Portugal. Math. (N.S.)
Vol. 68, Fasc. 3, 2011, 279–296
DOI 10.4171/PM/1892

Portugaliae Mathematica
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On the stabilization and controllability for a third order linear equation

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(Communicated by Enrique Zuazua)

Abstract. We analyze the stabilization and the exact controllability of a third order linear equation in a bounded interval. That is, we consider the following equation:

$$iu_t + i\gamma u_x + \alpha u_{xx} + i\beta u_{xxx} = 0,$$

where $u = u(x,t)$ is a complex valued function defined in $(0,L) \times (0,+\infty)$ and α, β and γ are real constants. Using multiplier techniques, HUM method and a special uniform continuation theorem, we prove the exponential decay of the total energy and the boundary exact controllability associated with the above equation. Moreover, we characterize a set of lengths L, named $\mathscr{X}$, in which it is possible to find non null solutions for the above equation with constant (in time) energy and we show it depends strongly on the parameters α, β and γ.

Mathematics Subject Classification (2010). Primary 35B30; Secondary 35Qxx.
Keywords. Exponential decay, stabilization, exact control.

1. Introduction

We consider the third order linear system in a bounded interval

$$\begin{aligned}
&iu_t + i\gamma u_x + \alpha u_{xx} + i\beta u_{xxx} = 0 \quad \text{in } (0,L) \times (0,+\infty)\\
&u(0,t) = u(L,t) = 0 \quad \text{for all } t \geq 0\\
&u_x(L,t) = 0 \quad \text{for all } t \geq 0\\
&u(x,0) = u_0 \quad \text{in } (0,L).
\end{aligned} \tag{1.1}$$

Where u is a complex valued function, u_0 belongs to $L^2(0,L)$, α and γ are non null real constants and β is a positive constant.

When $\gamma = 0$ in (1.1), the third order linear equation is related to a nonlinear Schrödinger equation proposed by Kodama [7] to model a pulse propagation in a long-distance and high-speed optical fiber transmission system. Kodama [7] considered the following (perturbed) nonlinear Schrödinger equation with higher-order terms (as the perturbation terms)

$$iu_t + \frac{1}{2}u_{xx} + |u|^2 u = i\epsilon(\beta_1 u_{xxx} + \beta_2 |u|^2 u_x + \beta_3 u^2 \bar{u}_x) - i\Gamma u, \tag{1.2}$$

where ϵ is a small real parameter ($|\epsilon| \ll 1$) and β_1, β_2, β_3 and Γ are real constants. In [5], Chu used equation (1.2) with $\beta_2 = \beta_3 = \Gamma = 0$ as a model for the soliton propagation in an optical fiber. He showed that the third order term can be used to reduce the mutual interaction between solitons when the fiber is operated nonlinearly. In [14], Oikawa numerically investigated the same equation under periodic boundary conditions. He found out the third order term can give rise to chaotic behavior.

When $\alpha = 0$ in (1.1), we have the linear Korteweg-de Vries (KdV) equation. If we add to the linear KdV equation the non-linear term uu_x, we obtain a model for propagation of surface water waves along a channel. Rosier [16] established boundary controllability results for the linear and non-linear KdV equation on bounded domains with various boundary conditions. The stabilization of KdV system on bounded domains was proved in Menzala et al. [13] for the linear and non-linear cases. There, a localized damping in the non-linear case was considered. Results on the global stabilization of the generalized KdV system have been obtained by Rosier-Zhang [17]. Linares-Pazoto [10] studied the stabilization of the generalized KdV system with critical exponents. Massarolo et al. [12] analyzed the uniform decay for the KdV system with a very weak localized dissipation.

Dispersive problems have been object of intensive research (see, for instance, the classical paper of Benjamin-Bona-Mahoni [1], Biagioni-Linares [2], Bona-Chen [3], and references therein).

For controllability problems involving dispersive systems, we can consider the work of Russel-Zhang [18] about the KdV system and the paper by Linares-Ortega [9], where the Benjamin-Ono equation has been analyzed. Vasconcellos-Silva [21] studied the existence, regularity of the solutions and stabilization for the Kawahara system. The stabilization and controllability for linear Kawahara system is proved in Vasconcellos-Silva [19] and [20].

We consider the inner product in $L^2(0, L)$ defined by:

$$(f \mid g) = \mathrm{Re} \int_0^L f(x)\overline{g(x)}\, dx$$

and the inner product in $H_0^1(0,L)$ by:

$$(f \mid g)_1 = \mathrm{Re} \int_0^L f_x(x)\overline{g_x(x)}\, dx.$$

This paper is devoted to study the rate of the decay of the energy, as $t \to +\infty$ and the exact boundary controllability associated to the system (1.1).

The energy is defined by:

$$E(t) = \frac{1}{2} \int_0^L |u(x,t)|^2\, dx = \frac{1}{2} \|u(t)\|^2.$$

Using the boundary conditions in (1.1) we prove that

$$\frac{dE}{dt} = -\frac{\beta}{2} |u_x(0,t)|^2, \qquad \forall t > 0.$$

Since $\beta > 0$, we observe that, according to the above energy dissipation law the energy $E(t)$ is a non increasing function of the time.

In Section 2, we study the global existence and uniqueness and some regularity results for solutions of the system (1.1). There, we consider semigroups theory and multipliers techniques.

In Section 3, using multipliers techniques and a special uniform continuation theorem, we analyze the decay of the energy associated to the linear problem. We prove the energy decays exponentially when the length of the interval L does not belong to a critical enumerable real set $\mathscr{X}$. We define this set precisely and we show it depends on the parameters α, β and γ. Moreover we obtain some observability results.

In Section 4, taking into account the observability results obtained in Section 3 and using the Hilbert Uniqueness Method (HUM) (see, Lions [11]), we prove a boundary exact controllability for the system (1.1).

Finally in Section 5, we present some remarks about the system (1.1) and we show explicitly the critical set $\mathscr{X}$ for the third order Schrödinger equation.

2. Existence, uniqueness and regularity

In this section we prove existence, uniqueness and regularity results of solutions for the system (1.1). Here u_0 belongs to $L^2(0,L)$, α and γ are non null real constants and β is a positive constant.

We shall use basically semigroups theory to prove the existence and uniqueness and for regularity of solutions we shall consider the multipliers techniques.

Theorem 2.1 (Existence, uniqueness and regularity). *If* u_0 *belongs to* $L^2(0,L)$, $\alpha \neq 0$, $\gamma \neq 0$ *and* $\beta > 0$, *then the problem* (1.1) *has a unique solution* u *belonging to* $C\big([0,+\infty);L^2(0,L)\big)$, *which satisfies:*

(i) $\|u\|_{C([0,+\infty);L^2(0,L))} \leq C\|u_0\|$.

(ii) $u_x(0,\cdot)$ *belongs to* $L^2(0,+\infty)$ *and* $\|u_x(0,\cdot)\|_{L^2(0,+\infty)} \leq C\|u_0\|$.

(iii) *For each* $T > 0$, u *belongs to* $L^2\big(0,T;H_0^1(0,L)\big)$ *and there exists* $C_1(L,T,\gamma,\beta) > 0$ *such that* $\|u\|_{L^2(0,T;H_0^1(0,L))} \leq C_1\|u_0\|$.

Moreover, the energy dissipation law,

$$\frac{dE}{dt} = -\frac{\beta}{2}|u_x(0,t)|^2 \leq 0, \qquad \forall t > 0, \tag{2.1}$$

holds.

Proof. Let A denote the closed linear operator $Av = -\gamma v' + i\alpha v'' - \beta v'''$ defined on the dense domain $D(A) \subset L^2(0,L)$, where

$$D(A) = \{v \in H^3(0,L) : v(0) = v(L) = v'(L) = 0\}.$$

Let $v \in D(A)$. Then, using integration by parts and definition of $D(A)$, we have:

$$(Av\,|\,v) = \operatorname{Re}\int_0^L(-\gamma v' + i\alpha v'' - \beta v''')\bar{v}\,dx = -\beta\operatorname{Re}\int_0^L v'''\bar{v}\,dx = -\frac{\beta}{2}|v'(0)|^2 \leq 0.$$

On the other hand, we see that the adjoint operator A^*, is defined by $A^*w = \gamma w' - i\alpha w'' + \beta w'''$, where w belongs to

$$D(A^*) = \{w \in H^3(0,L) : w(0) = w(L) = w'(0) = 0\}.$$

So, using integration by parts again, we obtain:

$$(A^*w\,|\,w) = \operatorname{Re}\int_0^L(\gamma w' - i\alpha w'' + \beta w''')\overline{w}\,dx = -\frac{\beta}{2}|w'(L)|^2 \leq 0.$$

Thus, we have that the operators A and A^* are dissipative operators.

Therefore, from classical results in semigroup theory we prove that there is a unique mild solution u of the problem (1.1). Furthermore, u belongs to $C\big([0,+\infty);L^2(0,L)\big)$ and is such that

$$\|u\|_{C([0,+\infty);L^2(0,L))} \leq C\|u_0\|. \tag{2.2}$$

To show item (ii) and the energy dissipation law, we first consider $u_0 \in D(A)$. Taking the inner product of the equation in (1.1) with iu, we have

$$\mathrm{Re}\int_0^L (iu_t + i\gamma u_x + \alpha u_{xx} + i\beta u_{xxx})\overline{iu} = 0. \tag{2.3}$$

So, integrating by parts in $(0, L)$ and using the boundary conditions, we obtain

$$\mathrm{Re}\int_0^L iu_t\overline{iu}\,dx = \int_0^L \frac{1}{2}\frac{d}{dt}|u(x,t)|^2\,dx = \frac{dE}{dt}, \tag{2.4}$$

$$\mathrm{Re}\int_0^L iu_x\overline{iu}\,dx = \mathrm{Re}\int_0^L u_{xx}\overline{iu}\,dx = 0, \tag{2.5}$$

$$\mathrm{Re}\int_0^L iu_{xxx}\overline{iu}\,dx = \frac{1}{2}|u_x(0,t)|^2. \tag{2.6}$$

Replacing (2.4), (2.5) and (2.6) in (2.3) and using the density of $D(A)$ in $L^2(0, L)$ we obtain,

$$\frac{dE}{dt} = -\frac{\beta}{2}|u_x(0,t)|^2. \tag{2.7}$$

On the other hand, integrating (2.7) in $(0, T)$, we have,

$$\frac{1}{2}\|u(T)\|^2 - \frac{1}{2}\|u_0\|^2 = -\frac{\beta}{2}\int_0^T |u_x(0,t)|^2\,dt. \tag{2.8}$$

By (2.7) and (2.8) we prove item (ii) and the energy dissipation law.

Now, to prove that u, the solution of the problem (1.1), belongs to $L^2(0, T; H_0^1(0, L))$, we consider again $u_0 \in D(A)$. Taking the inner product of each element of the equation in (1.1) with ixu, integrating by parts in $(0, L) \times (0, T)$ and using the boundary conditions, we have

$$\begin{aligned}\mathrm{Re}\int_0^T\int_0^L iu_t\overline{ixu}\,dx\,dt &= \int_0^L\int_0^T x\frac{1}{2}\frac{d}{dt}|u(x,t)|^2\,dt\,dx\\ &= \frac{1}{2}\int_0^L x|u(x,T)|^2\,dx - \frac{1}{2}\int_0^L x|u(x,0)|^2\,dx,\end{aligned} \tag{2.9}$$

$$\mathrm{Re}\,\gamma\int_0^T\int_0^L iu_x x\overline{iu}\,dx\,dt = -\frac{\gamma}{2}\int_0^T\int_0^L |u(x,t)|^2\,dx\,dt, \tag{2.10}$$

$$\operatorname{Re}\alpha\int_0^T\int_0^L u_{xx}\overline{ixu}\,dx\,dt=-\alpha\int_0^T\int_0^L \operatorname{Im}(u_x\bar{u})\,dx\,dt, \tag{2.11}$$

$$\operatorname{Re}\beta\int_0^T\int_0^L iu_{xxx}\overline{ixu}\,dx\,dt=\frac{3\beta}{2}\int_0^T\int_0^L |u_x(x,t)|^2\,dx\,dt. \tag{2.12}$$

Adding (2.9), (2.10), (2.11) and (2.12), by (1.1) we obtain:

$$\frac{1}{2}\int_0^L x|u(x,T)|^2\,dx-\frac{\gamma}{2}\int_0^T\int_0^L |u(x,t)|^2\,dx\,dt+\frac{3\beta}{2}\int_0^T\int_0^L |u_x(x,t)|^2\,dx\,dt$$
$$=\frac{1}{2}\int_0^L x|u(x,0)|^2\,dx+\alpha\int_0^T\int_0^L Im(u_x\bar{u})\,dx\,dt. \tag{2.13}$$

Hence, if γ is a negative constant, we have:

$$-\frac{\gamma}{2}\int_0^T\int_0^L |u(x,t)|^2\,dx\,dt+\frac{3\beta}{4}\int_0^T\int_0^L |u_x(x,t)|^2\,dx\,dt$$
$$\leq\frac{L}{2}\|u_0\|^2+\frac{\alpha^2}{3\beta}\int_0^T\int_0^L |u(x,t)|^2\,dx\,dt.$$

Otherwise, if γ is a positive constant, we have:

$$\frac{3\beta}{4}\int_0^T\int_0^L |u_x(x,t)|^2\,dx\,dt\leq\frac{L}{2}\|u_0\|^2+\left(\frac{\gamma}{2}+\frac{\alpha^2}{3\beta}\right)\int_0^T\int_0^L |u(x,t)|^2\,dx\,dt.$$

So, in both cases, by the density of $D(A)$ in $L^2(0,L)$ and using (2.2), we prove the item (iii) and we conclude the theorem. □

3. Exponential decay of the energy

In this subsection we shall prove the decay of the energy associated to the system (1.1). As in [13] and [21], we shall use multipliers techniques. The origin of this method can be found in Zuazua [23], see also Komornik [8]. In view of the energy dissipation law (see Theorem 2.1, (2.1)) and in order to analyze the rate of decay of solutions, as $t\to+\infty$, it is natural to study the observability problem.

Proposition 3.1 (A first observability result). *Under the assumptions of Theorem 2.1, for all $L>0$, $T>0$ and $u_0\in L^2(0,L)$, we have:*

$$\frac{1}{2}\|u_0\|^2\leq\frac{1}{2T}\int_0^T\|u(t)\|^2\,dt+\frac{\beta}{2}\int_0^T|u_x(0,t)|^2\,dt. \tag{3.1}$$

Proof. We first consider $u_0 \in D(A)$ and $T > 0$. We take the inner product of each element of the equation in (1.1) with $i(T-t)u$.

So, integrating by parts in $(0,L) \times (0,T)$ and using the boundary conditions, we obtain

$$\operatorname{Re}\int_0^T\int_0^L iu_t(T-t)\overline{iu}\,dx\,dt = \int_0^T (T-t)\frac{1}{2}\frac{d}{dt}\|u(t)\|^2\,dt$$
$$= -\frac{T}{2}\|u_0\|^2 + \frac{1}{2}\int_0^T \|u(t)\|^2\,dt, \tag{3.2}$$

$$\operatorname{Re}\gamma\int_0^T\int_0^L iu_x(T-t)\overline{iu}\,dx\,dt = \operatorname{Re}\alpha\int_0^T\int_0^L u_{xx}(T-t)\overline{iu}\,dx\,dt = 0 \tag{3.3}$$

and

$$\operatorname{Re}\beta\int_0^T\int_0^L iu_{xxx}(T-t)\overline{iu}\,dx\,dt = \beta\int_0^T (T-t)\left(\operatorname{Re}\int_0^L u_{xxx}\bar{u}\,dx\right)dt$$
$$= \frac{\beta}{2}\int_0^T (T-t)|u_x(0,t)|^2\,dt. \tag{3.4}$$

By (1.1), (3.2), (3.3) and (3.4) we have:

$$\frac{T}{2}\|u_0\|^2 = \frac{1}{2}\int_0^T \|u(t)\|^2\,dt + \frac{\beta}{2}\int_0^T (T-t)|u_x(0,t)|^2\,dt.$$

Therefore, by density of $D(A)$ in $L^2(0,L)$, we proved (3.1) and we conclude the Proposition 3.1. □

Remark 3.2. Inequality (3.1) provides the boundary observability result we need to prove the exponential decay of solutions of (1.1), up to a compact removable term $\int_0^T \|u(t)\|^2\,dt$. In order to get rid of this extra term we need to show that the following unique continuation property is fulfilled: if $u_x(0,t) = 0$ for $0 \le t \le T$, then $u \equiv 0$.

The next theorem shows that this uniqueness property may fail for a countable set of critical lengths L.

Inspired by the ideas developed in [16], [19] and [20], we have the following result:

Theorem 3.3. *Let $\mathcal{X}$ be the set of the values L of the length of interval which satisfies the following conditions:*

There exist $\lambda \in \mathbb{C}$ and a non trivial $u_0 \in H_0^2(0,L) \cap H^3(0,L)$ such that:

$$Au_0 = \lambda u_0. \tag{3.5}$$

If $\alpha^2 + 3\gamma\beta > 0$, then $\mathscr{X}$ is a countable set defined by:

$$\mathscr{X} = \left\{ L > 0 : L = 2\beta\pi\sqrt{\frac{k^2+kl+l^2}{\alpha^2+3\gamma\beta}}, k, l \in \mathbb{Z}, k, l > 0 \right\}.$$

Moreover, these conditions are necessary and sufficient.

Proof. Let u_0 be in $H_0^2(0,L) \cap H^3(0,L)$, satisfying (3.5) and define v by:

$$v(x) = \begin{cases} u_0(x) & \text{if } x \in [0,L] \\ 0 & \text{if } x \in \mathbb{R}\setminus[0,L] \end{cases}$$

It is easy to see that v belongs to $H^2(\mathbb{R})$ and satisfies the following equation in $\mathscr{D}'(\mathbb{R})$:

$$\lambda v - \gamma v' + i\alpha v'' - \beta v''' = \beta u_0''(L)\delta_L - \beta u_0''(0)\delta_0,$$

where δ_a is the Dirac measure at $x = a$.

The above problem is equivalent to the existence of complex numbers λ and $(a_1, a_2) \neq (0,0)$ and a function $v \in H^2(\mathbb{R})$ with compact support in $[0,L]$ satisfying

$$\lambda v - \gamma v' + i\alpha v'' - \beta v''' = \beta a_2\delta_L - \beta a_1\delta_0. \tag{3.6}$$

Taking Fourier Transform in (3.6) and setting $\lambda = ir$, we have

$$\hat{v}(\xi) = -i\beta\frac{a_2e^{-i\xi L} - a_1}{r - \gamma\xi - \alpha\xi^2 + \beta\xi^3}.$$

Then, by Paley-Wiener Theorem (see, Yosida [22]) and the characterization of $H^2(\mathbb{R})$ by means of the Fourier transforms, we see that (3.6) is equivalent to the existence of complex numbers r and $(a_1, a_2) \neq (0,0)$ such that the map

$$f(\xi) = \frac{a_1 - a_2e^{-i\xi L}}{r - \gamma\xi - \alpha\xi^2 + \beta\xi^3}$$

satisfies

(i) f is an entire function in $\mathbb{C}$;

(ii) there exist positive constants N, C such that $|f(\xi)| \leq C(1+|\xi|)^N e^{L|\mathrm{Im}\,\xi|}$;

(iii) $\int_{\mathbb{R}} |f(\xi)|^2(1+|\xi|^2)^2\, d\xi < +\infty$.

Since the roots of $N(\xi) = a_1 - a_2 e^{-iL\xi}$ are simple (unless $a_1 = a_2 = 0$), item (i) holds provided that the roots of $p(\xi) = r - \gamma\xi - \alpha\xi^2 + \beta\xi^3$ are simple and also roots of $N(\xi)$. Notice that if (i) holds, then (ii) and (iii) are satisfied.

Thus, we shall prove that there exist complex numbers r, μ_0 and positive integers k, l such that, if we set

$$\mu_1 = \mu_0 + \frac{2k\pi}{L} \qquad \text{and} \qquad \mu_2 = \mu_1 + \frac{2l\pi}{L}, \tag{3.7}$$

we have

$$p(\xi) = \beta\xi^3 - \alpha\xi^2 - \gamma\xi + r \equiv \beta(\xi - \mu_0)(\xi - \mu_1)(\xi - \mu_2). \tag{3.8}$$

Notice that by (3.7), we have

$$3\mu_0 + \frac{2(l+k)\pi}{L} = \mu_0 + \mu_1 + \mu_2 = \frac{\alpha}{\beta}.$$

Therefore, μ_0 is a real number. Thus, by (3.7) and (3.8), it follows $r = -\beta\mu_0\mu_1\mu_2 \in \mathbb{R}$.

Thus by (3.7) and (3.8), to prove f is an entire function in $\mathbb{C}$, we have to analyze when there exist real numbers r, μ_0 and positive integers k, l such that

$$\begin{cases} \mu_0 + \mu_1 + \mu_2 = \dfrac{\alpha}{\beta} \\ \mu_0\mu_1 + \mu_0\mu_2 + \mu_1\mu_2 = -\dfrac{\gamma}{\beta} \\ r = -\beta\mu_0\mu_1\mu_2 \end{cases}$$

From the first two equations it follows that

$$\mu_0 = -\frac{2(2k+l)\pi}{3L} + \frac{\alpha}{3\beta} \qquad \text{and} \qquad \frac{4(k^2+kl+l^2)\pi^2}{L^2} = \frac{\alpha^2 + 3\gamma\beta}{\beta^2}.$$

Thus, for f to be an entire function in $\mathbb{C}$, it is necessary and sufficient that the coefficients of the third order linear equation satisfy $\alpha^2 + 3\gamma\beta > 0$ and L is given by

$$L = 2\beta\pi\sqrt{\frac{k^2+kl+l^2}{\alpha^2+3\gamma\beta}}$$

where k and l are positive integers. □

Remark 3.4. Let $u_0 \in H_0^2(0,L) \cap H^3(0,L) \backslash \{0\}$ and $\lambda \in \mathbb{C}$ satisfying the equation (3.5). Then $u(x,t) = e^{\lambda t} u_0(x)$ solves the problem (1.1) with $u_x(0,t) = u_0'(0)e^{\lambda t} = 0$. In this case, by (2.7) it follows that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 = 0.$$

So, if L belongs to the enumerable set $\mathscr{X}$ we have non null solutions of the system (1.1) with energy constant in time.

Remark 3.5. When the constant γ is negative and $\alpha^2 + 3\gamma\beta \leq 0$, the set $\mathscr{X}$ is empty. So, in this case, we have decay of the energy associated to the system (1.1) for every length L. For further considerations, see Section 5 (Final Remarks).

Now, we can prove the following unique continuation property.

Proposition 3.6 (unique continuation). *For $T > 0$ and $L > 0$, let u be the solution of* (1.1) *satisfying $u_x(0,t) = 0$, for all t in* $(0,T)$. *Then, if L belongs to* $(0,+\infty)\backslash\mathscr{X}$, *we have $u \equiv 0$.*

Proof. For $T > 0$ and $L > 0$, let $\mathscr{U}_{T,L}$ be the vector space of the initial data $u_0 \in L^2(0,L)$ such that the solution u of (1.1) satisfies $u_x(0,t) = 0$, $\forall t \in (0,T)$.

At first, we prove that for any $T > 0$ and for any $L > 0$ the space $\mathscr{U}_{T,L}$ is finite dimensional.

In fact, if $\{w_n\}$ is a sequence in the unit ball of $\mathscr{U}_{T,L}$, we have a sequence $\{v_n\}$ of solutions of the system (1.1) with $v_n(x,0) = w_n(x)$, $x \in (0,L)$ and satisfying $(v_n)_x(0,t) = 0$, for all t in $(0,T)$ and n in $\mathbb{N}$. By Theorem (2.1), item (iii), there exists a constant C_1 such that $\|v_n\|_{L^2(0,T;H_0^1(0,L))} \leq C_1$, $n \in \mathbb{N}$.

So, $(v_n)_t = -\gamma(v_n)_x + i\alpha(v_n)_{xx} - \beta(v_n)_{xxx}$, $n \in \mathbb{N}$ is bounded in $L^2(0,T;H^{-2}(0,L))$.

Since the embedding $H_0^1(0,L) \hookrightarrow L^2(0,L)$ is compact, it follows by using classical compactness results, that v_n is relatively compact in $L^2(0,T;L^2(0,L))$.

Hence, by Proposition 3.1 (inequality (3.1)), the unit ball of $\mathscr{U}_{T,L}$ is compact and therefore it follows by Riesz Theorem that $\mathscr{U}_{T,L}$ is a finite dimensional space.

Now, to prove that if L belongs to $(0,+\infty)\backslash\mathscr{X}$ then the space $\mathscr{U}_{T,L} = \{0\}$, we use a similar method to that developed in Rosier [16] (Lemma 3.4) and the Theorem 3.3 above. □

Remark 3.7. It is important to observe that, as proved in the first part of Proposition 3.6, for any $T > 0$ and for any $L > 0$, the space $\mathscr{U}_{T,L}$ is finite dimensional. Therefore, if $L \in \mathscr{X}$, we obtain that the subspace of solutions of (1.1) satisfying $u_x(0,\cdot) \equiv 0$ is finite-dimensional. According to Theorem 3.3, this finite-dimensional subspace has the same dimension of the vector space generated by the eigenfunctions satisfying (3.5).

The next result is an important observability property associated to the system (1.1).

Proposition 3.8 (A second observability result). *Under the assumptions of Theorem 2.1 we have: for any $L \in (0, +\infty)\backslash\mathcal{X}$ and for any $T > 0$, there exists $C_2 = C_2(L, T) > 0$ such that for all u_0 in $L^2(0, L)$,*

$$\|u_0\|^2 \le C_2\beta\|u_x(0, \cdot)\|^2_{L^2(0,T)}. \tag{3.9}$$

Proof. By Proposition 3.1 (inequality (3.1)), it is suffices to prove that, for any $T > 0$,

$$\frac{1}{2}\int_0^T\int_0^L |u(x,t)|^2\,dx\,dt \le c_1\left\{\frac{\beta}{2}\int_0^T |u_x(0,t)|^2\,dt\right\} \tag{3.10}$$

for some constant $c_1 > 0$, independent of the solution u.

Suppose that (3.10) is not valid. Then, there exists a sequence of solutions u_n of (1.1) such that:

$$\lim_{n\to\infty}\frac{\int_0^T\int_0^L |u_n(x,t)|^2\,dx\,dt}{\beta\int_0^T |(u_n)_x(0,t)|^2\,dt} = +\infty. \tag{3.11}$$

Let $\lambda_n = \sqrt{\int_0^T\int_0^L |u_n(x,t)|^2\,dx\,dt}$ and $v_n(x,t) = \dfrac{u_n(x,t)}{\lambda_n}$. Clearly, $\{v_n\}$ solves the system (1.1) with initial data $v_n(x,0) = u_n(x,0)/\lambda_n$. Furthermore

$$\int_0^T\int_0^L |v_n(x,t)|^2\,dx\,dt = 1 \tag{3.12}$$

and by (3.11)

$$\lim_{n\to\infty}\left\{\beta\int_0^T |(v_n)_x(0,t)|^2\,dt\right\} = 0. \tag{3.13}$$

Using (3.12) and (3.13), it follows by (3.1) that $\{v_n(x,0)\}$ is a bounded sequence in $L^2(0,L)$. According to Theorem 2.1 item (iii), there exists $M(T,\gamma,\beta) = M > 0$ such that

$$\|v_n\|^2_{L^1(0,T;H_0^1(0,L))} \le M, \qquad \forall n \in \mathbb{N}. \tag{3.14}$$

Estimate (3.14) shows that

$$(v_n)_t = -\gamma(v_n)_x + i\alpha(v_n)_{xx} - \beta(v_n)_{xxx}, \qquad n \in \mathbb{N}$$

is bounded in $L^2\big(0,T;H^{-2}(0,L)\big)$. Since the embedding $H_0^1(0,L)\hookrightarrow L^2(0,L)$ is compact, it follows by using classical compactness results, that v_n is relatively compact in $L^2\big(0,T;L^2(0,L)\big)$. By extracting subsequences we obtain

$$v_n \rightharpoonup v \qquad \text{weakly in } L^1\big(0,T;H_0^1(0,L)\big)$$

and

$$v_n \to v \qquad \text{strongly in } L^2\big((0,L)\times(0,T)\big). \tag{3.15}$$

Here, to simplify the notation, we denote the subsequence by the same index n.

By (3.12), we have

$$\|v\|_{L^2((0,L)\times(0,T))} = 1. \tag{3.16}$$

Then by (3.13) and (3.15) we deduce

$$0 = \liminf_{n\to\infty}\left(\beta\int_0^T |(v_n)_x(0,t)|^2\,dt\right) \geq \left(\beta\int_0^T |v_x(0,t)|^2\,dt\right)$$

which guarantees that $v_x(0,t)=0$, for all $t\in(0,T)$. Then the limit v satisfies

$$\begin{aligned} &v_t + \gamma v_x - i\alpha v_{xx} + \beta v_{xxx} = 0 \qquad \text{in } (0,L)\times(0,T)\\ &v(0,t) = v(L,t) = 0 \qquad \text{for all } t\in(0,T)\\ &v_x(0,t) = v_x(L,t) = 0 \qquad \text{for all } t\in(0,T)\end{aligned}$$

Using the Proposition 3.6 we have $v\equiv 0$, which contradicts (3.16) and consequently, (3.10) holds. □

Our main theorem in this section, is a consequence of the above results.

Theorem 3.9 (A stabilization result). *If L does not belong to $\mathscr{X}$, then there exist $c>0$ and $\mu>0$ such that*

$$E(t) \leq c\|u_0\|^2 e^{-\mu t} \tag{3.17}$$

for all $t\geq 0$ and all solution of (1.1) *with $u_0\in L^2(0,L)$.*

Proof. By Proposition 3.8 (inequality (3.9)), we have:

$$E(0) = \frac{1}{2}\|u_0\|^2 \leq C\Big(\frac{\beta}{2}\int_0^T |u_x(0,t)|^2\,dt\Big). \tag{3.18}$$

The equation in (2.8) together with (3.18) produces the following inequalities:

$$\begin{aligned}(1+C)E(T) &= (1+C)\Big[E(0) - \frac{\beta}{2}\int_0^T |u_x(0,t)|^2\,dt\Big]\\ &\le CE(0) - \frac{\beta}{2}\int_0^T |u_x(0,t)|^2\,dt \le CE(0).\end{aligned}$$

Therefore,

$$E(T) \le \frac{C}{1+C}E(0).$$

So, by the semigroup property, the conclusion of Theorem 3.9 follows. □

4. Boundary exact controllability

In this section we study the boundary exact controllability problem associated to the system (1.1).

We begin considering the following problem:

Given the initial and final states (u_0, u_T) belonging to $L^2(0,L) \times L^2(0,L)$, is it possible to find a control function $h \in L^2(0,T)$ and a countable set $\mathscr{X}$ such that the solution w of the below system satisfies $w(x,T) = u_T(x)$, $\forall x \in (0,L)$, $\forall T > 0$ and $\forall L \in (0,+\infty)\backslash\mathscr{X}$?

$$\begin{aligned}&iw_t + i\gamma w_x + \alpha w_{xx} + i\beta w_{xxx} = 0 \quad \text{in } (0,L)\times(0,+\infty)\\ &w(0,t) = w(L,t) = 0 \quad \text{for all } t \ge 0\\ &w_x(L,t) = h(t) \quad \text{for all } t \ge 0\\ &w(x,0) = u_0 \quad \text{in } (0,L).\end{aligned} \tag{4.1}$$

At first, we need to show that the system (4.1) is well posed.

Theorem 4.1. *Under assumptions of the Theorem* 2.1 *and if* h *belongs to* $L^2(0,L)$, *then for any* $T > 0$, *the system* (4.1) *has a unique weak solution* w *belonging to* $X_T = C([0,T]; L^2(0,L)) \cap L^2(0,T; H_0^1(0,L))$.

For sake of completeness, we will give a sketch of the proof of the above Theorem.

Proof. Step 1: We solve the following problem:

$$\begin{aligned} &iv_t + i\gamma v_x + \alpha v_{xx} + i\beta v_{xxx} = 0 \quad \text{in } (0,L)\times(0,T)\\ &v(0,t) = v(L,t) = 0 \quad \text{for all } t \in (0,T)\\ &v_x(L,t) = h(t) \quad \text{for all } t \in (0,T)\\ &v(x,0) = 0 \quad \text{in } (0,L). \end{aligned} \tag{4.2}$$

Now, we consider h in $B_T = \{h \in C^1([0,T];\mathbb{R}) : h(0) = 0\}$ and consider $\varphi \in C^\infty[0,L]$ such that $\varphi(0) = \varphi(L) = 0$ and $\varphi'(L) = -1$.

We can write $v(x,t) = y(x,t) - h(t)\varphi(x)$, where y is a solution of the system

$$\begin{aligned} &iy_t + i\gamma y_x + \alpha y_{xx} + i\beta y_{xxx} = f(x,t) \quad \text{in } (0,L)\times(0,T)\\ &y(0,t) = y(L,t) = 0 \quad \text{for all } t \in (0,T)\\ &y_x(L,t) = 0 \quad \text{for all } t \in (0,T)\\ &y(x,0) = 0 \quad \text{in } (0,L), \end{aligned} \tag{4.3}$$

where $f(x,t) = ih'(t)\varphi(x) + h(t)\big(i\gamma\varphi'(x) + \alpha\varphi''(x) + i\beta\varphi'''\big)$.

Since f belongs to $C\big([0,T];L^2(0,L)\big)$, we have by Pazy [15] (Section 4.2), that y belongs to $C\big([0,T];D(A)\big) \cap C^1\big((0,T);L^2(0,L)\big)$ and it is the unique solution of (4.3). Hence, as h belongs to B_T, we have $v \in C\big([0,T];D(A)\big) \cap C^1\big((0,T);L^2(0,L)\big)$ and moreover v is a unique classical solution of (4.2).

Using integration by parts and boundary conditions, we deduce some a priori estimates for (4.2).

$$\|v\|_{C([0,T];L^2(0,L))} \leq \sqrt{\beta}\|h\|_{L^2(0,T)}, \tag{4.4}$$

$$\|v_x(0,\cdot)\|_{L^2(0,T)} \leq \|h\|_{L^2(0,T)} \tag{4.5}$$

and

$$\|v\|^2_{L^2(0,T;H_0^1(0,L))} \leq C_2\|h\|^2_{L^2(0,T)}, \tag{4.6}$$

where $C_2 = C_2(\gamma,\beta,L,T) > 0$.

Now, by (4.4) and (4.6), we see that the linear map $h \in B_T \to v \in X_T$ is continuous with $L^2(0,T)$-norm in B_T. So, by density of B_T in $L^2(0,T)$, the linear map may be extended in a unique way to obtain a linear and continuous map $\Gamma : L^2(0,T) \to X_T$.

Hence, for each h in $L^2(0,T)$, $v = \Gamma(h)$ is the weak solution of (4.2) in $\mathscr{D}'\big(0,T;H^{-2}(0,L)\big)$.

Step 2: We observe that the solution of the system (4.1) is defined in a unique manner by $w = u + v$, where u is the solution of (1.1) and $v = \Gamma(h)$ is the solution of (4.2). □

Theorem 4.2 (The exact boundary controllability for linear system (4.1)). *Let L be in $(0,+\infty)\backslash\mathscr{X}$. Then, for any $T > 0$ and for any $(u_0, u_T) \in L^2(0,L) \times L^2(0,L)$, there exists $h \in L^2(0,T)$ such that the solution w of* (4.1) *satisfies $w(\cdot,T) = u_T$.*

Proof. We are going to apply the HUM. We can observe that the Theorem 4.1 guarantees that the solution of the system (4.1) can be written in a unique way as sum of the solutions of systems (1.1) and (4.2). Then, without loss of generality we can consider $u_0 \equiv 0$.

Let $\phi_T \in \mathscr{D}(0,L)$ and let ϕ be the classical solution of the following problem:

$$\begin{aligned} & i\phi_t + i\gamma\phi_x - \alpha\phi_{xx} + i\beta\phi_{xxx} = 0 \qquad \text{in } (0,L)\times(0,T) \\ & \phi(0,t) = \phi(L,t) = 0 \qquad \text{for all } t \in (0,T) \\ & \phi_x(0,t) = 0 \qquad \text{for all } t \in (0,T) \\ & \phi(x,T) = \phi_T \qquad \text{in } (0,L). \end{aligned} \tag{4.7}$$

We observe that $\phi(x,t) = u(L-x, T-t)$, $(x,t) \in (0,L)\times(0,T)$, where u is a classical solution of (1.1) with $u(x,0) = \phi_T(L-x)$.

Then, by Theorem 2.1 and Proposition 3.8, there exist $C = C(L,\beta) > 0$ and $C_3 = C_3(L,T,\beta) > 0$ such that:

$$\frac{1}{C_3}\|\phi_T\| \le \|\phi_x(L,\cdot)\|_{L^2(0,T)} \le C\|\phi_T\|. \tag{4.8}$$

So, we can consider v as the classical solution of (4.2) with $v_x(L,t) = \phi_x(L,t)$, $t \in (0,T)$.

Now, multiplying the equation in (4.2) by $\overline{i\phi}$, integrating in $(0,L)\times(0,T)$, using inner product, integration by parts, boundary conditions and the equation in (4.7), we obtain:

$$\begin{aligned} & Re\int_0^L v(x,T)\overline{\phi_T(x)}\,dx - Re\int_0^T v_x(L,t)\overline{\phi_x(L,t)}\,dt \\ & \quad = Re\int_0^L v(x,T)\overline{\phi_T(x)}\,dx - \int_0^T |\phi_x(L,t)|^2\,dt = 0. \end{aligned} \tag{4.9}$$

By density of $\mathscr{D}(0,L)$ in $L^2(0,L)$, we have that (4.9) holds for $\phi_T \in L^2(0,L)$, which implies (see Theorem 2.1) that $h(\cdot) = \phi_x(L,\cdot)$ belongs to $L^2(0,T)$.

Let us be consider the linear map $\Lambda : L^2(0,L) \to L^2(0,L)$ defined by $\Lambda(\phi_T) = v(\cdot,T)$. By (4.8) and (4.9) we deduce that the map Λ is an isomorphism.

Hence, given u_T in $L^2(0,L)$ there exists a unique ϕ_T belonging to $L^2(0,L)$ such that $v(\cdot,T) = \Lambda(\phi_T) = u_T$, which concludes the proof of the Theorem. □

5. Final remarks

As we have seen previously, in Section 3, when $L \in \mathscr{X}$ the decay of solutions of (1.1) failed only because of the existence of a finite-dimensional subspace of undamped solutions. In the case $L \notin \mathscr{X}$ we show that solutions of the system decay exponentially to zero. Under these circumstances it is natural to add an extra damping term to the system (1.1) to prove the exponential decay of the energy for any length L. For instance, we can consider a damping term $a(x)u$ effectively acting on an open subset of $(0, L)$. More specifically, we can assume that $a = a(x)$ is a non-negative function belonging to $L^\infty(0, L)$ and $a \geq a_0 > 0$ a.e. in an open non-empty subset ω of $(0, L)$. The system with the damping term $a(x)u$ is also well posed in $L^2(0, L)$. This can be easily proved considering it as a perturbation of the system (1.1).

Now, we can prove that the energy associated to damped system has exponential decay, using similar methods developed in the Proposition 3.8 above and in [13] and [19]. In this case the unique continuation principle is the Holmgren's Uniqueness Theorem.

We can also observe that, when $\alpha = 0$ and $\beta = \gamma = 1$, the system (1.1) is the linear KdV system and in this case the critical enumerable set is defined by:

$$\mathscr{X} = \left\{ L > 0 : L = 2\pi\sqrt{\frac{k^2 + kl + l^2}{3}}, k, l \in \mathbb{Z}, k, l > 0 \right\},$$

as has been proved by Rosier [16].

When, $\alpha = 0$, $\beta = 1$ and $\gamma = 0$, we have again the KdV linear system without the slope u_x and in this case we can see, by Theorem 3.3, that the set $\mathscr{X}$ is empty and therefore the energy decays for all lengths L.

So, for the linear KdV system, we conclude that the term u_x has influence in the existence or not of the critical set $\mathscr{X}$, and therefore in decay of the energy and in the exact controllability.

When we consider $\gamma = 0$, $\beta > 0$ and $\alpha \neq 0$, we have the third order Schrödinger equation and the enumerable set $\mathscr{X}$ is:

$$\mathscr{X} = \left\{ L > 0 : L = 2\pi\beta\sqrt{\frac{k^2 + kl + l^2}{\alpha^2}}, k, l \in \mathbb{Z}, k, l > 0 \right\}.$$

According to Theorem 3.3 and Proposition 3.6, see also Remark 3.7, we can conclude that it is possible to obtain the exponential decay of the energy associated to system (1.1), even when the length L belongs to critical set $\mathscr{X}$, for instance, it is sufficient to consider the initial data u_0 in a subspace orthogonal to the finite dimensional subspace generated by the eigenfunctions satisfying (3.5).

In the case of the exact boundary controllability, it is also necessary to consider the final state u_T in a subspace orthogonal to the finite dimensional subspace. For further considerations see Cerpa-Crépeau [4] and Coron-Crépeau [6].

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Received August 1, 2010; revised January 12, 2011

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