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# Sums of seventh powers in the polynomial ring $\mathbb{F}_{2^m}[T]$

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**Abstract.** Let $F$ be a finite field with even characteristic and $q \geq 16$ elements. We study representations of polynomials $P \in F[T]$ as sums $P = X_1^7 + \cdots + X_s^7$.

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## 1. Introduction

Let $F$ be a finite field of characteristic $p$ with $q = p^m$ elements and let $k > 1$ be an integer. Analogues of the Waring's problem for the polynomial ring $F[T]$ have been investigated, ([20], [11], [17], [5], [18], [7], [4], [13], [9], [8], [2], [3]). Roughly speaking, Waring's problem over $F[T]$ consists of representing a polynomial $M \in F[T]$ as a sum

$$M = M_1^k + \cdots + M_s^k \tag{1.1}$$

with $M_1, \ldots, M_s \in F[T]$. Some obstructions to that may occur ([16]), leading to consider Waring's problem over the subring $\mathscr{S}(F[T], k)$ formed by the polynomials of $F[T]$ which are sums of $k$-th powers. Without degree conditions in (1.1), the problem of representing $M$ as sum (1.1) is close to the so called easy Waring's problem for $\mathbb{Z}$. In order to have a problem close to the non-easy Waring's problem, the degree conditions

$$k \deg M_i < \deg M + k \tag{1.2}$$

are required. A representation (1.1) satisfying degree conditions (1.2) is called a *strict representation* in opposition to representations without degree conditions. For the strict Waring's problem, the analogue of the classical Waring numbers

$g_{\mathbb{N}}(k)$ and $G_{\mathbb{N}}(k)$ have been defined as follows. Let $g(p^m, k)$, respectively $G(p^m, k)$, denote the least integer $s$, if it exists, such that every polynomial $M \in \mathscr{S}(F[T], k)$, respectively every polynomial $M \in \mathscr{S}(F[T], k)$ of sufficiently large degree, may be written as a sum (1.1) satisfying the degree conditions (1.2). Otherwise, $g(p^m, k)$, respectively $G(p^m, k)$ is equal to $\infty$. This notation is possible since these numbers only depend on $p^m$ and $k$. Gallardo's method for cubes ([7] and [4]) was generalized in [1] or in [10] where bounds for $g(p^m, k)$ and $G(p^m, k)$ were established when $p^m$ and $k$ satisfy some conditions. One of the conditions required in [1] is that every $a \in F$ may be written as a sum of $k$-th powers of elements of $F$. For such a field, called a *k-Waring field*, $\ell(p^m, k)$ is defined to be the least integer $s$ such that every $a \in F$ may be written as a sum of $s$ $k$-th powers of elements of $F$.

When $F$ is a $k$-Waring field satisfying one of the two conditions

(i) $p > k$,

(ii) $p^n > k = hp^\nu - 1$, for some integers $\nu > 0$ and $0 < h \leq p$, it is possible to bound the Waring's number $g(p^m, k)$, ([1]). The smallest exponent $k$ satisfying this last condition is $k = 3$, see [7], [4], [8], [9]. In the case of even characteristic, the second smallest exponent $k$ satisfying condition (ii) is $k = 7$. The case $k = 7$, $q = 2^m$ with $m \notin \{1, 2, 3\}$ is covered by ([1], Theorems 1.2 and 1.3) or by ([10], Theorem 1.4). Proposition 4.2 in [1] gives that $\mathscr{S}(F[T], k) = F[T]$. Theorem 1.4 in [10] gives that if $m \not\equiv 0 \pmod 3$, then $G(2^m, 7) \leq 28$ and that if $m \equiv 0 \pmod 3$ and $m \geq 12$, then $G(2^m, 7) \leq 35$. Theorem 1.2 in [1] joint to ([1], Proposition 3.1) gives the following bounds: If $m \not\equiv 0 \pmod 3$, then $G(2^m, 7) \leq 21$. If $m \equiv 0 \pmod 3$ and $m \geq 12$, then $G(2^m, 7) \leq 27$; $G(2^9, 7) \leq 34$, $G(64, 7) \leq 41$. For almost all $q = 2^m$, these bounds are comparable with the bound $G_{\mathbb{N}}(7) \leq 33$, known for the corresponding Waring's number for the integers ([19]). The case of the numbers $g(2^m, 7)$ is different. In the case when $m \notin \{1, 2, 3\}$ ([1], Theorem 1.3) as well as ([10], Theorem 1.4) gives $g(2^m, 7) \leq 239\ell(2^m, 7)$, when, for the integers, it is known that $g_{\mathbb{N}}(7) = 143$, ([6]).

In this article we obtain better bounds for the numbers $g(2^m, 7)$ in the case when $m \notin \{1, 2, 3\}$. The method gives also better bounds for some numbers $G(2^m, 7)$. Results obtained in the cases $m = 2$ or $m = 3$ will appear in separate papers. The main results proved in this article are summarized by the two following theorems.

**Theorem 1.1.** (i) *If* $q \not\equiv 1 \pmod 7$ *and* $q \geq 16$, *then* $G(q, 7) \leq 21$.
(ii) *If* $q \equiv 1 \pmod 7$ *and* $q > 14^{16}$, *then* $G(q, 7) \leq 22$.
(iii) *If* $q \equiv 1 \pmod 7$ *and* $512 \leq q < 14^{16}$, *then* $G(q, 7) \leq 27$.
(iv) *If* $q = 64$, *then* $G(q, 7) \leq 31$.

This theorem is a consequence of Corollary 4.2 below. Corollary 4.2 also gives that $G(q,7) \le 21$ when $q \ge R(7)$, a non-effective constant expressed in [18] by means of Ramsey numbers.

**Theorem 1.2.** (i) *If* $q \not\equiv 1 \pmod 7$ *and* $q \equiv 1 \pmod 3$, *then* $g(q,7) \le 42$.
(ii) *If* $q \not\equiv 1 \pmod 7$ *and* $q \not\equiv 1 \pmod 3$, *then* $g(q,7) \le 46$.
(iii) *If* $q \equiv 1 \pmod{21}$ *and* $q \ne 64$, *then* $g(q,7) \le 57$.
(iv) *If* $q \equiv 1 \pmod 7$ *and* $q \not\equiv 1 \pmod 3$, *then* $g(q,7) \le 59$.
(v) $g(64,7) \le 74$.

This theorem is a consequence of Corollary 4.6 and Theorem 5.7 below.

Proving that polynomials of small degree are sums or strict sums of seventh powers require some results on the solvability of systems of algebraic equations over the finite field $F$. This is done at Section 2. In this section we also compute the exact values of the numbers $\ell(q,7)$. Representations of polynomials of small degree is done in Section 3. In Section 4 we use a descent process described in [1] and [10] and we obtain an upper bound for $G(q,7)$. In Section 5 we describe an other descent process from which we deduce a bound for $g(q,7)$. We use two types of numbering. Pairs (X.Y) will be used to number formulae occuring in definitions, propositions and theorems, single numbers (z) will be used for formulae only used in the course of a proof.

We fix an algebraic closure $E$ of the field $F$ and we denote by $\mathbb{F}_{2^\nu}$ the subfield of $E$ with $2^\nu$ elements. We shall suppose that the field $F$ has $q \ge 16$ elements.

## 2. Algebraic equations and identities

**Proposition 2.1.** *We have*

(i) $\ell(q,7) = 1$ *if* $q \not\equiv 1 \pmod 7$,

(ii) $\ell(q,7) = 2$ *if* $q \equiv 1 \pmod 7$ *and* $q \ge 512$,

(iii) $\ell(64,7) = 3$.

*Proof.* The first part is obvious. If $q \equiv 1 \pmod 7$, then $\ell(q,7) > 1$. From [14], $\ell(q,7) \le 2$ if $q > 512$, so that $\ell(q,7) = 2$ if $q > 512$. It remains to prove that $\ell(512,7) = 2$ and $\ell(64,7) = 3$. Let $\omega$ be such that $\mathbb{F}_{64} = \mathbb{F}_2(\omega)$ with $\omega^6 = \omega + 1$. Then $\omega$ is primitive. Since 7 and 9 are coprime, the multiplicative group of $\mathbb{F}_{64}$ is the direct product of the group formed by the 9-th powers and the group formed by the 7-th powers. Let $a \in \mathbb{F}_{64}$ be $\ne 0$. Then $a$ may be written as a product

$$a = (\omega^i)^9 \times (\omega^j)^7, \qquad \text{with } 0 \le i \le 6,\ 0 \le j \le 8.$$

We have

$$\omega^9 = \omega^4 + \omega^3,$$
$$(\omega^3)^9 = \omega^3 + \omega^2 + \omega,$$
$$(\omega^5)^9 = \omega^4 + \omega^3 + 1,$$

so that

$$(\omega^3)^9 = \omega^{14} + \omega^{49},$$
$$(\omega^5)^9 = \omega^7 + \omega^{56},$$

and

$$\omega^9 = 1 + \omega^7 + \omega^{56}.$$

Thus, for every $i = 0, \dots, 6$, $(\omega^i)^9$ is a sum of at most 3 seventh powers, so that, every $a \in \mathbb{F}_{64}$ is a sum at most 3 seventh powers. We conclude after observing that $\omega^9$ is not a sum of 2 seventh powers.

Let $\gamma$ be such that $\mathbb{F}_{512} = \mathbb{F}_2(\gamma)$ with $\gamma^9 = \gamma^4 + 1$. Then $\gamma$ is primitive. Since 7 and 73 are coprime, the multiplicative group of $\mathbb{F}_{512}$ is the direct product of the group formed by the 73-th powers and the group formed by the seventh powers. We have

$$\gamma^{73} = (\gamma^2)^7 + (\gamma^{69})^7,$$
$$(\gamma^3)^{73} = (\gamma^{66})^7 + (\gamma^{70})^7,$$

so that

$$(\gamma^5)^{73} = ((\gamma^3)^{73})^4.$$

Thus, for $1 \le i \le 6$, $(\gamma^i)^{73}$ is a sum of 2 seventh powers. Let $a \in \mathbb{F}_{512}$ be $\neq 0$. Then $a$ may be written as a product

$$a = (\gamma^i)^{73} \times (\gamma^j)^7, \qquad \text{with } 0 \le i \le 6,\ 0 \le j \le 72.$$

Thus, $a$ is either a seventh power, or a sum of 2 seventh powers. □

**Proposition 2.2.** *Let $\beta \in \mathbb{F}_8$ be such that $\beta^3 = \beta + 1$. Then*

$$T = \sum_{i=0}^{6} \beta^i (T + \beta^i)^7. \tag{2.1}$$

*Proof.* A verification. □

The parameter $v(q,k)$ was defined in [1] as the least integer $s$ such that $T$ is a strict sum of $s$ $k$-th powers. From ([1], Proposition 4.2-(i)),

$$v(q,7) \leq 7/\gcd(q-1,7) + \ell(q,7)\big(7 - 7/\gcd(q-1,7)\big). \tag{2.2}$$

In some cases, (2.2) may be improved.

**Proposition 2.3.** (i) *We have*

$$v(64,7) \leq 16. \tag{2.3}$$

(ii) *For $r$ a positive integer, we have*

$$v(2^{21r},7) \leq 7. \tag{2.4}$$

(iii) *There exists a constant $R(7)$ such that $v(q,7) \leq 7$ whenever $q \geq R(7)$.*
(iv) *If $q \geq 7^{16}$, then $v(q,7) \leq 8$.*

*Proof.* Suppose $\mathbb{F}_8 \subset F$.

(i) Let $\omega$ be such that $\mathbb{F}_{64} = \mathbb{F}_2(\omega)$ with $\omega^6 = \omega + 1$. Let $\beta = \omega^{27}$, so that $\beta^3 = \beta + 1$. We have $\beta = \omega^{14} + \omega^{49}$. Thus, $\beta$, $\beta^2$ and $\beta^4$ are sums of two seventh powers. Moreover, $\beta^3 = \beta + 1 = \omega^{14} + \omega^{49} + 1$, so that $\beta^3$, $\beta^6$ and $\beta^5 = \beta^{12}$ are sums of 3 seventh powers. By (2.1), $T$ is a strict sum of 16 seventh powers.

(ii) From ([12], Theorem 3.75, p. 117), the polynomial $T^7 + \beta$ is irreducible in $\mathbb{F}_8[T]$; it splits in linear factors over the field $\mathbb{F}_{8^7}$. Thus, each $\beta^i$ is a seventh power in the field $\mathbb{F}_{2^{21}}$. By (2.1), $T$ is a strict sum of 7 seventh powers in each polynomial ring $\mathbb{F}_{2^{21r}}[T]$.

(iii) See ([18], Theorem 4).

(iv) See ([13], Theorem 1). □

**Proposition 2.4.** *For every $(a,b) \in F^2$, the system*

$$\begin{cases} a = x_1 + x_2 + x_3, \\ b = x_1^3 + x_2^3 + x_3^3. \end{cases} \tag{$\mathscr{T}(a,b)$}$$

*has a solution $(x_1,x_2,x_3) \in F^3$ such that $(x_1,x_2,x_3) \neq (0,0,0)$.*

*Proof.* From [15], $\big(\mathscr{T}(a,b)\big)$ has a solution $(x_1,\dots,x_3) \in F^3$. If $(a,b) \neq 0$, such a solution is $\neq (0,0,0)$. If $(a,b) = (0,0)$, then, $(1,1,0)$ is solution of $\big(\mathscr{T}(a,b)\big)$. □

## 3. Strict sums of small degree

From ([1], Proposition 4.2-(ii)), there exists a positive integer $s$ such that for each $\mathbf{a} = (a_1, \dots, a_7) \in F^7$, there exists $(x_1, y_1, \dots, x_s, y_s) \in F^{2s}$ such that

$$\deg\Big(\sum_{i=0}^{7} a_i T^i - \sum_{r=1}^{s} (x_r T + y_r)^7\Big) \leq 0. \tag{3.1}$$

Let $\sigma = \sigma(q)$ denote the least integer $s$ with this property. From ([1], Proposition 4.2-(ii)),

$$\sigma(q) \leq 7\ell(q, 7). \tag{3.2}$$

The same proposition gives that every polynomial $P \in F[T]$ such that $7(n-1) < \deg P \leq 7n$ is a strict sum of $(7n+1)\ell(q, 7)$ seventh powers.

This last bound, obtained by induction, increases the effect of the number $\ell = \ell(q, 7)$. In what follows, we try to reduce this effect. For that, we prove two propositions.

**Proposition 3.1.** *Let $P \in F[T]$ with $\deg P \leq 21$. Then there are polynomials $Q_1, \dots, Q_{4+\ell} \in F[T]$ with degree $\leq 3$ such that*

$$\deg\Big(P + \sum_{i=1}^{4+\ell} (Q_i)^7\Big) \leq 14. \tag{3.3}$$

*Proof.* Let

$$P = \sum_{i=0}^{21} a_i T^i. \tag{1}$$

We prove that we can solve (3.3) with $Q_1$, $Q_2$, $Q_3$, $Q_4$ of degree 3 and $Q_1$ and $Q_2$ monic polynomials. We note that for $(u, x, y) \in F^2$,

$$\begin{aligned}\deg\big((uT^3 + xT^2 + yT + z)^7 &+ u^7T^{21} + u^6xT^{20} + (u^6y + u^5x^2)T^{19} \\ &+ (u^6z + u^4x^3)T^{18} + (u^5y^2 + u^4x^2y + u^4x^4)T^{17} \\ &+ \big(u^4(x^2z + xy^2) + u^2x^5\big)T^{16} + (u^5z^2 + u^4y^3 + u^2x^4y)T^{15}\big) \leq 14. \end{aligned} \tag{2}$$

Let $x_1 \neq 0$ and let

$$x_2 = a_{20} + x_1. \tag{3}$$

Let $z_1$ be defined by

$$a_{16} = x_1^5 + x_2^5 + x_1^2 z_1 \tag{4}$$

and let $z_2 = 0$. Set

$$b_{19} = a_{19} + a_{20}^2, \tag{5}$$
$$b_{18} = a_{18} + x_1^3 + x_2^3 + z_1 + z_2, \tag{6}$$
$$b_{17} = a_{17} + a_{20}^4, \tag{7}$$
$$b_{15} = a_{15} + x_1^6 + x_2^6 + z_1^2 + z_2^2. \tag{8}$$

Since $F \neq \mathbb{F}_8$, there exists $(u_3, u_4) \in F^2$ such that $u_3 u_4 \neq 0$ and $u_3^7 \neq u_4^7$. Then the matrix

$$\begin{pmatrix} u_3^{12} & u_4^{12} \\ u_3^5 & u_4^5 \end{pmatrix}$$

is invertible. Let $(\eta_3, \eta_4) \in F^2$ be defined by

$$b_{19}^2 = u_3^{12}\eta_3 + u_4^{12}\eta_4,$$
$$b_{17} = u_3^5\eta_3 + u_4^5\eta_4$$

and let $(y_3, y_4)$ be defined by

$$\eta_3 = y_3^2, \qquad \eta_4 = y_4^2.$$

Then, from (3) and (5),

$$a_{19} = x_1^2 + x_2^2 + u_3^6 y_3 + u_4^6 y_4, \tag{9}$$

and from (3) and (7),

$$a_{17} = x_1^4 + x_2^4 + u_3^5 y_3^2 + u_4^5 y_4^2. \tag{10}$$

Let $(\zeta_3, \zeta_4) \in F^2$ be defined by

$$b_{18}^2 = u_3^{12}\zeta_3 + u_4^{12}\zeta_4,$$
$$b_{15} + u_3^4 y_3^3 + u_4^4 y_4^3 = u_3^5\zeta_3 + u_4^5\zeta_4$$

and let $(z_3, z_4)$ be defined by

$$\zeta_3 = z_3^2, \qquad \zeta_4 = z_4^2.$$

Then, from (3) and (6),

$$a_{18} = x_1^3 + x_2^3 + z_1 + z_2 + u_3^6 z_3 + u_4^6 z_4, \tag{11}$$

and from (3) and (8),

$$a_{15} = x_1^6 + x_2^6 + z_1^2 + z_2^2 + u_3^4 y_3^3 + u_4^4 y_4^3 + u_3^5 z_3^2 + u_4^5 z_4^2. \tag{12}$$

By $(1), \ldots, (11)$ and (12),

$$\begin{aligned}(T^3 + x_1 T^2 + z_1)^7 &+ (T^3 + x_2 T^2 + z_2)^7 \\ &+ (u_3 T^3 + y_3 T + z_3)^7 + (u_4 T^3 + y_4 T + z_4)^7 \\ &= P + (a_{21} + u_3^7 + u_4^7) T^{21} + R,\end{aligned}$$

with $\deg R \leq 14$. In the field $F$, $a_{21} + u_3^7 + u_4^7$ is a sum of $\ell = \ell(q, 7)$ seventh powers, say,

$$a_{21} + u_3^7 + u_4^7 = u_5^7 + \cdots + u_{4+\ell}^7,$$

so that

$$\deg\Big(P + (T^3 + x_1 T^2 + z_1)^7 + (T^3 + x_2 T^2 + z_2)^7 + (u_3 T^3 + y_3 T + z_3)^7 + (u_4 T^3 + y_4 T + z_4)^7 + \sum_{i=5}^{4+\ell} (u_i T^3)^7\Big) \leq 14. \qquad \square$$

**Proposition 3.2.** *Let $P \in F[T]$ with $\deg P \leq 14$. Then there are polynomials $Q_1, \ldots, Q_{3+4\ell} \in F[T]$ with degree $\leq 2$ such that*

$$\deg\Big(P + \sum_{i=1}^{3+4\ell} (Q_i)^7\Big) \leq 7. \tag{3.4}$$

*Proof.* We note that for $(x, y) \in F^2$,

$$\deg\big((T^2 + yT + z)^7 + T^{14} + yT^{13} + (y^2 + z)T^{12} + y^3 T^{11} + (y^4 + y^2 z + z^2)T^{10} + (y^5 + yz^2)T^9 + (y^6 + y4z + z^3)T^8\big) \leq 7. \tag{1}$$

Let

$$P = \sum_{i=0}^{14} a_i T^i. \tag{2}$$

From Proposition 2.4, there exists $(y_1, y_2, y_3) \in F^3$ such that

$$a_{13} = y_1 + y_2 + y_3,$$

$$a_{11} = y_1^3 + y_2^3 + y_3^3,$$

and such that $y_1 \neq 0$. Let $z_1 \in F$ be defined by the condition

$$y_1 z_1^2 = a_9 + y_1^5 + y_2^5 + y_3^5$$

and let

$$z_2 = z_3 = 0.$$

From (1),

$$\deg\Bigl(P + \sum_{i=0}^{3} (T^2 + y_i T + z_i)^7 + b_{14}T^{14} + b_{12}T^{11} + b_{10}T^{10} + b_8 T^8\Bigr) \leq 7, \quad (3)$$

with

$$b_{14} = a_{14} + 1,$$

$$b_{12} = a_{12} + \sum_{i=1}^{3} (y_i^2 + z_i),$$

$$b_{10} = a_{10} + \sum_{i=1}^{3} (y_i^4 + y_i^2 z_i + z_i^2),$$

$$b_8 = a_8 + \sum_{i=1}^{3} (y_i^6 + y_i^4 z_i + z_i^3).$$

If $(b_{14}, b_{12}, b_{10}, b_8) = (0,0,0,0)$, then $\deg\bigl(P + \sum_{i=1}^{3} (T^2 + y_i T + z_i)^7\bigr) \leq 7$. We suppose $(b_{14}, b_{12}, b_{10}, b_8) \neq (0,0,0,0)$. Let $\zeta_1$, $\zeta_2$, $\zeta_3$, $\zeta_4$ be distinct elements in $F$. Then the Van der Monde matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 \\ \zeta_1^2 & \zeta_2^2 & \zeta_3^2 & \zeta_4^2 \\ \zeta_1^3 & \zeta_3^2 & \zeta_3^3 & \zeta_4^3 \end{pmatrix}$$

is invertible. Let $(\eta_1, \eta_2, \eta_3, \eta_4) \in F^4$ be defined by:

$$\begin{aligned} b_{14} &= \eta_1 + \eta_2 + \eta_3 + \eta_4, \\ b_{12} &= \zeta_1\eta_1 + \zeta_2\eta_2 + \zeta_3\eta_3 + \zeta_4\eta_4, \\ b_{10} &= \zeta_1^2\eta_1 + \zeta_2^2\eta_2 + \zeta_3^2\eta_3 + \zeta_4^2\eta_4, \\ b_8 &= \zeta_1^3\eta_1 + \zeta_2^2\eta_2 + \zeta_3^3\eta_3 + \zeta_4^3\eta_4. \end{aligned}$$

Then, $(\eta_1, \eta_2, \eta_3, \eta_4) \neq (0,0,0,0)$. Suppose that $\eta_{i_1}, \dots, \eta_{i_r}$ are $\neq 0$ and that $\eta_i = 0$ if $i \notin \{i_1, \dots, i_r\}$. Each non-zero $\eta_{i_j}$ is a sum of $\ell$ seventh powers. For $j = 1, \dots, r$, there are non-zero elements $x_{j,1}, \dots, x_{j,v_j}$ in $F$ with $v_j \leq \ell$ such that,

$$\eta_{i_j} = \sum_{n=1}^{v_j} (x_{j,n})^7,$$

so that

$$\begin{cases} b_{14} = \sum_{j=1}^{r} \sum_{n=1}^{v_j} (x_{j,n})^7, \\ b_{12} = \sum_{j=1}^{r} \sum_{n=1}^{v_j} (x_{j,n})^6 \left(\frac{\zeta_{i_j}}{x_{j,n}}\right), \\ b_{10} = \sum_{j=1}^{r} \sum_{n=1}^{v_j} (x_{j,n})^5 \left(\frac{\zeta_{i_j}}{x_{j,n}}\right)^2, \\ b_8 = \sum_{j=1}^{r} \sum_{n=1}^{v_j} (x_{j,n})^4 \left(\frac{\zeta_{i_j}}{x_{j,n}}\right)^3. \end{cases} \tag{4}$$

From (3) and (4),

$$\deg\left(P + \sum_{i=1}^{3} (T^2 + y_i T + z_i)^7 + \sum_{j=1}^{r} \sum_{n=1}^{v_j} \left(x_{j,n}T^2 + \frac{\zeta_{i_j}}{x_{j,n}}\right)^7\right) \leq 7.$$

Let $s = 3 + v_1 + \cdots + v_r$. We have proved the existence of polynomials $Q_1, \dots, Q_s$ with $\deg Q_i \leq 2$ such that

$$\deg(P + Q_1^7 + \cdots + Q_s^7) \leq 7.$$

We conclude by observing that $s \leq 3 + 4\ell$. □

**Proposition 3.3.** (i) *Every $P \in F[T]$ with degree $\leq 7$ is a strict sum of $8\ell(q,7)$ seventh powers.*

(ii) *Every $P \in F[T]$ with degree $\leq 14$ is a strict sum of $12\ell(q,7) + 3$ seventh powers.*

(iii) *Every* $P \in F[T]$ *with degree* $\leq 21$ *is a strict sum of* $13\ell(q,7)+7$ *seventh powers.*

*Proof.* (i) Let $\sigma = \sigma(q)$. Let $P \in F[T]$ with degree $\leq 7$. There exist polynomials $Q_1, \dots, Q_\sigma$ with degree $\leq 1$ such that $P + \sum_{i=1}^{\sigma} (Q_i)^7$ is a constant, thus, a sum of $\ell$ seventh powers in the field $F$. Therefore, $P$ is a strict sum of $\sigma + \ell$ seventh powers. We conclude with (3.2). We obtain (ii), then (iii), using Proposition 3.2, then Proposition 3.1. □

## 4. The old descent

A general process which works when $T$ is sum of $k$-th powers has been discribed in [1] as well as in [10]. Taking $k = 7$ in ([10], Theorem 1.4-(ii), (iii)) gives that every polynomial $A \in F[T]$ of degree $\geq 2346$ is a strict sum of $7\ell(q,7)+15$ seventh powers and that every polynomial $A \in F[T]$ of degree $\geq 239$ is a strict sum of $7\ell(q,7)+42$ seventh powers. Using the process discribed in [1], we prove the following theorem.

**Theorem 4.1.** *Every polynomial* $A \in F[T]$ *of degree* $\geq 29411$ *is a strict sum of* $13 + \max\big(\ell(q,7)-1, 1\big) + v(q,7)$ *seventh powers.*

*Proof.* Let $\gamma = \gamma(q,7) = \max\big(\ell(q,7)-1, 1\big)$. Let $A \in F[T]$ with $7(n-1) < \deg A \leq 7n$. From ([1], Lemma 5.1), there exist $B_1, \dots, B_\gamma, H \in F[T]$ such that

$$A = B_1^7 + \cdots + B_\gamma^7 + H, \tag{4.1}$$

$$\deg B_i \leq n \quad \text{for } i = 1, \dots, \gamma, \qquad \deg H = 7n, \tag{4.2}$$

the leading coefficient of $H$ being a seventh power.

From ([1], Lemma 5.2-(ii)), there is a sequence $H_0, H_1, \dots, H_i, \dots,$ of polynomials of $F[T]$ of degree $7n_0, 7n_1, \dots, 7n_i, \dots,$ and a sequence $X_0, X_1, \dots, X_i, \dots,$ of polynomials of degree $n_0, n_1, \dots, n_i, \dots,$ such that $H = H_0$ and such that for each index $i$,

$$H_i = X_i^7 + H_{i+1}, \tag{4.3}$$

$$6n_i \leq 7n_{i+1} < 6n_i + 7. \tag{4.4}$$

Moreover, for each index $i$ there is a polynomial $Y_i \in F[T]$ of degree $n_i$ such that

$$\deg(H_i + Y_i^7) < 6n_i. \tag{4.5}$$

Let $r$, if it exists, be the smallest index such that $6n_r - 1 \leq n$. Using identity (4.3) for $i = 0, \ldots, r-1$, then, using identity (4.5) one time with $i = r$, we get

$$H = X_0^7 + \cdots + X_{r-1}^7 + Y_r^7 + R, \tag{4.6}$$

with $\deg R \leq n$. Now, with $v = v(q, 7)$, there exist $R_1, \ldots R_v \in F[T]$, of degree $\leq \deg R$ such that

$$R = R_1^7 + \cdots + R_v^7,$$

so that

$$H = X_1^7 + \cdots + X_r^7 + Y_r^7 + R_1^7 + \cdots + R_v^7, \tag{4.7}$$

with $\deg X_i = n_i \leq n_0 = n$, $\deg Y_r = n_r \leq n_0 = n$, $\deg R_j \leq \deg R \leq n$. Thus, (4.7) is a strict sum of $r + 1 + v$ seventh powers. From (4.2), we get that

$$7^i n_i \leq 6^i n + 6^i + \sum_{j=1}^{i-1} 7^j 6^{i-j}.$$

Thus, for each $r$,

$$6n_r - 1 \leq 6\left(\frac{6}{7}\right)^r n + 35 - 36\left(\frac{6}{7}\right)^r. \tag{4.8}$$

For $r \geq 12$, we have $\left(\frac{6}{7}\right)^r < \frac{1}{6}$. Suppose $r = 12$. If $n \geq 421$, then

$$6\left(\frac{6}{7}\right)^{12} n + 35 - 36\left(\frac{6}{7}\right)^{12} \leq n. \qquad \square$$

**Corollary 4.2.** (i) *If $q$ satisfies one of the conditions*

(1) $q \not\equiv 1 \pmod 7$,

(2) $q = 2^{21r}$ *with $r$ a positive integer,*

(3) $q \geq R(7)$,

*then* $G(q, 7) \leq 21$.

(ii) *If* $q \equiv 1 \pmod 7$ *and* $q > 14^{16}$, *then* $G(q, 7) \leq 22$.

(iii) *If* $q \equiv 1 \pmod 7$ *and* $512 \leq q < 14^{16}$, *then* $G(q, 7) \leq 27$.

(iv) *If* $q = 64$, *then* $G(q, 7) \leq 31$.

*Proof.* We have

$$G(q, 7) \leq 13 + \max\big(\ell(q, 7) - 1, 1\big) + v(q, 7).$$

For $\ell(q,7) \le 2$, we have

$$G(q,7) \le 14 + v(q,7).$$

From (2.2) and (2.4), if $\gcd(q-1,7) = 1$ or if $q$ is a power of $2^{21}$, then $v(q,7) \le 7$; from Proposition 2.3, if $q \ge R(7)$, then $v(q,7) \le 7$ and if $q > 14^{16}$, then $v(q,7) \le 7$. On the other hand, if $q$ is a power of $2^{21}$, or if $q \ge 14^{16}$, then $\ell(q,7) \le 2$. This gives (i) and (ii). If $q \equiv 1 \pmod 7$ and $q \ge 512$, from Proposition 2.1 and (2.2), $v(q,7) \le 13, \ell(q,7) = 2$, so that $G(q,7) \le 27$. If $q = 64$, from (2.3) and Proposition 2.1, $G(64,7) \le 31$. □

**Remark.** Adding polynomials in the descent process allows one to get strict representations of polynomials of lower degree. For instance, see ([1], Proposition 5.3), it is possible to prove

**Proposition 4.3.** (i) *Every polynomial* $P \in F[T]$ *with degree* $\ge 246$ *is a strict sum of* $21 + \max(\ell(q,7) - 1, 1) + v(q,7)$ *seventh powers.*

(ii) *Every polynomial* $P \in F[T]$ *with degree* $\ge 239$ *is a strict sum of* $38 + \max(\ell(q,7) - 1, 1) + v(q,7)$ *seventh powers.*

In order to get a bound for $g(q,7)$ we want to write every polynomial of degree $\le 238$ as a strict sum of seventh powers. For that we use

**Proposition 4.4.** *Let* $P \in F[T]$ *such that* $7(n-1) < \deg P \le 7n$ *with* $n \ge 3$. *Then* $P$ *is a strict sum of* $(4 + \ell(q,7))n + 10\ell(q,7) - 5$ *seventh powers*

*Proof.* Let $P \in F[T]$ such that $7(n-1) < \deg P \le 7n$ with $n \ge 3$. If $n = 3$, there is nothing to prove. Suppose $n > 3$. By euclidean division,

$$P = T^{7(n-3)}Q + R, \qquad \text{with } \deg Q \le 21,\ \deg R < 7(n-3).$$

Proposition 3.1 gives the existence of polynomials $Q_{1,1}, \dots, Q_{1,4+\ell}$ of degree $\le 3$ such that

$$\deg\Big(Q + \sum_{i=1}^{4+\ell} (Q_{1,i})^7\Big) \le 14,$$

so that

$$P = \sum_{i=1}^{4+\ell} (T^{n-3}Q_{1,i})^7 + P_1,$$

with $\deg P_1 \le 7(n-1)$. By induction, we get the existence of polynomials $Q_{r,1}, \dots, Q_{r,4+\ell}$ of degree $\le 3$, $1 \le r \le n-2$, such that

$$P = \sum_{r=1}^{n-2} \sum_{i=1}^{4+\ell} (T^{n-2-r} Q_{r,i})^7 + P_{n-2}$$

with $\deg P_{n-2} \le 14$. Propositions 3.1 and 3.3 give the existence of polynomials $Q_1, \dots, Q_\tau$ of degree $\le 2$ such that

$$P_{n-2} = \sum_{i=1}^{\tau} Q_i^7,$$

with $\tau = 3 + 12\ell$. Thus $P$ is a strict sum of $(4+\ell)(n-2)+\tau$ seventh powers. □

**Theorem 4.5.** *We have*

$$g(q,7) \le 44\ell(q,7) + 131.$$

*Proof.* From the above proposition, every polynomial of degree $\le 238$ is a strict sum of $44\ell + 131$ seventh powers. From Proposition 4.3, every polynomial of degree $\ge 239$ is a strict sum of $38 + \max(\ell - 1, 1) + v(q,7)$ seventh powers. Thus, $g(q,7) \le \max\big(44\ell + 131, 38 + \max(\ell-1,1) + v(q,7)\big) = 44\ell + 131$. □

This theorem improves the bounds deduced from ([1], Theorem 7.1) or from ([10], Theorem 1.4-(iv)). In particular we have the

**Corollary 4.6.** (i) *If* $q \not\equiv 1 \pmod 7$ *and* $q \ge 16$, *then* $g(q,7) \le 175$.
(ii) *If* $q \equiv 1 \pmod 7$ *and* $q \ge 512$, *then* $g(q,7) \le 219$.
(iii) *If* $q = 64$, *then* $g(q,7) \le 263$.

## 5. The new descent

The process described in this section yields improvements in the treatment of the numbers $g(q,7)$. It is known from ([1], Proposition 4.2-(ii)), that every $P \in F[T]$ of degree 4 is a strict sum of $8\ell(q,7)$ seventh powers. We prove the existence of linear polynomials of degree 4 which are strict sum of 5 seventh powers.

**Proposition 5.1.** *If* $F \ne \mathbb{F}_2$, *there exists* $(x,y,z) \in F^3$ *such that*

$$(T+x)^7 + (T+y)^7 + (T+x+y)^7 + T^7 + z^7 = T^4 + u(x,y)T^2 + T, \quad (5.1)$$

*with* $u(x,y) \in F$. *Moreover, if* $\alpha \in \mathbb{F}_4$ *is such that* $\alpha^2 = \alpha + 1$, *then*

$$T^4 + T = (T+1)^7 + (T+\alpha)^7 + (T+\alpha+1)^7 + T^7, \quad (5.2)$$

*and if* $\beta \in \mathbb{F}_8$ *is such that* $\beta^3 = \beta + 1$, *then*

$$T^4 + T^2 + T = (T + \beta)^7 + (T + \beta^2)^7 + (T + \beta^2 + \beta)^7 + T^7 + 1. \quad (5.3)$$

*Proof.* A simple verification gives (5.2) and (5.3). Therefore, if $q$ is a power of 4, then (5.1) is true with $(x, y, z) = (1, \alpha, 0)$ and $u(x, y) = 0$ and if $q$ is a power of 8, then (5.1) is true with $(x, y, z) = (\beta, \beta^2, 1)$ and $u(x, y) = 1$. We suppose that $q$ is neither a power of 4, nor a power of 8, so that every $b \in F$ is a third power and a seventh power. Let $\xi \in F$ be $\neq 0$ and such that $\mathrm{tr}(\xi) = 0$. Then there is $\lambda \in F$ such that $\lambda^2 + \lambda + \xi = 0$. Let $x \in F$ be such that $1/\xi = x^3$ and let $z \in F$ be such that $x^7\big(1 + \lambda^7 + (1 + \lambda)^7\big) = z^7$. We have

$$x^3 + (\lambda x)^3 + (x + \lambda x)^3 = x^3(\lambda + \lambda^2) = 1,$$

so that (5.1) is true with $y = \lambda x$ and $u(x, y) = x^5(\lambda + \lambda^4)$. □

In what follows, we fix $(x, y, z) \in F^3$ satisfying (5.1) and we set $u = u(x, y)$. Moreover, if $q$ is a power of 4, we choose $(x, y, z) = (1, \alpha, 0)$ and $u(x, y) = 0$.

**Proposition 5.2.** *For $i$ a non-negative integer and $X \in F[T]$, let*

$$L_i(X) = X^4T^{3i} + uX^2T^{5i} + XT^{6i}. \quad (5.4)$$

*Then the map $L_i$ is $\mathbb{F}_2$-linear and we have*

$$L_i(X) = \sum_{r=1}^{\rho(q)} (X + b_r T^i)^7, \quad (5.5)$$

*with* $b_1, \ldots, b_{\rho(q)} \in F$

$$\rho(q) = \begin{cases} 4 & \text{if } q \text{ is a power of } 4, \\ 5 & \text{otherwise.} \end{cases} \quad (5.6)$$

*Proof.* Immediate. □

We shall make use of the following corollary.

**Corollary 5.3.** *Let $n$ be a non-negative integer and let $a \in F$. Then we have*

$$a^4T^{4n} = L_0(aT^n) + ua^2T^{2n} + aT^n, \quad (5.7)$$

$$a^4T^{4n+3} = L_1(aT^n) + ua^2T^{2n+5} + aT^{n+6}. \quad (5.8)$$

*If* $n > 0$, *then*

$$a^4T^{4n+2} = L_2(aT^{n-1}) + ua^2T^{2n+8} + aT^{n+11}. \tag{5.9}$$

*If* $n > 1$, *then*

$$a^4T^{4n+1} = L_3(aT^{n-2}) + ua^2T^{2n+11} + aT^{n+16}. \tag{5.10}$$

*Proof.* (5.7) and (5.8) are immediate. We get (5.9) and (5.10) noting that $T^{4n+2} = T^{4(n-1)+6}$ and that $T^{4n+1} = T^{4(n-2)+9}$. □

Roughly speaking, the new descent process uses the following idea. Let $X = x_NT^N + x_{N-1}T^{N-1} + \cdots + x_1T + x_0$ be a polynomial of $F[T]$. Making use of (5.7), ... (5.10), we replace a monomial $x_kT^k$ by the sum of an appropriate $L_i(T^j)$ and a monomial of lower degree. We begin with the monomial $x_NT^N$, then, following decreasing degrees, we replace each monomial one after the other, as long as the process gives monomials of lower degree.

**Proposition 5.4.** *Let* $X = x_NT^N + x_{N-1}T^{N-1} + \cdots + x_1T + x_0$ *be a polynomial of* $F[T]$. *Then there exist* $Y_0, \ldots, Y_3$, $Y \in F[T]$ *such that*

$$X = L_0(Y_0) + L_1(Y_1) + L_2(Y_2) + L_3(Y_3) + Y, \tag{5.11}$$

*with*

$$\begin{aligned}
&\deg Y \le 21.\\
&\deg Y_0 \le n,\quad \deg Y_1 \le n-1,\quad \deg Y_2 \le n-2,\quad \deg Y_3 \le n-3,\quad \textit{if } N = 4n,\\
&\deg Y_0 \le n,\quad \deg Y_1 \le n-1,\quad \deg Y_2 \le n-2,\quad \deg Y_3 \le n-2,\quad \textit{if } N = 4n+1,\\
&\deg Y_0 \le n,\quad \deg Y_1 \le n-1,\quad \deg Y_2 \le n-1,\quad \deg Y_3 \le n-2,\quad \textit{if } N = 4n+2,\\
&\deg Y_0 \le n,\quad \deg Y_1 \le n,\quad \deg Y_2 \le n-1,\quad \deg Y_3 \le n-2\quad \textit{if } N = 4n+3.
\end{aligned} \tag{5.12}$$

*Proof.* If $\deg X < 22$, then (5.11) is true with $Y_0 = Y_1 = Y_2 = Y_3 = 0$ and $Y = X$. We suppose now that $\deg X \ge 22$. We use the descent process. We use (5.7) as long as we meet monomials $x_{4n}T^{4n}$ with $n > 0$. Respectively, we use (5.8), (5.9), (5.10) as long as we meet monomials $x_{4n+3}T^{4n+3}$ with $4n + 3 > \max(2n + 5, n + 6)$, that is $n > 1$, monomials $x_{4n+2}T^{4n+3}$ with $4n + 2 > \max(2n + 8, n + 11)$, that is $n > 3$, monomials $x_{4n+1}T^{4n+1}$ with $4n + 1 > \max(2n + 11, n + 16)$, that is $n > 5$. Doing this, we write $X$ as a sum

$$X = L_0(Y_0) + L_1(Y_1) + L_2(Y_2) + L_3(Y_3) + Y,$$

with $\deg Y$, $\deg Y_0$, $\deg Y_1$, $\deg Y_2$, $\deg Y_3$ satisfying (5.12). □

**Proposition 5.5.** *Let $H \in F[T]$ with degree $7n \geq 56$ and leading coefficient a seventh power.*

*If* $\deg H \notin \{91, 105\}$, *then there exist* $X_0, X_1, X_2, X_3, Y_0, Y_1, Y_2, Y_3, Z \in F[T]$ *with* $\deg X_i \leq n$, $\deg Y_j \leq n$ *and* $\deg Z \leq 21$ *such that*

$$H = X_0^7 + X_1^7 + X_2^7 + X_3^7 + L_0(Y_0) + L_1(Y_1) + L_2(Y_2) + L_3(Y_3) + Z. \quad (5.13)$$

*If* $\deg H \in \{91, 105\}$, *then there exist* $X_0, X_1, X_2, X_3, X_4, Y_0, Y_1, Y_2, Y_3, Z \in F[T]$ *with* $\deg X_i \leq n$, $\deg Y_j \leq n$ *and* $\deg Z \leq 21$ *such that*

$$\begin{aligned} H = {} & X_0^7 + X_1^7 + X_2^7 + X_3^7 + X_4^7 \\ & + L_0(Y_0) + L_1(Y_1) + L_2(Y_2) + L_3(Y_3) + Z. \end{aligned} \quad (5.13')$$

*Proof.* We proceed as for the proof of Theorem 4.1 and we keep the same notations. Let $r$, if it exists, be the least index such that $6n_r - 1 \leq 4n + 3$. From identity (4.8), we have $r = 3$ if $n \geq 43$. Using identity (4.3) for $i = 0, 1, 2$, then identity (4.5) for $i = 3$, we get

$$H = X_0^7 + X_1^7 + X_2^7 + X_3^7 + R, \quad (1)$$

with $\deg R \leq 4n + 3$. For $8 \leq n \leq 42$, the sequence $(n, 4n + 3, n_1, n_2, n_3, 6n_3 - 1)$ is given by:

| $n$ | $4n+3$ | $n_1$ | $n_2$ | $n_3$ | $6n_3-1$ | $n$ | $4n+3$ | $n_1$ | $n_2$ | $n_3$ | $6n_3-1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 42 | 171 | 36 | 31 | 27 | 161 | 41 | 167 | 36 | 31 | 27 | 161 |
| 40 | 163 | 35 | 30 | 26 | 155 | 39 | 159 | 34 | 30 | 26 | 155 |
| 38 | 155 | 33 | 29 | 25 | 149 | 37 | 151 | 32 | 28 | 24 | 143 |
| 36 | 147 | 31 | 27 | 24 | 143 | 35 | 143 | 30 | 26 | 23 | 137 |
| 34 | 139 | 30 | 26 | 23 | 137 | 33 | 135 | 29 | 25 | 22 | 131 |
| 32 | 131 | 28 | 24 | 21 | 125 | 31 | 127 | 27 | 24 | 21 | 125 |
| 30 | 123 | 26 | 23 | 20 | 119 | 29 | 119 | 25 | 22 | 19 | 113 |
| 28 | 115 | 24 | 21 | 18 | 107 | 27 | 111 | 24 | 21 | 18 | 107 |
| 26 | 107 | 23 | 20 | 18 | 107 | 25 | 103 | 22 | 19 | 17 | 101 |
| 24 | 99 | 21 | 18 | 16 | 95 | 23 | 95 | 20 | 18 | 16 | 95 |
| 22 | 91 | 19 | 17 | 15 | 89 | 21 | 87 | 18 | 16 | 14 | 83 |
| 20 | 83 | 18 | 16 | 14 | 83 | 19 | 79 | 17 | 15 | 13 | 77 |
| 18 | 75 | 16 | 14 | 12 | 71 | 17 | 71 | 15 | 13 | 12 | 71 |
| 16 | 67 | 14 | 12 | 11 | 65 | 15 | 63 | 13 | 12 | 11 | 65 |
| 14 | 59 | 12 | 11 | 10 | 59 | 13 | 55 | 12 | 11 | 10 | 59 |
| 12 | 51 | 11 | 10 | 9 | 53 | 11 | 47 | 10 | 9 | 8 | 47 |
| 10 | 43 | 9 | 8 | 7 | 41 | 9 | 39 | 8 | 7 | 6 | 35 |
| 8 | 35 | 7 | 6 | 6 | 35 | | | | | | |

Observe that with the exceptions $n = 15, 13$, we have $6n_3 - 1 \le 4n + 3$. For $n = 7$, the sequence is $(7, 31, 6, 6, 6, 35)$ and for $n \le 6$ the sequence is $(n, 4n + 3, n, n, n, 6n - 1)$. We suppose $n \ge 8$ and $n \ne 13, 15$. From the previous proposition, there exist $Y_0, \dots, Y_3 \in F[T]$ such that

$$R = L_0(Y_0) + L_1(Y_1) + L_2(Y_2) + L_3(Y_3) + Z, \tag{2}$$

with $\deg Z \le 21$ and $\deg Y_i \le n$. For $n = 13$ or $n = 15$ we add a step in the descent process. We have $n_4 = 9$ if $n = 13$ and $n_4 = 10$ if $n = 15$, so that $6n_4 - 1 \le 4n$. Instead of (1), we have

$$H = X_0^7 + X_1^7 + X_2^7 + X_3^7 + X_4^7 + R, \tag{1'}$$

with $\deg R \le 4n + 3$. □

**Theorem 5.6.** (i) *Every polynomial $P \in F[T]$ with degree $\ge 105$ is a strict sum of $13\ell(q, 7) + 11 + 4\rho(q) + \max(\ell(q, 7) - 1, 1)$ seventh powers.*

(ii) *If $P \in F[T]$ is such that $50 \le \deg P \le 84$ or such that $92 \le \deg P \le 98$, then $P$ is a strict sum of $13\ell(q, 7) + 11 + 4\rho(q) + \max(\ell(q, 7) - 1, 1)$ seventh powers.*

(iii) *If $P \in F[T]$ is such that $95 \le \deg P \le 91$ or such that $99 \le \deg P \le 105$, then $P$ is a strict sum of $13\ell(q, 7) + 12 + 4\rho(q) + \max(\ell(q, 7) - 1, 1)$ seventh powers.*

*Proof.* Proceeding as for the proof of Theorem 4.1, it is sufficient to prove that every $H \in F[T]$ with $\deg H = 7n \ge 56$ and leading coefficient a seventh power, is a strict sum of $13\ell + 11 + 4\rho(q)$ seventh powers. Let $H$ be such a polynomial. We suppose $n \ne 13, 15$. From Proposition 5.5 and Corollary 5.3, there exists $Z \in F[T]$ with $\deg Z \le 21$ such that $H + Z$ is sum of $4 + 4\rho(q)$ seventh powers of polynomials of degree $\le n$. Let $s = 13\ell + 7$. From Proposition 3.3-(iii), there exist $Z_1, \dots, Z_s$ with $\deg Z_i \le 3$, such that

$$Z = \sum_{i=1}^{s} Z_i^7.$$

Thus, $H$ is a strict sum of $s + 4 + 4\rho(q)$ seventh powers. If $n = 13, 15$ we have to add a seventh power. □

The following theorem improves the bounds deduced from ([1], Theorem 7.1).

**Theorem 5.7.** (i) *If $q \not\equiv 1 \pmod 7$ and $q \equiv 1 \pmod 3$, then $g(q, 7) \le 42$.*

(ii) *If $q \not\equiv 1 \pmod 7$ and $q \not\equiv 1 \pmod 3$, then $g(q, 7) \le 46$.*

(iii) *If $q \equiv 1 \pmod{21}$ and $q \ne 64$, then $g(q, 7) \le 57$.*

(iv) *If $q \equiv 1 \pmod 7$ and $q \not\equiv 1 \pmod 3$, then $g(q, 7) \le 59$.*

(v) *$g(64, 7) \le 74$.*

*Proof.* From Proposition 4.4, every polynomial of degree $\leq 49$ is a strict sum of less than $17\ell + 23$ seventh powers. From Theorem 5.6, every polynomial of degree $\geq 50$ is a strict sum of $13\ell + 12 + 4\rho(q) + \max(\ell - 1, 1)$ seventh powers. Thus,

$$g(q,7) \leq \max\bigl(17\ell + 23, 13\ell + 12 + 4\rho(q) + \max(\ell - 1, 1)\bigr)$$

with

$$\rho(q) = \begin{cases} 4 & \text{if } q \text{ is a power of } 4, \\ 5 & \text{otherwise.} \end{cases}$$

We conclude with Proposition 2.1. □

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