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Null controllability of degenerate parabolic cascade systems

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Abstract. In this paper, we study the null controllability of degenerate semilinear cascade parabolic systems with one control force. The key tool is the Carleman estimates developed recently for degenerate one dimension parabolic equations. We develop a Carleman estimate for these systems and then an observability inequality for the linear adjoint system. We conclude by linearization and fixed point arguments.

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1. Introduction

This paper is concerned with null controllability for the following coupled degenerate semilinear cascade parabolic system

$$u_t - \big(a_1(x)u_x\big)_x + F_1(t,x,u) = h(t,x)1_\omega, \qquad (t,x) \in (0,T)\times(0,1), \tag{1}$$

$$v_t - \big(a_2(x)v_x\big)_x + F_2(t,x,u,v) = 0, \qquad (t,x) \in (0,T)\times(0,1), \tag{2}$$

$$u(t,0) = u(t,1) = v(t,0) = v(t,1) = 0, \qquad t \in (0,T), \tag{3}$$

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \qquad x \in (0,1), \tag{4}$$

where ω is an open subset of $(0,1)$, $h_i \in L^2\big((0,T)\times(0,1)\big)$, $u_0, v_0 \in L^2(0,1)$ and $T > 0$, and a_1, a_2 are two (different) diffusion coefficients. By a linearization technique, one reduces this question to the study of the null controllability of the linear cascade system

$$u_t - \big(a_1(x)u_x\big)_x + c_1(t,x)u = h(t,x)1_\omega, \qquad (t,x) \in (0,T)\times(0,1), \tag{5}$$

$$v_t - \big(a_2(x)v_x\big)_x + c_2(t,x)v + b(t,x)u = 0, \qquad (t,x) \in (0,T)\times(0,1), \tag{6}$$

$$u(t,0) = u(t,1) = v(t,0) = v(t,1) = 0, \qquad t \in (0,T), \tag{7}$$

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \qquad x \in (0,1), \tag{8}$$

where the coefficients b, c_1, c_2 are given in terms of F_1 and F_2. Several works were done in the null controllability of degenerate parabolic equation (9), i.e. the diffusion coefficient a_1 can vanish at $x = 0$ or at both ends $x = 0$, $x = 1$, see [1], [9], [10], [8], [20]. The main result in these works is the development of new Carleman inequalities for degenerate parabolic equations, which are used in the standard way to show observability inequalities of the adjoint degenerate problems, and then obtain the null controllability of these degenerate parabolic equations.

The null controllability of parabolic systems was studied in [4], [3], [16], [17], [15], [18] in the nondegenerate case, i.e., a_1, a_2 are positive in $[0,1]$. In [19], Liu et al. considered parabolic cascade systems, with degeneracy in only one equation, using the nondegenerate Carleman estimate of Fursikov and Imanuvilov [14] and an approximation argument as in [12]. Recently, using the degenerate Carleman estimate of Alabau-Boussouira et al. [1], Cannarsa and De Tereza studied in [7] cascade degenerate linear systems (9)–(12) with the same degeneracies $a_1 = a_2$, and with particular coupling term $b = 1_O$ for some open set O.

In this paper, we consider the degenerate semilinear system (1)–(4) with two different degeneracies a_1 and a_2. To study its null controllability, by the linearization argument, we consider the linear degenerate system (9)–(12) with general coupling terms b. To study the null controllability of the last system, we show an observability inequality for its adjoint system. As in the above litterature, to do this, we develop a Carleman estimate for these systems. To get this aim, we apply as in [7], the Carleman estimates developped in ([1], Theorem 3.1, Corollary 3.2) to the both degenerate equations (17) and (18). These lead to two inequalities with two different weight functions φ_1 and φ_2. We choose carefully some constants in the weight functions to get the comparaison $\varphi_1 \le \varphi_2$, which provides the desired Carleman estimate. Note that, we do not require any comparaison between a_1 and a_2.

At the end, the null controllability of the semilinear system (1)–(4) will follow by a standard linearization argument and a fixed point theroem.

2. Null controllability of linear systems

Consider the linear cascade system

$$u_t - \big(a_1(x)u_x\big)_x + c_1(t,x)u = h(t,x)1_\omega, \qquad (t,x) \in (0,T)\times(0,1), \tag{9}$$

$$v_t - \big(a_2(x)v_x\big)_x + c_2(t,x)v + b(t,x)u = 0, \qquad (t,x) \in (0,T)\times(0,1), \tag{10}$$

$$u(t,0) = u(t,1) = v(t,0) = v(t,1) = 0, \qquad t \in (0,T), \tag{11}$$

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \qquad x \in (0,1). \tag{12}$$

We assume the following assumptions:

(H1) $b, c_1, c_2 \in L^\infty\big((0,T)\times(0,1)\big)$,

(H2) $\begin{cases} \text{(i) } a_i \in \mathscr{C}([0,1]) \cap \mathscr{C}^1\big((0,1]\big) : a_i(0) = 0;\ a_i > 0 \text{ on } (0,1], \\ \text{(ii) } \exists K \in [0,1) : x a_i'(x) \le K a_i(x),\ \forall x \in (0,1]. \end{cases}$

Under (H2), the above systems can be called as in [1] weakly degenerate systems. Analogously, one can obtain the same results in the cases where the two equations are strongly degenerate or one is weakly and the other is strongly degenerate, see [1] for the definition.

2.1. Well-posedness. In order to study the well-posedness of system (9)–(12), we introduce the following weighted spaces

$$H^1_{a_i}(0,1) := \{u \in L^2(0,1) : u \text{ locally absol. cont. in } [0,1], \\ \sqrt{a_i}u_x \in L^2(0,1) \text{ and } u(0) = u(1) = 0\}$$

with the norm $\|u\|^2_{H^1_{a_i}(0,1)} := \|u\|^2_{L^2(0,1)} + \|\sqrt{a_i}u_x\|^2_{L^2(0,1)}$ and

$$H^2_{a_i}(0,1) := \{u \in H^1_{a_i}(0,1) : a_i u_x \in H^1(0,1)\}$$

with the norm

$$\|u\|^2_{H^2_{a_i}(0,1)} := \|u\|^2_{H^1_{a_i}(0,1)} + \|(a_i u_x)_x\|^2_{L^2(0,1)}.$$

We define the operator $\big(A_i, D(A_i)\big)$ by

$$A_i u := (a_i u_x)_x, \qquad u \in D(A_i) = H^2_{a_i}(0,1).$$

We recall the following properties of $\big(A_i, D(A_i)\big)$.

Proposition 2.1 ([6]). *The operator $A_i : D(A_i) \to L^2(0,1)$ is a closed self-adjoint negative operator with dense domain.*

In the Hilbert space $\mathbb{H} = L^2(0,1) \times L^2(0,1)$, the system (9)–(12) can be transformed into the inhomogeneous Cauchy problem

$$(CP) \qquad \begin{cases} X'(t) = \mathscr{A} X(t) + B(t)X(t) + f(t), \\ X(0) = \binom{u_0}{v_0}, \end{cases}$$

where $X(t) = \binom{u(t)}{v(t)}$, $\mathscr{A} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$; $D(\mathscr{A}) = D(A_1) \times D(A_2)$, $f(t) = \binom{h(t,\cdot)1_\omega}{0}$,

$$B(t) = \begin{pmatrix} M_{c_1(t)} & 0 \\ M_{b(t)} & M_{c_2(t)} \end{pmatrix}, \qquad \text{where } M_{c_i(t)}u = c_i(t)u \text{ and } M_{b(t)}u = b(t)u.$$

As the operator $\mathscr{A}$ is diagonal and since $B(t)$ is a bounded perturbation, the following wellposedness and regularity results hold.

Proposition 2.2. (i) *The operator $\mathscr{A}$ generates a contraction strongly continuous semigroup.*

(ii) *For all $h \in L^2\big((0,T) \times (0,1)\big)$ and $u_0, v_0 \in L^2(0,1)$ there exists a unique mild solution $(u,v) \in X_T := C\big([0,T], \big(L^2(0,1)\big)^2\big) \cap L^2(0,T; H^1_{a_1} \times H^1_{a_2})$ of* (9)–(12). *Moreover, for $(u_0, v_0) \in H^1_{a_1} \times H^1_{a_2}$, $(u,v) \in C([0,T]; H^1_{a_1} \times H^1_{a_2}) \cap H^1\big(0,T; \big(L^2(0,1)\big)^2\big) \cap L^2(0,T; H^2_{a_1} \times H^2_{a_2})$ and there exists a constant $C_T > 0$ such that*

$$\sup_{[0,T]} \|(u,v)(t)\|^2_{H^1_{a_1} \times H^1_{a_2}} + \int_0^T \left(\|(u_t,v_t)\|^2_{L^2} + \left\|\big((a_1u_x)_x, (a_2v_x)_x\big)\right\|^2_{L^2}\right) dt$$
$$\leq C_T\left(\|(u_0,v_0)\|^2_{H^1_{a_1} \times H^1_{a_2}} + \|h\|^2_{L^2((0,T)\times(0,1))}\right).$$

As in [1], [8], we can show also the well posedness of the semilinear cascade systems.

Proposition 2.3. *For all $Y_0, Z_0 \in L^2(0,1)$, the system*

$$Y_t - \big(a_1(x)Y_x\big)_x + F_1(t,x,Y) = 0, \qquad (t,x) \in (0,T) \times (0,1), \tag{13}$$
$$Z_t - \big(a_2(x)Z_x\big)_x + F_2(t,x,Y,Z) = 0, \qquad (t,x) \in (0,T) \times (0,1), \tag{14}$$
$$Y(t,0) = Y(t,1) = 0, \quad Z(t,0) = Z(t,1) = 0, \qquad t \in (0,T), \tag{15}$$
$$Y(0,x) = Y_0(x), \quad Z(0,x) = Z_0(x), \qquad x \in (0,1), \tag{16}$$

has a solution $(Y,Z) \in X_T$.

2.2. Carleman estimates for the adjoint cascade system. This subsection is devoted to show Carleman estimates for the adjoint cascade system

$$U_t - \big(a_1(x)U_x\big)_x + c_1(t,x)U + b(t,x)V = 0, \qquad (t,x) \in (0,T)\times(0,1), \tag{17}$$

$$V_t - \big(a_2(x)V_x\big)_x + c_2(t,x)V = 0, \qquad (t,x) \in (0,T)\times(0,1), \tag{18}$$

$$U(t,1) = U(t,0) = V(t,1) = V(t,0) = 0, \qquad t \in (0,T), \tag{19}$$

$$U(0,x) = U_0(x), \quad V(0,x) = V_0(x), \qquad x \in (0,1). \tag{20}$$

For this purpose, we define the weight functions $\varphi_i(t,x) = \Theta(t)\psi_i(x)$, $\Theta(t) := \frac{1}{[t(T-t)]^4}$, $\psi_i(x) := \lambda_i\Big(\int_0^x \frac{y}{a_i(y)}dy - d_i\Big)$, $\Phi_i(t,x) := \Theta(t)\Psi_i(x)$, $\Psi_i(x) := e^{2\rho_i} - e^{r_i\zeta_i(x)}$, $\zeta_i(x) := \int_x^1 \frac{1}{\sqrt{a_i(y)}}dy$, $\rho_i := r_i\zeta_i(0)$, where the positive constants d_i, r_i and λ_i are chosen such that $d_i \geq \max\Big\{\frac{1}{a_i(1)(2-K)}, 4\int_0^1 \frac{y}{a_i(y)}dy\Big\}$, $e^{\rho_2} \geq 4\frac{d_2 - \int_0^1 \frac{y}{a_2(y)}dy}{d_2 - 4\int_0^1 \frac{y}{a_2(y)}dy}$, $\rho_1 = 2\rho_2$, $\lambda_1 = \frac{e^{2\rho_1}-1}{d_1 - \int_0^1 \frac{y}{a_1(y)}dy}$ and $\lambda_2 = \frac{4}{3d_2}(e^{2\rho_2} - e^{\rho_2})$.

One can show that these weight functions and their parameters satisfy the following properties which are needed in the sequel.

Lemma 2.4. We have $\frac{\lambda_1}{\lambda_2} \geq \frac{d_2}{d_1 - \int_0^1 \frac{y}{a_1(y)}dy}$, $e^{2\rho_1} - e^{\rho_1} \geq e^{2\rho_2} - 1$, $\lambda_i \geq \frac{e^{2\rho_i}-1}{d_i - \int_0^1 \frac{y}{a_i(y)}dy}$, $\lambda_2 \leq \frac{4}{3d_2}(e^{2\rho_2} - e^{\rho_2})$ and $\varphi_1 \leq \varphi_2$, $-\Phi_1 \leq -\Phi_2$, $\varphi_i \leq -\Phi_i$, $4\Phi_2 + 3\varphi_2 \geq 0$.

We now announce a result on an intermediate Carleman estimate, which could be used to obtain the null controllability of the linear cascade systems with two control forces.

Theorem 2.5. *Let $T > 0$ be given. There exist two positive constants C and s_0 such that every solution (U,V) of* (17)–(20) *satisfies*

$$\begin{aligned}\int_0^T\int_0^1 \left[s\Theta a_1 U_x^2 + s^3\Theta^3 \frac{x^2}{a_1(x)} U^2\right] e^{2s\varphi_1}\,dx\,dt \\ + \int_0^T\int_0^1 \left[s\Theta a_2 V_x^2 + s^3\Theta^3 \frac{x^2}{a_2(x)} V^2\right] e^{2s\varphi_2}\,dx\,dt \\ \leq C\int_0^T\int_\omega [U^2 + V^2] e^{-2s\Phi_2}\,dx\,dt.\end{aligned}$$

Proof. Let us choose an arbitrary open subset $\omega' := (\alpha',\beta')$ such that $\omega' \Subset \omega := (\alpha,\beta)$, and consider the cut-off function $\xi \in \mathscr{C}^\infty(0,1)$ such that

$$\begin{cases} 0 \leq \xi(x) \leq 1, & x \in (0,1), \\ \xi(x) = 1, & 0 \leq x \leq \alpha', \\ \xi(x) = 0, & \beta' \leq x \leq 1. \end{cases}$$

Let $w = \xi U$ and $z = \xi V$ where (U, V) is the solution of (17)–(20). Then w and z satisfy the following system

$$\begin{aligned} &w_t - (a_1 w_x)_x + c_1 w = -\xi b z - \xi_x a_1 U_x - (a_1 \xi_x U)_x =: f, \\ &\qquad (t,x) \in (0,T) \times (0,1), \qquad (21)\\ &z_t - (a_2 z_x)_x + c_2 z = -\xi_x a_2 V_x - (a_2 \xi_x V)_x, \qquad (t,x) \in (0,T) \times (0,1), \qquad (22)\\ &w(t,1) = w(t,0) = z(t,1) = z(t,0) = 0, \qquad t \in (0,T), \qquad (23)\\ &w(0,x) = w_0(x), \quad z(0,x) = z_0(x), \qquad x \in (0,1). \qquad (24)\end{aligned}$$

Applying the Carleman estimate estabished in ([1], Corollary 3.2) to the equation (21), we obtain

$$\begin{aligned} &\int_0^T \int_0^1 \left[s\Theta(t) a_1(x) w_x^2(t,x) + s^3 \Theta^3(t) \frac{x^2}{a_1(x)} w^2(t,x) \right] e^{2s\varphi_1} \, dx \, dt \\ &\le C \Big(\int_0^T \int_0^1 |f|^2 e^{2s\varphi_1(t,x)} \, dx \, dt + s a_1(1) \int_0^T \Theta(t) e^{2s\varphi_1(t,1)} w_x^2(t,1) \, dt \Big) \\ &\le \bar{C} \int_0^T \int_0^1 \big[\xi^2 b^2 z^2(t,x) + \big(\xi_x a_1 U_x(t,x) + (a_1 \xi_x U)_x(t,x) \big)^2 \big] e^{2s\varphi_1} \, dx \, dt, \qquad (25)\end{aligned}$$

since $w_x(t,1) = 0$. From the definition of ξ, we have

$$\begin{aligned} \int_0^1 \big(\xi_x a_1 U_x + (a_1 \xi_x U)_x \big)^2 e^{2s\varphi_1} \, dx &\le \int_{\omega'} \big(8 (a_1 \xi_x)^2 U_x^2 + 2 \big((a_1 \xi_x)_x \big)^2 U^2 \big) e^{2s\varphi_1} \, dx \\ &\le C \int_{\omega'} [U^2 + U_x^2] e^{2s\varphi_1} \, dx. \qquad (26)\end{aligned}$$

In the other hand, using the fact that $\frac{x^2}{a_2(x)}$ is non-decreasing and Hardy–Poincaré inequality ([1], Proposition 2.1), we have

$$\int_0^1 \xi^2 b^2 z^2 e^{2s\varphi_1} \, dx \le \frac{\|b\|_\infty^2}{a_2(1)} \int_0^1 \frac{a_2(x)}{x^2} (z e^{s\varphi_2})^2 \, dx \le C \frac{\|b\|_\infty^2}{a_2(1)} \int_0^1 a_2(x) \big((z e^{s\varphi_2})_x \big)^2 \, dx.$$

Hence, since $\psi_{2,x}(x) = \frac{x}{a_2(x)}$, one obtains

$$\int_0^1 \xi^2 b^2 z^2 e^{2s\varphi_1} \, dx \le C \int_0^1 a_2(x) z_x^2 e^{2s\varphi_2} \, dx + C \int_0^1 s^2 \Theta^2 \frac{x^2}{a_2(x)} z^2 e^{2s\varphi_2} \, dx,$$

and for large s, we obtain the estimate

$$\bar{C}\int_0^1 \xi^2 b^2 z^2 e^{2s\varphi_1}\,dx \le \frac{1}{2}\int_0^1 s\Theta a_2(x) z_x^2 e^{2s\varphi_2}\,dx + \frac{1}{2}\int_0^1 s^3\Theta^3 \frac{x^2}{a_2(x)} z^2 e^{2s\varphi_2}\,dx. \quad (27)$$

Combining (25), (26) and (27) we obtain for s large enough

$$\begin{aligned}\int_0^T\int_0^1 \left[s\Theta(t)a_1(x)w_x^2(t,x) + s^3\Theta^3(t)\frac{x^2}{a_1(x)}w^2(t,x)\right]e^{2s\varphi_1}\,dx\,dt \\ \le C\int_0^T\int_{\omega'}[U^2+U_x^2]e^{2s\varphi_1}\,dx + \frac{1}{2}\int_0^T\int_0^1 s\Theta a_2(x)z_x^2 e^{2s\varphi_2}\,dx \\ + \frac{1}{2}\int_0^T\int_0^1 s^3\Theta^3\frac{x^2}{a_2(x)}z^2e^{2s\varphi_2}\,dx. \quad (28)\end{aligned}$$

Arguing as before we obtain for the second component z

$$\int_0^T\int_0^1\left[s\Theta a_2(x)z_x^2 + s^3\Theta^3\frac{x^2}{a_2(x)}z^2\right]e^{2s\varphi_2}\,dx\,dt \le C\int_0^T\int_{\omega'}[V_x^2+V^2]e^{2s\varphi_2}\,dx\,dt. \quad (29)$$

Consequently, from Lemma 2.4 and the estimates (28)–(29), we obtain for s large enough

$$\begin{aligned}\int_0^T\int_0^1\left[s\Theta a_1 w_x^2 + s^3\Theta^3\frac{x^2}{a_1(x)}w^2\right]e^{2s\varphi_1}\,dx\,dt \\ + \int_0^T\int_0^1\left[s\Theta a_2 z_x^2 + s^3\Theta^3\frac{x^2}{a_2(x)}z^2\right]e^{2s\varphi_2}\,dx\,dt \\ \le C\int_0^T\int_{\omega'}[U_x^2+V_x^2+U^2+V^2]e^{2s\varphi_2}\,dx\,dt. \quad (30)\end{aligned}$$

Using Cacciopoli's inequality, see Lemma 4.1, the estimate (30) becomes

$$\begin{aligned}\int_0^T\int_0^1\left[s\Theta a_1 w_x^2 + s^3\Theta^3\frac{x^2}{a_1(x)}w^2\right]e^{2s\varphi_1}\,dx\,dt \\ + \int_0^T\int_0^1\left[s\Theta a_2 z_x^2 + s^3\Theta^3\frac{x^2}{a_2(x)}z^2\right]e^{2s\varphi_2}\,dx\,dt \\ \le C\int_0^T\int_{\omega}[U^2+V^2]e^{2s\varphi_2}\,dx\,dt. \quad (31)\end{aligned}$$

Let now $W := \eta U$ and $Z := \eta V$, where $\eta = 1-\xi$. Then W, Z are supported in $(\alpha, 1)$ and satisfy

$$W_t - (a_1 W_x)_x + c_1 W = -\eta b Z - \eta_x a_1 U_x - (a_1 \eta_x U)_x =: g,$$
$$(t,x) \in (0,T) \times (0,1), \tag{32}$$
$$Z_t - (a_2 Z_x)_x + c_2 Z = -\eta_x a_2 V_x - (a_2 \eta_x V)_x, \qquad (t,x) \in (0,T) \times (0,1), \tag{33}$$
$$W(t,1) = W(t,0) = 0, \quad Z(t,1) = Z(t,0) = 0, \qquad t \in (0,T), \tag{34}$$
$$W(0,x) = W_0(x), \quad Z(0,x) = Z_0(x), \qquad x \in (0,1). \tag{35}$$

On $(\alpha,1)$, equations in system (32)–(35) are uniformly parabolic, and the function a_i can be replaced by a positive function in $\mathscr{C}^1(0,1)$ which coincides with a_i in $(\alpha,1)$ denoted again by a_i, $i = 1,2$. Hence, using a classical Carleman estimate for nondegenerate problems, see [1], [5], to the equation (32), and since $-[\sigma_1(t,x)e^{-2s\Phi_1(t,x)}a_1(x)W_x^2(t,x)]_{x=0}^{x=1} \le 0$ one obtains

$$\begin{aligned}
&\int_0^T \int_0^1 s^3\Theta^3 e^{3r_1\zeta_1(x)} W^2(t,x) e^{-2s\Phi_1(t,x)}\,dx\,dt + \int_0^T \int_0^1 s\Theta e^{r_1\zeta_1(x)} W_x^2(t,x) e^{-2s\Phi_1(t,x)}\,dx\,dt \\
&\quad \le C \int_0^T \int_0^1 g^2(t,x) e^{-2s\Phi_1(t,x)}\,dx\,dt - C\int_0^T [\sigma_1(t,x)e^{-2s\Phi_1(t,x)}a_1(x)W_x^2(t,x)]_{x=0}^{x=1}\,dt \\
&\quad \le C \int_0^T \int_0^1 [\eta b Z + \eta_x a_1(x) U_x + (a_1\eta_x U)_x]^2 e^{-2s\Phi_1}\,dx\,dt \\
&\quad \le C \int_0^T \int_0^1 Z^2 e^{-2s\Phi_2}\,dx\,dt + C\int_0^T \int_{\omega'} [U^2 + U_x^2] e^{-2s\Phi_1}\,dx\,dt,
\end{aligned} \tag{36}$$

where $\sigma_i(t,x) = r_i s\Theta(t)e^{r_i\zeta_i(x)}$ and ζ_i, Φ_i are defined above. Similarly for the second component Z we obtain

$$\begin{aligned}
&\int_0^T \int_0^1 s^3\Theta^3 e^{3r_2\zeta_2(x)} Z^2 e^{-2s\Phi_2}\,dx\,dt + \int_0^T \int_0^1 s\Theta e^{r_2\zeta_2(x)} Z_x^2 e^{-2s\Phi_2}\,dx\,dt \\
&\quad \le C\int_0^T \int_{\omega'} [V^2 + V_x^2] e^{-2s\Phi_2}\,dx\,dt.
\end{aligned} \tag{37}$$

Thus, combining the estimates (36)–(37) and using the Caccioppoli's inequality, we obtain for s large enough

$$\begin{aligned}
&\int_0^T \int_0^1 s^3\Theta^3 e^{3r_1\zeta_1(x)} e^{-2s\Phi_1} W^2\,dx\,dt + \int_0^T \int_0^1 s^3\Theta^3 e^{3r_2\zeta_2(x)} e^{-2s\Phi_2} Z^2\,dx\,dt \\
&\quad + \int_0^T \int_0^1 s\Theta e^{r_1\zeta_1(x)} e^{-2s\Phi_1} W_x^2\,dx\,dt + \int_0^T \int_0^1 s\Theta e^{r_2\zeta_2(x)} e^{-2s\Phi_2} Z_x^2\,dx\,dt \\
&\quad \le \tilde{C} \int_0^T \int_0^1 Z^2 e^{-2s\Phi_2}\,dx\,dt + C\int_0^T \int_{\omega'} [U^2 + V^2 + U_x^2 + V_x^2] e^{-2s\Phi_2}\,dx \\
&\quad \le \tilde{C} \int_0^T \int_0^1 Z^2 e^{-2s\Phi_2}\,dx\,dt + C\int_0^T \int_{\omega} [U^2 + V^2] e^{-2s\Phi_2}\,dx\,dt.
\end{aligned}$$

For s large enough, we have $\tilde{C} \le \frac{1}{2} s^3 \Theta^3 e^{3r_i\zeta_i(x)}$ (it suffices that $s \ge \frac{T^8}{256}\sqrt[3]{2\tilde{C}}$). Thus,

$$\begin{aligned}
&\int_0^T \int_0^1 s^3\Theta^3 e^{3r_1\zeta_1(x)} e^{-2s\Phi_1} W^2\,dx\,dt + \int_0^T \int_0^1 s^3\Theta^3 e^{3r_2\zeta_2(x)} e^{-2s\Phi_2} Z^2\,dx\,dt \\
&\quad + \int_0^T \int_0^1 s\Theta e^{r_1\zeta_1(x)} e^{-2s\Phi_1} W_x^2\,dx\,dt + \int_0^T \int_0^1 s\Theta e^{r_2\zeta_2(x)} e^{-2s\Phi_2} Z_x^2\,dx\,dt \\
&\le C \int_0^T \int_\omega [U^2 + V^2] e^{-2s\Phi_2}\,dx\,dt.
\end{aligned}$$

By Lemma 2.4, there exists a constant $C > 0$ such that

$$\frac{x^2}{a_i(x)} e^{2s\varphi_i(t,x)} \le C e^{3r_i\zeta_i(x)} e^{-2s\Phi_i(t,x)} \quad \text{and} \quad a_i(x) e^{2s\varphi_i(t,x)} \le C e^{r_i\zeta_i(x)} e^{-2s\Phi_i(t,x)}$$

for all $(t,x) \in [0,T] \times (\alpha, 1)$. As a consequence we have

$$\begin{aligned}
&\int_0^T \int_0^1 \left[s^3\Theta^3 \frac{x^2}{a_1(x)} W^2 + s\Theta a_1 W_x^2 \right] e^{2s\varphi_1}\,dx\,dt \\
&\quad + \int_0^T \int_0^1 \left[s^3\Theta^3 \frac{x^2}{a_2(x)} Z^2 + s\Theta a_2 Z_x^2 \right] e^{2s\varphi_2}\,dx\,dt \\
&\le C \int_0^T \int_\omega [U^2 + V^2] e^{-2s\Phi_2}\,dx\,dt. \qquad (38)
\end{aligned}$$

Since $U = w + W$ and $V = z + Z$, we have

$$U^2 \le 2(w^2 + W^2), \quad V^2 \le 2(z^2 + Z^2), \quad U_x^2 \le 2(w_x^2 + W_x^2), \quad V_x^2 \le 2(z_x^2 + Z_x^2).$$

Thus by (31), (38) and Lemma 2.4 one has

$$\begin{aligned}
&\int_0^T \int_0^1 \left[s\Theta a_1 U_x^2 + s^3\Theta^3 \frac{x^2}{a_1(x)} U^2 \right] e^{2s\varphi_1}\,dx\,dt \\
&\quad + \int_0^T \int_0^1 \left[s\Theta a_2 V_x^2 + s^3\Theta^3 \frac{x^2}{a_2(x)} V^2 \right] e^{2s\varphi_2}\,dx\,dt \\
&\le C \int_0^T \int_\omega [U^2 + V^2] e^{-2s\Phi_2}\,dx\,dt.
\end{aligned}$$

This ends the proof. □

For cascade systems (9)–(12) with one control force, we need to show the following Carleman estimate.

Theorem 2.6. *Let $T > 0$ be given. Assume that*

$$b \geq \gamma \quad \text{on } [0, T] \times \omega_1 \quad \text{for some } \omega_1 \Subset \omega \text{ and } \gamma > 0. \tag{39}$$

Then there exist positive constants C and s_0 such that every solution (U, V) of (17)–(20) *satisfies*

$$\begin{aligned}\int_0^T \int_0^1 & \left[s\Theta a_1 U_x^2 + s^3\Theta^3 \frac{x^2}{a_1(x)} U^2 \right] e^{2s\varphi_1}\, dx\, dt \\ &+ \int_0^T \int_0^1 \left[s\Theta a_2 V_x^2 + s^3\Theta^3 \frac{x^2}{a_2(x)} V^2 \right] e^{2s\varphi_2}\, dx\, dt \\ &\leq C \int_0^T \int_\omega U^2\, dx\, dt \end{aligned} \tag{40}$$

for all $s \geq s_0$.

Remark 2.7. 1. The assumption (39) can be replaced by $b \leq -\gamma$ on $[0, T] \times \omega_1$ for $\omega_1 \Subset \omega$ and $\gamma > 0$.

2. If $b = 1_O$ for some open set O then the assumption (39) is equivalent to $O \cap \omega \neq \emptyset$, assumed in [7].

Theorem 2.6 is a consequence of Theorem 2.5 applied to ω_1 and the following lemma, see also the proofs of ([7], Theorem 3.2) and [19].

Lemma 2.8. *For all $\epsilon > 0$ there is $C_\epsilon > 0$ such that*

$$\int_0^T \int_{\omega_1} V^2 e^{-2s\Phi_2}\, dx\, dt \leq \epsilon J(V) + C_\epsilon \int_0^T \int_\omega U^2\, dx\, dt, \tag{41}$$

where

$$J(V) := \int_0^T \int_0^1 \left[s\Theta a_2 V_x^2 + s^3\Theta^3 \frac{x^2}{a_2(x)} V^2 \right] e^{2s\varphi_2}\, dx\, dt.$$

Proof. Let $\chi \in \mathscr{C}^\infty(0, 1)$ be such that $\operatorname{supp} \chi \subset \omega$ and $\chi \equiv 1$ on ω_1. Multiplying the equation (17) by $\chi e^{-2s\Phi_2} V$, we obtain

$$\int_0^T\int_0^1 b\chi e^{-2s\Phi_2}V^2\,dx\,dt = -\int_0^T\int_0^1 \chi e^{-2s\Phi_2}VU_t\,dx\,dt + \int_0^T\int_0^1 \chi e^{-2s\Phi_2}V(a_1U_x)_x\,dx\,dt - \int_0^T\int_0^1 c_1\chi e^{-2s\Phi_2}UV\,dx\,dt. \tag{42}$$

Integrating by parts and using (18), we obtain

$$\int_0^T\int_0^1 \chi e^{-2s\Phi_2}VU_t\,dx\,dt = \int_0^T\int_0^1 a_2\chi e^{-2s\Phi_2}U_xV_x\,dx\,dt + \int_0^T\int_0^1 a_2(\chi e^{-2s\Phi_2})_xUV_x\,dx\,dt + \int_0^T\int_0^1 (2s\dot{\Theta}\Psi_2\chi + c_2\chi)e^{-2s\Phi_2}UV\,dx\,dt, \tag{43}$$

and

$$\int_0^T\int_0^1 \chi e^{-2s\Phi_2}V(a_1U_x)_x\,dx\,dt = -\int_0^T\int_0^1 a_1\chi e^{-2s\Phi_2}U_xV_x\,dx\,dt + \int_0^T\int_0^1 a_1(\chi e^{-2s\Phi_2})_xUV_x\,dx\,dt + \int_0^T\int_0^1 \big(a_1(\chi e^{-2s\Phi_2})_x\big)_xUV\,dx\,dt. \tag{44}$$

So combining the identities (42)–(44), we get

$$\int_0^T\int_0^1 b\chi e^{-2s\Phi_2}V^2\,dx\,dt = -\underbrace{\int_0^T\int_0^1 (a_1+a_2)\chi e^{-2s\Phi_2}U_xV_x\,dx\,dt}_{I_1} + \underbrace{\int_0^T\int_0^1 (a_1-a_2)(\chi e^{-2s\Phi_2})_xUV_x\,dx\,dt}_{I_2} - \underbrace{\int_0^T\int_0^1 \left[(2s\dot{\Theta}\Psi_2+c_1+c_2)\chi - \big(a_1(\chi e^{-2s\Phi_2})_x\big)_x e^{2s\Phi_2}\right]e^{-2s\Phi_2}UV\,dx\,dt}_{I_3}.$$

For the integral I_1, we have

$$\left|\int_0^T\int_0^1 a_i\chi e^{-2s\Phi_2}U_xV_x\,dx\,dt\right| = \left|\int_0^T\int_0^1 [(s\Theta a_2)^{1/2}e^{s\varphi_2}V_x][(s\Theta a_2)^{-1/2}a_i\chi e^{-s(2\Phi_2+\varphi_2)}U_x]\,dx\,dt\right| \le \varepsilon\int_0^T\int_0^1 s\Theta a_2e^{2s\varphi_2}V_x^2\,dx\,dt + \frac{1}{4\varepsilon}\underbrace{\int_0^T\int_0^1 s^{-1}\Theta^{-1}\frac{a_i^2}{a_2}\chi^2e^{-2s(2\Phi_2+\varphi_2)}U_x^2\,dx\,dt}_{L}. \tag{45}$$

The last integral L should be estimated by an integral in U^2. Multiplying the equation (17) by $s^{-1}\Theta^{-1}\frac{a_i^2}{a_1a_2}\chi^2 e^{-2s(2\Phi_2+\varphi_2)}U$ and integrating by parts, we obtain

$$\begin{aligned}
L = & \underbrace{-\frac{1}{2}\int_0^T\int_0^1 \left(s^{-1}\dot{\Theta}\Theta^{-2} + 2\dot{\Theta}\Theta^{-1}(2\Psi_2+\psi_2)\right)\frac{a_i^2}{a_1a_2}\chi^2 e^{-2s(2\Phi_2+\varphi_2)}U^2\,dx\,dt}_{L_1} \\
& \underbrace{+\frac{1}{2}\int_0^T\int_0^1 s^{-1}\Theta^{-1}\left(a_1\left(\chi^2\frac{a_i^2}{a_1a_2}e^{-2s(2\Phi_2+\varphi_2)}\right)_x\right)_x U^2\,dx\,dt}_{L_2} \\
& \underbrace{-\int_0^T\int_0^1 s^{-1}\Theta^{-1}c_1\chi^2\frac{a_i^2}{a_1a_2}e^{-2s(2\Phi_2+\varphi_2)}U^2\,dx\,dt}_{L_3} \\
& \underbrace{-\int_0^T\int_0^1 s^{-1}\Theta^{-1}b\chi^2\frac{a_i^2}{a_1a_2}e^{-2s(2\Phi_2+\varphi_2)}UV\,dx\,dt}_{L_4}.
\end{aligned}$$

Since $|\dot{\Theta}| \le C\Theta^{5/4}$ and $\operatorname{supp}\chi \subset \omega$, the functions a_i, $\frac{1}{a_i}$, χ, ψ_i, Ψ_i and their derivatives are bounded on ω and also c_i and b. Then

$$|L_1| \le C\int_0^T\int_\omega \Theta^{1/4}e^{-2s(2\Phi_2+\varphi_2)}U^2\,dx\,dt, \qquad |L_2| \le C\int_0^T\int_\omega s\Theta e^{-2s(2\Phi_2+\varphi_2)}U^2\,dx\,dt,$$

$$|L_3| \le C\int_0^T\int_\omega s^{-1}\Theta^{-1}e^{-2s(2\Phi_2+\varphi_2)}U^2\,dx\,dt,$$

$$\begin{aligned}
|L_4| &= \int_0^T\int_0^1 \left[s^{3/2}\Theta^{3/2}\frac{x}{\sqrt{a_2}}e^{s\varphi_2}V\right]\left[s^{-5/2}\Theta^{-5/2}b\chi^2\frac{a_i^2}{xa_1\sqrt{a_2}}e^{-s(4\Phi_2+3\varphi_2)}U\right]dx\,dt \\
&\le \varepsilon^2\int_0^T\int_\omega s^3\Theta^3\frac{x^2}{a_2}e^{2s\varphi_2}V^2\,dx\,dt \\
&\quad + \frac{1}{4\varepsilon^2}\int_0^T\int_0^1 s^{-5}\Theta^{-5}b^2\chi^4\frac{a_i^4}{x^2a_1^2a_2}e^{-2s(4\Phi_2+3\varphi_2)}U^2\,dx\,dt \\
&\le \varepsilon^2\int_0^T\int_\omega s^3\Theta^3\frac{x^2}{a_2}e^{2s\varphi_2}V^2\,dx\,dt + C_\varepsilon\int_0^T\int_\omega e^{-2s(4\Phi_2+3\varphi_2)}U^2\,dx\,dt.
\end{aligned}$$

So

$$|L| \le \varepsilon^2\int_0^T\int_\omega s^3\Theta^3\frac{x^2}{a_2}e^{2s\varphi_2}V^2\,dx\,dt + C_\varepsilon\int_0^T\int_\omega e^{-2s(4\Phi_2+3\varphi_2)}U^2\,dx\,dt. \tag{46}$$

From (45)–(46), we deduce

$$|I_1| \le 2\varepsilon \int_0^T \int_0^1 s\Theta a_2 e^{2s\varphi_2} V_x^2 \,dx\,dt + \frac{\varepsilon}{2}\int_0^T \int_\omega s^3\Theta^3 \frac{x^2}{a_2} e^{2s\varphi_2} V^2 \,dx\,dt$$
$$+ C_\varepsilon \int_0^T \int_\omega e^{-2s(4\Phi_2+3\varphi_2)} U^2 \,dx\,dt. \tag{47}$$

Similarly, we obtain

$$|I_2| \le \varepsilon \int_0^T \int_0^1 s\Theta a_2 e^{2s\varphi_2} V_x^2 \,dx\,dt + C_\varepsilon \int_0^T \int_\omega s\Theta e^{-2s(2\Phi_2+\varphi_2)} U^2 \,dx\,dt, \tag{48}$$
$$|I_3| \le \varepsilon \int_0^T \int_0^1 s^3\Theta^3 \frac{x^2}{a_2} e^{2s\varphi_2} V^2 \,dx\,dt + C_\varepsilon \int_0^T \int_\omega s\Theta e^{-2s(2\Phi_2+\varphi_2)} U^2 \,dx\,dt. \tag{49}$$

Consequently, from the estimates (47)–(49) and Lemma 2.4 we conclude that

$$\int_0^T \int_0^1 b\chi e^{-2s\Phi_2} V^2 \,dx\,dt \le 3\varepsilon J(V) + C_\varepsilon \int_0^T \int_\omega U^2 \,dx\,dt.$$

Finally, since $\chi \equiv 1$ on ω_1, (39) achieves the proof. □

2.3. Observability Inequality. In this subsection we use Carleman estimate (40) to prove the following observability inequality for the adjoint system (17)–(20).

Theorem 2.9. *Let $T > 0$ be given. There exists a positive constant C_T such that every solution (U, V) of* (17)–(20) *satisfies*

$$\int_0^1 [U^2(T,x) + V^2(T,x)]\,dx \le C_T \int_0^T \int_\omega U^2(t,x)\,dx\,dt. \tag{50}$$

Proof. Multiplying the first and the second equations in the system (17)–(20) respectively by U_t and V_t. Integrating over $(0,1)$ the sum of the new equations, we have

$$0 = \int_0^1 [U_t^2 + V_t^2]\,dx - [a_1 U_x U_t]_0^1 - [a_2 V_x V_t]_0^1 + \int_0^1 c_1 U U_t \,dx$$
$$+ \int_0^1 c_2 V V_t + \int_0^1 b V U_t \,dx + \frac{1}{2}\frac{d}{dt}\int_0^1 [a_1 U_x^2 + a_2 V_x^2]\,dx.$$

Using Young's inequality and the monotony of the function $\frac{x^2}{a_i(x)}$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1 [a_1 U_x^2 + a_2 V_x^2]\,dx &\le \int_0^1 c_1^2 U^2\,dx + \int_0^1 (c_2^2 + b^2) V^2\,dx \\ &\le C\int_0^1 \big(U^2(t,x) + V^2(t,x)\big)\,dx \\ &\le \frac{C}{\mathrm{Min}\big(a_1(1), a_2(1)\big)}\int_0^1 \left(\frac{a_1}{x^2} U^2(t,x) + \frac{a_2}{x^2} V^2(t,x)\right) dx.\end{aligned}$$

Using Hardy–Poincaré inequality, we obtain

$$\frac{d}{dt}\int_0^1 [a_1 U_x^2 + a_2 V_x^2]\,dx \le C_0 \int_0^1 [a_1 U_x^2 + a_2 V_x^2]\,dx\,dt.$$

Hence

$$\frac{d}{dt}\left\{e^{-C_0 t}\int_0^1 [a_1 U_x^2 + a_2 V_x^2]\,dx\right\} \le 0.$$

Consequently, the function $t \mapsto e^{-C_0 t}\int_0^1 [a_1 U_x^2 + a_2 V_x^2]\,dx$ is non-increasing. Thus,

$$\int_0^1 [a_1(x) U_x^2(T,x) + a_2(x) V_x^2(T,x)]\,dx \le e^{C_0 T}\int_0^1 [a_1(x) U_x^2(t,x) + a_2(x) V_x^2(t,x)]\,dx.$$

Integrating over $\left[\frac{T}{4}, \frac{3T}{4}\right]$, one has

$$\begin{aligned}&\int_0^1 [a_1(x) U_x^2(T,x) + a_2(x) V_x^2(T,x)]\,dx \\ &\quad\le \frac{2e^{C_0 T}}{T}\int_{T/4}^{3T/4}\int_0^1 [a_1(x) U_x^2 + a_2(x) V_x^2]\,dx\,dt \\ &\quad\le C_T \int_{T/4}^{3T/4}\int_0^1 s\Theta[a_1 e^{2s\varphi_1} U_x^2 + a_2 e^{2s\varphi_2} V_x^2]\,dx\,dt \\ &\quad\le C_T \int_0^T\int_\omega U^2(t,x)\,dx\,dt,\end{aligned}$$

by the Carleman estimate (40). Consequently using the Hardy–Poincaré inequality and the fact that the function $\frac{x^2}{a_i(x)}$ is non-decreasing, we deduce the estimate (50). □

3. Null controllability for semilinear systems

The aim of this section is to prove null controllability of the semilinear system

$$u_t - (a_1(x)u_x)_x + F_1(t,x,u) = h(t,x)1_\omega, \qquad (t,x) \in (0,T) \times (0,1), \tag{51}$$

$$v_t - (a_2(x)v_x)_x + F_2(t,x,u,v) = 0, \qquad (t,x) \in (0,T) \times (0,1), \tag{52}$$

$$u(t,0) = u(t,1) = v(t,0) = v(t,1) = 0, \qquad t \in (0,T), \tag{53}$$

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \qquad x \in (0,1). \tag{54}$$

We will use a standard strategy, as in [2], [8], [21], which consists in using the linearization technique, the approximate null controllability, the variational approach and the Schauder fixed point theorem.

Here and henceforth we assume in addition to (H2), that the functions $F_1(t,x,y)$ and $F_2(t,x,y,z)$ satisfy the assumptions

$$F_1(\cdot,\cdot,y) \text{ and } F_2(\cdot,\cdot,y,z) \text{ are measurable}, \qquad (y,z) \in \mathbb{R} \times \mathbb{R}, \tag{55}$$

$$F_1(t,x,0) = 0, \quad F_2(t,x,0,0) = 0, \qquad (t,x) \in [0,T] \times [0,1], \tag{56}$$

$$F_1(t,x,\cdot) \in C^1(\mathbb{R}), \quad F_2(t,x,\cdot,\cdot) \in C^1(\mathbb{R} \times \mathbb{R}), \qquad (t,x) \in [0,T] \times [0,1]. \tag{57}$$

Suppose moreover that there exist $M > 0$, $\gamma > 0$ and an open set $\omega_1 \Subset \omega$ such that

$$\frac{\partial F_i}{\partial y} \le M, \qquad \frac{\partial F_2}{\partial z} \le M, \tag{58}$$

$$\frac{\partial F_2}{\partial y}(t,x,y,z) \ge \gamma, \qquad x \in \omega_1,\ t \in [0,T],\ (y,z) \in \mathbb{R}^2. \tag{59}$$

Thus the semilinear system (51)–(54) can be written as

$$\begin{aligned}
&u_t - (a_1(x)u_x)_x + c_1^{u,v}(t,x)u = h(t,x)1_\omega, \qquad (t,x) \in (0,T) \times (0,1),\\
&v_t - (a_2(x)v_x)_x + c_2^{u,v}(t,x)v + b^{u,v}(t,x)u = 0, \qquad (t,x) \in (0,T) \times (0,1),\\
&u(t,0) = u(t,1) = v(t,0) = v(t,1) = 0, \qquad t \in (0,T),\\
&u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \qquad x \in (0,1),
\end{aligned}$$

where

$$\begin{aligned}
c_1^{u,v}(t,x) &:= \int_0^1 \frac{\partial F_1}{\partial y}(t,x,\lambda u(t,x))\,d\lambda,\\
c_2^{u,v}(t,x) &:= \int_0^1 \frac{\partial F_2}{\partial z}(t,x,\lambda u(t,x),\lambda v(t,x))\,d\lambda,\\
b^{u,v}(t,x) &:= \int_0^1 \frac{\partial F_2}{\partial y}(t,x,\lambda u(t,x),\lambda v(t,x))\,d\lambda.
\end{aligned}$$

Let the space

$$X_T = C\big(0,T;\big(L^2(0,1)\big)^2\big) \cap L^2(0,T;H^1_{a_1} \times H^1_{a_2}),$$

be equipped with the norm

$$\|(y,z)\|^2_{X_T} := \sup_{t\in[0,T]} \big(\|y(t)\|^2_{L^2} + \|z(t)\|^2_{L^2}\big) + \int_0^T (\|\sqrt{a_1}\,y_x\|^2_{L^2} + \|\sqrt{a_2}z_x\|^2_{L^2})\,dt.$$

For a fixed $(y,z) \in X_T$, consider the associated linear system

$$u_t - \big(a_1(x)u_x\big)_x + c_1^{y,z}(t,x)u = h(t,x)1_\omega, \qquad (t,x) \in (0,T) \times (0,1), \tag{60}$$
$$v_t - \big(a_2(x)v_x\big)_x + c_2^{y,z}(t,x)v + b^{y,z}(t,x)u = 0, \qquad (t,x) \in (0,T) \times (0,1), \tag{61}$$
$$u(t,0) = u(t,1) = v(t,0) = v(t,1) = 0, \qquad t \in (0,T), \tag{62}$$
$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \qquad x \in (0,1), \tag{63}$$

and its adjoint system

$$U_t - \big(a_1(x)U_x\big)_x + c_1^{y,z}(t,x)U + b^{y,z}(t,x)V = 0, \qquad (t,x) \in (0,T) \times (0,1), \tag{64}$$
$$V_t - \big(a_2(x)V_x\big)_x + c_2^{y,z}(t,x)V = 0, \qquad (t,x) \in (0,T) \times (0,1), \tag{65}$$
$$U(t,0) = U(t,1) = V(t,0) = V(t,1) = 0, \qquad t \in (0,T), \tag{66}$$
$$U(0,x) = U_0(x), \quad V(0,x) = V_0(x), \qquad x \in (0,1). \tag{67}$$

By assumptions (58) and (59) one has the uniform estimates

$$\|c_i^{y,z}\|_\infty, \qquad \|b^{y,z}\|_\infty \le M, \tag{68}$$
$$b^{y,z} \ge \gamma > 0 \quad \text{on } [0,T] \times \omega_1. \tag{69}$$

To build an adequate fixed point operator, we need first to show the uniqueness of the control with minimal norm. For a given $\epsilon > 0$ and $(u_0, v_0) \in L^2(0,1) \times L^2(0,1)$, introduce the quadratic functionals

$$J_{\epsilon,y,z}(h) = \frac{1}{2}\int_0^T \int_0^1 h^2\,dx\,dt + \frac{1}{2\epsilon}\|(u,v)(T)\|^2_{L^2}, \tag{70}$$

$$J^*_{\epsilon,y,z}(U_0,V_0) = \frac{1}{2}\int_0^T \int_\omega U^2\,dx\,dt + \frac{\epsilon}{2}\|(U_0,V_0)\|^2_{L^2} + \int_0^1 \big(U(T)u_0 + V(T)v_0\big)\,dx, \tag{71}$$

where (u, v) is the solution of the linear system (60)–(63) with initial data (u_0, v_0) and (U, V) is the solution of the adjoint system with given $(U_0, V_0) \in L^2(0,1) \times L^2(0,1)$. By classical arguments, the minimization problems

$$\min\{J_{\epsilon,y,z}(h), h \in L^2((0,T) \times (0,1))\} \quad \text{and}$$
$$\min\{J^*_{\epsilon,y,z}(U_0, V_0), (U_0, V_0) \in L^2(0,1) \times L^2(0,1)\}$$

have unique solutions $h^{\epsilon,y,z}$ and $(U_0^{\epsilon,y,z}, V_0^{\epsilon,y,z})$ such that

$$h^{\epsilon,y,z} = U^{\epsilon,y,z} 1_\omega \tag{72}$$

$$(U_0^{\epsilon,y,z}, V_0^{\epsilon,y,z}) = -\frac{1}{\epsilon}(u^{\epsilon,y,z}, v^{\epsilon,y,z})(T), \tag{73}$$

where $(u^{\epsilon,y,z}, v^{\epsilon,y,z})$ is the solution of (60)–(63) with the control $h^{\epsilon,y,z}$ and $(U^{\epsilon,y,z}, V^{\epsilon,y,z})$ the one of the adjoint system (64)–(67) with initial data $(U_0^{\epsilon,y,z}, V_0^{\epsilon,y,z})$. Since $J^*_\epsilon(U_0^{\epsilon,y,z}, V_0^{\epsilon,y,z}) \leq 0$, by (71) and (73), one has

$$\begin{aligned}\frac{1}{2}\int_0^T \int_\omega |U^{\epsilon,y,z}|^2\,dx\,dt + \frac{1}{2\epsilon}\|(u^{\epsilon,y,z}, v^{\epsilon,y,z})(T)\|^2_{L^2} \\ \leq \|(U^{\epsilon,y,z}, V^{\epsilon,y,z})(T)\|_{L^2}\|(u_0, v_0)\|_{L^2}.\end{aligned} \tag{74}$$

On the other hand by the observability inequality (50), we have

$$\|(U^{\epsilon,y,z}, V^{\epsilon,y,z})(T)\|^2_{L^2} \leq C_T \int_0^T \int_\omega |U^{\epsilon,y,z}|^2\,dx\,dt. \tag{75}$$

The estimates (72), (74) and (75) provide the essential estimate

$$\frac{1}{2}\int_0^T \int_0^1 |h^{\epsilon,y,z}|^2\,dx\,dt + \frac{1}{2\epsilon}\|(u^{\epsilon,y,z}, v^{\epsilon,y,z})(T)\|^2_{L^2} \leq C_T\|(u_0, v_0)\|^2_{L^2}. \tag{76}$$

The uniqueness of the control $h^{\epsilon,y,z}$ defines an operator $K_\epsilon : X_T \to X_T$, $(y,z) \mapsto (u^{\epsilon,y,z}, v^{\epsilon,y,z})$. It is clear that every fixed point (u^ϵ, v^ϵ) of K_ϵ is a solution of the semilinear system (51)–(54) associated to $h^{\epsilon,u^\epsilon,v^\epsilon}$ and satisfies

$$\|(u^\epsilon, v^\epsilon)(T)\|^2_{L^2} \leq \epsilon C.$$

That is, the semilinear system (51)–(54) will be approximately null controllable. To prove this, let us suppose first that $(u_0, v_0) \in H^1_{a_1} \times H^1_{a_2}$. By Proposition 2.2 and (68), we have the uniform estimate

$$\sup_{t\in[0,T]} \|(u^{\epsilon,y,z}, v^{\epsilon,y,z})(t)\|^2_{H^1_{a_1}\times H^1_{a_2}} + \int_0^T \left(\|(u_t^{\epsilon,y,z}, v_t^{\epsilon,y,z})\|^2_{L^2(0,1)} + \left\|\left((a_1 u_x^{\epsilon,y,z})_x, (a_2 v_x^{\epsilon,y,z})_x\right)\right\|^2_{L^2(0,1)}\right) dt$$
$$\le C_T\left(\|(u_0,v_0)\|^2_{H^1_{a_1}\times H^1_{a_2}} + \|h^{\epsilon,y,z}\|^2_{L^2((0,T)\times(0,1))}\right).$$

This with (76) yield the uniform estimates

$$\|(u^{\epsilon,y,z}, v^{\epsilon,y,z})\|_{X_T} \le C_T \|(u_0,v_0)\|_{H^1_{a_1}\times H^1_{a_2}}, \tag{77}$$
$$\|(u^{\epsilon,y,z}, v^{\epsilon,y,z})\|_{Y_T} \le C_T \|(u_0,v_0)\|_{H^1_{a_1}\times H^1_{a_2}}, \tag{78}$$

where $Y_T := H^1\left(0,T;\left(L^2(0,1)\right)^2\right) \cap L^2(0,T;H^2_{a_1}\times H^2_{a_2})$ with the norm

$$\|(y,z)\|^2_{Y_T} := \int_0^T \left(\|(y,z)\|^2_{H^1_{a_1}\times H^1_{a_2}} + \|(y_t,z_t)\|^2_{L^2} + \left\|\left((a_1 y_x)_x, (a_2 z_x)_x\right)\right\|^2_{L^2}\right) dt.$$

Thus the range of K_ϵ is a subset of the ball $B(0,R)$ of X_T with $R := C_T\|(u_0,v_0)\|_{H^1_{a_1}\times H^1_{a_2}}$, where C_T is the constant of (77). In particular $K_\epsilon(B(0,R)) \subset B(0,R)$. Prove now that K_ϵ is continuous and compact. The compactness follows from the compactness of the embedding

$$Y_T \overset{c}{\hookrightarrow} X_T, \tag{79}$$

see [8]. Let $(y_n,z_n) \to (\bar y,\bar z)$ in X_T as $n\to+\infty$. For simplicity set $u_n := u^{\epsilon,y_n,z_n}$, $v_n := v^{\epsilon,y_n,z_n}$ and $h_n := h^{\epsilon,y_n,z_n}$. By (78), the set (u_n,v_n) is bounded in Y_T. Hence, up to a subsequence it converges weakly in Y_T to a limit $(\bar u,\bar v)$ and strongly in X_T thanks to (79). Similarly, by (76), up to a subsequence, h^{ϵ,y_n,z_n} converges weakly to a limit $\bar h$. One can show that $(\bar u,\bar v)$ is the solution of the system (60)–(63) corresponding to $(\bar y,\bar z)$ and $\bar h$. So to prove that $K_\epsilon(\bar y,\bar z) = (\bar u,\bar v)$ it is sufficient to show that $\bar h = h^{\epsilon,\bar y,\bar z}$. By definition of h_n one has for all $h\in L^2\left((0,T)\times(0,1)\right)$

$$\frac12\int_0^T\int_0^1 |h_n|^2\,dx\,dt + \frac{1}{2\epsilon}\|(u_n,v_n)(T)\|^2_{L^2} \le \frac12\int_0^T\int_0^1 h^2\,dx\,dt + \frac{1}{2\epsilon}\|(u^{y_n,z_n,h}, v^{y_n,z_n,h})(T)\|^2$$

where $(u^{y_n,z_n,h}, v^{y_n,z_n,h})$ is the solution of the linear system (60)–(63) associated to (y_n,z_n) and h. Passing to the limit in the last inequality, one has for all $h\in L^2\left((0,T)\times(0,1)\right)$

$$\frac12\int_0^T\int_0^1 |\bar h|^2\,dx\,dt + \frac{1}{2\epsilon}\|(\bar u,\bar v)(T)\|^2_{L^2} \le \frac12\int_0^T\int_0^1 h^2\,dx\,dt + \frac{1}{2\epsilon}\|(u^{\bar y,\bar z,h}, v^{\bar y,\bar z,h})(T)\|^2_{L^2}.$$

That is, $\bar{h}$ minimizes $J_{\epsilon,\bar{y},\bar{z}}$. Thus $K_\epsilon(\bar{y},\bar{z}) = (\bar{u},\bar{v})$, and thus K_ϵ is continuous. The following result is then proved.

Theorem 3.1. *For every* $(u_0, v_0) \in H^1_{a_1} \times H^1_{a_2}$, *the semilinear system* (51)–(54) *is approximately null controllable, i.e., for all* $\epsilon > 0$ *there exists* $h_\epsilon \in L^2\big((0,T)\times(0,1)\big)$ *such that the associated solution* $(u^{h_\epsilon}, v^{h_\epsilon})$ *satisfies*

$$\|(u^{h_\epsilon}, v^{h_\epsilon})(T)\|_{L^2(0,1)} \le \epsilon. \tag{80}$$

Moreover there exists a constant C_T *such that*

$$\int_0^T \int_0^1 |h_\epsilon(t,x)|^2\,dx\,dt \le C_T \|(u_0,v_0)\|^2_{L^2(0,1)}.$$

As a consequence of this theorem, we have the following null controllability result.

Theorem 3.2. *For every* $(u_0, v_0) \in H^1_{a_1} \times H^1_{a_2}$, *the semilinear degenerate system* (51)–(54) *is null controllable, that is, there exists* $h \in L^2\big((0,T)\times(0,1)\big)$, *i.e., the associated solution* (u^h, v^h) *satisfies*

$$(u^h, v^h)(T,x) = (0,0), \qquad \forall x \in (0,1).$$

Moreover, there exists a constant C_T *such that*

$$\int_0^T \int_0^1 |h(t,x)|^2\,dx\,dt \le C_T \|(u_0,v_0)\|^2_{L^2(0,1)}.$$

Proof. By Theorem 3.1, the set h_ϵ has a subsequence converging weakly in $L^2\big((0,T)\times(0,1)\big)$ to a limit h_0 satisfying

$$\int_0^T \int_0^1 |h_0|^2(t,x)\,dx\,dt \le C_T \|(u_0,v_0)\|^2_{L^2(0,1)}.$$

Furthermore, one can prove that $(u^{h_\epsilon}, v^{h_\epsilon})$ converges to (u^{h_0}, v^{h_0}) strongly in X_T as $\epsilon \to 0$. Moreover, (u^{h_0}, v^{h_0}) solves (51)–(54) with $h = h_0$ and by (80) one has for all $x \in (0,1)$

$$(u^{h_0}, v^{h_0})(T,x) = (0,0).$$

This concludes the proof. □

Now we are in the position to state the main null controllability result for semilinear degenerate cascade systems.

Theorem 3.3. *For every* $(u_0, v_0) \in L^2(0,1) \times L^2(0,1)$, *the semilinear degenerate parabolic system* (51)–(54) *is null controllable.*

Proof. By Proposition 2.3, the system (13)–(16) considered on $\left(0, \frac{T}{2}\right) \times (0,1)$ with initial data $(u_0, v_0) \in L^2(0,1) \times L^2(0,1)$ has a solution $(Y, Z) \in X_{T/2}$. So, for some $t_0 \in (0, T/2)$, $(Y, Z)(t_0) \in H_0^1 \times H_0^1$. Consider now the following system

$$U_t - \left(a_1(x) V_x\right)_x + F_1(t, x, U) = h(t,x) 1_\omega, \qquad (t,x) \in (t_0, T) \times (0,1), \tag{81}$$

$$V_t - \left(a_2(x) V_x\right)_x + F_2(t, x, U, V) = 0, \qquad (t,x) \in (t_0, T) \times (0,1), \tag{82}$$

$$U(t,0) = U(t,1) = V(t,0) = V(t,1) = 0, \qquad t \in (t_0, T), \tag{83}$$

$$U(t_0, x) = Y(t_0, x), \quad V(t_0, x) = Z(t_0, x), \qquad x \in (0,1). \tag{84}$$

By Theorem 3.2 there exists a control $h_1 \in L^2\left((t_0, T) \times (0,1)\right)$ such that the system (81)–(84) has a solution (U, V) satisfying $(U, V)(T, x) = 0$ for all $x \in (0,1)$. One defines

$$u := \begin{cases} Y & \text{in } [0, t_0], \\ U & \text{in } [t_0, T], \end{cases} \qquad v := \begin{cases} Z & \text{in } [0, t_0], \\ V & \text{in } [t_0, T], \end{cases} \qquad \text{and} \qquad h := \begin{cases} 0 & \text{in } [0, t_0], \\ h_1 & \text{in } [t_0, T]. \end{cases}$$

Hence (u, v) is a solution of the system (51)–(54) and satisfies $(u, v)(T, x) = 0$ for all $x \in (0,1)$. This completes the proof. □

4. Appendix

As in [1], [7], we give the proof of the Caccioppoli's inequality for linear cascade systems with two degeneracies.

Lemma 4.1. *Let* $\omega' \Subset \omega$. *Then there exists a positive constant* C *such that*

$$\int_0^T \int_{\omega'} [U_x^2(t,x) + V_x^2(t,x)] e^{2s\varphi_i} \, dx \, dt \le C \int_0^T \int_\omega [U^2(t,x) + V^2(t,x)] e^{2s\varphi_i} \, dx \, dt.$$

Proof. Let $\chi \in \mathscr{C}^\infty(0,1)$ such that $\operatorname{supp} \chi \subset \omega$ and $\chi \equiv 1$ on ω'. We have

$$\begin{aligned} 0 &= \int_0^T \frac{d}{dt} \left[\int_0^1 \chi^2 e^{2s\varphi_i} (U^2 + V^2) \, dx \right] dt \\ &= 2 \int_0^T \int_0^1 s\dot{\varphi}_i \chi^2 e^{2s\varphi_i} (U^2 + V^2) \, dx \, dt - 2 \int_0^T \int_0^1 \chi^2 e^{2s\varphi_i} a_1(x) (U_x)^2 \, dx \, dt \\ &\quad - 2 \int_0^T \int_0^1 (\chi^2 e^{2s\varphi_i})_x a_1(x) U U_x \, dx - 2 \int_0^T \int_0^1 \chi^2 e^{2s\varphi_i} c_2 V^2 \, dx \, dt \, dt \end{aligned}$$

$$- 2\int_0^T \int_0^1 \chi^2 e^{2s\varphi_i} c_1 U^2 \,dx\,dt - 2\int_0^T \int_0^1 \chi^2 e^{2s\varphi_i} bUV \,dx\,dt$$
$$- 2\int_0^T \int_0^1 \chi^2 e^{2s\varphi_i} a_2(x)(V_x)^2 \,dx\,dt - 2\int_0^T \int_0^1 (\chi^2 e^{2s\varphi_i})_x a_2(x) V V_x \,dx\,dt.$$

Therefore

$$\int_0^T \int_0^1 \chi^2 e^{2s\varphi_i}(a_1 U_x^2 + a_2 V_x^2)\,dx\,dt$$
$$= \int_0^T \int_0^1 s\dot{\varphi}_i \chi^2 e^{2s\varphi_i}(U^2 + V^2)\,dx\,dt - \int_0^T \int_0^1 (\chi^2 e^{2s\varphi_i})_x (a_1 U U_x + a_2 V V_x)\,dx\,dt$$
$$- \int_0^T \int_0^1 \chi^2 e^{2s\varphi_i}(c_1 U^2 + c_2 V^2)\,dx\,dt - \int_0^T \int_0^1 \chi^2 e^{2s\varphi_i} bUV \,dx\,dt.$$

Then, using (H1) and the fact that $2UU_x = (U^2)_x$, $UV \le U^2 + V^2$, a_i, φ_i, ξ and their derivatives are bounded and that ξ is supported in ω and $\xi \equiv 1$ in ω', we deduce the estimate

$$\min_{x\in\omega'}\{a_1(x), a_2(x)\} \int_0^T \int_{\omega'} e^{2s\varphi_i}(U_x^2 + V_x^2)\,dx\,dt \le C \int_0^T \int_{\omega} e^{2s\varphi_i}(U^2 + V^2)\,dx\,dt$$

This ends the proof. □

5. Conclusion

In this paper, we studied the null controllability of cascade semilinear systems with two different degeneracies, using the Carleman estimate obtained in [1]. It is known in the literature that general coupled non degenerate systems can be transformed to cascade systems. This fact is also true for degenerate coupled systems with the same degeneracy. But this is not true in the case of two different degeneracies, because the exponential of weight functions can be compared as done in this paper, but will not never be equivalent. In a forthcoming paper, we deal with this problem by a different way.

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