Global existence of small solutions to the Kerr–Debye model for the three-dimensional Cauchy problem

Mohamed Kanso*

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Abstract. We consider the Kerr–Debye model, describing the electromagnetic wave propagation in a nonlinear medium exhibiting a finite response time. This model is quasilinear hyperbolic and endowed with a dissipative entropy. We consider the Cauchy problem in the three-dimensional case and show that, if the initial data are sufficiently small, the solutions are global in time.

1. Introduction

The domain of nonlinear optics involves activities of physical modelling, experimentations, mathematical analysis, and numerical simulations (see [5] and [11] for instance). Some interesting applications can be found in the domains of lasers, propagation through optic fibers, design of optic devices, and interactions between lasers and plasmas. A model for the nonlinear optical phenomena in isotropic crystal is the following nonlinear Maxwell's system.

$$\partial_t D - \operatorname{curl} H = 0, \qquad \partial_t B + \operatorname{curl} E = 0, \qquad \operatorname{div} D = \operatorname{div} B = 0,$$

where the electromagnetic field (E, H) is linked to the electric and magnetic displacements D and B by the constitutive relations

$$B = \mu_0 H, \qquad D = \varepsilon_0 E + P.$$

The polarization P is nonlinear and μ_0, ε_0 are the free space permeability and permittivity.

If the medium exhibits an instantaneous response we have a Kerr model:

$$P = P_K = \varepsilon_0 \varepsilon_r |E|^2 E.$$

If the medium exhibits a finite response time τ we consider the Kerr–Debye model in which P is given by

$$P = P_{KD} = \varepsilon_0 \chi E,$$

where

$$\partial_t \chi + \frac{1}{\tau} \chi = \frac{1}{\tau} \varepsilon_r |E|^2.$$

See for example [22] or [26] for details.

In this paper we are interested in studying the existence of smooth solutions for the Cauchy problem of the Kerr–Debye model. For the convenience of the reader we study this problem with $\mu_0 = \varepsilon_0 = \varepsilon_r = 1$ and $\tau = 1$. That is we deal with the Cauchy problem

$$\begin{cases} \partial_t D - \operatorname{curl} H = 0, \\ \partial_t H + \operatorname{curl} E = 0, \\ \partial_t \chi = |E|^2 - \chi, \\ D = (1 + \chi) E, \end{cases} \tag{1.1}$$

for $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3$ together with the initial data

$$(D, H, \chi)(0, x) = (D^0, H^0, \chi^0)(x), \tag{1.2}$$

with the divergence free relations

$$\operatorname{div} D = \operatorname{div} H = 0 \quad \text{for } t \geq 0. \tag{1.3}$$

We note that if the initial data are divergence free, then so are (D, H). Moreover, if χ is initially positive, then χ remains positive for all positive times.

The energy density given by

$$\mathscr{E}_{KD}(D, H, \chi) = \frac{1}{2}(1 + \chi)^{-1}|D|^2 + \frac{1}{2}|H|^2 + \frac{1}{4}\chi^2$$

is a strictly convex entropy in the domain $\{\chi \geq 0\}$ (with associated flux $E \times H = (1 + \chi)^{-1} D \times H$). So (1.1) is a quasilinear hyperbolic symmetrizable system.

Therefore the classical existence results in Sobolev spaces in [15] ensure that the problem (1.1)–(1.2) has a unique local (in time) smooth solution for smooth data, and we have the following result (see also [18]).

Proposition 1.1. *Let* $V^0 = (E^0, H^0, \chi^0) \in W^{s,2}(\mathbb{R}^3)$, $s \in \mathbb{N}$, $s > \frac{3}{2} + 1$. *We assume that* $\chi^0 \geq 0$. *Then there exists a maximal smooth solution* (E, H, χ) *to the Kerr–Debye problem* (1.1)–(1.2), *whose lifespan is denoted by* $T^\star$, *and such that*

$$V = (E, H, \chi) \in \mathscr{C}^0([0, T^\star); W^{s,2}) \cap \mathscr{C}^1([0, T^\star); W^{s-1,2}),$$

where we denote by $W^{m,p} := W^{m,p}(\mathbb{R}^3)$ *the usual Sobolev space,* $m \in \mathbb{N}$, $1 \leq p \leq \infty$, *with norm* $\|\cdot\|_{m,p}$.

We are interested in the problem of globality of this solution, i.e., do we have $T^\star = +\infty$?

For some general hyperbolic symmetrisable n-dimensional systems, the local smooth solutions may develop singularities in finite time, even when the initial data are smooth and small (see [21], for example). Despite these general considerations, sometimes dissipative mechanisms due to the source term can prevent the formation of singularities, at least for some restricted classes of initial data.

We remark that the Kerr–Debye system is partially dissipative with the property (see [12]):

$$\frac{d}{dt} \int_{\mathbb{R}^3} \mathscr{E}_{KD}(D, H, \chi) \, dx = -\frac{1}{2} \int_{\mathbb{R}^3} (|E|^2 - \chi)^2 \, dx.$$

In the multidimensional case of partially dissipative hyperbolic systems, many results are known about the global existence of solutions: see, for instance, [13] for the one-dimensional case and [25] for the general case. The authors study the interplay between the source term and the flux under the so-called Shizuta–Kawashima ([SK]) condition introduced first by Shizuta and Kawashima in [23]. This condition concerns the linearized system around a constant equilibrium state. If it is satisfied, the global existence is obtained in [13] and [25]. Also, it is possible to obtain informations about the asymptotic behavior of solutions, see [4]. If the linearized system is of the form

$$\partial_t U + \sum_{j=1}^m A^j \partial_j U = BU, \tag{1.4}$$

where $U = (u, v) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, $n = n_1 + n_2$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$ with $D \in \mathbb{R}^{n_1 \times n_2}$ is negative-definite, the [SK] condition writes: no vector $(X, 0) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is eigenvector of $(I + \sum_{j=1}^m \xi_j A^j)$.

In the 3-D case of the Kerr–Debye model (1.1) the linearized system around the null constant equilibrium writes:

$$\partial_t \begin{pmatrix} E \\ H \end{pmatrix} + \begin{pmatrix} 0 & -\mathrm{curl} \\ \mathrm{curl} & 0 \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} = 0, \qquad \partial_t \chi = -\chi,$$

i.e., the dissipative variable χ and the variable (E, H) are completely uncoupled. Therefore the Shizuta–Kawashima condition does not hold. In [3], K. Beauchard and E. Zuazua prove the existence of global small solutions assuming the existence of a family of constant equilibria fulfilling [SK] converging to zero. This assumption is not satisfied in our case. Consequently, we can not apply neither the results in [25] nor the results in [3] to our model (1.1).

Recently, in [24], a global existence result for smooth solutions to the Euler–Maxwell system was obtained. This system verifies a stability condition which is a modified version of the original one formulated in [23] (see [24] and references therein for details). Nevertheless, this condition is not satisfied by our system. For the same system, see also [10] for other global existence results and decay estimates.

Finally, we point out that, in [19], C. Mascia and R. Natalini started a general study in the one-dimensional case for relaxation hyperbolic systems that violate the [SK] condition. This investigation is motivated by the fact that this condition is not satisfied by various physical systems, still possessing dissipative entropies, as the one-dimensional Kerr–Debye system (see [9]). In the present work, we deal with the three-dimensional Kerr–Debye system so that the methods used in [19] are not relevant.

For our model (1.1), we expect that we could consider the influence of other factors, like the existence of linearly degenerate fields or, in several space dimensions, the well-known faster time decay of the linear system, even for the nondissipative case. In this framework we can mention the work of Reinhard Racke [20], in which he proves an existence result of global small solutions for some nonlinear wave equations. This result is mainly based on a decay estimates for the linear wave equation. This general result yields the global existence of solutions for the Kerr model in the three-dimensional case; see [20] §11.6.

Using both the partial dissipative character of the Kerr–Debye model (1.1) and the dispersion of the Maxwell equations in the 3-D case, we obtain, in the present paper, the following theorem of existence of global solutions.

Theorem 1.2 (Global existence). *There exist an integer $s \geq 7$ and a $\delta > 0$ such that the following holds:*
if the initial data $V^0 = (E^0, H^0, \chi^0)$ satisfies

$$\|V^0\|_{s,2} + \|V^0\|_{s,6/5} < \delta, \quad \text{with } \chi^0 \geq 0 \quad \text{and} \quad \mathrm{div}\, H^0 = \mathrm{div}[(1+\chi^0)E^0] = 0,$$

then there exists a unique solution V for the Cauchy problem (1.1)–(1.3), *with*

$$V = (E, H, \chi) \in \mathscr{C}^0\big([0, \infty), W^{s,2}\big) \cap \mathscr{C}^1\big([0, \infty), W^{s,2}\big).$$

Moreover, we have

$$\|V(t)\|_\infty + \|V(t)\|_6 = O(t^{-2/3}), \quad \|V(t)\|_{s,2} = O(1) \quad \text{as } t \to \infty.$$

As in [16], [17] and [20], our proof consists in combining the local existence theorem given in Proposition 1.1 with a priori estimate in appropriate L^p-norm. Therefore we proceed in two principal steps: the first is to get a high energy estimate, by using variational methods on a symmetric form. The second step is to obtain a weighted a priori estimate, based on $L^p - L^q$ decay estimates for the linear wave equation.

The main difficulty here, is that the degree of vanishing of the nonlinearity near zero is not great enough for the Kerr–Debye model (1.1) in its three variables. To overcome this difficulty we use the following new ideas. First, we treat the model (1.1) by splitting it into two parts: the Maxwell equations and the ordinary differential equation (ODE) satisfied by χ. The Maxwell part is estimated by classical variational method while χ is estimated by solving the ODE. Secondly, we remark that in this step, the weighted norm appearing in the weighted a priori estimate is used to control χ. In previous related papers [16], [17] and [20], the introduction of a weight in the high energy estimates was not necessary.

Third new idea: we use different variables for the high energy estimate and for the weighted a priori estimate. On the one hand we use (E, H) to obtain a symmetric form for Maxwell equations, so we are able to perform an energy estimate. On the other hand, with the variable (D, H) we transform the Maxwell equations in a wave equation using the divergence free conditions. This transformation is crucial to obtain a weighted a priori estimate. At this step again, the key point is the choice of adapted weights for the variable χ.

There are some mathematical studies on the Kerr–Debye Model. The Kerr–Debye system is a quasilinear hyperbolic system with source term and it is totally linearly degenerate, i.e., each characteristic field is linearly degenerate. So we can expect that, if the lifespan $T^\star$ is finite, the behavior of the smooth solution is analogous to the semilinear case. Indeed this result has been proved in the 1-D case in [7]: if $T^\star$ is finite then the solution and its gradient explode, so no shock wave can appear. In fact, using more precise dissipative properties for the Kerr–Debye model it is proved in [9], without smallness condition, that $T^\star = +\infty$ for the one-dimensional Cauchy problem.

For the initial-boundary value problem (IBVP), a result of global existence of smooth solution, without smallness condition, in the 1-D and 2-D TE cases for the

Kerr–Debye model is proved in [8]. By adapting the proof of this result on the Cauchy problem in the 2-D TE case, a similar global existence result can be also easily obtained.

However in the 2-D TM and the 3-D cases of both the Cauchy problem and the IBVP, the situation is different: we are unable to obtain similar properties as in the 1-D and the 2-D TE cases. Nevertheless, in the 3-D case of the Cauchy problem we obtain, in the present work, a result of global existence of smooth solution with small initial data, while the global existence of solutions for the 2-D TM Kerr–Debye model in all cases remains an open problem.

Concerning the study of the behavior of the smooth solutions when the response time τ tends to zero, the convergence of smooth solutions of Kerr–Debye system towards a smooth solution of Kerr system is proved, in [12] for the Cauchy problem and in [8] for the IBVP.

Besides, there are related results. Recently, in [2] Aregba and Hanouzet studied the Kerr–Debye shock profiles for the 1-D and 3-D Kerr system. They determine the plane discontinuities of the full vector 3-D Kerr system and their admissibility in the sense of Liu and the sense of Lax. Then they characterize the large amplitude Kerr shocks giving rise to the existence of Kerr–Debye relaxation profiles.

In the domain of numerical methods, we can mention the work of P. Huynh [14] around a finite element method in a nonlinear Kerr medium, and the recent work of Aregba–Berthon [1], which presents a 1-D finite volume schemes for Kerr–Debye model.

This paper is organized as follows. In Section 2 and Section 3 we prove respectively a high energy estimate and a weighted a priori estimate for small data. Section 4 is devoted to end the proof of Theorem 1.2.

Notations. We denote by $x = (x_1, x_2, x_3)$ the cartesian variables in $\mathbb{R}^3$. The partial derivative is denoted in the following way: $\partial_i = \partial/\partial_{x_i}$, $\partial_t = \partial/\partial_t$, and for $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$, $\nabla^\alpha = \partial^{|\alpha|}/(\partial_1)^{\alpha_1}(\partial_2)^{\alpha_2}(\partial_3)^{\alpha_3}$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$.

In this paper we use c to denote various positive constants without confusion.

2. High energy estimate

In this section we prove an energy estimate for small solutions of (1.1)–(1.2). This result will be proved by classical variational estimates. For that we use the variable $V = (E, H, \chi)$. Indeed with E and H we transform the nonlinear Maxwell equations in symmetric hyperbolic form. We estimate χ by solving the third equation in (1.1). Therefore we have to define for $u \in W^{m,p}$ the norm: $|u|_{m,p}(t) = \|u\|_{L^\infty(0,t;W^{m,p})}$.

We begin this section with two preliminary results.

Lemma 2.1. *Let $m \in \mathbb{N}$ and $p \geq 2$. Then there exists a constant $c = c(m, n) > 0$ such that for all $f, g \in W^{m,p} \cap L^\infty$ and $\alpha \in \mathbb{N}_0^n$, $|\alpha| = m$, the following inequalities hold:*

$$\|\nabla^\alpha(fg)\|_p \leq c(\|f\|_\infty \|\nabla^m g\|_p + \|\nabla^m f\|_p \|g\|_\infty), \qquad (2.1)$$

$$\|\nabla^\alpha(fg) - f\nabla^\alpha g\|_p \leq c(\|\nabla f\|_\infty \|\nabla^{m-1} g\|_p + \|\nabla^m f\|_p \|g\|_\infty). \qquad (2.2)$$

Proof. See [18], Section 2.1, Proposition 2.1. □

This lemma will be used repeatedly in the sequel.

Lemma 2.2. *Let $a > 0$, $f, g \in \mathscr{C}^0([0, a])$ such that $f, g \geq 0$ and let $v : [0, a] \to \mathbb{R}$, $v \geq 0$. If*

$$v^2(t) \leq v_0^2 + \int_0^t \big(f(r)v^2(r) + g(r)v(r)\big)\, dr \quad \text{for all } t \in [0, a],$$

then

$$v(t) \leq \frac{1}{2}\left[v_0 \exp\left(\int_0^t f(r)\, dr\right) + \int_0^t g(r) \exp\left(\int_r^t f(\tau)\, d\tau\right)\right] dr.$$

This result is classical, so we omit the proof.

In the rest of this paper, we use the notation

$$M_{s_1}(t) := \max_{0 \leq \tau \leq t} (1 + \tau)^{2/3} \|(E, H)(\tau)\|_{s_1, 6},$$

where $s_1 \in \mathbb{N}$ will be defined below.

Proposition 2.3. *Let $V = (U, \chi) = (E, H, \chi)$ be the local solution of the initial value problem (1.1)–(1.2) with the initial data $V^0 = (E^0, H^0, \chi^0)$. We denote by $T^\star$ the lifespan of this solution. Let $s, s_1 \in \mathbb{N}$ such that $2 \leq s_1 \leq s - 1$. Then there exist a constant c independent of $T^\star$ and a $\delta \leq \inf\{1, 1/c\}$ sufficiently small such that if*

$$\|V^0\|_{s, 2} \leq \delta/2 \quad \text{and} \quad \chi^0 \geq 0, \qquad (2.3)$$

then

$$\|V(t)\|_{s, 2} \leq c\|V^0\|_{s, 2}\big[(1 + M_{s_1}^2(t)) \exp\{cM_{s_1}^2(t)\}\big] \quad \text{for all } t \in [0, \tilde{T}], \qquad (2.4)$$

where $\tilde{T} > 0$ is defined by

$$\tilde{T} = \max\{T < T^\star \text{ such that } |V|_{s, 2}(T) \leq \delta\}.$$

Proof. Using the variable $V = (U, \chi) = (E, H, \chi)$, the system (1.1) becomes

$$
\begin{cases}
(1 + \chi)\partial_t E + (\partial_t \chi)E - \operatorname{curl} H = 0, & \text{(i)} \\
\partial_t H + \operatorname{curl} E = 0, & \text{(ii)} \\
\partial_t \chi = |E|^2 - \chi, & \text{(iii)}
\end{cases}
\qquad (2.5)
$$

with the initial data

$$
(E, H, \chi)(0, x) = (E^0, H^0, \chi^0)(x) \qquad \text{for } x \in \mathbb{R}^3.
$$

We apply ∇^α, $|\alpha| = m \leq s$ to system (2.5) and then take the inner product of (2.5.i) with E and (2.5.ii) with H. Thus, summing up the two terms, we obtain

$$
\int_{\mathbb{R}^3} (1 + \chi)\partial_t \nabla^\alpha E \cdot \nabla^\alpha E \, dx + \int_{\mathbb{R}^3} \partial_t \nabla^\alpha H \cdot \nabla^\alpha H \, dx
$$

$$
= -\int_{\mathbb{R}^3} \nabla^\alpha(\partial_t \chi E) \cdot \nabla^\alpha E \, dx
$$

$$
- \int_{\mathbb{R}^3} \left\{ \nabla^\alpha\big((1 + \chi)\partial_t E\big) - (1 + \chi)\partial_t \nabla^\alpha E \right\} \cdot \nabla^\alpha E \, dx. \qquad (2.6)
$$

We have

$$
\int_{\mathbb{R}^3} (1 + \chi)\partial_t \nabla^\alpha E \cdot \nabla^\alpha E \, dx = \frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^3} (1 + \chi)|\nabla^\alpha E|^2 \, dx - \frac{1}{2}\int_{\mathbb{R}^3} \partial_t \chi |\nabla^\alpha E|^2 \, dx.
$$

So replacing it in (2.6) we obtain:

$$
\frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^3} \left\{ (1 + \chi)|\nabla^\alpha E|^2 + |\nabla^\alpha H|^2 \right\} dx
$$

$$
= \frac{1}{2}\int_{\mathbb{R}^3} \partial_t \chi |\nabla^\alpha E|^2 \, dx - \int_{\mathbb{R}^3} \nabla^\alpha(\partial_t \chi E) \cdot \nabla^\alpha E \, dx
$$

$$
- \int_{\mathbb{R}^3} \left\{ \nabla^\alpha\big((1 + \chi)\partial_t E\big) - (1 + \chi)\partial_t \nabla^\alpha E \right\} \nabla^\alpha E \, dx
$$

$$
= I_1 + I_2 + I_3. \qquad (2.7)
$$

Let us now estimate the right-hand side terms in (2.7) for all $t \in [0, \tilde{T}]$: first we have

$$
|I_1| \leq \frac{1}{2}\|\partial_t \chi\|_\infty \cdot \|\nabla^\alpha E\|_2^2.
$$

Solving (2.5.iii) we get

$$\chi(t) = \chi^0 e^{-t} + \int_0^t e^{(s-t)} |E(s)|^2 \, ds \tag{2.8}$$

and

$$\partial_t \chi(t) = |E|^2(t) - \chi^0 e^{-t} - \int_0^t e^{(s-t)} |E(s)|^2 \, ds. \tag{2.9}$$

This implies

$$\|\partial_t \chi\|_\infty(t) \leq \| |E(t)|^2 \|_\infty + \|\chi^0\|_\infty e^{-t} + \int_0^t e^{(\tau-t)} \| |E(\tau)|^2 \|_\infty \, d\tau.$$

Observing that for $2 \leq s_1 \leq s - 1$, by Sobolev's inequalities, we have

$$W^{s,2} \hookrightarrow W^{s_1,6} \hookrightarrow W^{1,\infty}. \tag{2.10}$$

From this we get for all $t \in [0, \tilde{T}]$,

$$\|U(t)\|_{1,\infty} \leq c\|U(t)\|_{s_1,6} \leq c(1+t)^{-2/3} M_{s_1}(t), \tag{2.11}$$

and

$$\|\chi^0\|_{1,\infty} \leq \max_{0 \leq \tau \leq t} \|V(\tau)\|_{1,\infty} \leq c|V|_{s,2}(t) \leq 1. \tag{2.12}$$

From that we get

$$\|\partial_t \chi\|_\infty(t) \leq (1+t)^{-4/3} M_{s_1}^2(t) + e^{-t} + \int_0^t e^{(\tau-t)}(1+\tau)^{-4/3} M_{s_1}^2(\tau) \, d\tau$$

$$\leq (1+t)^{-4/3} M_{s_1}^2(t) + e^{-t} + M_{s_1}^2(t) \int_0^t e^{(\tau-t)}(1+\tau)^{-4/3} \, d\tau,$$

and from the fact that, for $r \leq 0$,

$$\int_0^t e^{(\tau-t)}(1+\tau)^r \, d\tau \leq c(1+t)^r, \tag{2.13}$$

we arrive at

$$\|\partial_t \chi\|_\infty(t) \leq c\big(e^{-t} + (1+t)^{-4/3} M_{s_1}^2(t)\big).$$

This estimate yields

$$|I_1|(t) \leq c\big(e^{-t} + (1+t)^{-4/3} M_{s_1}^2(t)\big) |E|_{m,2}^2(t). \tag{2.14}$$

In the same way we have

$$|I_2| \leq \|\nabla^\alpha(\partial_t\chi E)\|_2 \|\nabla^\alpha E\|_2.$$

By (2.1) we have

$$\|\nabla^\alpha(\partial_t\chi E)\|_2 \leq c(\|\nabla^\alpha\partial_t\chi\|_2 \cdot \|E\|_\infty + \|\nabla^\alpha E\|_2 \cdot \|\partial_t\chi\|_\infty).$$

From (2.9) and (2.1) we get

$$\|\nabla^\alpha\partial_t\chi\|_2(t) \leq \||\nabla^\alpha|E(t)|^2\|_2 + \|\nabla^\alpha\chi^0\|_2 e^{-t} + \int_0^t e^{(\tau-t)} \||\nabla^\alpha|E(\tau)|^2\|_2 \, d\tau$$

and

$$\||\nabla^\alpha|E|^2\|_2 \leq c\|E\|_\infty \|\nabla^\alpha E\|_2.$$

Thus,

$$\|\nabla^\alpha\partial_t\chi\|_2(t) \leq c\|E\|_\infty \|\nabla^\alpha E\|_2 + \|\nabla^\alpha\chi^0\|_2 e^{-t} + c\int_0^t e^{(\tau-t)} \|E(\tau)\|_\infty \|\nabla^\alpha E(\tau)\|_2 \, d\tau$$

$$\leq c(1+t)^{-2/3} M_{s_1}(t)|\nabla^\alpha E|_2(t) + \|\nabla^\alpha\chi^0\|_2 e^{-t}$$

$$+ c|\nabla^\alpha E|_2(t) \int_0^t e^{(\tau-t)}(1+\tau)^{-2/3} M_{s_1}(\tau) \, d\tau$$

$$\leq c\big(\|\nabla^\alpha\chi^0\|_2 e^{-t} + (1+\tau)^{-2/3} M_{s_1}(t)\big)|\nabla^\alpha E|_2(t).$$

Thus

$$\|\nabla^\alpha\partial_t\chi\|_2 \cdot \|E\|_\infty(t) \leq \|\nabla^\alpha\chi^0\|_2 e^{-t}\|E\|_\infty(t) + (1+t)^{-2/3} M_{s_1}(t)|\nabla^\alpha E|_2(t)\|E(t)\|_\infty$$

$$\leq \|\nabla^\alpha\chi^0\|_2 e^{-t} + (1+t)^{-4/3} M_{s_1}^2(t)|\nabla^\alpha E|_2(t).$$

Here we have applied inequalities (2.11) and (2.12) on $\|E\|_\infty(t)$.

The second term in the right-hand side is the same as in the estimate of I_1, so we obtain

$$|I_2|(t) \leq c\big[\big(e^{-t} + (1+t)^{-4/3} M_{s_1}^2(t)\big)|E|_{m,2}^2(t) + \|\chi^0\|_{m,2} e^{-t}|E|_{m,2}(t)\big]. \tag{2.15}$$

In the same way we have

$$|I_3| \leq \left\| \nabla^\alpha \big((1+\chi)\partial_t E \big) - (1+\chi)\partial_t \nabla^\alpha E \right\|_2 \|\nabla^\alpha E\|_2.$$

By (2.2) we obtain

$$\left\| \nabla^\alpha \big((1+\chi)\partial_t E \big) - (1+\chi)\partial_t \nabla^\alpha E \right\|_2$$
$$\leq c \big(\|\nabla(1+\chi)\|_\infty \|\nabla^{m-1}\partial_t E\|_2 + \|\nabla^m (1+\chi)\|_2 \|\partial_t E\|_\infty \big).$$

From (2.8), (2.12) and (2.1) we get

$$\|\nabla(1+\chi)\|_\infty \leq \|\nabla\chi^0\|_\infty e^{-t} + (1+t)^{-4/3} M_{s_1}^2(t) \leq e^{-t} + (1+t)^{-4/3} M_{s_1}^2(t).$$

Equation (2.5.i) yields

$$\partial_t E = \operatorname{curl} H - \partial_t \chi E - \chi \partial_t E$$

and

$$\|\nabla^{m-1}\partial_t E\|_2 \leq \|\nabla^m H\|_2 + \|\nabla^{m-1}(\partial_t \chi E)\|_2 + \|\nabla^{m-1}(\chi \partial_t E)\|_2.$$

Using (2.12) in the same way as above, we prove

$$\|\nabla^{m-1}(\partial_t \chi E)\|_2 \leq c(\|\nabla^{m-1}\chi^0\|_2 \|E\|_\infty + \|\nabla^{m-1} E\|_2)$$
$$\leq c(\|\nabla^{m-1}\chi^0\|_2 + \|E\|_{m,2})$$
$$\leq c|U|_{m,2}(t)$$

and

$$\|\nabla^{m-1}(\chi \partial_t E)\|_2 \leq c\big((\|\nabla^{m-1}\chi^0\|_2 + \|\nabla^{m-1} E\|_2)\|\partial_t E\|_\infty + \|\nabla^{m-1}\partial_t E\|_2 \|\chi\|_\infty \big)$$
$$\leq c(\|\nabla^{m-1}\chi^0\|_2 + \|\nabla^{m-1} E\|_2 + \|\nabla^{m-1}\partial_t E\|_2 \|\chi\|_\infty)$$
$$\leq c\big(|U|_{m,2}(t) + \|\nabla^{m-1}\partial_t E\|_2 \|\chi\|_\infty \big).$$

Now taking

$$\delta \leq \inf\{1, 1/2c\},$$

by (2.12) we get

$$\|\chi\|_\infty \leq \delta \leq 1/2c.$$

Then the last inequalities yield

$$\|\nabla^{m-1}\partial_t E\|_2 \le c|U|_{m,2}(t). \tag{2.16}$$

So we obtain

$$
\begin{aligned}
|I_3|(t) \le c\big[&\big(e^{-t} + (1+t)^{-4/3}M_{s_1}^2(t)\big)|U|_{m,2}^2(t) \\
&+ \big(e^{-t} + (1+t)^{-4/3}M_{s_1}^2(t)\big)\|\chi^0\|_{m,2}|U|_{m,2}(t)\big].
\end{aligned} \tag{2.17}
$$

Combining (2.7), (2.14), (2.15) and (2.17), we obtain the estimate

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}&\int_{\mathbb{R}^3}\{(1+\chi)|\nabla^\alpha E|^2 + |\nabla^\alpha H|^2\}\,dx \\
&\le c\big(e^{-t} + (1+t)^{-4/3}M_{s_1}^2(t)\big)\big(|U|_{m,2}^2(t) + \|\chi^0\|_{m,2}|U|_{m,2}(t)\big).
\end{aligned}
$$

We sum up these inequalities for $|\alpha| = m \le s$ and we integrate in time. Since $\chi \ge 0$, we obtain

$$|U|_{s,2}^2(t) \le c\|U^0\|_{s,2}^2 + c\int_0^t \big(g(\tau)|U|_{s,2}^2(\tau) + g(\tau)\|\chi^0\|_{s,2}|U|_{s,2}(\tau)\big)\,d\tau, \tag{2.18}$$

where $g(t) = \big(e^{-t} + (1+t)^{-4/3}M_{s_1}^2(t)\big)$.

We apply Lemma 2.2 to (2.18) (replacing v with $|U|_{s,2}$). Next we use (2.1) to estimate $|\chi|_{s,2}$. So we obtain (2.4). $\qquad\square$

3. Weighted a priori estimate

In this section we prove a weighted a priori estimate which will be combined with the energy estimate (2.4) to obtain an a priori bound in the $W^{s,2}$-norm of the solution of (1.1)–(1.3) for small data. The proof is based on a decay estimate for the linear wave equation. For that we will use the variable (D, H, χ). In fact with the divergence free conditions (1.3) we can transform the Maxwell equations into nonlinear wave equation.

Proposition 3.1. *Let* $V = (U, \chi) = (E, H, \chi)$ *be the local solution of the initial value problem* (1.1)–(1.3) *and let* $s_0, s_1 \in \mathbb{N}$ *satisfy*

$$3 \le s_1 \le s_0 - 4.$$

We assume that

$$V^0 \in W^{s_0,2} \cap W^{s_1+3,6/5}.$$

Then for all $M_0 > 0$ there exists $0 < \delta_1(M_0) \leq \delta/2$, independent of $\tilde{T}$ (δ and $\tilde{T}$ defined in Proposition 2.3), such that the following holds:
   If

$$\|V^0\|_{s_0, 2} + \|V^0\|_{s_1+3, 6/5} \leq \delta_1 \quad \text{and} \quad \|\chi^0\|_{s_0, 3/2} \leq \delta_1,$$

with $\chi^0 \geq 0$ and $\operatorname{div} H^0 = \operatorname{div}[(1 + \chi^0)E^0] = 0$, then

$$M_{s_1}(t) = \max_{0 \leq \tau \leq t} (1 + \tau)^{2/3} \|U(\tau)\|_{s_1, 6} \leq M_0 \quad \text{for all } t \in [0, \tilde{T}].$$

Proof. Using the variable $v = (D, H, \chi)$, the first two equations of the system (1.1) become

$$\begin{align}
\text{(i)} \quad & \partial_t D - \operatorname{curl} H = 0, \\
\text{(ii)} \quad & \partial_t H + \operatorname{curl} D = \operatorname{curl}(\chi E).
\end{align} \tag{3.1}$$

We recall that, by these equations we get $\operatorname{div} D = \operatorname{div} H = 0$ for all $t \geq 0$.
   We denote by $f(t) = (0, \operatorname{curl}(\chi E))$. Let $u = (D, H)$ be the solution of (3.1). Using the representation of Duhamel, we can write u as

$$u(t) = e^{t\Lambda} u^0 + \int_0^t e^{(t-\tau)\Lambda} f(\tau) \, d\tau := u_1(t) + u_2(t), \quad 0 \leq t < \tilde{T}, \tag{3.2}$$

where Λ is the operator defined by

$$\Lambda = \begin{pmatrix} 0 & \operatorname{curl} \\ -\operatorname{curl} & 0 \end{pmatrix}. \tag{3.3}$$

In order to obtain $\|U\|_{s_1, 6}$, we must first estimate u in the norm $\| \ \|_{s_1, 6}$. We need to use the following two lemmas.

Lemma 3.2 (emigroup estimate). *Let $\bar{U} = (\bar{D}, \bar{H})$ the solution of the linear problem*

$$\begin{cases} \partial_t \bar{U} = \Lambda \bar{U}, \\ \bar{U}(t = 0) = \bar{U}^0, \\ \operatorname{div} \bar{D}^0 = \operatorname{div} \bar{H}^0 = 0, \end{cases} \tag{3.4}$$

where Λ is the operator defined in (3.3), and let $1 < p \leq 2 \leq q < \infty$, $1/p + 1/q = 1$, $N_q > 3(1 - 2/q)$. Then there is a constant $c = c(p)$ such that for all $\bar{U}^0 \in W^{N_q, p}$, and for all $t \geq 0$:

$$\|\bar{U}(t)\|_q \leq c(1 + t)^{-(1-2/q)} \|\bar{U}^0\|_{N_q, p}.$$

Proof. $L^2 - L^2$ estimate: From a classical result on Maxwell equations we have

$$\|(\bar{D}, \bar{H})\|_2 = \|(\bar{D}^0, \bar{H}^0)\|_2, \qquad t \geq 0. \tag{3.5}$$

$L^\infty - W^{3,1}$ estimate: Using the divergence free condition we transform easily (3.4) in the linear wave equation

$$\begin{cases} \bar{U}_{tt} - \Delta \bar{U} = 0, \\ \bar{U}^0 = (\bar{D}^0, \bar{H}^0), \\ \partial_t \bar{U}(t = 0) = \bar{U}^1 = (\operatorname{curl} \bar{H}^0, -\operatorname{curl} \bar{D}^0). \end{cases} \tag{3.6}$$

So each component $\bar{U}_i$ (for $i = 1, \ldots, 6$) of $\bar{U}$ can be written in the form

$$\bar{U}_i(t) = w(t) \bar{U}_i^1 + \partial_t \big(w(t) \bar{U}_i^0 \big),$$

where the operator $w(t)$ is defined by (see [20], p. 15)

$$\big(w(t) g \big)(x) := \bar{u}(t, x),$$

with $\bar{u}$ the solution of

$$\begin{cases} \bar{u}_{tt} - \Delta \bar{u} = 0, \\ \bar{u}(t = 0) = 0, \quad \partial_t \bar{u}(t = 0) = g. \end{cases}$$

Thus, by the Kirchoff representation formula of $w(t)$ (see for instance [20], Theorem 2.1), we get for $i = 1, \ldots, 6$,

$$\|\bar{U}_i(t)\|_\infty \leq c(1 + t)^{-1} (\|\bar{U}_i^0\|_{3,1} + \|\bar{U}_i^1\|_{2,1}) \qquad \text{for all } t \geq 0,$$

and using (3.6), we obtain

$$\|\bar{U}(t)\|_\infty \leq c(1 + t)^{-1} \|\bar{U}^0\|_{3,1} \qquad \text{for all } t \geq 0. \tag{3.7}$$

So by interpolation (see for example [20], Theorem A.10) we obtain from (3.5) and (3.7)

$$\|\bar{U}(t)\|_q \leq c(1 + t)^{-(1-2/q)} \|\bar{U}^0\|_{N_q, p} \qquad \text{for all } t \geq 0. \qquad \square$$

Now we have to estimate the nonlinear term, $f(t) = \big(0, \operatorname{curl}(\chi E) \big)$.

Lemma 3.3. *There exists $c > 0$ such that*

$$\|f(t)\|_{s_1+3,6/5} \le c\big[\|\chi^0\|_{s_0,3/2}e^{-t}M_{s_1}(t) + \big(e^{-t} + (1+t)^{-4/3}M_{s_1}^2(t)\big)|U|_{s_0,2}(t)\big].$$

Proof. Using the special form of f we obtain

$$\begin{aligned}
\|f(t)\|_{s_1+3,6/5} &= \|\text{curl}\,(\chi E)\|_{s_1+3,6/5} \\
&\le c\|\chi E\|_{s_1+4,6/5} \\
&\le c \sum_{0 \le |\alpha| \le s_1+4} \|\nabla^\alpha(\chi E)\|_{6/5} \\
&\le c \sum_{0 \le |\alpha|+|\beta| \le s_1+4} \|\nabla^\alpha\chi\nabla^\beta E\|_{6/5}.
\end{aligned}$$

We have the three following cases to consider:

- First let $|\alpha| \ge s_1 + 1$. Then $|\beta| \le s_1$.
  This implies that

$$\|\nabla^\alpha\chi\nabla^\beta E\|_{6/5} \le \|\nabla^\alpha\chi\|_{3/2}\|\nabla^\beta E\|_6.$$

By (2.8), we have

$$\|\nabla^\alpha\chi\|_{3/2} \le \|\nabla^\alpha\chi^0\|_{3/2}e^{-t} + \int_0^t \|\nabla^\alpha(|E|^2)\|_{3/2}e^{(\tau-t)}\,d\tau, \tag{3.8}$$

with

$$\begin{aligned}
\|\nabla^\alpha(|E|^2)\|_{3/2} &\le c \sum_{0 \le |\beta|+|\gamma| \le |\alpha|} \|\nabla^\beta E\nabla^\gamma E\|_{3/2} \\
&\le c\|E\|_{s_1,6}\|E\|_{s_0,2} \le c(1+t)^{-2/3}M_{s_1}(t)|E|_{s_0,2}(t).
\end{aligned}$$

Then, plugging this in (3.8) and using (2.13), we obtain in this case

$$\begin{aligned}
&\|f(t)\|_{s_1+3,6/5} \\
&\quad \le c\big[\|\chi^0\|_{s_0,3/2}e^{-t}(1+t)^{-2/3}M_{s_1}(t) + (1+t)^{-4/3}M_{s_1}^2(t)|E|_{s_0,2}(t)\big]. \tag{3.9}
\end{aligned}$$

- Secondly let $|\beta| \ge s_1 + 1$. Then $|\alpha| \le s_1$ and

$$\|\nabla^\alpha\chi\nabla^\beta E\|_{6/5} \le \|\nabla^\alpha\chi\|_3\|\nabla^\beta E\|_2.$$

In the same way as in the first case we have

$$\|\nabla^\alpha \chi\|_3 \le \|\nabla^\alpha \chi^0\|_3 e^{-t} + \int_0^t \|\nabla^\alpha (|E|^2)\|_3 e^{(s-t)}\, ds$$

and

$$\|\nabla^\alpha (|E|^2)\|_3 \le c(1+t)^{-4/3} M_{s_1}^2(t).$$

We remark that, by (2.10), we have

$$\|V^0\|_{s_1, \infty} + \|V^0\|_{s_1, 6/5} \le c\delta \le 1,$$

so, by interpolating

$$\|\nabla^\alpha \chi^0\|_3 \le 1,$$

we then get

$$\|f(t)\|_{s_1+3, 6/5} \le c[e^{-t} + (1+t)^{-4/3} M_{s_1}^2(t)]|E|_{s_0, 2}(t). \qquad (3.10)$$

- Finally let $|\alpha| \le s_1$ and $|\beta| \le s_1$. Here we can proceed exactly as in two previous cases.

From (3.9) and (3.10) we end the proof of Lemma 5. $\qquad\square$

End of the proof of Proposition 3.1. According to Lemma 3.2, let $q = 6$ ($p = 6/5$). Then by (3.2) u_1 satisfies

$$\|u_1(t)\|_{s_1, 6} \le c(1+t)^{-(1-2/6)}\|u^0\|_{s_1+3, 6/5}$$
$$\le c(1+t)^{-(1-2/6)}\|U^0\|_{s_1+3, 6/5} \le c(1+t)^{-2/3}\delta_1, \qquad (3.11)$$

and, using Lemma 3.3, u_2 satisfies

$$\|u_2(t)\|_{s_1, 6} \le c\int_0^t (1+t-r)^{-2/3}\|f(r)\|_{s_1+3, 6/5}\, dr$$
$$\le c\int_0^t (1+t-r)^{-2/3}\|\chi^0\|_{s_0, 3/2} e^{-r} M_{s_1}(r)|U|_{s_0, 2}(r)$$
$$+ c\int_0^t (1+t-r)^{-2/3}\big(e^{-r} + (1+r)^{-4/3} M_{s_1}^2(r)\big)|U|_{s_0, 2}(r). \qquad (3.12)$$

According to Proposition 2.3, we get

$$\|u_2(t)\|_{s_1,6}$$

$$\leq c \int_0^t (1+t-r)^{-2/3} e^{-r} \|\chi^0\|_{s_0,3/2} M_{s_1}(r)\, dr$$

$$+ c \int_0^t (1+t-r)^{-2/3} e^{-r} \|u^0\|_{s_0,2} \left(1+M_{s_1}^2(r)\right) \exp\{cM_{s_1}^2(r)\}\, dr$$

$$+ c \int_0^t (1+t-r)^{-2/3} (1+r)^{-4/3} M_{s_1}^2(r) \|u^0\|_{s_0,2} \left(1+M_{s_1}^2(r)\right) \exp\{cM_{s_1}^2(r)\}\, dr$$

$$\leq c(1+t)^{-2/3}\delta_1 M_{s_1}(t) \int_0^t (1+t-r)^{-2/3} e^{-r} (1+t)^{2/3}\, dr$$

$$+ c(1+t)^{-2/3}\delta_1 \left(1+M_{s_1}^2(t)\right) \exp\{cM_{s_1}^2(t)\}$$

$$\cdot \int_0^t (1+t-r)^{-2/3}(1+t)^{2/3}\left(e^{-r} + (1+r)^{-4/3} M_{s_1}^2(r)\right)\, dr.$$

Now we remark that there exists $k < \infty$ such that

$$\int_0^t (1+t-r)^{-2/3} e^{-r}(1+t)^{2/3}\, dr + \int_0^t (1+t-r)^{-2/3}(1+t)^{2/3}(1+r)^{-4/3}\, dr \leq k.$$

Thus

$$\|u_2(t)\|_{s_1,6} \leq c(1+t)^{-2/3}\delta_1 \left(M_{s_1}(t) + \left(1+M_{s_1}^2(t)\right)^2 \exp\{cM_{s_1}^2(t)\}\right).$$

Combining (3.11) and (3.12) we get the following estimate for u:

$$\|u(t)\|_{s_1,6} \leq c(1+t)^{-2/3}\delta_1 + c(1+t)^{-2/3}\delta_1 \left(M_{s_1}(t) + \left(1+M_{s_1}^2(t)\right)^2 \exp\{cM_{s_1}^2(t)\}\right).$$

This implies that

$$\max_{0\leq\tau\leq t} (1+\tau)^{2/3}\|u(\tau)\|_{s_1,6}$$

$$\leq c\delta_1 \left(1+M_{s_1}(t) + \left(1+M_{s_1}^2(t)\right)^2 \exp\{cM_{s_1}^2(t)\}\right), \qquad 0\leq t\leq \tilde{T}. \qquad (3.13)$$

Writing

$$E = D - \chi E,$$

and using the same technique as in the estimate (2.16), we obtain

$$\max_{0\leq\tau\leq t} (1+\tau)^{2/3}\|E(\tau)\|_{s_1,6} \leq c \max_{0\leq\tau\leq t} (1+\tau)^{2/3}\|D(\tau)\|_{s_1,6}.$$

This leads to

$$\max_{0 \le \tau \le t} (1+\tau)^{2/3} \|U(\tau)\|_{s_1,6} = M_{s_1}(t) \le c \max_{0 \le \tau \le t} (1+\tau)^{2/3} \|u(\tau)\|_{s_1,6},$$

and by (3.13) we get

$$M_{s_1}(t) \le c\delta_1 \left(1 + M_{s_1}(t) + \left(1 + M_{s_1}^2(t)\right)^2 \exp\{cM_{s_1}^2(t)\}\right), \quad 0 \le t \le \tilde{T}. \quad (3.14)$$

We intoduce $x = M_{s_1}(t)$ and the real-valued function φ defined for $x \ge 0$ by

$$\varphi(x) := c\delta_1 \left(1 + x + (1+x^2)^2 e^{cx^2}\right) - x.$$

We have

$$\varphi(0) = c\delta_1 > 0, \quad \varphi'(0) = c\delta_1 - 1.$$

So φ has a first positive zero at x_0 with $\varphi'(x_0) < 0$ if δ_1 is sufficiently small $(\delta_1 = \delta_1(c))$ and

$$0 = \varphi(x_0) = c\delta_1 \left(1 + x_0 + (1+x_0^2)^2 e^{cx_0^2}\right) - x_0.$$

This implies that

$$\delta_1 = \frac{x_0}{c\left(1 + x_0 + (1+x_0^2)^2 e^{cx_0^2}\right)} < \frac{x_0}{c},$$

whence (without loss of generality we can take $k < c$)

$$M_{s_1}(0) = \|u_0\|_{s_1,6} \le k\|u_0\|_{s_0,2} \le k\delta_1 < x_0. \quad (3.15)$$

The relation (3.14) implies that

$$\varphi\left(M_{s_1}(t)\right) \ge 0, \quad 0 \le t \le \tilde{T},$$

which together with (3.15) and a continuous dependence argument leads to

$$M_{s_1}(t) \le x_0, \quad 0 \le t \le \tilde{T}. \quad (3.16)$$

Then we conclude the proof with

$$M_0 := x_0 = x_0(\delta_1). \qquad \square$$

4. Proof of Theorem 1.2

The results in Proposition 2.3 and Proposition 3.1 easily lead to the following a priori bound:

Proposition 4.1. *Let $V = (E, H, \chi)$ be the maximal small solution of the initial value problem (1.1)–(1.3) on $[0, T^\star)$. Then there exist a constant $c \geq 1$, an integer $s \geq 7$ and $\delta > 0$ sufficiently small such that if $\|V^0\|_{s,2} + \|V^0\|_{s,6/5} \leq \delta/2$, $\|\chi^0\|_{s,3/2} \leq \delta/2$, with $\chi^0 \geq 0$ and $\operatorname{div} H^0 = \operatorname{div}[(1 + \chi^0)E^0] = 0$, then*

$$\|V(t)\|_{s,2} \leq c\|V^0\|_{s,2}[(1 + M_0^2)\exp\{cM_0^2\}] \quad \textit{for all } t \in [0, \tilde{T}],$$

where $\tilde{T}$ is defined by

$$\tilde{T} = \max\{T < T^\star \textit{ such that } |V|_{s,2}(T) \leq \delta\}.$$

Remark 4.2. We can write, without loss of generality,

$$\|V(t)\|_{s,2} \leq cM_0\|V^0\|_{s,2} \quad \text{for all } t \in [0, \tilde{T}]. \tag{4.1}$$

Remark 4.3. The condition

$$\|\chi^0\|_{s,3/2} \leq \delta/2$$

is automatically satisfied by interpolation between $W^{s,2}$ and $W^{s,6/5}$ spaces.

Therefore we have an a priori estimate in the $W^{s,2}$-norm of the solution of (1.1)–(1.3).

We suppose that $T^\star < +\infty$. Then from the definition of $\tilde{T}$ we get

$$|V|_{s,2}(\tilde{T}) = \delta. \tag{4.2}$$

Choosing

$$M_0 = 1/c,$$

and using (4.1) we obtain

$$\|V(t)\|_{s,2} \leq cM_0\delta \leq \frac{1}{2}\delta \quad \text{for all } t \in [0, \tilde{T}],$$

which is contradictory with (4.2). So $T^\star = +\infty$.

In particular, we obtain

$$\|V(t)\|_{s,2} < cM_0^2 \quad \text{for all } 0 \leq t < \infty,$$

and with (2.11) and Proposition 3.1,

$$\|U(t)\|_{\infty} \le c(1+t)^{-2/3} M_{s_1}(t) \le c M_0 (1+t)^{-2/3} \quad \text{for all } 0 \le t < \infty.$$

We prove that χ satisfies this last estimate using (2.8) and the fact that $e^{-t} \le (1+t)^{-2/3}$ for $t \ge 0$. This concludes the proof of Theorem 1.2.

Remark 4.4. In the one and two dimensional cases of the Kerr and Kerr–Debye models, this method of proof is not applicable. Indeed, we do not have enough decay in the linear wave equation to ensure the convergence of integrals.

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M. Kanso, Institut de Mathématiques de Bordeaux, UMR 5251, Université Bordeaux 1, 351 cours de la Libération, 33405 Talence cedex, France

E-mail: Mohamed.Kanso@math.u-bordeaux1.fr