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A universal enveloping algebra of Malcev superalgebras

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Abstract. In this paper an enveloping superalgebra is presented for Malcev superalgebra. An extension of the Poincaré–Birkhoff–Witt Theorem to this class of superalgebras is obtained.

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1. Introduction

Given a superalgebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$, let us denote by A^- the superalgebra obtained from A replacing the product xy by the super-commutator $[x, y] = xy - (-1)^{\bar{x}\bar{y}} yx$, for homogeneous elements $x \in A_{\bar{x}}$, $y \in A_{\bar{y}}$. It is known that if A is an associative superalgebra one obtains a Lie superalgebra A^-, and conversely, superizing the arguments used in the Lie algebra case, the Poincaré–Birkhoff–Witt Theorem establishes that any Lie superalgebra is isomorphically embedded into an algebra A^-, for a suitable associative superalgebra A [4].

If we start with an alternative superalgebra A (alternativity is a weaker form of associativity) then A^- is a Malcev superalgebra. It remains an open problem whether any Malcev superalgebra is isomorphic to a subalgebra of A^-, for some alternative superalgebra A. In [3], Pérez-Izquierdo and Shestakov presented an enveloping algebra of Malcev algebras (constructed in a more general way), showing that this generalizes the classical notion of enveloping algebra for the particular case of Lie algebras. They prove that for every Malcev algebra M there exist an algebra $U(M)$ and an injective Malcev algebra homomorphism $\iota : M \to U(M)^-$ such that the image is contained in the generalized alternative nucleus $N_{\mathrm{alt}}(U(M))$, being $U(M)$ a universal object with respect to such

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homomorphisms. The algebra $U(M)$ is in general not alternative, but it has a basis of Poincaré–Birkhoff–Witt Theorem type over M and it inherits the good properties of the universal enveloping algebra of Lie algebras.

It is worth noting that enveloping superalgebras for Akivis superalgebras have been recently studied in [1] by Albuquerque and Santana, superizing the work of Shestakov in the Akivis algebra case [5]. It is our goal to present a universal enveloping superalgebra of Malcev superalgebras, superizing the theory exposed by Pérez-Izquierdo and Shestakov in [3]. Our approach is similar to the one employed in [3], but the superization of the results implies more elaborated calculations and arguments.

2. Preliminaries

A *superalgebra* A is a $\mathbb{Z}_2$-graded algebra (meaning that we consider an underlying $\mathbb{Z}_2$-graded vector space $A = A_{\bar{0}} \oplus A_{\bar{1}}$ and $A_\alpha A_\beta \subset A_{\alpha+\beta}$, for all $\alpha, \beta \in \mathbb{Z}_2$). We write $x \in A_{\bar{x}}$ to mean that x is a homogeneous element of the superalgebra A of degree $\bar{x}$, with $\bar{x} \in \mathbb{Z}_2$. We recall that a *superalgebra* $M = M_{\bar{0}} \oplus M_{\bar{1}}$ endowed with the multiplication $[\,,]$ is called a *Malcev superalgebra* if it satisfies the following two conditions: $\forall x \in M_{\bar{x}},\ y \in M_{\bar{y}},\ z \in M_{\bar{z}},\ t \in M_{\bar{t}}$

(i) super-anticommutativity: $[x, y] = -(-1)^{\bar{x}\bar{y}}[y, x]$

(ii) super-Malcev identity:

$$(-1)^{\bar{y}\bar{z}}[[x, z], [y, t]] = [[[x, y], z], t] + (-1)^{\bar{x}(\bar{y}+\bar{z}+\bar{t})}[[[y, z], t], x] \\ + (-1)^{(\bar{x}+\bar{y})(\bar{z}+\bar{t})}[[[z, t], x], y] + (-1)^{\bar{t}(\bar{x}+\bar{y}+\bar{z})}[[[t, x], y], z].$$

Let V be a vector space of countable dimension. The *Grassmann* (or *exterior*) *algebra* over V, usually denoted by $\Lambda(V)$, is the quotient of the tensor algebra over the ideal generated by the symmetric tensors $\{x \otimes x : x \in V\}$. If $\{e_1, e_2, e_3, \ldots\}$ is a basis of V then the elements $1,\ e_{i_1} \cdot \ldots \cdot e_{i_n}$, with $i_1 < \cdots < i_n$, constitute a basis for $\Lambda(V)$ satisfying $e_i^2 = 0$ and $e_i e_j = -e_j e_i$. The algebra $\Lambda(V)$ is associative with identity, and it is a $\mathbb{Z}_2$-graded algebra $\Lambda(V) = \Lambda(V)_{\bar{0}} \oplus \Lambda(V)_{\bar{1}}$, where its even part $\Lambda(V)_{\bar{0}}$ is the linear span of all tensors of even length and the odd part $\Lambda(V)_{\bar{1}}$ is the linear span of all tensors of odd length.

Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be a superalgebra. The *Grassmann enveloping algebra* of A is the algebra $G(A) = \left(A_{\bar{0}} \otimes \Lambda(V)_{\bar{0}}\right) \oplus \left(A_{\bar{1}} \otimes \Lambda(V)_{\bar{1}}\right)$, with the multiplication defined by

$$(x \otimes e_\alpha)(y \otimes e_\beta) = xy \otimes e_\alpha e_\beta, \qquad \forall (x \otimes e_\alpha) \in A_{\bar{x}} \otimes \Lambda(V)_{\bar{x}}, \\ (y \otimes e_\beta) \in A_{\bar{y}} \otimes \Lambda(V)_{\bar{y}}.$$

If $\mathscr{V}$ is a type of algebras defined by homogeneous identities, a superalgebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$ is a $\mathscr{V}$-*superalgebra* if its Grassmann enveloping algebra $G(A)$ is in $\mathscr{V}$.

The associator (x, y, z) of elements x, y, z in the superalgebra A is defined in terms of the associator defined in its Grassmann enveloping algebra $G(A)$ in the following way:

$$(x, y, z) \otimes (e_\alpha e_\beta e_\gamma) = (x \otimes e_\alpha, y \otimes e_\beta, z \otimes e_\gamma).$$

Making some simple computations we see that

$$\begin{aligned}(x, y, z) \otimes (e_\alpha e_\beta e_\gamma) &= \big((x \otimes e_\alpha)(y \otimes e_\beta)\big)(z \otimes e_\gamma) - (x \otimes e_\alpha)\big((y \otimes e_\beta)(z \otimes e_\gamma)\big) \\ &= \big((xy)z - x(yz)\big) \otimes (e_\alpha e_\beta e_\gamma).\end{aligned}$$

In this way the associator will be $(x, y, z) = (xy)z - x(yz)$ for elements x, y, z in A.

The *superjacobian* of homogeneous elements $x \in A_{\bar{x}}$, $y \in A_{\bar{y}}$, $z \in A_{\bar{z}}$ is given by

$$SJ(x, y, z) = [[x, y], z] + (-1)^{\bar{x}(\bar{y}+\bar{z})}[[y, z], x] + (-1)^{\bar{z}(\bar{x}+\bar{y})}[[z, x], y].$$

It is easy to see that the superjacobian is super-skewsymmetric. The super-Malcev identity is equivalent to the condition

$$SJ(x, y, z) = 6(x, y, z), \qquad \forall x \in A_{\bar{x}},\ y \in A_{\bar{y}},\ z \in A_{\bar{z}}. \tag{2.1}$$

3. Generalized alternative nucleus

The notion of generalized alternative nucleus of an arbitrary algebra presented in [2] and used in [3] can be straightforward extended to the super case.

Definition 3.1. Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be a superalgebra. The *generalized alternative nucleus* of A, which we denote by $N_{\mathrm{alt}}(A)$, is the set of A generated by the elements

$$\{a \in A_{\bar{a}} : (a, x, y) = -(-1)^{\bar{a}\bar{x}}(x, a, y) = (-1)^{\bar{a}(\bar{x}+\bar{y})}(x, y, a), \forall x \in A_{\bar{x}}, y \in A_{\bar{y}}\}.$$

Next we shall prove that the generalized alternative nucleus $N_{\mathrm{alt}}(A)^-$ is a Malcev superalgebra endowed with the super-commutator $[a, b] = ab - (-1)^{\bar{a}\bar{b}}ba$, for homogeneous elements $a \in N_{\mathrm{alt}}(A)_{\bar{a}}$, $b \in N_{\mathrm{alt}}(A)_{\bar{b}}$.

We recall that a *ternary superderivation* of a superalgebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$ is a triple (D_1, D_2, D_3) in $\big(\mathrm{End}(A)\big)_\alpha \times \big(\mathrm{End}(A)\big)_\beta \times \big(\mathrm{End}(A)\big)_\gamma$ such that

$$D_1(xy) = D_2(x)y + (-1)^{\gamma\bar{x}}xD_3(y), \qquad \forall x \in A_{\bar{x}},\ y \in A_{\bar{y}}.$$

The Lie superalgebra of ternary superderivations is denoted by $\mathrm{Tsder}(A)$. If $D = D_1 = D_2 = D_3$ then we have that D is a homogeneous superderivation of A of degree α.

For $a \in A_{\bar{a}}$, we define as usual the linear maps $L_a : A \to A$ and $R_a : A \to A$ by $L_a(x) = ax$ and $R_a(x) = (-1)^{\bar{a}\bar{x}} xa$, for all $x \in A_{\bar{x}}$, respectively. The maps L_a and R_a are homogeneous of degree $\bar{a}$. Let $T_a = L_a + R_a$. We can show that $a \in N_{\mathrm{alt}}(A)$ if and only if

$$(L_a, T_a, -L_a) \text{ and } (R_a, -R_a, T_a) \in \mathrm{Tsder}(A). \tag{3.1}$$

Lemma 3.2. *Let* $a \in \left(N_{\mathrm{alt}}(A)\right)_{\bar{a}}$, $b \in \left(N_{\mathrm{alt}}(A)\right)_{\bar{b}}$ *and* $x \in A_{\bar{x}}$. *Then*

(i) $L_{ax} = L_a L_x + [R_a, L_x]$ *and* $L_{xa} = L_x L_a + [L_x, R_a]$;

(ii) $(-1)^{\bar{a}\bar{x}} R_{ax} = R_x R_a + [R_x, L_a]$ *and* $(-1)^{\bar{a}\bar{x}} R_{xa} = R_a R_x + [L_a, R_x]$;

(iii) $[L_a, R_b] = [R_a, L_b]$;

(iv) $[L_a, L_b] = L_{[a,b]} - 2[R_a, L_b]$ *and* $[R_a, R_b] = -R_{[a,b]} - 2[L_a, R_b]$.

Proof. The proof is a superization of that of [2], Lemma 4.2. We easily see that (i), (ii) and (iii) follows from the identities: $(a, x, y) = (-1)^{\bar{a}(\bar{x}+\bar{y})}(x, y, a)$, $(x, a, y) = -(-1)^{\bar{a}\bar{y}}(x, y, a)$, $(a, y, x) = -(-1)^{\bar{a}\bar{y}}(y, a, x)$, $(a, y, x) = (-1)^{\bar{a}(\bar{x}+\bar{y})}(y, x, a)$, and $(a, x, b) = (-1)^{\bar{a}(\bar{b}+\bar{x})+\bar{b}\bar{x}}(b, x, a)$, $\forall a \in A_{\bar{a}},\ b \in A_{\bar{b}},\ x \in A_{\bar{x}},\ y \in A_{\bar{y}}$. As $[L_a, L_b] = L_a L_b - (-1)^{\bar{a}\bar{b}} L_b L_a$ and $[R_a, R_b] = R_a R_b - (-1)^{\bar{a}\bar{b}} R_b R_a$, for all $a \in A_{\bar{a}}$, $b \in A_{\bar{b}}$, then (iv) is a consequence of (i), (ii) and (iii), finishing the proof. □

Proposition 3.3. *If* $a \in \left(N_{\mathrm{alt}}(A)\right)_{\bar{a}}$ *and* $b \in \left(N_{\mathrm{alt}}(A)\right)_{\bar{b}}$ *then* $[a, b] \in \left(N_{\mathrm{alt}}(A)\right)_{\bar{a}+\bar{b}}$.

Proof. Consider $a \in \left(N_{\mathrm{alt}}(A)\right)_{\bar{a}}$ and $b \in \left(N_{\mathrm{alt}}(A)\right)_{\bar{b}}$. From Lemma 3.2(iv) we know that $L_{[a,b]} = [L_a, L_b] + 2[R_a, L_b]$ and $R_{[a,b]} = -[R_a, R_b] - 2[L_a, R_b]$. Moreover, by (3.1) we have that $(L_a, T_a, -L_a)$, $(L_b, T_b, -L_b)$, $(R_a, -R_a, T_a)$, and $(R_b, -R_b, T_b) \in \mathrm{Tsder}(A)$. Consequently $(L_{[a,b]}, [T_a, T_b] + 2[-R_a, T_b], [L_a, L_b] + 2[T_a, -L_b])$ and $(R_{[a,b]}, -[R_a, R_b] + 2[T_a, R_b], -[T_a, T_b] + 2[L_a, T_b])$ are ternary superderivation of A. As

$$-L_{[a,b]} = -[L_a, L_b] - 2[R_a, L_b] = [L_a, L_b] + 2[T_a, -L_b],$$
$$-R_{[a,b]} = [R_a, R_b] + 2[L_a, R_b] = -[R_a, R_b] + 2[T_a, R_b],$$

and

$$T_{[a,b]} = [L_a, L_b] - [R_a, R_b] = [T_a, T_b] + 2[-R_a, T_b] = -[T_a, T_b] + 2[L_a, T_b],$$

we conclude that $(L_{[a,b]}, T_{[a,b]}, -L_{[a,b]})$ and $(R_{[a,b]}, -R_{[a,b]}, T_{[a,b]}) \in \mathrm{Tsder}(A)$, and so $[a, b] \in \left(N_{\mathrm{alt}}(A)\right)_{\bar{a}+\bar{b}}$ as desired. □

Using the relation between a superalgebra A and its Grassmann enveloping algebra $G(A)$, the next result shows that $N_{\mathrm{alt}}(A)^-$ is a Malcev superalgebra.

Proposition 3.4. *If A is a superalgebra then $G\big(N_{\mathrm{alt}}(A)\big) = N_{\mathrm{alt}}\big(G(A)\big)$. Furthermore, $N_{\mathrm{alt}}(A)^-$ is a Malcev superalgebra.*

Proof. Let $x \otimes e_\alpha \in N_{\mathrm{alt}}\big(G(A)\big)$. Let us assume that $x \otimes e_\alpha \in A_{\bar{x}} \otimes \Lambda(V)_{\bar{x}}$. For $y \otimes e_\beta \in A_{\bar{y}} \otimes \Lambda(V)_{\bar{y}}$ and $z \otimes e_\gamma \in A_{\bar{z}} \otimes \Lambda(V)_{\bar{z}}$ we have

$$
\begin{aligned}
&(a \otimes e_\alpha, x \otimes e_\beta, y \otimes e_\gamma) = -(x \otimes e_\beta, a \otimes e_\alpha, y \otimes e_\gamma) = (x \otimes e_\beta, y \otimes e_\gamma, a \otimes e_\alpha)\\
\iff\quad &(a, x, y) \otimes (e_\alpha e_\beta e_\gamma) = -(x, a, y) \otimes (e_\beta e_\alpha e_\gamma) = (x, y, a) \otimes (e_\beta e_\gamma e_\alpha)\\
\iff\quad &(a, x, y) \otimes (e_\alpha e_\beta e_\gamma) = -(-1)^{\bar{a}\bar{x}}(x, a, y) \otimes (e_\alpha e_\beta e_\gamma)\\
&\qquad\qquad = (-1)^{\bar{a}(\bar{x}+\bar{y})}(x, y, a) \otimes (e_\alpha e_\beta e_\gamma),
\end{aligned}
$$

so $a \otimes e_\alpha \in G\big(N_{\mathrm{alt}}(A)\big)$, because $(a, x, y) = -(-1)^{\bar{a}\bar{x}}(x, a, y) = (-1)^{\bar{a}(\bar{x}+\bar{y})}(x, y, a)$, for $x \in A_{\bar{x}}$ and $y \in A_{\bar{y}}$, which means that $a \in N_{\mathrm{alt}}(A)$, showing the equality of the sets. It is proved that the generalized alternative nucleus $N_{\mathrm{alt}}(A)$, of an arbitrary algebra A, is a Malcev algebra when endowed with the commutator product $[x, y] = xy - yx$, for elements $x, y \in A$, where the initial multiplication in A is defined by juxtaposition (see [2], [3]). As $G\big(N_{\mathrm{alt}}(A)\big) = N_{\mathrm{alt}}\big(G(A)\big)$ and $N_{\mathrm{alt}}\big(G(A)\big)$ is a Malcev superalgebra endowed with the commutator product

$$[x \otimes e_\alpha, y \otimes e_\beta] = xy \otimes e_\alpha e_\beta - yx \otimes e_\beta e_\alpha = \{xy - (-1)^{\bar{x}\bar{y}} yx\} \otimes e_\alpha e_\beta,$$

we infer that $G\big(N_{\mathrm{alt}}(A)\big)^-$ is a Malcev algebra, consequently $N_{\mathrm{alt}}(A)^-$ is a Malcev superalgebra when endowed with the super-commutator $[x, y] = xy - (-1)^{\bar{x}\bar{y}} yx$, for homogeneous elements $x \in A_{\bar{x}}$, $y \in A_{\bar{y}}$, as desired. □

4. Enveloping superalgebra of a Malcev superalgebra

In this section an enveloping superalgebra of a Malcev superalgebra is presented and some of its properties are studied.

Definition 4.1. Let M be a Malcev superalgebra. A *universal enveloping superalgebra* of M is a pair (U, ι), where U is a unital superalgebra and $\iota : M \to U^-$ is a Malcev homomorphism, with the image of M inside the generalized alternative nucleus $N_{\mathrm{alt}}(U)$, satisfying the following condition: for any unital superalgebra A and any Malcev homomorphism $\varphi : M \to A^-$, with the image of M inside the alternative nucleus $N_{\mathrm{alt}}(A)$, there exists a unique superalgebra homomorphism $\tilde{\varphi} : U \to A$ of degree 0 such that $\tilde{\varphi}(1) = 1$ and $\varphi = \tilde{\varphi} \circ \iota$.

Next we will construct the universal enveloping superalgebra of a Malcev superalgebra $(M = M_{\bar{0}} \oplus M_{\bar{1}}, [\,,])$ over a commutative and associative ring $\mathbb{K}$ with $\frac{1}{2}, \frac{1}{3} \in \mathbb{K}$, which is a free module over $\mathbb{K}$. We consider the nonassociative $\mathbb{Z}$-graded tensor algebra of M,

$$\tilde{T}(M) = \bigoplus_{n \in \mathbb{Z}} \tilde{T}^n(M),$$

where $\tilde{T}^n(M) = \{0\}$ if $n < 0$, $\tilde{T}^0(M) = \mathbb{K}$, $\tilde{T}^1(M) = M$ and $\tilde{T}^n(M) = \bigoplus_{i=1}^{n-1} \tilde{T}^i(M) \otimes \tilde{T}^{n-i}(M)$ if $n \geq 2$, with the multiplication $xy := x \otimes y$, whenever $x, y \in \tilde{T}(M)$. For example, $\tilde{T}^3(M) = \left[\tilde{T}^1(M) \otimes \left(\tilde{T}^1(M) \otimes \tilde{T}^1(M)\right)\right] \oplus \left[\left(\tilde{T}^1(M) \otimes \tilde{T}^1(M)\right) \otimes \tilde{T}^1(M)\right]$, where the two summands are different, meaning that, in general $x(yz) \neq (xy)z$, for $x, y, z \in M$. We also have that $\tilde{T}(M)$ is a superalgebra where the $\mathbb{Z}_2$-gradation of it is induced by the $\mathbb{Z}_2$-gradation of M and is defined by

$$\tilde{T}(M) = \Big(\bigoplus_{n \geq 0} \tilde{T}^n(M)_{\bar{0}}\Big) \oplus \Big(\bigoplus_{n \geq 0} \tilde{T}^n(M)_{\bar{1}}\Big),$$

where $\tilde{T}^n(M)_\gamma = \bigoplus_{i=1}^{n-1} \bigoplus_{\alpha+\beta=\gamma} \left(\tilde{T}^i(M)_\alpha \otimes \tilde{T}^{n-i}(M)_\beta\right)$, for $\gamma \in \mathbb{Z}_2$. Consider the $\mathbb{Z}_2$-graded ideal I in $\tilde{T}(M)$ generated by the set of all homogeneous elements

$$ab - (-1)^{\bar{a}\bar{b}} ba - [a,b],\ (a,x,y) + (-1)^{\bar{a}\bar{x}}(x,a,y),\ (a,x,y) - (-1)^{\bar{a}(\bar{x}+\bar{y})}(x,y,a),$$

with $a \in M_{\bar{a}}$, $b \in M_{\bar{b}}$, $x \in \tilde{T}(M)_{\bar{x}}$, $y \in \tilde{T}(M)_{\bar{y}}$. The quotient algebra $U(M) = \tilde{T}(M)/I$ is a superalgebra with the natural $\mathbb{Z}_2$-gradation induced by the graded ideal I, with the usual multiplication $(x+I)(y+I) = xy + I$, for every $x, y \in \tilde{T}(M)$. Consider the map $\imath : M \to \tilde{T}(M) \to U(M)$ given by $\imath(a) = a + I$, for all $a \in M$, composition of the canonical injection with the quotient map.

Proposition 4.2. *The pair $\left(U(M), \imath\right)$ is a universal enveloping superalgebra of M. Moreover, M is isomorphic to a subalgebra of $N_{\mathrm{alt}}(A)^-$, for some superalgebra A, if and only if the map $\imath$ is injective.*

Proof. Let $a \in M_{\bar{a}}$. As for all $x \in \tilde{T}(M)_{\bar{x}}$, $y \in \tilde{T}(M)_{\bar{y}}$, $(a,x,y) + (-1)^{\bar{a}\bar{x}}(x,a,y)$ and $(a,x,y) - (-1)^{\bar{a}(\bar{x}+\bar{y})}(x,y,a)$ belong to $I(M)$ then $\imath(a)$ is contained in generalized alternative nucleus of $U(M)$. The assertion $ab - (-1)^{\bar{a}\bar{b}} ba - [a,b] \in I(M)$, for every $a \in M_{\bar{a}}$, $b \in M_{\bar{b}}$, ensures that $\imath : M \to N_{\mathrm{alt}}\left(U(M)\right)^-$ is a Malcev homomorphism.

Now we take care about the universal property. Let A be an unital superalgebra and $\varphi : M \to A^-$ a Malcev homomorphism, with the image of M inside the generalized alternative nucleus $N_{\mathrm{alt}}(A)$. We have to prove that there exists a unique superalgebra homomorphism $\tilde{\varphi} : U(M) \to A$ such that $\tilde{\varphi}(1) = 1$ and $\varphi = \tilde{\varphi} \circ \imath$. Taking in account the universal property of the tensor algebra, there exists a unique superalgebra homomorphism $\bar{\varphi} : \tilde{T}(M) \to A$ of degree 0 verifying $\bar{\varphi}(x) = \varphi(x)$, whenever $x \in M$. Since φ is a Malcev homomorphism and $\varphi(a) \in N_{\mathrm{alt}}(A)$, for all $a \in M_{\bar{a}}$, we infer that $I \subseteq \operatorname{Ker} \bar{\varphi}$. Consequently, there exists a superalgebra homomorphism $\tilde{\varphi} : U(M) \to A$ of degree 0, such that $\tilde{\varphi}(\imath(x)) = \bar{\varphi}(x)$, with $x \in M$. Since $U(M)$ is generated by $\mathbb{K}$ and $\imath(M)$, then we guarantee that $\tilde{\varphi}$ is unique, as desired. □

It is our goal to find a basis of the vector space $U(M)$ for a Malcev superalgebra M. Let $\{a_i : i \in \Lambda\}$ be a basis of $M = M_{\bar{0}} \oplus M_{\bar{1}}$ indexed by the totally ordered set $\Lambda = \Lambda_{\bar{0}} \cup \Lambda_{\bar{1}}$ verifying the conditions: $\{a_i : i \in \Lambda_\alpha\}$ is a basis of M_α $(\alpha = \bar{0}, \bar{1})$ and $i_p < i_q$ if $i_p \in \Lambda_{\bar{0}}$, $i_q \in \Lambda_{\bar{1}}$. Take

$$\Omega = \{(i_1, \dots, i_n) : n \geq 0, i_1, \dots, i_n \in \Lambda \text{ with } i_1 \leq \cdots \leq i_n \text{ and } i_p < i_{p+1} \text{ if } a_{i_p} \in M_{\bar{1}}\}.$$

To simplify, if $I = (i_1, \dots, i_n) \in \Omega$ we will abbreviate $\imath(a_I) = \imath(a_{i_1})\big(\imath(a_{i_2}) \cdot \big(\dots\big(\imath(a_{i_{n-1}})\imath(a_{i_n})\big)\dots\big)\big)$. If $n = 0$ then $I = \emptyset$ and we use the convention that $\imath(a_I) = 1$. We will denote by $|I|$ the size n of I and $I' = (i_2, \dots, i_n)$ if $|I| \geq 1$.

Theorem 4.3. *The set $\{\imath(a_I) : I \in \Omega\}$ is a basis of the vector space $U(M)$.*

The proof of Theorem 4.3 is very laborious, so we will do it in a staged manner and it will take the rest of the paper. First we are going to prove that $U(M)$ is spanned by the set $\{\imath(a_I) : I \in \Omega\}$ and in next section we will take care about the linear independence.

Proposition 4.4. *$U(M)$ is spanned by the monomials $\{\imath(a_I) : I \in \Omega\}$.*

Proof. Denote by U the vector space spanned by $\{\imath(a_I) : I \in \Omega\}$ and U_n an auxiliary vector space spanned by $\{\imath(a_I) : I \in \Omega \text{ and } |I| \leq n\}$. As $U(M)$ is generated by $\imath(M)$ and $\imath(M) \subset U$, it remains to show that U is a subalgebra of $U(M)$. We shall prove the proposition by induction on n. Let us assume that $\imath(a)U_{n-1} \subseteq U_n$ and $[U_{n-1}, \imath(a)] \subseteq U_{n-1}$. For $a \in M$ and $I \in \Omega$ such that $|I| = n$, we have

$$\begin{aligned}
[\imath(a_I), \imath(a)] = [\imath(a_{i_1})\imath(a_{I'}), \imath(a)] &= (-1)^{\overline{\imath(a)}\,\overline{\imath(a_{I'})}}[\imath(a_{i_1}), \imath(a)]\imath(a_{I'}) + \imath(a_{i_1})[\imath(a_{I'}), \imath(a)] \\
&\quad + \big(\imath(a_{i_1}), \imath(a_{I'}), \imath(a)\big) - (-1)^{\overline{\imath(a)}\,\overline{\imath(a_{I'})}}\big(\imath(a_{i_1}), \imath(a), \imath(a_{I'})\big) \\
&\quad + (-1)^{\overline{\imath(a)}(\overline{\imath(a_{i_1})}+\overline{\imath(a_{I'})})}\big(\imath(a), \imath(a_{i_1}), \imath(a_{I'})\big) \\
&= (-1)^{\overline{\imath(a)}\,\overline{\imath(a_{I'})}}[\imath(a_{i_1}), \imath(a)]\imath(a_{I'}) + \imath(a_{i_1})[\imath(a_{I'}), \imath(a)] + 3\big(\imath(a_{i_1}), \imath(a_{I'}), \imath(a)\big)
\end{aligned}$$

$$= (-1)^{\overline{\iota(a)}\,\overline{\iota(a_{I'})}} \underbrace{[\iota(a_{i_1}),\iota(a)]\iota(a_{I'})}_{U_n} + \underbrace{\iota(a_{i_1})[\iota(a_{I'}),\iota(a)]}_{U_n}$$
$$+\frac{1}{2}\Big\{ \underbrace{[[\iota(a_{i_1}),\iota(a_{I'})],\iota(a)]}_{U_{n-1}} - (-1)^{\overline{\iota(a)}\,\overline{\iota(a_{I'})}} \underbrace{[[\iota(a_{i_1}),\iota(a)],\iota(a_{I'})]}_{U_{n-1}}$$
$$- \underbrace{[\iota(a_{i_1}),[\iota(a_{I'}),\iota(a)]]}_{U_{n-1}} \Big\} \in U_n,$$

because $\iota(a), \iota(a_{i_1}) \in N_{\text{alt}}\big(U(M)\big)$ and we use the two following assertions:

$$[xy,z] - (-1)^{\bar{y}\bar{z}}[x,z]y - x[y,z] = (x,y,z) - (-1)^{\bar{y}\bar{z}}(x,z,y) + (-1)^{\bar{z}(\bar{x}+\bar{y})}(z,x,y)$$

and

$$\begin{aligned}&[[x,y],z] - (-1)^{\bar{y}\bar{z}}[[x,z],y] - [x,[y,z]]\\ &\quad= (x,y,z) - (-1)^{\bar{y}\bar{z}}(x,z,y) + (-1)^{\bar{z}(\bar{x}+\bar{y})}(z,x,y)\\ &\quad\quad - (-1)^{\bar{x}\bar{y}}(y,x,z) + (-1)^{\bar{x}(\bar{y}+\bar{z})}(y,z,x) - (-1)^{\bar{x}(\bar{y}+\bar{z})+\bar{y}\bar{z}}(z,y,x)\end{aligned} \tag{4.1}$$

holding in any superalgebra, where x, y and z are homogeneous elements. In conclusion, we obtain $[U_n, \iota(a)] \subseteq U_n$. Now we take care about the other condition. For $a_{i_0} \in M$ we have

$$\begin{aligned}\iota(a_{i_0})\iota(a_I) &= \iota(a_{i_0})\big(\iota(a_{i_1})\iota(a_{I'})\big) = \big(\iota(a_{i_0})\iota(a_{i_1})\big)\iota(a_{I'}) - \big(\iota(a_{i_0}),\iota(a_{i_1}),\iota(a_{I'})\big)\\
&= (-1)^{\overline{\iota(a_{i_0})}\,\overline{\iota(a_{i_1})}}\big(\iota(a_{i_1})\iota(a_{i_0})\big)\iota(a_{I'}) + [\iota(a_{i_0}),\iota(a_{i_1})]\iota(a_{I'})\\
&\quad + (-1)^{\overline{\iota(a_{i_1})}\,\overline{\iota(a_{I'})}}\big(\iota(a_{i_0}),\iota(a_{I'}),\iota(a_{i_1})\big)\\
&\equiv (-1)^{\overline{\iota(a_{i_0})}\,\overline{\iota(a_{i_1})}}\iota(a_{i_1})\big(\iota(a_{i_0})\iota(a_{I'})\big)\\
&\quad + 2(-1)^{\overline{\iota(a_{i_1})}\,\overline{\iota(a_{I'})}}\big(\iota(a_{i_0}),\iota(a_{I'}),\iota(a_{i_1})\big) \bmod U_n\\
&\equiv (-1)^{\overline{\iota(a_{i_0})}\,\overline{\iota(a_{i_1})}}\iota(a_{i_1})\big(\iota(a_{i_0})\iota(a_{I'})\big)\\
&\quad + \frac{1}{3}(-1)^{\overline{\iota(a_{i_1})}\,\overline{\iota(a_{I'})}}\Big\{ \underbrace{[[\iota(a_{i_0}),\iota(a_{I'})],\iota(a_{i_1})]}_{U_{n-1}}\\
&\quad - (-1)^{\overline{\iota(a_{i_1})}\,\overline{\iota(a_{I'})}} \underbrace{[[\iota(a_{i_0}),\iota(a_{i_1})],\iota(a_{I'})]}_{U_{n-1}} - \underbrace{[\iota(a_{i_0}),[\iota(a_{I'}),\iota(a_{i_1})]]}_{U_{n-1}} \Big\} \bmod U_n\\
&\equiv (-1)^{\overline{\iota(a_{i_0})}\,\overline{\iota(a_{i_1})}}\iota(a_{i_1})\big(\iota(a_{i_0})\iota(a_{I'})\big) \bmod U_n\end{aligned}$$

where we take in account that $\imath(a_{i_0}), \imath(a_{i_1}) \in N_{\text{alt}}(U(M))$ and (4.1). Hence $\imath(a)U_n \subseteq U_{n+1}$. By induction we proved that $[U_n, \imath(a)] \subseteq U_n$ and $\imath(a)U_n \subseteq U_{n+1}$, for all n. From $U_n\imath(a) \subseteq \imath(a)U_n + [\imath(a), U_n] \subseteq U_{n+1} + U_n \subseteq U_{n+1}$ we also have $U_n\imath(a) \subseteq U_{n+1}$. Therefore $\imath(a)U + U\imath(a) \subseteq U$. We shall proceed by induction again. Let us now suppose that $\imath(a_I)U \subseteq U$, for $I \in \Omega$ with $|I| < n$. Take $I \in \Omega$ with $|I| = n$ and $x \in U$, we have

$$\begin{aligned}
\imath(a_I)x &= \big(\imath(a_{i_1})\imath(a_{I'})\big)x = \imath(a_{i_1})\big(\imath(a_{I'})x\big) + \big(\imath(a_{i_1}), \imath(a_{I'}), x\big) \\
&= \imath(a_{i_1})\big(\imath(a_{I'})x\big) + (-1)^{\overline{\imath(a_{i_1})}(\overline{\imath(a_{I'})}+\bar{x})}\big(\imath(a_{I'}), x, \imath(a_{i_1})\big) \\
&= \imath(a_{i_1})\underbrace{\big(\imath(a_{I'})x\big)}_{U} + (-1)^{\overline{\imath(a_{i_1})}(\overline{\imath(a_{I'})}+\bar{x})}\underbrace{\big(\imath(a_{I'})x\big)}_{U}\imath(a_{i_1}) \\
&\quad - (-1)^{\overline{\imath(a_{i_1})}(\overline{\imath(a_{I'})}+\bar{x})}\underbrace{\imath(a_{I'})\big(x\imath(a_{i_1})\big)}_{U} \subseteq U
\end{aligned}$$

which guarantees that $\imath(a)U \subseteq U$ for $|I| = n$. Hence U is a subalgebra and consequently $U(M) = U$. □

Now we show that the universal enveloping superalgebra of a Malcev superalgebra generalizes the classical notion of the universal enveloping of a Lie superalgebra (see [4]).

Corollary 4.5. *If M is a Lie superalgebra then $U(M)$ and the universal enveloping superalgebra of M as Lie superalgebra are isomorphic.*

Proof. Consider (U, θ) the universal enveloping superalgebra of M as Lie superalgebra. Since U is associative, by universal property of $\big(U(M), \imath\big)$ there exists an epimorphism of superalgebras $\psi : U(M) \to U$ such that $\psi\big(\imath(a)\big) = \theta(a)$, for $a \in M$. Since $U(M)$ is spanned by the monomials $\{\imath(a_I) : I \in \Omega\}$, and this generator set is mapped into a basis of U, the epimorphism $\psi : U(M) \to U$ is an isomorphism. □

5. Linear independence of $\{\imath(a_I) : I \in \Omega\}$

This section is devoted to ensure the linear independence of the monomials $\{\imath(a_I) : I \in \Omega\}$. To obtain our goal we use the relation between Malcev superalgebras and Lie super-triple system.

We recall that a *Lie super-triple system* is a $\mathbb{Z}_2$-graded vector space V equipped with a trilinear product $[\,,\,,\,] : V \times V \times V \to V$ satisfying the following conditions: $\forall a \in V_{\bar{a}}$, $b \in V_{\bar{b}}$, $c \in V_{\bar{c}}$, $u \in V_{\bar{u}}$, $v \in V_{\bar{v}}$

(i) $\overline{[a,b,c]} \equiv (\bar{a} + \bar{b} + \bar{c}) \bmod 2$;

(ii) $[a,b,c] = -(-1)^{\bar{a}\bar{b}}[b,a,c]$;

(iii) $(-1)^{\bar{a}\bar{c}}[a,b,c] + (-1)^{\bar{a}\bar{b}}[b,c,a] + (-1)^{\bar{b}\bar{c}}[c,a,b] = 0$;

(iv) $[u,v,[a,b,c]] = [[u,v,a],b,c] + (-1)^{\bar{a}(\bar{u}+\bar{v})}[a,[u,v,b],c] + (-1)^{(\bar{a}+\bar{b})(\bar{u}+\bar{v})}[a,b,[u,v,c]]$.

Given a Malcev superalgebra $(M = M_{\bar{0}} \oplus M_{\bar{1}}, [\,,\,])$ we can construct the Lie super-triple system with underlying $\mathbb{Z}_2$-graded vector space M endowed with the trilinear product $M \otimes M \otimes M \to M$ defined by: $\forall a \in M_{\bar{a}},\ b \in M_{\bar{b}},\ c \in M_{\bar{c}}$

$$[a,b,c] = \frac{1}{3}\{2[[a,b],c] - (-1)^{\bar{a}(\bar{b}+\bar{c})}[[b,c],a] - (-1)^{\bar{c}(\bar{a}+\bar{b})}[[c,a],b]\}.$$

From the Lie super-triple system defined just above we construct a Lie superalgebra $L(M,[\,,\,,\,]) = L(M) \oplus M$, where $L(M)$ is the Lie superalgebra generated by the adjoint operators $\mathrm{ad}_a(x) = [a,x]$, for $a \in M_{\bar{a}},\ x \in M_{\bar{x}}$, with the multiplication defined in the following way: we consider in $L(M)$ its own multiplication and

$$[\varphi, a] = \varphi(a), \qquad \forall a \in M_{\bar{a}},\ \varphi \in L(M)_{\bar{\varphi}},$$
$$[a,b] \in L(M), \qquad \forall a \in M_{\bar{a}},\ b \in M_{\bar{b}},\ \text{defined by } [a,b](x) = [a,b,x],\ \forall x \in M_{\bar{x}}.$$

We can easily see that the Lie superalgebra $L(M,[\,,\,,\,])$ possesses a $\mathbb{Z}_2$-gradation with even and odd parts $L(M)$ and M respectively. This superalgebra is not large enough for our purposes, but it will help us to find the structure that we need.

We denote by $\mathscr{L}(M)$ the Lie superalgebra generated by $\{\lambda_a, \rho_a : a \in M_{\bar{a}}\}$ where the degree of the generators is given by $\overline{\lambda_a} = \overline{\rho_a} = \bar{a},\ a \in M_{\bar{a}}$, satisfying the relations

$$\begin{aligned}
&\lambda_{\alpha a+\beta b} = \alpha\lambda_a + \beta\lambda_b, \qquad \rho_{\alpha a+\beta b} = \alpha\rho_a + \beta\rho_b,\\
&[\lambda_a, \lambda_b] = \lambda_{[a,b]} - 2[\lambda_a, \rho_b], \qquad [\rho_a, \rho_b] = -\rho_{[a,b]} - 2[\lambda_a, \rho_b],\\
&[\lambda_a, \rho_b] = [\rho_a, \lambda_b], \qquad \forall a \in M_{\bar{a}},\ b \in M_{\bar{b}},\ \alpha, \beta \in \mathbb{K}.
\end{aligned} \tag{5.1}$$

Proposition 5.1. *The even linear map* $\varphi : \mathscr{L}(M) \to L(M,[\,,\,,\,])$ *defined by* $\varphi(\lambda_a) = \frac{1}{2}(\mathrm{ad}_a + a)$ *and* $\varphi(\rho_a) = \frac{1}{2}(-\mathrm{ad}_a + a)$, *whenever* $a \in M_{\bar{a}}$, *is an epimorphism of Lie superalgebras.*

To abbreviate we define in $\mathscr{L}(M)$ the elements

$$\begin{aligned}
\mathrm{ad}_a &= \lambda_a - \rho_a, \qquad T_a = \lambda_a + \rho_a,\\
D_{a,b} &= [\lambda_a, \lambda_b] + [\rho_a, \rho_b] + [\lambda_a, \rho_b]\\
&= \mathrm{ad}_{[a,b]} - 3[\lambda_a, \rho_b], \qquad \forall a \in M_{\bar{a}},\ b \in M_{\bar{b}}.
\end{aligned} \tag{5.2}$$

Proposition 5.2. *The Lie superalgebra $\mathscr{L}(M)$ is provided with a $\mathbb{Z}_2$-gradation $\mathscr{L}(M) = \mathscr{L}_+ \oplus \mathscr{L}_-$, where $\mathscr{L}_+ = \mathrm{span}\langle \mathrm{ad}_a, D_{a,b} : a \in M_{\bar{a}}, b \in M_{\bar{b}}\rangle$ and $\mathscr{L}_- = \mathrm{span}\langle T_a : a \in M_{\bar{a}}\rangle$. In particular, M is identified with $\mathscr{L}_-$.*

Proof. Let $a \in M_{\bar{a}}$, $b \in M_{\bar{b}}$. From (5.1) we obtain

$$[T_a, T_b] = [\lambda_a, \lambda_b] + [\rho_a, \rho_b] + 2[\lambda_a, \rho_b] = \mathrm{ad}_{[a,b]} - 2[\lambda_a, \rho_b]. \tag{5.3}$$

Inserting (5.2) in (5.3) leads to

$$3[T_a, T_b] = \mathrm{ad}_{[a,b]} + 2D_{a,b}. \tag{5.4}$$

Using again (5.1), we get

$$[\mathrm{ad}_a, T_b] = [\lambda_a, \lambda_b] - [\rho_a, \rho_b] = \lambda_{[a,b]} + \rho_{[a,b]} = T_{[a,b]}. \tag{5.5}$$

Adding (5.1) to (5.2), it follows that

$$\begin{aligned}[\mathrm{ad}_a, \mathrm{ad}_b] &= [\lambda_a, \lambda_b] + [\rho_a, \rho_b] - 2[\lambda_a, \rho_b] \\ &= \mathrm{ad}_{[a,b]} - 6[\lambda_a, \rho_b] = -\mathrm{ad}_{[a,b]} + 2D_{a,b},\end{aligned} \tag{5.6}$$

so $2D_{a,b} = \mathrm{ad}_{[a,b]} + [\mathrm{ad}_a, \mathrm{ad}_b]$. Using this last relation, (5.5), super-Jacobi identity, and linearity condition $T_{\alpha a+\beta b} = \alpha T_a + \beta T_b$, $\forall a \in M_{\bar{a}}$, $b \in M_{\bar{b}}$, $\alpha, \beta \in \mathbb{K}$, we get

$$\begin{aligned}2[D_{a,b}, T_c] &= [\mathrm{ad}_{[a,b]} + [\mathrm{ad}_a, \mathrm{ad}_b], T_c] \\ &= [\mathrm{ad}_{[a,b]}, T_c] - (-1)^{\bar{a}(\bar{b}+\bar{c})}[[\mathrm{ad}_b, T_c], \mathrm{ad}_a] - (-1)^{\bar{c}(\bar{a}+\bar{b})}[[T_c, \mathrm{ad}_a], \mathrm{ad}_b] \\ &= T_{[[a,b],c]} + T_{[a,[b,c]]} + (-1)^{\bar{b}\bar{c}} T_{[[a,c],b]} \\ &= T_{[[a,b],c]+(-1)^{\bar{b}\bar{c}}[[a,c],b]+[a,[b,c]]}.\end{aligned} \tag{5.7}$$

For $a \in M_{\bar{a}}$, $b \in M_{\bar{b}}$, we define the linear map $D_{a,b} : M \to M$ by

$$D_{a,b}(c) = \frac{1}{2}\{[[a,b],c] + (-1)^{\bar{b}\bar{c}}[[a,c],b] + [a,[b,c]]\}, \qquad \forall c \in M_{\bar{c}}. \tag{5.8}$$

So (5.7) may be written as

$$[D_{a,b}, T_c] = T_{(1/2)\{[[a,b],c]+(-1)^{\bar{b}\bar{c}}[[a,c],b]+[a,[b,c]]\}} = T_{D_{a,b}(c)}. \tag{5.9}$$

Combine (5.4), (5.5), (5.6), super-Jacobi identity, and linearity condition $\mathrm{ad}_{\alpha a+\beta b} = \alpha\,\mathrm{ad}_a + \beta\,\mathrm{ad}_b$, $\forall a \in M_{\bar{a}}$, $b \in M_{\bar{b}}$, $\alpha, \beta \in \mathbb{K}$, we get

$$
\begin{aligned}
2[D_{a,b}, \mathrm{ad}_c] &= [3[T_a, T_b] - \mathrm{ad}_{[a,b]}, \mathrm{ad}_c] \\
&= -3(-1)^{\bar{a}(\bar{b}+\bar{c})}[[T_b, \mathrm{ad}_c], T_a] - 3(-1)^{\bar{c}(\bar{a}+\bar{b})}[[\mathrm{ad}_c, T_a], T_b] - [\mathrm{ad}_{[a,b]}, \mathrm{ad}_c] \\
&= 3(-1)^{\bar{b}\bar{c}}[T_{[a,c]}, T_b] + 3[T_a, T_{[b,c]}] + \mathrm{ad}_{[[a,b],c]} - 2D_{[a,b],c} \\
&= \mathrm{ad}_{[[a,b],c]+(-1)^{\bar{b}\bar{c}}[[a,c],b]+[a,[b,c]]} \\
&\quad + 2\{D_{a,[b,c]} + (-1)^{\bar{a}(\bar{b}+\bar{c})}D_{b,[c,a]} + (-1)^{\bar{c}(\bar{a}+\bar{b})}D_{c,[a,b]}\},
\end{aligned}
$$

and so

$$
\begin{aligned}
2[D_{a,b}, \mathrm{ad}_c] &= 2\,\mathrm{ad}_{D_{a,b}(c)} \\
&\quad + 2\{D_{a,[b,c]} + (-1)^{\bar{a}(\bar{b}+\bar{c})}D_{b,[c,a]} + (-1)^{\bar{c}(\bar{a}+\bar{b})}D_{c,[a,b]}\}. \qquad (5.10)
\end{aligned}
$$

We shall prove that in a Malcev superalgebra M (with $a \in M_{\bar{a}},\ b \in M_{\bar{b}}$) the linear map $D_{a,b} : M \to M$ is a homogeneous superderivation of degree $\bar{a} + \bar{b}$.

Now, we show that

$$
2D_{a,b}(c) = 2[[a,b],c] - SJ(a,b,c), \qquad (5.11)
$$

where $SJ(a,b,c)$ is the superjacobian of the homogeneous elements a, b, and c. Indeed,

$$
\begin{aligned}
2[[a,b],c] - SJ(a,b,c) &= [[a,b],c] - (-1)^{\bar{a}(\bar{b}+\bar{c})}[[b,c],a] - (-1)^{\bar{c}(\bar{a}+\bar{b})}[[c,a],b] \\
&= [[a,b],c] + (-1)^{\bar{b}\bar{c}}[[a,c],b] + [a,[b,c]] = 2D_{a,b}(c).
\end{aligned}
$$

By (5.11) and the super-skewsymmetry of the superjacobian

$$
\begin{aligned}
&2\{D_{a,b}(c) + (-1)^{\bar{a}(\bar{b}+\bar{c})}D_{b,c}(a) + (-1)^{\bar{c}(\bar{a}+\bar{b})}D_{c,a}(b)\} \\
&\quad = 2\{[[a,b],c] + (-1)^{\bar{a}(\bar{b}+\bar{c})}[[b,c],a] + (-1)^{\bar{c}(\bar{a}+\bar{b})}[[c,a],b]\} \\
&\qquad - SJ(a,b,c) - (-1)^{\bar{a}(\bar{b}+\bar{c})}SJ(b,c,a) - (-1)^{\bar{c}(\bar{a}+\bar{b})}SJ(c,a,b) \\
&\quad = -SJ(a,b,c). \qquad (5.12)
\end{aligned}
$$

We are going to prove that $D_{a,[b,c]} + (-1)^{\bar{a}(\bar{b}+\bar{c})}D_{b,[c,a]} + (-1)^{\bar{c}(\bar{a}+\bar{b})}D_{c,[a,b]} = 0$. As $\mathscr{L}(M)$ is a Lie superalgebra then $SJ(\mathrm{ad}_a, \mathrm{ad}_b, \mathrm{ad}_c) = 0$. On the other hand, from (5.6), (5.10), (5.12), as $D_{b,a} = -(-1)^{\bar{a}\bar{b}}D_{a,b}$ and $\mathrm{ad}_{\alpha a+\beta b} = \alpha\,\mathrm{ad}_a + \beta\,\mathrm{ad}_b$, $\forall a \in M_{\bar{a}}$, $b \in M_{\bar{b}}$, $\alpha, \beta \in \mathbb{K}$,

$$\begin{aligned}
SJ(\mathrm{ad}_a, \mathrm{ad}_b, \mathrm{ad}_c) &= [-\mathrm{ad}_{[a,b]} + 2D_{a,b}, \mathrm{ad}_c] \\
&\quad + (-1)^{\bar{a}(\bar{b}+\bar{c})}[-\mathrm{ad}_{[b,c]} + 2D_{b,c}, \mathrm{ad}_a] \\
&\quad + (-1)^{\bar{c}(\bar{a}+\bar{b})}[-\mathrm{ad}_{[c,a]} + 2D_{c,a}, \mathrm{ad}_b] \\
&= \mathrm{ad}_{SJ(a,b,c)} + 2\{\mathrm{ad}_{D_{a,b}(c)+(-1)^{\bar{a}(\bar{b}+\bar{c})}D_{b,c}(a)+(-1)^{\bar{c}(\bar{a}+\bar{b})}D_{c,a}(b)}\} \\
&\quad + 8\{D_{a,[b,c]} + (-1)^{\bar{a}(\bar{b}+\bar{c})}D_{b,[c,a]} + (-1)^{\bar{c}(\bar{a}+\bar{b})}D_{c,[a,b]}\}.
\end{aligned}$$

Thus $D_{a,[b,c]} + (-1)^{\bar{a}(\bar{b}+\bar{c})}D_{b,[c,a]} + (-1)^{\bar{c}(\bar{a}+\bar{b})}D_{c,[a,b]} = 0$ which simplifies expression (5.10) as

$$[D_{a,b}, \mathrm{ad}_c] = \mathrm{ad}_{D_{a,b}(c)}. \tag{5.13}$$

By (5.6), (5.13), super-Jacobi identity, $\mathrm{ad}_{\alpha a+\beta b} = \alpha\,\mathrm{ad}_a + \beta\,\mathrm{ad}_b$, $\forall a \in M_{\bar{a}}$, $b \in M_{\bar{b}}$, $\alpha, \beta \in \mathbb{K}$,

$$\begin{aligned}
2[D_{a,b}, D_{c,d}] &= [D_{a,b}, [\mathrm{ad}_c, \mathrm{ad}_d]] + [D_{a,b}, \mathrm{ad}_{[c,d]}] \\
&= (-1)^{\bar{c}(\bar{a}+\bar{b})}[\mathrm{ad}_c, \mathrm{ad}_{D_{a,b}(d)}] - (-1)^{\bar{d}(\bar{a}+\bar{b}+\bar{c})}[\mathrm{ad}_d, \mathrm{ad}_{D_{a,b}(c)}] + \mathrm{ad}_{D_{a,b}([c,d])} \\
&= \mathrm{ad}_{D_{a,b}([c,d])-[D_{a,b}(c),d]-(-1)^{\bar{c}(\bar{a}+\bar{b})}[c,D_{a,b}(d)]} \\
&\quad + 2\{D_{D_{a,b}(c),d} + (-1)^{\bar{c}(\bar{a}+\bar{b})}D_{c,D_{a,b}(d)}\}.
\end{aligned}$$

As $D_{a,b}$ is a homogeneous superderivation of degree $\bar{a}+\bar{b}$, then

$$[D_{a,b}, D_{c,d}] = D_{D_{a,b}(c),d} + (-1)^{\bar{c}(\bar{a}+\bar{b})}D_{c,D_{a,b}(d)}. \qquad \square$$

Now let $T(M)$ be the classical $\mathbb{Z}$-graded associative tensor algebra of the Malcev superalgebra M,

$$T(M) = \bigoplus_{n\in\mathbb{Z}} T^n(M),$$

where $T^n(M) = \{0\}$ if $n < 0$, $T^0(M) = \mathbb{K}$ and $T^n(M) = M \otimes M \otimes \cdots \otimes M$ (n times) if $n > 0$. The $\mathbb{Z}_2$-gradation of M induces a $\mathbb{Z}_2$-gradation of $T(M)$ in a way that the canonical injection $M \to T(M)$ is an even linear map and $T(M)$ is a superalgebra with gradation

$$T(M)_\alpha = \bigoplus_{n\in\mathbb{Z}}\Big(\bigoplus_{\alpha_1+\cdots+\alpha_n=\alpha} M_{\alpha_1} \otimes \cdots \otimes M_{\alpha_n}\Big),$$

for $\alpha \in \mathbb{Z}_2$. Let J be the $\mathbb{Z}_2$-graded ideal of $T(M)$ generated by the homogeneous elements

$$ab - (-1)^{\bar{a}\bar{b}} ba, \qquad \forall a \in M_{\bar{a}},\, b \in M_{\bar{b}}.$$

The quotient algebra $S(M) = T(M)/J$ which is a superalgebra with the natural $\mathbb{Z}_2$-gradation induced by the graded ideal J, is called the *supersymmetric* superalgebra of M. As M can be identified with $\mathscr{L}_-$ (by Proposition 5.2) we also identify the supersymmetric tensor superalgebra $S(M)$ on M with $S(\mathscr{L}_-)$. Using the $\mathbb{Z}_2$-gradation on $\mathscr{L}(M)$ we define an $\mathscr{L}(M)$-module structure on $S(M)$. Let us consider a Lie superalgebra $\mathscr{L} = \mathscr{L}_{\bar{0}} \oplus \mathscr{L}_{\bar{1}}$ endowed with a $\mathbb{Z}_2$-gradation $\mathscr{L} = \mathscr{L}_+ \oplus \mathscr{L}_-$ being $\mathscr{L}_+$ the even part and $\mathscr{L}_-$ the odd part, its universal enveloping superalgebra $U(\mathscr{L})$, the left ideal K of $U(\mathscr{L})$ generated by $\mathscr{L}_+$, $K = U(\mathscr{L})\mathscr{L}_+$, and the $\mathscr{L}$-module $U(\mathscr{L})/K$. Consider a basis $\{x_i : i \in \Lambda_-\}$ of $\mathscr{L}_-$ such that $\Lambda_- = (\Lambda_-)_{\bar{0}} \oplus (\Lambda_-)_{\bar{1}}$ and let $\leq$ be an order in Λ_- verifying $\{x_i : i \in (\Lambda_-)_\alpha\}$ is a basis of $(\mathscr{L}_-)_\alpha$, for $\alpha = \bar{0}, \bar{1}$, and $i_p < i_q$ if $i_p \in (\Lambda_-)_{\bar{0}}$ and $i_q \in (\Lambda_-)_{\bar{1}}$. Applying Poincaré–Birkhoff–Witt Theorem we have a basis of $U(\mathscr{L})/K$ defined by:

$$\{x_{i_1} \dots x_{i_n} + K : n \geq 0, i_1, \dots, i_n \in \Lambda_- \text{ with } i_1 \leq \dots \leq i_n \text{ and } i_p < i_{p+1} \text{ if } x_{i_p} \in (\mathscr{L}_-)_{\bar{1}}\}.$$

If $n = 0$ we have $x_{i_1} \dots x_{i_n} = 1$ by convention. Taking in account the basis of $S(\mathscr{L}_-)$ defined by

$$\{x_{i_1} \dots x_{i_n} : n \geq 0, i_1, \dots, i_n \in \Lambda_- \text{ with } i_1 \leq \dots \leq i_n \text{ and } i_p < i_{p+1} \text{ if } x_{i_p} \in (\mathscr{L}_-)_{\bar{1}}\}$$

we have an even linear isomorphism $\theta : S(\mathscr{L}_-) \to U(\mathscr{L})/K$ defined by

$$x_{i_1} \dots x_{i_n} \mapsto x_{i_1} \dots x_{i_n} + K.$$

We can construct an $\mathscr{L}$-module of $S(\mathscr{L}_-)$ by pull backing the $\mathscr{L}$-module structure of $U(\mathscr{L})/K$ by means of θ in the following way: for $\lambda \in \mathscr{L}$ and $x \in S(\mathscr{L}_-)$ we define $\lambda \circ x = \theta^{-1}(\lambda\theta(x))$, where $\lambda\theta(x)$ is the action of $\mathscr{L}$ in $U(\mathscr{L})/K$. Let us take the natural gradation on $S(\mathscr{L}_-) = \bigoplus_{i=0}^{\infty} S(\mathscr{L}_-)^i$. Associated to this gradation we have the filtration $S(\mathscr{L}_-) = \bigcup_{n=0}^{\infty} S(\mathscr{L}_-)_n$, where $S(\mathscr{L}_-)_n = \bigoplus_{i=0}^{n} S(\mathscr{L}_-)^i$. Next lemma shows us how $\mathscr{L}$ acts in the components of the filtration of $S(\mathscr{L}_-)$.

Lemma 5.3. *We have the following assertions:*

(i) $\mathscr{L}_+ \circ S(\mathscr{L}_-)_n \subseteq S(\mathscr{L}_-)_n$ *and* $\mathscr{L}_- \circ S(\mathscr{L}_-)_n \subseteq S(\mathscr{L}_-)_{n+1}$*;*

(ii) *If* $i_1 \leq \dots \leq i_k \leq \dots \leq i_{n+1}$ *and* $x_{i_k} \in (\mathscr{L}_-)_{\bar{x}_{i_k}}$ *therefore*

$$x_{i_k} \circ (x_{i_1} \dots \hat{x}_{i_k} \dots x_{i_{n+1}}) \equiv (-1)^{\bar{x}_{i_k}(\bar{x}_{i_1} + \dots + \bar{x}_{i_{n+1}})} x_{i_1} \dots x_{i_{n+1}} \bmod S(\mathscr{L}_-)_{n-1},$$

where "$\hat{x}_{i_k}$" means that we omit this factor.

Proof. We use induction in n to show (i). Taking $x_{i_k} \in (\mathscr{L}_-)_{\bar{x}_{i_k}}$, with $i_1 \le \cdots \le i_k \le \cdots \le i_{n+1}$, we have

$$\begin{aligned}\theta\big(x_{i_k} \circ (x_{i_1} \ldots \hat{x}_{i_k} \ldots x_{i_{n+1}})\big) &= x_{i_k}(x_{i_1} \ldots \hat{x}_{i_k} \ldots x_{i_{n+1}} + K) \\ &= x_{i_k} x_{i_1} \ldots \hat{x}_{i_k} \ldots x_{i_{n+1}} + K \\ &= ([x_{i_k}, x_{i_1}] \ldots \hat{x}_{i_k} \ldots x_{i_{n+1}} + K) \\ &\quad + \big((-1)^{\bar{x}_{i_k}\bar{x}_{i_1}} x_{i_1} x_{i_k} x_{i_2} \ldots \hat{x}_{i_k} \ldots x_{i_{n+1}} + K\big).\end{aligned}$$

Since $[x_{i_k}, x_{i_1}] \in \mathscr{L}_+$ we can apply the hypothesis to conclude that the first term of the sum is in $\theta\big(S(\mathscr{L}_-)_{n-1}\big)$. Repeating the process we obtain $\mathscr{L}_- \circ S(\mathscr{L}_-)_n \subseteq S(\mathscr{L}_-)_{n+1}$. Now, we show the former inclusion in (i). For arbitrary $\lambda_+ \in \mathscr{L}_+$ and $x_{i_1} \ldots x_{i_n} \in S(\mathscr{L}_-)_n$ we infer that

$$\begin{aligned}\theta\big(\lambda_+ \circ (x_{i_1} \ldots x_{i_n})\big) &= \lambda_+(x_{i_1} \ldots x_{i_n}) + K = [\lambda_+, x_{i_1} \ldots x_{i_n}] + K \\ &= \sum_{j=1}^{n} (-1)^{\bar{\lambda}_+(\bar{x}_{i_1} + \cdots + \bar{x}_{i_{j-1}})} x_{i_1} \ldots [\lambda_+, x_{i_j}] \ldots x_{i_n} + K,\end{aligned}$$

because $(x_{i_1} \ldots x_{i_n})\lambda_+$ is in the left ideal K generated by $\mathscr{L}_+$. The $\mathbb{Z}_2$-gradation $\mathscr{L} = \mathscr{L}_+ \oplus \mathscr{L}_-$ yields $[\lambda_+, x_{i_j}] \in \mathscr{L}_-$, hence any term of the sum is in $\theta\big(S(\mathscr{L}_-)_n\big)$, as required. □

We note that the pair $\lambda'_a = T_a$, $\rho'_a = -\rho_a$, as well the pair $\lambda''_a = -\lambda_a$, $\rho''_a = T_a$, satisfies relations (5.1) defining $\mathscr{L}(M)$. We can define two endomorphisms $\xi, \eta : \mathscr{L}(M) \to \mathscr{L}(M)$ by $\xi(\lambda_a) = T_a$, $\xi(\rho_a) = -\rho_a$ and $\eta(\lambda_a) = -\lambda_a$, $\eta(\rho_a) = T_a$, for $a \in M_{\bar{a}}$, respectively, which are automorphisms because $\xi^2 = \eta^2 = \mathrm{id}_{\mathscr{L}(M)}$. Consider an $\mathscr{L}(M)$-module $S(M)$ and an automorphism ε of $S(M)$. We define the twisted $\mathscr{L}(M)$-module $S(M)_\varepsilon$ in the following way: for all $\lambda \in \mathscr{L}(M)$ and $x \in S(M)$ we have $\lambda x := \varepsilon(\lambda) \circ x$. In particular, using the automorphism ζ and η referred before, we define the $\mathscr{L}(M)$-modules $S(M)_\xi$ and $S(M)_\eta$, respectively.

Proposition 5.4. *If there exists an $\mathscr{L}(M)$-module homomorphism $* : S(M)_\xi \otimes S(M)_\eta \to S(M)$ verifying*

(i) *$a * x = 2\lambda_a \circ x$ and $x * a = 2(-1)^{\bar{a}\bar{x}}\rho_a \circ x$, with $a \in M_{\bar{a}}$, $x \in S(M)_{\bar{x}}$,*

(ii) *$1 * x = x * 1 = x$, for $x \in S(M)$,*

then Theorem 4.3 *is true.*

Proof. Let us assume that $S(M)$ is an algebra with multiplication $*$. From (i) and since $*$ is an $\mathscr{L}(M)$-module homomorphism, we infer that for $a \in M_{\bar{a}}$, $x \in S(M)_{\bar{x}}$, $y \in S(M)_{\bar{y}}$

$$\begin{aligned} a*(x*y) &= 2\lambda_a\circ(x*y) = 2\big(\xi(\lambda_a)\circ x\big)*y + 2(-1)^{\bar{a}\bar{x}}x*\big(\eta(\lambda_a)\circ y\big)\\ &= 2(T_a\circ x)*y - 2(-1)^{\bar{a}\bar{x}}x*(\lambda_a\circ y). \end{aligned}$$

We also have,

$$\begin{aligned} &(a*x)*y + (-1)^{\bar{a}\bar{x}}(x*a)*y - (-1)^{\bar{a}\bar{x}}x*(a*y)\\ &\quad= (2\lambda_a\circ x)*y + (2\rho_a\circ x)*y - (-1)^{\bar{a}\bar{x}}x*(2\lambda_a\circ y)\\ &\quad= 2(T_a\circ x)*y - 2(-1)^{\bar{a}\bar{x}}x*(\lambda_a\circ y), \end{aligned}$$

thus $(x,a,y) = -(-1)^{\bar{a}\bar{x}}(a,x,y)$. Similar, we observe that

$$\begin{aligned} (x*y)*a &= 2(-1)^{\bar{a}(\bar{x}+\bar{y})}\rho_a\circ(x*y)\\ &= 2(-1)^{\bar{a}(\bar{x}+\bar{y})}\big(\xi(\rho_a)\circ x\big)*y + 2(-1)^{\bar{a}\bar{y}}x*\big(\eta(\rho_a)\circ y\big)\\ &= -2(-1)^{\bar{a}(\bar{x}+\bar{y})}(\rho_a\circ x)*y + 2(-1)^{\bar{a}\bar{y}}x*(T_a\circ y). \end{aligned}$$

We also get,

$$\begin{aligned} &x*(y*a) - (-1)^{\bar{a}\bar{y}}(x*a)*y + (-1)^{\bar{a}\bar{y}}x*(a*y)\\ &\quad= (-1)^{\bar{a}\bar{y}}x*(2\rho_a\circ y) - (-1)^{\bar{a}(\bar{x}+\bar{y})}(2\rho_a\circ x)*y + (-1)^{\bar{a}\bar{y}}x*(2\lambda_a\circ y)\\ &\quad= -2(-1)^{\bar{a}(\bar{x}+\bar{y})}(\rho_a\circ x)*y + 2(-1)^{\bar{a}\bar{y}}x*(T_a\circ y), \end{aligned}$$

hence $(x,y,a) = -(-1)^{\bar{a}\bar{y}}(x,a,y)$, we conclude that $M \subseteq N_{\mathrm{alt}}\big((S(M),*)\big)$. Consider the basis $\{a_i : i\in\Lambda\}$ of M. Since $T_a + \mathrm{ad}_a = 2\lambda_a$, $\forall a\in M_{\bar{a}}$, we obtain that

$$\begin{aligned} a_{i_1}*(a_{i_2}\dots a_{i_n}) &= 2\lambda_{a_{i_1}}\circ(a_{i_2}\dots a_{i_n}) = T_{a_{i_1}}\circ(a_{i_2}\dots a_{i_n}) + \mathrm{ad}_{a_{i_1}}\circ(a_{i_2}\dots a_{i_n})\\ &\equiv a_{i_1}\dots a_{i_n} \bmod S(M)_{n-1}, \end{aligned}$$

because $\mathrm{ad}_{a_{i_1}}\circ(a_{i_2}\dots a_{i_n}) \in \mathscr{L}_+\circ S(M)_{n-1} \subseteq S(M)_{n-1}$ (from Lemma 5.3). Repeating this argument we get that

$$a_{i_1}*\big(a_{i_2}*\big(\dots(a_{i_{n-1}}*a_{i_n})\big)\big) \equiv a_{i_1}\dots a_{i_n} \bmod S(M)_{n-1},$$

consequently,

$$\big\{a_{i_1}*\big(a_{i_2}*\big(\dots(a_{i_{n-1}}*a_{i_n})\big)\big) : (i_1,\dots,i_n)\in\Omega\big\} \tag{5.14}$$

is a basis of $S(M)$. From the universal property of the enveloping algebra $\big(U(M),\imath\big)$, there exists a superalgebra homomorphism $U(M)\to\big(S(M),*\big)$ of degree 0 which send a linear generator set $\{\imath(a_I) : I\in\Omega\}$ of $U(M)$ onto a basis (5.14) of $S(M)$, therefore it is indeed an isomorphism. □

Now, it is our task to define a product $*$ in $S(M)$ satisfying the conditions of Proposition 5.4. Using an argument similar to the one used in proof just above, we show that given $a_I = a_{i_1} \dots a_{i_n}$ in $S(M)$ we get that $r_I = a_I - 2\lambda_{a_{i_1}} \circ a_{I'} \in S(M)_{|I|-1}$ and $\overline{r_I} = \overline{a_I}$. We define recursively the product $*$ in $S(M)$ in the following way: set that $1 * x = x$, for $x \in S(M)$. Assuming by hypothesis that $a_J * x$ is given for $|J| < |I|$ we define, by induction, that

$$a_I * x = 2T_{a_{i_1}} \circ (a_{I'} * x) - 2(-1)^{\overline{a_{i_1}}\,\overline{a_{I'}}} a_{I'} * (\rho_{a_{i_1}} \circ x) + r_I * x. \tag{5.15}$$

Proposition 5.5. *Given $\lambda \in \mathscr{L}(M)_{\bar{\lambda}}$, $x \in S(M)_{\bar{x}}$ and $y \in S(M)_{\bar{y}}$ it is verified that*

$$\lambda \circ (x * y) = \big(\xi(\lambda) \circ x\big) * y + (-1)^{\bar{\lambda}\bar{x}} x * \big(\eta(\lambda) \circ y\big).$$

Proof. We shall prove that $\lambda \circ (a_I * x) = \big(\xi(\lambda) \circ a_I\big) * x + (-1)^{\bar{\lambda}\overline{a_I}} a_I * \big(\eta(\lambda) \circ x\big)$ by induction on the size $|I|$ of I. If $|I| = 0$, this is $a_I = 1$, we have to ensure that $\lambda \circ x = \big(\xi(\lambda) \circ 1\big) * x + \eta(\lambda) \circ x$. For $\lambda = D + \lambda_a + \rho_b$, with $D = \Sigma_i \alpha_i D_{a_i, b_i}$ ($\alpha_i \in \mathbb{K}$), we get

$$\lambda - \eta(\lambda) = D + \lambda_a + \rho_b - \eta(D) + \lambda_a - T_b = 2\lambda_a - \lambda_b = \lambda_{2a-b}.$$

Since $D_{a,b}$ is a derivation then $D_{a,b} \circ 1 = 0$. As $\lambda_a \circ 1 = \rho_a \circ 1 = \frac{1}{2}a$ and $T_a \circ 1 = a$ then $\xi(\lambda) \circ 1 = \xi(D + \lambda_a + \rho_b) \circ 1 = (D + T_a - \rho_b) \circ 1 = a - \frac{1}{2}b$. Therefore

$$\big(\xi(\lambda) \circ 1\big) * x = 2\lambda_{a-(1/2)b} \circ x = \lambda_{2a-b} \circ x = \big(\lambda - \eta(\lambda)\big) \circ x,$$

the required formula.

Relatively to the induction step, first we will show that we can reduce our study just to the case that $\lambda = T_{a_{i_0}}$, for a suitable a_{i_0}. Indeed, using definitions, applying the hypothesis and doing some calculations we have that

$$\begin{aligned}
&\lambda \circ (a_I * x) - \big(\xi(\lambda) \circ a_I\big) * x - (-1)^{\bar{\lambda}\overline{a_I}} a_I * \big(\eta(\lambda) \circ x\big) \\
&\quad \overset{\text{def}}{=} \lambda \circ \{2T_{a_{i_1}} \circ (a_{I'} * x) - 2(-1)^{\overline{a_{i_1}}\,\overline{a_{I'}}} a_{I'} * (\rho_{a_{i_1}} \circ x) + r_I * x\} \\
&\qquad - \{\xi(\lambda) \circ (2\lambda_{a_{i_1}} \circ a_{I'} + r_I)\} * x - 2(-1)^{\bar{\lambda}\overline{a_I}} T_{a_{i_1}} \circ \big(a_{I'} * \big(\eta(\lambda) \circ x\big)\big) \\
&\qquad + 2(-1)^{\bar{\lambda}\overline{a_I} + \overline{a_{i_1}}\,\overline{a_{I'}}} a_{I'} * \big(\rho_{a_{i_1}} \eta(\lambda) \circ x\big) - (-1)^{\bar{\lambda}\overline{a_I}} r_I * \big(\eta(\lambda) \circ x\big) \\
&\quad \overset{\text{hyp}}{=} 2\lambda T_{a_{i_1}} \circ (a_{I'} * x) - 2(-1)^{\overline{a_{i_1}}\,\overline{a_{I'}}} \lambda \circ \big(a_{I'} * (\rho_{a_{i_1}} \circ x)\big) - 2\big(\xi(\lambda)\lambda_{a_{i_1}} \circ a_{I'}\big) * x \\
&\qquad - 2(-1)^{\bar{\lambda}\overline{a_I}} T_{a_{i_1}} \circ \big(a_{I'} * \big(\eta(\lambda) \circ x\big)\big) + 2(-1)^{\bar{\lambda}\overline{a_I} + \overline{a_{i_1}}\,\overline{a_{I'}}} a_{I'} * \big(\rho_{a_{i_1}} \eta(\lambda) \circ x\big) \\
&\quad = 2[\lambda, T_{a_{i_1}}] \circ (a_{I'} * x) + 2(-1)^{\bar{\lambda}\overline{a_{i_1}}} T_{a_{i_1}} \lambda \circ (a_{I'} * x)
\end{aligned}$$

$$
\begin{aligned}
&- 2(-1)^{\overline{a_{i_1}}\,\overline{a_{I'}}}\lambda \circ \big(a_{I'} * (\rho_{a_{i_1}} \circ x)\big) - 2\big(\xi(\lambda)\lambda_{a_{i_1}} \circ a_{I'}\big) * x \\
&- 2(-1)^{\bar{\lambda}\overline{a_I}} T_{a_{i_1}} \circ \big(a_{I'} * \big(\eta(\lambda) \circ x\big)\big) + 2(-1)^{\bar{\lambda}\overline{a_I}+\overline{a_{i_1}}\,\overline{a_{I'}}} a_{I'} * \big(\rho_{a_{i_1}}\eta(\lambda) \circ x\big) \\
\overset{\text{hyp}}{=}\ & 2[\lambda, T_{a_{i_1}}] \circ (a_{I'} * x) + 2(-1)^{\bar{\lambda}\overline{a_{i_1}}} T_{a_{i_1}}\lambda \circ (a_{I'} * x) \\
&- 2(-1)^{\overline{a_{i_1}}\,\overline{a_{I'}}}\big(\xi(\lambda) \circ a_{I'}\big) * (\rho_{a_{i_1}} \circ x) - 2(-1)^{\overline{a_{i_1}}\,\overline{a_{I'}}+\bar{\lambda}\overline{a_I}} a_{I'} * \big(\eta(\lambda)\rho_{a_{i_1}} \circ x\big) \\
&- 2\big(\xi(\lambda)\lambda_{a_{i_1}} \circ a_{I'}\big) * x - 2(-1)^{\bar{\lambda}\overline{a_I}} T_{a_{i_1}} \circ \big(a_{I'} * \big(\eta(\lambda) \circ x\big)\big) \\
&+ 2(-1)^{\bar{\lambda}\overline{a_I}+\overline{a_{i_1}}\,\overline{a_{I'}}} a_{I'} * \big(\rho_{a_{i_1}}\eta(\lambda) \circ x\big) \\
\overset{\text{hyp}}{=}\ & 2\big([\xi(\lambda), \lambda_{a_{i_1}}] \circ a_{I'}\big) * x + 2(-1)^{\bar{\lambda}\overline{a_{i_1}}} T_{a_{i_1}}\lambda \circ (a_{I'} * x) \\
&- 2(-1)^{\overline{a_{i_1}}\,\overline{a_{I'}}}\big(\xi(\lambda) \circ a_{I'}\big) * (\rho_{a_{i_1}} \circ x) - 2\big(\xi(\lambda)\lambda_{a_{i_1}} \circ a_{I'}\big) * x \\
&- 2(-1)^{\bar{\lambda}\overline{a_I}} T_{a_{i_1}} \circ \big(a_{I'} * \big(\eta(\lambda) \circ x\big)\big) \\
=\ & -2(-1)^{\bar{\lambda}\overline{a_{i_1}}}\big(\lambda_{a_{i_1}} \circ \xi(\lambda) \circ a_{I'}\big) * x + 2(-1)^{\bar{\lambda}\overline{a_{i_1}}} T_{a_{i_1}}\lambda \circ (a_{I'} * x) \\
&- 2(-1)^{\overline{a_{i_1}}\,\overline{a_{I'}}}\big(\xi(\lambda) \circ a_{I'}\big) * (\rho_{a_{i_1}} \circ x) - 2(-1)^{\bar{\lambda}\overline{a_I}} T_{a_{i_1}} \circ \big(a_{I'} * \big(\eta(\lambda) \circ x\big)\big) \\
\overset{\text{hyp}}{=}\ & -2(-1)^{\bar{\lambda}\overline{a_{i_1}}}\big(\lambda_{a_{i_1}} \circ \xi(\lambda) \circ a_{I'}\big) * x + 2(-1)^{\bar{\lambda}\overline{a_{i_1}}} T_{a_{i_1}} \circ \big(\big(\xi(\lambda) \circ a_{I'}\big) * x\big) \\
&- 2(-1)^{\overline{a_{i_1}}\,\overline{a_{I'}}}\big(\xi(\lambda) \circ a_{I'}\big) * (\rho_{a_{i_1}} \circ x).
\end{aligned}
$$

On the other hand, using hypothesis with $T_{a_{i_1}}$ we obtain

$$
\begin{aligned}
&2(-1)^{\bar{\lambda}\overline{a_{i_1}}} T_{a_{i_1}} \circ \big(\big(\xi(\lambda) \circ a_{I'}\big) * x\big) \\
&\quad = 2(-1)^{\bar{\lambda}\overline{a_{i_1}}}\big(\xi(T_{a_{i_1}}) \circ \xi(\lambda) \circ a_{I'}\big) * x \\
&\qquad + 2(-1)^{\bar{\lambda}\overline{a_{i_1}}+\overline{a_{i_1}}(\bar{\lambda}+\overline{a_{I'}})}\big(\xi(\lambda) \circ a_{I'}\big) * \big(\eta(T_{a_{i_1}}) \circ x\big) \\
&\quad = 2(-1)^{\bar{\lambda}\overline{a_{i_1}}}\big(\lambda_{a_{i_1}} \circ \xi(\lambda) \circ a_{I'}\big) * x + 2(-1)^{\overline{a_{i_1}}\,\overline{a_{I'}}}\big(\xi(\lambda) \circ a_{I'}\big) * (\rho_{a_{i_1}} \circ x),
\end{aligned}
$$

so we may assume $\lambda = T_{a_{i_0}}$, for some a_{i_0}. Using hypothesis of induction we can show that we may assume that $i_0 \leq i_1$. So, denoting $a_{(i_0,I)} = a_{i_0}a_{i_1}\dots a_{i_n}$, with $i_0 \leq i_1 \leq \cdots \leq i_n$, we have that

$$
\begin{aligned}
2(\lambda_{a_{i_0}} \circ a_I) * x &= a_{(i_0,I)} * x - r_{(i_0,I)} * x \\
&= 2T_{a_{i_0}} \circ (a_I * x) - 2(-1)^{\overline{a_{i_0}}\,\overline{a_I}} a_I * (\rho_{a_{i_0}} \circ x) + r_{(i_0,I)} * x - r_{(i_0,I)} * x \\
&= 2T_{a_{i_0}} \circ (a_I * x) - 2(-1)^{\overline{a_{i_0}}\,\overline{a_I}} a_I * (\rho_{a_{i_0}} \circ x),
\end{aligned}
$$

thus

$$T_{a_{i_0}} \circ (a_I * x) = (\lambda_{a_{i_0}} \circ a_I) * x + (-1)^{\overline{a_{i_0}}\,\overline{a_I}} a_I * (\rho_{a_{i_0}} \circ x),$$

completing the proof. □

Proposition 5.6. *For the product defined by* (5.15) *the following statements hold:* $1 * x = x * 1 = x$, $a * x = 2\lambda_a \circ x$ *and* $x * a = 2(-1)^{\bar{a}\bar{x}} \rho_a \circ x$, *for any* $a \in M_{\bar{a}}$, $x \in S(M)_{\bar{x}}$.

Proof. If we fix $\delta_a = \eta(\mathrm{ad}_a)$ then $\xi(\delta_a) = \delta_a$. In fact, $\delta_a = \eta(\mathrm{ad}_a) = \eta(\lambda_a - \rho_a) = -\lambda_a - T_a$. On the other hand, $\xi(\delta_a) = \xi\big(\eta(\mathrm{ad}_a)\big) = \xi(-\lambda_a - T_a) = \xi(-2\lambda_a - \rho_a) = -2T_a + \rho_a = -2\lambda_a - \rho_a = -\lambda_a - T_a$ as required. As $\eta(\delta_a) = \mathrm{ad}_a$ and $\mathrm{ad}_a \circ 1 = 0$, we have that

$$\begin{aligned}
\delta_a \circ (x * 1 - x) &= \big(\xi(\delta_a) \circ x\big) * 1 + (-1)^{\bar{a}\bar{x}} x * \big(\eta(\delta_a) \circ 1\big) - \delta_a \circ x \\
&= (\delta_a \circ x) * 1 + (-1)^{\bar{a}\bar{x}} x * (\mathrm{ad}_a \circ 1) - \delta_a \circ x \\
&= (\delta_a \circ x) * 1 - \delta_a \circ x.
\end{aligned}$$

More general, we infer that $\delta_{a_1} \ldots \delta_{a_n} \circ (x * 1 - x) = (\delta_{a_1} \ldots \delta_{a_n} \circ x) * 1 - \delta_{a_1} \ldots \delta_{a_n} \circ x$, for $a_i \in M_{\overline{a_i}}$ $(1 \le i \le n)$. Denote by S the vector space spanned by $\delta_{a_1} \ldots \delta_{a_n} \circ 1$, with $a_i \in M_{\overline{a_i}}$ $(1 \le i \le n)$. Using this last condition we show that $x * 1 - x = 0$, for all $x \in S$. In fact

$$(\delta_{a_1} \ldots \delta_{a_n} \circ 1) * 1 - \delta_{a_1} \ldots \delta_{a_n} \circ 1 = \delta_{a_1} \ldots \delta_{a_n} \circ (1 * 1 - 1) = 0.$$

We will show that $S(M) = S$ by induction in n. Since $\delta_a \circ 1 = (-2\lambda_a - \rho_a) \circ 1 = -2\frac{1}{2}a - \frac{1}{2}a = -\frac{3}{2}a$, then $S(M)_1 (= M) \subseteq S$. Let us now assume that $S(M)_{n-1} \subseteq S$. Let $a_I \in S(M)_n$ with $I = (i_1, \ldots, i_n)$. Since $\delta_a = -\lambda_a - T_a$ we have $2\delta_a = -3T_a - \mathrm{ad}_a$, hence

$$\begin{aligned}
\delta_{a_1} \circ a_{I'} &= \left(-\frac{3}{2} T_{a_1} - \frac{1}{2} \mathrm{ad}_{a_1}\right) \circ a_{I'} = -\frac{3}{2} T_{a_1} \circ a_{I'} - \frac{1}{2} \mathrm{ad}_{a_1} \circ a_{I'} \\
&= -\frac{3}{2} a_I - \frac{1}{2} \underbrace{\mathrm{ad}_{a_1}}_{\in \mathscr{L}_+} \circ \underbrace{a_{I'}}_{\in S(M)_{n-1}}.
\end{aligned}$$

By hypothesis of induction we infer that $\delta_{a_1} \circ a_{I'} \in S$, by Lemma 5.3 we get that $\mathrm{ad}_{a_1} \circ a_{I'}$ is in $S(M)_{n-1}$, consequently $S(M)_n \subseteq S$. Finally, we ensure the last condition of the proposition (the other is similar). Since

$$\rho_a \circ x = \rho_a \circ (x * 1) = \big(\xi(\rho_a) \circ x\big) * 1 + (-1)^{\bar{a}\bar{x}} x * \big(\eta(\rho_a) \circ 1\big)$$
$$= -\rho_a \circ x + (-1)^{\bar{a}\bar{x}} x * (T_a \circ 1) = -\rho_a \circ x + (-1)^{\bar{a}\bar{x}} x * a,$$

therefore $x * a = 2(-1)^{\bar{a}\bar{x}} \rho_a \circ x$, as desired. □

We summarize as follows: Proposition 5.6 shows that the product $*$ in $S(M)$ defined recursively by (5.15) satisfies the conditions of Proposition 5.4. Which guarantees that Theorem 4.3 holds, providing the vector space $U(M)$ with a basis $\{\imath(a_I) : I \in \Omega\}$. So, the universal enveloping algebra $U(M)$ has a basis of Poincaré–Birkhoff–Witt Theorem type over the Malcev superalgebra M.

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