \leq M_0$ and each a.c. function $w : [0, T] \to \mathbb{R}^n$ satisfying

$$w(0) = x, \qquad I^g(0, T, w) \leq \sigma^g(0, T, x) + 4$$

the inequality (5.15) is valid.

Let

$$T_0 > 1, \qquad \varepsilon \in (0, 1). \tag{5.16}$$

By Proposition 3.2 there exist numbers $q > 0$, $\delta \in (0, \varepsilon)$ such that the following property holds:

(P7) for each $h_1, h_2 \in \mathbb{R}^n$ satisfying $d\big(h_i, H(f)\big) \leq \delta$, $i = 1, 2$, and each $T \geq q$ there exists an a.c. function $\xi : [0, T] \to \mathbb{R}^n$ which satisfies

$$\xi(0) = h_1, \qquad \xi(T) = h_2, \qquad \Gamma^f(0, T, \xi) \leq \varepsilon/16.$$

By Proposition 3.6 there exist a neighborhood $\mathscr{U}_2$ of f in $\mathscr{A}$ and a positive number S_0 such that the following property holds:

(P8) for each $g \in \mathscr{U}_2$, each $T \geq 2S_0$ and each a.c. function $w : [0, T] \to \mathbb{R}^n$ which satisfies

$$|w(0)|, |w(T)| \leq M_1, \qquad I^g(0, T, w) = U^g\big(0, T, w(0), w(T)\big)$$

we have

$$d\big(w(t), H(f)\big) \le \delta/2, \qquad t \in [S_0, T - S_0].$$

Since v is an (f)-good function there is

$$S_1 > T_0 + S_0$$

such that

$$d\big(v(t), H(f)\big) \le \delta/2 \qquad \text{for all } t \ge S_1. \tag{5.17}$$

By Proposition 3.7 there exists a neighborhood $\mathscr{U}_3$ of f in $\mathscr{A}$ such that the following property holds:

(P9) for each $g \in \mathscr{U}_3$ and each $y, z \in \mathbb{R}^n$ satisfying $|y|, |z| \le M_1$ we have

$$|U^f(0, S_1 + q, y, z) - U^g(0, S_1 + q, y, z)| \le \varepsilon/16.$$

By Proposition 3.8 there exists a neighborhood $\mathscr{U}_4$ of f in $\mathscr{A}$ such that the following property holds:

(P10) for each $g \in \mathscr{U}_4$ and each a.c. function $w : [0, S_1 + q] \to \mathbb{R}^n$ satisfying

$$\begin{aligned} &\min\{I^f(0, S_1 + q, w), I^g(0, S_1 + q, w)\} \\ &\quad \le 1 + (S_1 + q)|\mu(f)| + 2\sup\{|\pi^f(z)| \mid z \in \mathbb{R}^n \text{ and } |z| \le M_1\} \end{aligned}$$

the inequality

$$|I^f(0, S_1 + q, w) - I^g(0, S_1 + q, w)| \le \varepsilon/16$$

holds.

Put

$$\mathscr{U} = \bigcap_{i=1}^{4} \mathscr{U}_i, \qquad T_\varepsilon = T_0 + S_1 + 2S_0 + q. \tag{5.18}$$

Assume that

$$T \ge T_\varepsilon, \qquad g \in \mathscr{U}. \tag{5.19}$$

By Proposition 3.5 there exists an a.c. function $w : [0, T] \to \mathbb{R}^n$ such that

$$w(0) = v(0), \qquad I^g(0, T, w) = \sigma^g\big(0, T, v(0)\big). \tag{5.20}$$

In view of (5.14), (5.18), (5.19), (5.20) and (P6)

$$|w(t)| \le M_1, \qquad t \in [0, T]. \tag{5.21}$$

Property (P8), (5.18), (5.19), (5.20) and (5.21) imply that

$$d\big(w(t), H(f)\big) \le \delta/2 \qquad \text{for all } t \in [S_0, T - S_0]. \tag{5.22}$$

By (5.17)

$$d\big(v(S_1), H(f)\big) \le \delta/2. \tag{5.23}$$

It follows from (5.18), (5.22) and the inequality $S_1 > S_0$ that

$$d\big(w(S_1 + q), H(f)\big) \le \delta/2. \tag{5.24}$$

By (5.23), (5.24) and (P7) there exists an a.c. function $\xi : [S_1, S_1 + q] \to \mathbb{R}^n$ which satisfies

$$\xi(S_1) = v(S_1), \qquad \xi(S_1 + q) = w(S_1 + q), \qquad \Gamma^f(S_1, S_1 + q, \xi) \le \varepsilon/16. \tag{5.25}$$

Define

$$\begin{aligned} u(t) = v(t), \quad t \in [0, S_1], \qquad u(t) = \xi(t), \quad t \in (S_1, S_1 + q], \\ u(t) = w(t), \quad t \in (S_1 + q, T]. \end{aligned} \tag{5.26}$$

Clearly, the a.c. function u is well-defined. In view of (5.20) and (5.26)

$$u(0) = w(0), \qquad u(T) = w(T). \tag{5.27}$$

We will estimate

$$I^g(0, T, w) - I^g(0, T, u) = \sigma^g\big(0, T, v(0)\big) - I^g(0, T, u). \tag{5.28}$$

It follows from (5.20) that

$$\sigma^g\big(0, T, v(0)\big) = I^g(0, T, w) = U^g\big(0, T, w(0), w(T)\big). \tag{5.29}$$

By (5.26) and (5.29)

$$\begin{aligned} I^g(0, T, w) - I^g(0, T, u) &= I^g(0, S_1 + q, w) - I^g(0, S_1 + q, u) \\ &= U^g\big(0, S_1 + q, w(0), w(S_1 + q)\big) \\ &\quad - I^g(0, S_1 + q, u). \end{aligned} \tag{5.30}$$

By (2.3), (5.26), (5.27), (5.21) and Theorem 2.1,

$$\begin{aligned}
I^f(0, S_1 + q, u) &= \Gamma^f(0, S_1 + q, u) + (S_1 + q)\mu(f) + \pi^f\big(u(0)\big) - \pi^f\big(u(S_1 + q)\big) \\
&\le \Gamma^f(0, S_1, v) + \Gamma^f(S_1, S_1 + q, \xi) + (S_1 + q)|\mu(f)| \\
&\quad + 2\sup\{|\pi^f(z)| \mid z \in \mathbb{R}^n \text{ and } |z| \le M_1\} \\
&\le 1 + (S_1 + q)|\mu(f)| \\
&\quad + 2\sup\{|\pi^f(z)| \mid z \in \mathbb{R}^n \text{ and } |z| \le M_1\}. \qquad (5.31)
\end{aligned}$$

By (5.31), (5.18), (5.19), (2.1) and (P10),

$$|I^f(0, S_1 + q, u) - I^g(0, S_1 + q, u)| \le \varepsilon/16. \qquad (5.32)$$

It follows from (5.30), (5.21), (5.19), (5.18), (P9), (5.25)–(5.27), (2.3), the representation formula (2.1), the nonnegativity of θ^f_T and Theorem 2.1 that

$$\begin{aligned}
I^g(0, T, w) - I^g(0, T, u) &\ge U^f\big(0, S_1 + q, w(0), w(S_1 + q)\big) - \varepsilon/16 \\
&\quad - I^f(0, S_1 + q, u) - \varepsilon/16 \\
&= U^f\big(0, S_1 + q, u(0), u(S_1 + q)\big) - I^f(0, S_1 + q, u) - \varepsilon/8 \\
&\ge (S_1 + q)\mu(f) + \pi^f\big(u(0)\big) - \pi^f\big(u(S_1 + q)\big) \\
&\quad - I^f(0, S_1 + q, u) - \varepsilon/8 \\
&\ge -\varepsilon/8 - \Gamma^f(0, S_1 + q, u) \\
&= -\varepsilon/8 - \Gamma^f(0, S_1, v) - \Gamma^f(S_1, S_1 + q, \xi) \\
&\ge -\varepsilon/8 - \varepsilon/16 > -\varepsilon/4. \qquad (5.33)
\end{aligned}$$

In view of (5.26) and the inequality $S_1 > T_0$

$$u(t) = v(t), \qquad t \in [0, T_0].$$

By (5.29) and (5.33)

$$I^g(0, T, u) \le I^g(0, T, w) + \varepsilon/4 = \sigma^g\big(0, T, v(0)\big) + \varepsilon/4.$$

Therefore the function v is strongly (f)-agreeable, assertion 3 implies assertion 1 and Theorem 1.4 is proved.

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Received July 22, 2010; revised January 23, 2011

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