Portugal. Math. (N.S.)
Vol. 68, Fasc. 4, 2011, 381–387
DOI 10.4171/PM/1897

Portugaliae Mathematica

A note on *n*-gerbes and transgressions

Mauro Spera*

(Communicated by Rui Loja Fernandes)

Abstract. In this note we provide an explicit interpretation of a class of $(q-1)$-gerbes with multi-layered connections in terms of transgression of a fibrewise closed q-form on a fibration to a closed $(q+1)$-form on the base manifold, with the basic example of the Euler class of an oriented vector bundle in mind ($q \geq 0$). Picken's and Ferreira–Gothen's n-gerbopoles are discussed from this point of view. Furthermore, string structures (à la Cocquereaux–Pilch and à la Spera–Wurzbacher) are briefly addressed and recast within the proposed framework.

Mathematics Subject Classification (2010). 53C08, 55N30, 57R22, 55R25, 57R20, 81S10, 81T30.

Keywords. Gerbes, transgression, Euler class, geometric quantization, string structures.

1. Introduction

The concept of (abelian) *n-gerbe* together with its related differential geometric apparatus of *multi-layered connections* and *curvature* has been introduced by N. Hitchin and K. Chatterjee ([9], [3]) and further investigated, among others, by R. Picken and A. Ferreira and P. Gothen ([12], [7]). It is essentially cohomological (an integral $(n+2)$-cohomology class on a manifold M). However, one would like to understand it in geometric terms. This is the original motivation of the groundbreaking work of J. L. Brylinski aimed at generalizing the Weil–Kostant theorem, the cornerstone of geometric quantization ([2]), to 3-cohomology classes.

In this note we give a construction of $(q-1)$-gerbes via transgressing a fibrewise closed q-form ω on a fibration to a closed $(q+1)$-form τ on the base manifold, emphasizing the basic example of the Euler class of an oriented sphere bundle (see [1], and Section 2 as well). Indeed, the various elements of the associated

*The author is indebted to MIUR (grant ex 60% funds) for financial support. He is also grateful to a referee for the careful reading and critical observations.

Čech–de Rham complex successively arising onward to a D-cocycle (the final obstruction to the latter being τ), actually yield a multi-layered connection, with (generalized) curvature given by (pull-back of) τ itself. When viewed in this way, a $(q-1)$-gerbe may arise as the obstruction to patching a family of $(q-2)$-gerbes together; the *collating formula* in [1] then yields an explicit expression for a q-form ψ restricting fibrewise to ω. The q-form ψ is in fact the last and most relevant part of the multi-layered connection. This interpretation stems from the fact that the first Chern class of a complex line bundle, i.e., the Euler class of the associated real (oriented) vector bundle, arises as the obstruction to gluing together fibrewise angular forms on the corresponding principal S^1-bundle (cf. [1]). A similar interpretation (see [2], [13]) can be devised for the GLSW-construction of fibrations in pre-symplectic manifolds (with the pre-symplectic forms defining the same cohomology class), yielding a gerbe (1-gerbe in the new terminology), i.e., a closed 3-form on the base manifold measuring the failure of constructing a closed 2-form on the total space restricting fibrewise to the given forms, or, equivalently, via the Weil–Kostant theorem, to find a complex line bundle on the total space restricting fibrewise to the WK-line bundles (one assumes the fibres simply connected, so their first cohomology vanishes). An interpretation of *string structures* in these terms has been given in [13] and it will be briefly reviewed in Section 4 of the present note, together with the Cocquereaux–Pilch one ([5]). The latter goes through the interpretation of the Chern–Simons form on a principal G-bundle again as a generalized connection (actually the last piece of a multi-layered connection). In Section 3 we address the issue of constructing n-gerbopoles in the above manner, this leading directly to the question of realizing the (positive) generator of $H^{n+2}(S^{n+2}, \mathbb{Z})$ as the Euler class of an oriented vector bundle. The answer to this is known, and we get, in particular, a "gerby" interpretation of the three basic Hopf fibrations.

2. Gerbes via transgression

All objects are assumed to be smooth. We closely follow [1]. One can formulate the concepts below in condensed form in terms of Leray's spectral sequence, but we stick to a more concrete description which turns out to be perfectly tailored to our purposes. Let $\pi : E \to M$ be a fibration over a manifold with base M (equipped with a good covering $\mathscr{U}$, yielding the covering $\pi^{-1}\mathscr{U}$ on E), typical fibre F, total space E. Recall (see e.g. [1]) that a (de Rham) q-cohomology class $\omega \in H^q(F)$ is *transgressive* if it is represented by the restriction of a global q-form ψ on E such that

$$d\psi = \pi^*\tau$$

for some $(q+1)$-form τ on the base M. Since π^* is injective, τ is actually closed and thus defines a de Rham cohomology class in $H^{q+1}(M)$. Now, if ω is trangressive, it can be extended to a cochain

$$\alpha = \alpha_0 + \alpha_1 + \cdots + \alpha_q$$

(with $\alpha_0 = \omega$) in the double complex $C^*(\pi^{-1}\mathscr{U}, \Omega^*)$ (Čech–de Rham) such that

$$D\alpha = \pi^*\beta$$

for a Čech cocycle β (where $D = D' + D'' = \delta + (-1)^\bullet d$ is of course the total differential). The *collating formula* ([1]) then yields an explicit expression for the sought for τ, related to β:

$$\psi = \sum_{i=0}^{q} (-1)^i (D''K)^i \alpha_i + (-1)^{q+1} K (D''K)^q \pi^*\beta,$$

where K is a homotopy operator relative to δ, i.e., $\delta K + K\delta = 1$, defined in [1] and

$$d\psi = (-1)^{q+1}(D''K)^{q+1}\pi^*\beta \equiv \pi^*\tau,$$

where $\tau = (-D''K)^{q+1}\beta$ is a closed global form on M (this explicit Čech–de Rham isomorphism is given in [1], Prop. 9.8).

We can treat ψ directly as a generalized (abelian) connection (q-connection), with curvature $d\psi = \pi^*\tau$, i.e., the pull-back of a $(q+1)$-form on the base. Here α_0 is ω: more precisely, extend ω to ω_U on $U \times F$, $U \in \mathscr{U}$, requiring that on non empty overlaps, the cohomology classes defined by the ω_U's agree (a sort of generalized orientability condition). Now, upon examining the constructions of [12], [7], we see immediately that the following holds:

Theorem 2.1. (i) *In the above notation, given a transgressive element $\omega \in H^q(F)$, the cochain α is actually a $(q-1)$-gerbe equipped with a multi-layered connection, whose curvature is τ.*

(ii) *In particular, if the j-th cohomology of F vanishes for $j = 1, 2, \ldots, q-1$, any $\omega \in H^q(F)$ is transgressive, so it gives rise to a $(q-1)$-gerbe with multi-layered connection.*

(iii) *In particular,* (*minus*) *the Euler class $e(E)$ of an oriented sphere bundle E of rank q becomes the curvature* (*and Chern class*) *of a $(q-1)$-gerbe with multi-layered connection, manufactured from the angular form on the fibre and terminating with the global angular form.*

Proof. This is just a matter of "bookkeeping": given a good covering of M, pulling back to a covering of E, and given local forms $\omega^{0,q}$ in the ensuing Čech-de Rham complex yielding the same cohomology classes on non trivial overlaps (actually, $[\omega]$), one forms the "zig-zag" cochain

$$\omega^{0,q} + \omega^{1,q-1} + \cdots + \omega^{q,0}$$

(we shifted to the notation $\omega^{j,q-j} := \alpha_j$ for further clarity) and $\delta\omega^{q,0}$ yields a class β in $\check{H}(M, R)$ (or in $\check{H}(M, \mathbb{Z})$, had we started from an integral class). Comparison with [12] and [7] yields (shorthand notation, and up to minor differences)

$$\omega^{q,0} = \log g, \qquad \omega^{j,q-j} = A^j$$

i.e.,

$$\mathscr{G} = \exp \omega^{q,0} + \omega^{1,q-1} + \cdots + \omega^{j,q-j} + \cdots + \omega^{q,0} - \pi^*\tau \qquad (F - G)$$

or

$$\mathscr{G} = \omega^{q,0} + \omega^{1,q-1} + \cdots, \omega^{j,q-j} + \cdots, \omega^{q,0} - \pi^*\tau. \qquad (P)$$

In particular, a posteriori, the $\omega^{0,q} = A^q$ can be taken to be the restrictions of ψ. Part (iii) is then clear from [1].

It is important to observe that the Picken and Ferreira-Gothen forms are defined on the base manifold M, whereas those pertaining to the Čech-de Rham complex live on the total space E; therefore, in order to properly compare the two constructions, the latter are to be pulled back to M via local trivializing sections. This way of proceeding is akin to the construction of local connection forms on the base manifold from a principal connection form on the total space.

Notice that since β becomes τ, via the Čech–de Rham isomorphism, the latter is actually the *Chern class* of the $(q-1)$-gerbe with connection above ([12], [7]). □

The cohomology vanishing condition (ii) is fulfilled in many relevant examples (see also below and Section 4).

3. Gerbopoles via Euler classes

According to Picken and Ferreira-Gothen (see also [2]) an *n-gerbopole* is an n-gerbe with connection realizing the positive generator of $H^{n+2}(S^{n+2}, \mathbb{Z})$; the cases $n = 0$ and $n = 1$ yield, respectively, the *monopole* and the *gerbopole* discussed by Picken ([12], see also [2])). Now examination of the inductive procedure set up in [7], leads to the idea of interpreting it within the above formalism. In particular, one may ask the question of realizing the (positive) generator of $H^{n+2}(S^{n+2},\mathbb{Z})$ as the *Euler class of an oriented vector bundle of rank* $n+2$ (after considering its associated sphere bundle of rank $n+1$ via the introduction of a bundle metric). The answer to this is known, and we get, in particular:

Theorem 3.1 (Realization of gerbopoles via Euler classes of oriented vector bundles). *Let* τ_0 *be the standard generator of* $H^{n+2}(S^{n+2},\mathbb{Z})$. *Then:*

(i) *No realization of* τ_0 *is possible on odd-dimensional spheres. This is true in particular for the original* 1*-gerbopole of Picken* ([12]).

On even-dimensional spheres:

(ii) τ_0 *can be realized only for* $n = 0, 2, 6$ (*this corresponding to the three basic Hopf fibrations*).

(iii) *The same holds for* odd *multiples of* τ_0.

(iv) Even *multiples of* τ_0 *can always be realized.*

Proof. The proof is just a rephrasal of the discussion of the Euler class in [11] or [8]. □

Theorem 3.1 yields, in particular, a possibly interesting "gerby" intepretation of the basic Hopf fibrations $S^1 \to S^3 \to S^2$, $S^3 \to S^7 \to S^4$, $S^7 \to S^{15} \to S^8$; see also Section 4.

4. String structures revisited

In this section we briefly review well known examples of gerbes emphasizing the transgression point of view.

4.1. The first Pontrjagin class as a 2-gerbe. As a further application of the basic construction in Section 2, we consider a principal bundle $G \to P \to M$, with G a compact, connected, simple, simply connected Lie group. Then $\pi_1(G) = \pi_2(G) = 0$ and $\pi_3(G) = \mathbb{Z}$. Therefore $H^i(G,\mathbb{Z}) = 0$ for $i = 1, 2$, and $H^3(G,\mathbb{Z}) = \mathbb{Z}$ is generated by a canonical 3-form v. Transgression to M is then possible and yields the *first Pontrjagin class* $p_1(P)$ as the *curvature of a multi-layered connection terminating with the Chern-Simons form* $CS(\Theta)$ (Θ being a connection on $G \to P \to M$ with curvature Ω, cf. [4], [5]):

$$dCS(\Theta) = p_1(\Omega)$$

(shorthand notation, both forms live in the total space P). In particular, the first two Hopf fibrations can be read within this framework as the *monopole* and the *instanton* bundles, with $G = \mathrm{U}(1)$ and $G = \mathrm{SU}(2) \cong S^3$, respectively. Also, for $\dim M = 3$, the Chern Simons action is the holonomy of the Chern-Simons form viewed as a gerbe connection (folklore).

4.2. String structures à la Cocquereaux–Pilch. This comes from pulling back the above setting to $\mathscr{L}M$ via the map $E := \int_{S^1} ev$, given by evaluation of a loop at $t \in S^1$ followed by fibre integration. "Loopification" of all objects determines a transition from the above 2-gerbe to the 1-gerbe *obstructing the emergence of a string structure* ([10]—see [5] for full details). Schematically, one gets a principal bundle $\mathscr{L}G \to \mathscr{L}P \to \mathscr{L}M$ (up to subtleties extensively discussed in [13]), and fibrewise one has the canonical 2-form ω defining a central extension $1 \to S^1 \to \tilde{\mathscr{L}}G \to \mathscr{L}G \to 1$, which eventually becomes the (fibrewise restriction) of a "Chern–Simons loop form" $\mathscr{L}CS$ on the total space whose differential $d\mathscr{L}CS$ is the pull-back of a 3-form on $\mathscr{L}M$ cohomologous to the pull-back $E^*\big(p_1(\Omega)\big)$ of $p_1(\Omega)$ (viewed as a 4-form on M), ([5]).

4.3. String structures à la Spera–Wurzbacher. Consider the (restricted) isotropic Grassmannian fibration over $\mathscr{L}M$ (see [13]). The typical fibre $\mathscr{I}^0_{\mathrm{res}}$ is simply connected, so *we can transgress the canonical symplectic form thereon*, yielding an obstruction in $H^3(\mathscr{L}M, \mathbb{Z})$ potentially forbidding the existence of a closed 2-form on the total space restricting fibrewise to the latter or equivalently, the existence of a global complex line bundle restricting fibrewise to the Pfaffian bundles over fibres (this is indeed a GLSW point of view [6], [2]). Actually, as it was already shown in [13], everything applies, in general, to a smooth fibration $\pi : Y \to X$ with presymplectic 1-connected fibres Y_x whose presymplectic forms define the same cohomology class (in order to ensure that $H^1(Y_x) = 0$ and to start the transgression procedure). An analogous GLSW portrait can actually be depicted for the previous example as well.

References

[1] R. Bott and L. W. Tu, *Differential forms in algebraic topology*. Graduate Texts in Math. 82, Springer-Verlag, New York 1982. Zbl 0496.55001 MR 658304

[2] J.-L. Brylinski, *Loop spaces, characteristic classes and geometric quantization*. Progr. Math. 107. Birkhäuser, Boston 1993. Zbl 0823.55002 MR 1197353

[3] D. S. Chatterjee, On gerbs. PhD thesis, University of Cambridge, Cambridge 1998.

[4] S. S. Chern and J. Simons, Characteristic forms and geometric invariants. *Ann. of Math.* (2) **99** (1974), 48–69. Zbl 0283.53036 MR 0353327

[5] R. Coquereaux and K. Pilch, String structures on loop bundles. *Comm. Math. Phys.* **120** (1989), 353–378. Zbl 0672.55008 MR 981209

[6] M. J. Gotay, R. Lashof, J. Śniatycki, and A. Weinstein, Closed forms on symplectic fibre bundles. *Comment. Math. Helv.* **58** (1983), 617–621. Zbl 0536.53040 MR 728456

[7] A. Ferreira and P. Gothen, A cohomological approach to n-gerbes with connections. CMUP preprint 2006-13. http://cmup.fc.up.pt/cmup/pbgothen/n-gerbes-2006-03-15.pdf

[8] A. Hatcher, Vector bundles and K-theory. http://www.math.cornell.edu/~hatcher/VBKT/VBpage.html

[9] N. Hitchin, Lectures on special Lagrangian submanifolds. In *Winter school on mirror symmetry, vector bundles and Lagrangian submanifolds*, AMS/IP Stud. Adv. Math. 23, Amer. Math. Soc., Providence, RI, 2001, 151–182. Zbl 1079.14522 MR 1876068

[10] T. P. Killingback, World-sheet anomalies and loop geometry. *Nuclear Phys. B* **288** (1987), 578–588. MR 892061

[11] J. W. Milnor and J. D. Stasheff, *Characteristic classes*. Ann. of Math. Stud. 76, Princeton University Press, Princeton, N.J., 1974. Zbl 0298.57008 MR 0440554

[12] R. Picken, A cohomological description of Abelian bundles and gerbes. In *Twenty years of Bialowieza: a mathematical anthology*, World Scientific Monograph Series in Mathematics 8, World Scientific, Singapore 2005. Zbl 1110.53035 MR 2180790

[13] M. Spera and T. Wurzbacher, Twistor spaces and spinors over loop spaces. *Math. Ann.* **338** (2007), 801–843. Zbl 1147.58019 MR 2317752

Received November 18, 2010; revised July 7, 2011

M. Spera, Dipartimento di Informatica, Università degli Studi di Verona, Ca' Vignal 2, Strada le Grazie 15, 37134 Verona, Italy

E-mail: mauro.spera@univr.it