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Scarf lattice ideals

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Abstract. This paper deals with the Scarf property of lattice ideals initiated by Peeva and Sturmfels [10], [11]. We will present a Scarf lattice ideal that is neither generic nor of codimension 2 and show that this property gives rise to several algebraic and combinatorial properties. In particular, we prove that for monomial curves, this property coincides with the notion of genericity, and that certain Scarf lattice ideals can have certain Scarf initial ideals.

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1. Introduction

Let $B = (b_{ij})$ be an integer $n \times m$-matrix of rank m. Let L be the lattice spanned in $\mathbb{Z}^n$ by the columns of B. Let I_L be the lattice ideal in $S = \Bbbk[\boldsymbol{x}] := \Bbbk[x_1, \dots, x_n]$, $\Bbbk$ a field, generated by all pure binomials $\boldsymbol{x}^{\boldsymbol{u}^+} - \boldsymbol{x}^{\boldsymbol{u}^-}$ where $\boldsymbol{u} = \boldsymbol{u}^+ - \boldsymbol{u}^-$ runs over L. If L is saturated, that is, the Abelian group $\mathbb{Z}^n/L$ is torsion-free, then I_L is prime and there exists an integer $d \times n$-matrix $A = (a_{ij})$ of rank $d (= n - m)$ such that $L = \ker_{\mathbb{Z}} A$. In this case, I_L is called the toric ideal of A and is denoted by I_A.

Throughout this paper, we assume that the matrix B is homogeneous with respect to a strictly positive integer vector $\boldsymbol{w} = (w_1, \dots, w_n)$, that is, the following equivalent conditions are satisfied (cf. [12], Proposition 2.1):

- $\boldsymbol{w}B = 0$.
- $L \cap \mathbb{N}^n = \{0\}$, i.e., L contains no non-negative vectors.
- For each $\boldsymbol{u} \in \mathbb{R}^n$, the body $P_{\boldsymbol{u}} := \{\boldsymbol{v} \in \mathbb{R}^m : B\boldsymbol{v} \leq \boldsymbol{u}\}$ is a polytope.
- Both rings S and S/I_L are $\mathbb{Z}$-graded by $\deg(x_i) = w_i$.

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Here we recall some definitions and results from [11]. The rings S and S/I_L are graded by the abelian group $\Gamma := \mathbb{Z}^n/L$ via $\deg(\boldsymbol{x}^{\boldsymbol{u}}) := \boldsymbol{u} + L$. When L is saturated, then $\Gamma \simeq \mathbb{Z}^d$, and we can equivalently define $\deg(\boldsymbol{x}^{\boldsymbol{u}}) := A\boldsymbol{u}$. Notice that our assumption on the matrix B allows us to choose the matrix A non-negative integer. The set of all monomials of a fixed degree in Γ is called a fiber, and $\mathbb{N}^n/L$ is the set of all fibers. The fiber containing a particular monomial $\boldsymbol{x}^{\boldsymbol{u}}$ can be identified with the lattice points in the polytope $P_{\boldsymbol{u}}$ via the map $\boldsymbol{v} \mapsto \boldsymbol{u} - B\boldsymbol{v}$. Two polytopes $P_{\boldsymbol{u}}$ and $P_{\boldsymbol{u}'}$ are lattice translates of each other if $\boldsymbol{u} - \boldsymbol{u}' \in L$. Disregarding lattice equivalence, we set $P_C := P_{\boldsymbol{u}}$ for all monomials $\boldsymbol{x}^{\boldsymbol{u}}$ in a fibre C. This polytope is called the polytope of the fiber $C \in \mathbb{N}^n/L$. A fiber C is called basic if $\gcd(C) = 1$ and $\gcd(C \backslash \{\boldsymbol{x}^{\boldsymbol{u}}\}) \neq 1$ for all $\boldsymbol{x}^{\boldsymbol{u}} \in C$ where $\gcd(C)$ denotes the greatest common divisor of all monomials in C. If C is a basic fiber and $\boldsymbol{x}^{\boldsymbol{u}}$ a monomial in C, then the monomials in $C \backslash \{\boldsymbol{x}^{\boldsymbol{u}}\}$ divided by their greatest common divisor form a basic fiber. For any finite subset $J \subset L$, let $\max(J)$ be the vector which is coordinatewise maximum of J. Let

$$\Delta_L := \{J \subset L : \max(J) \neq \max(J') \text{ for all finite subsets } J' \subset L \text{ other than } J\}\cdot$$

Δ_L is an infinite simplicial complex of dimension at most $n-1$ which has L as its vertex set. Since the lattice L acts naturally on Δ_L via $(\boldsymbol{u}, J) \mapsto \boldsymbol{u} + J$, we can form the finite simplicial complex

$$\Delta_L^0 := \{J \subset L \backslash \{0\} \mid \bar{J} := J \cup \{0\} \in \Delta_L\},$$

modulo the action by L. The simplicial complex Δ_L^0 is called the *linked Scarf complex*. We have the one to one correspondence $J \mapsto C_J := \{\boldsymbol{x}^{\max(\bar{J}) - \boldsymbol{u}} : \boldsymbol{u} \in \bar{J}\}$ between the faces of Δ_L^0 and the set of all basic fibers, and that $\#J = \#C_J - 1$. The Γ-graded module $\boldsymbol{F}_L := \bigoplus_{J \in \Delta_L^0} S(-\boldsymbol{e}_{C_J})$ equipped with the differential given in [11] is the algebraic Scarf complex where each basis element $\boldsymbol{e}_{C_J}$ is in homological degree $\#J$ and Γ-degree C_J, i.e., Γ-degree of a monomial in C_J. In general the complex $\boldsymbol{F}_L$ is contained in the minimal free resolution of S/I_L over S, and if the equality occurs we say that I_L is a *Scarf lattice ideal*.

It follows from the definition that the minimal free resolution of a Scarf lattice ideal I_L is a monomial resolution which does not depend on the characteristic of the field $\Bbbk$, and the quotient $\sum_{J \in \Delta_L^0} (-1)^{\#J} \cdot \boldsymbol{x}^{\max(\bar{J})} / \prod_{i=1}^n (1 - x_i)$ is the Γ-graded Hilbert series of S/I_L, where we identify all monomials in a fiber.

All codimension 1, non-complete intersection codimension 2 and generic lattice ideals, i.e., lattice ideals generated by binomials with full supports, are the well-known examples of Scarf lattice ideals [10], [11]. However, as we will see in Section 5, we can have other types of Scarf lattice ideals.

This paper is organized as follows. In Section 2, we will describe minimal generators of a Scarf lattice ideal (cf. Theorem 2.2). We will see that a Scarf lattice ideal defining a monomial curve must be generic (cf. Theorem 2.4). In Section 3, we will see that the initial ideals of a Scarf lattice ideal may also be minimally resolved by a certain kind of Scarf complexes (cf. Theorem 3.1). We will also present a proof of an unpublished result due to Yanagawa which states that all initial monomial ideals of a non-complete intersection codimension 2 lattice ideal are Scarf (cf. Corollary 3.2). In Section 4, we will see that for a Scarf lattice ideal being Cohen–Macaulay (resp. being Gorenstein) is equivalent to satisfying S_2 condition (resp. being principal) (cf. Theorem 4.2 and Theorem 4.3). Moreover, if a Scarf lattice ideal is k-Buchsbaum ($k > 0$), then the length of its minimal free resolution is maximal (cf. Theorem 4.4). We will also see that like the generic lattice ideals, Cohen–Macaulay codimension 2 lattice ideals have always a Cohen–Macaulay initial ideal (cf. Corollary 4.10). In Section 5, we will provide some examples of Scarf lattice ideals. In particular, we will present a Scarf lattice ideal that is neither generic nor of codimension 2.

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2. Minimal generators

For the homogeneous matrix B, the set of neighbors of the origin (cf. [2]) and its Hilbert basis (cf. [10]) are defined by

$$N(B) := \{\boldsymbol{u} \in \mathbb{Z}^m \mid \boldsymbol{u} \neq 0, \operatorname{int} P_{(B\boldsymbol{u})^+} \cap \mathbb{Z}^m = \emptyset\},$$

and

$$H(B) := \{\boldsymbol{u} \in \mathbb{Z}^m \mid \boldsymbol{u} \neq 0, \#(P_{(B\boldsymbol{u})^+} \cap \mathbb{Z}^m) = 2\},$$

respectively. Clearly $H(B) \subseteq N(B)$ and both of them are 0-symmetric. Therefore we can identify antipodal pairs in them. Since B is homogeneous, $N(B) \neq \emptyset$ (cf. [2]). However, $H(B)$ may or may not be empty.

Lemma 2.1. *If for each $\boldsymbol{u} \in N(B)$ we have $\# \operatorname{supp}(B\boldsymbol{u}) = n$, then $H(B) = N(B)$.*

Proof. Let $\boldsymbol{u} \in N(B)$. By definition of $P_{(B\boldsymbol{u})^+}$, each facet of $P_{(B\boldsymbol{u})^+}$ goes either from the origin or from $\boldsymbol{u}$. Suppose the contrary that $P_{(B\boldsymbol{u})^+}$ has a lattice point $\boldsymbol{v}_0$ other than 0 and $\boldsymbol{u}$. We consider two following cases:

Case 1: $\boldsymbol{v}_0$ is on the facet passing from the origin. This case is not possible because there exists a Gale vector b_i, i.e., a row vector of the matrix B, such that $b_i \cdot \boldsymbol{v}_0 = 0$ which is a contradiction by $\# \operatorname{supp}(B\boldsymbol{v}_0) = n$ (notice that if $\boldsymbol{u} \in N(B)$ and $\boldsymbol{v} \in P_{(B\boldsymbol{u})^+} \cap \mathbb{Z}^m$, then $\boldsymbol{v} \in N(B)$).

Case 2: $\boldsymbol{v}_0$ is on the facet passing from $\boldsymbol{u}$. In this case, we consider the polytope $P_{(B\boldsymbol{u})^+} - \boldsymbol{u} = P_{(B\boldsymbol{u})^-} = P_{(B(-\boldsymbol{u}))^+}$. Then $\boldsymbol{v}_0 - \boldsymbol{u}$ is on the facet of $P_{(B(-\boldsymbol{u}))^+}$ passing from the origin. Thus, there exists a Gale vector b_i such that $b_i \cdot (\boldsymbol{v}_0 - \boldsymbol{u}) = 0$. Since $\boldsymbol{v}_0 - \boldsymbol{u} \in N(B)$, this contradicts $\#\operatorname{supp}\big(B(\boldsymbol{v}_0 - \boldsymbol{u})\big) = n$. □

Theorem 2.2. *Let B be an integer $n \times m$-matrix of rank m which is homogeneous with respect to a strictly positive integer vector $\boldsymbol{w} = (w_1, \dots, w_n)$. Consider the following statements:*

(1) $\boldsymbol{u} \in H(B)$.

(2) $\{\boldsymbol{x}^{(B\boldsymbol{u})^+}, \boldsymbol{x}^{(B\boldsymbol{u})^-}\}$ *is a 2-element fiber.*

(3) $\{0, B\boldsymbol{u}\} \in \Delta_L$.

(4) $\{B\boldsymbol{u}\} \in \Delta_L^0$.

(5) $\boldsymbol{x}^{(B\boldsymbol{u})^+} - \boldsymbol{x}^{(B\boldsymbol{u})^-}$ *is an indispensable binomial.*

(6) $\boldsymbol{u} \in N(B)$.

Then the first five statements are equivalent and they imply (6). *Moreover, if B is generic, then all of them are equivalent. Consequently, if I_L is a Scarf lattice ideal, then it has a unique minimal set of Γ-homogeneous binomial generators which correspond to the elements of $H(B)$.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3): The implications follow from correspondences $\boldsymbol{v} \mapsto \boldsymbol{u} - B\boldsymbol{v}$ and $J \mapsto C_J$ mentioned in Section 1, respectively.

(3) $\Leftrightarrow$ (4): Follows from the definition of Δ_L^0.

(2) $\Rightarrow$ (5): By definition of an indispensable binomial, we have to show that every system of binomial generators of I_L contains $\boldsymbol{x}^{(B\boldsymbol{u})^+} - \boldsymbol{x}^{(B\boldsymbol{u})^-}$ up to a sign. This follows from the fact that the Betti number corresponding to the fiber $\{\boldsymbol{x}^{(B\boldsymbol{u})^+}, \boldsymbol{x}^{(B\boldsymbol{u})^-}\}$ is equal to 1 (cf. [10], Lemma 2.1).

(5) $\Rightarrow$ (2): Let $\mathscr{G}$ be an arbitrary set of minimal generators of I_L. Since $\boldsymbol{x}^{(B\boldsymbol{u})^+} - \boldsymbol{x}^{(B\boldsymbol{u})^-}$ is an indispensable binomial, then we may assume that $\boldsymbol{x}^{(B\boldsymbol{u})^+} - \boldsymbol{x}^{(B\boldsymbol{u})^-} \in \mathscr{G}$. Suppose, on the contrary, that $\{\boldsymbol{x}^{(B\boldsymbol{u})^+}, \boldsymbol{x}^{(B\boldsymbol{u})^-}\}$ is not a 2-element fiber. Then it has a monomial $\boldsymbol{x}^{\boldsymbol{a}}$ other than $\boldsymbol{x}^{(B\boldsymbol{u})^+}$ and $\boldsymbol{x}^{(B\boldsymbol{u})^-}$. We can replace $\boldsymbol{x}^{(B\boldsymbol{u})^+} - \boldsymbol{x}^{(B\boldsymbol{u})^-} \in \mathscr{G}$ by two binomials $\boldsymbol{x}^{(B\boldsymbol{u})^+} - \boldsymbol{x}^{\boldsymbol{a}}$ and $\boldsymbol{x}^{\boldsymbol{a}} - \boldsymbol{x}^{(B\boldsymbol{u})^-}$ and reduce the new set of generators to the minimal one by eliminating a superfluous element. This contradicts that the binomial $\boldsymbol{x}^{(B\boldsymbol{u})^+} - \boldsymbol{x}^{(B\boldsymbol{u})^-}$ is indispensable.

(1) $\Rightarrow$ (6): Follows from the definitions of $H(B)$ and $N(B)$.

If B is generic, then Lemma 2.1 implies the result. □

Remark 2.3. The Scarf property of I_L does not imply that $N(B) = H(B)$. To see this, suppose that

$$B^T = \begin{bmatrix} 1 & -1 & 1 & -1 & 2 & -2 \\ 0 & 1 & -1 & 0 & 1 & -1 \end{bmatrix}.$$

Since the Gale diagram intersects each of the four open quadrants, by [10], Proposition 4.1, I_L is not Cohen–Macaulay and therefore is a Scarf lattice ideal. We can see easily that $\boldsymbol{u} = (1,1) \in N(B)$, but $\boldsymbol{u} \notin H(B)$.

Theorem 2.4. *If B is of size $n \times (n-1)$, then I_L is a Scarf lattice ideal if and only if it is generic. In particular, the result is true for the defining ideal of a monomial curve in $\mathbb{A}^n$.*

Proof. If I_L is generic, the result is obvious. Conversely, suppose that I_L is a Scarf lattice ideal and let $\boldsymbol{u} \in N(B)$. By relaxing (cf. the proof of [8], Proposition 2.6.1, to see what "relaxing" means) the facets of the simplex $P_{(B\boldsymbol{u})^+}$ (if it is necessary), we can find a maximal lattice point free polytope Q which has $\boldsymbol{u}$ as one of its lattice points. Here we recall that a polytope is said to be maximal lattice point free if it contains no lattice points in its interior, but every facet of it contains at least one lattice point in its relative interior. It is easy to show that all homogeneous matrices of size $n \times (n-1)$ are Cohen–Macaulay. Therefore the matrix B is Cohen–Macaulay, and we can apply [12], Theorem 3.2, to see that Q corresponds to a basic fiber of the degree of a highest minimal syzygy of S/I_L over S. Since I_L is a Scarf lattice ideal, Q has exactly n lattice points, i.e., each facet of Q has a unique lattice point. This implies that $0 \neq \boldsymbol{u}$ is not on the facet passed from the origin and consequently $\#\operatorname{supp}(B\boldsymbol{u}) = n$. Since $\boldsymbol{u}$ is an arbitrary element of $N(B)$, we get the result. □

Remark 2.5. An important problem in combinatorial commutative algebra is to characterize face numbers (resp. total Betti numbers) of Scarf complex Δ_L^0 (resp. of Scarf lattice ideal I_L). In the non-complete intersection codimension 2 case, we know by [10] that $f(\Delta_L^0) = \big(f_0, 2(f_0-2), f_0-3\big)$. For a generic $(n+1) \times n$-matrix B, Björner [3] proved that the h-vector $h(\Delta_L^0) = (h_0, \dots, h_n)$ satisfies the equalities $h_0 = h_{n-1} = 1$, $h_n = 0$ and $h_i = h_{n-1-i}$ for all $0 \le i \le \lfloor \frac{n-1}{2} \rfloor$. Then using this observation he also showed that $f_0, f_1, \dots, f_{\lfloor (n-3)/2 \rfloor}$ completely determine $f(\Delta_L^0)$. Here, by Theorem 2.4, the problem is solved in the case of Scarf monomial curves.

3. Initial ideals

Let M be a monomial ideal in S minimally generated by monomials $\boldsymbol{x}^{\boldsymbol{u}_1}, \dots, \boldsymbol{x}^{\boldsymbol{u}_r}$ and

$$\Delta_M := \{J \subseteq \{\boldsymbol{u}_1, \dots, \boldsymbol{u}_r\} \mid \max(J) \neq \max(J') \text{ for all } J' \subseteq \{\boldsymbol{u}_1, \dots, \boldsymbol{u}_r\} \text{ other than } J\}.$$

Recall from [9], Chapter 6 that Δ_M is a simplicial complex and the $\mathbb{N}^n$-graded module $\boldsymbol{F}_M := \bigoplus_{J \in \Delta} S(-\boldsymbol{e}_J)$ equipped with the differential given in [9], Chapter 6 is the monomial Scarf complex where each basis element $\boldsymbol{e}_J$ is in the homological degree $\#J$ and $\mathbb{N}^n$-degree $\max(J) \in \mathbb{N}^n$. In general, the complex $\boldsymbol{F}_M$ is contained in the minimal free resolution of S/M over S, and if the equality occurs we say that M is a *Scarf monomial ideal*.

Theorem 3.1. *Let I_L be an x_i-full Scarf lattice ideal, i.e., the variable x_i appears in each of its minimal binomial generators. Then the reverse lexicographic initial ideal of I_L with x_i smallest is a Scarf monomial ideal.*

Proof. We may assume that $\mathscr{G} = \{\boldsymbol{x}^{\boldsymbol{u}_1^+} - \boldsymbol{x}^{\boldsymbol{u}_1^-}, \ldots, \boldsymbol{x}^{\boldsymbol{u}_r^+} - \boldsymbol{x}^{\boldsymbol{u}_r^-}\}$ is the unique minimal set of Γ-homogeneous binomial generators of I_L so that x_i divides each monomial $\boldsymbol{x}^{\boldsymbol{u}_i^-}$ for $i = 1, \ldots, r$. Then by [10], Lemma 8.4, $M = \langle \boldsymbol{x}^{\boldsymbol{u}_1^+}, \ldots, \boldsymbol{x}^{\boldsymbol{u}_r^+} \rangle$ is the reverse lexicographic initial ideal of I_L with x_i smallest. It is easy to show that $I_L + \langle x_i \rangle = M + \langle x_i \rangle$. Therefore using the properties of tensor product, we can show that $\mathbb{k}[x_i] \otimes_{\mathbb{k}} (S/I_L + \langle x_i \rangle) \simeq S/M$. Since x_i is a nonzero divisor on S/I_L, it follows that the minimal free resolution of S/M over S is obtained from the minimal free resolution of S/I_L by setting $x_i = 0$ in the matrices of differential. If we prove that the face poset of Δ_M is isomorphic to the face poset of Δ_L^0, we get the result. To this end, we note that since the Scarf complex $\boldsymbol{F}_M$ is contained in the minimal free resolution of S/M over S, the above argument shows that $f(\Delta_L^0) \geq f(\Delta_M)$, where the inequality is component-wise comparison of f-vectors. The vertex sets of Δ_L^0 and Δ_M are $\mathscr{V} := \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r\}$ and $\mathscr{V}^+ := \{\boldsymbol{u}_1^+, \ldots, \boldsymbol{u}_r^+\}$, respectively. If $J \in \Delta_L^0$, then $\max(\bar{J}) \neq \max(\bar{J}')$ for all $J' \subseteq \mathscr{V}$ other than J, or equivalently $\max(J^+) \neq \max(J'^+)$ for all $J'^+ \subseteq \mathscr{V}^+$ other than J^+, which is also equivalent to $J^+ \in \Delta_M$. So we have the inclusion $\Delta_L^0 \hookrightarrow \Delta_M$ defined by $J \mapsto J^+$. In view of $f(\Delta_L^0) \geq f(\Delta_M)$ this gives us the result. □

Corollary 3.2 (Yanagawa). *All initial monomial ideals of a non-complete intersection codimension* 2 *lattice ideal are Scarf monomial ideals.*

Proof. Let I_L be a non-complete intersection codimension 2 lattice ideal and M be an initial ideal of I_L with respect to any term order represented by a generic weight vector λ. Following Peeva and Sturmfels [10], Algorithm 8.2, we construct a lattice ideal $I_{\tilde{L}}$ in $S[t] = \mathbb{k}[x_1, \ldots, x_n, t]$ which is the flat deformation of I_L with respect to λ and whose image under substitution $t = 1$ and $t = 0$ are I_L and M. Now the ideal $I_{\tilde{L}}$ is a t-full Scarf lattice ideal. Thus by Theorem 3.1, we get the result. □

Example 3.3. The codimension 2 lattice ideal $I_L = \langle x_1x_3 - x_2^2, x_1x_4 - x_2x_3, x_2x_4 - x_3^2 \rangle \subset S = \mathbb{k}[x_1, x_2, x_3, x_4]$ is the defining ideal of the twisted cubic curve $(s, t) \mapsto (s^3, s^2t, st^2, t^3)$ in $\mathbb{P}^3$ and is x_2-full Scarf lattice ideal. It has eight distinct initial ideals [13], and seven of them are not generic (in the sense of [9], Definition 6.5). If λ denotes the degree reverse lexicographic order with x_2 smallest, then $M = \mathrm{in}_\lambda(I_L) = \langle x_1x_3, x_1x_4, x_3^2\rangle$, which is not generic. Setting $\boldsymbol{u}_1 = (1,-2,1,0)$, $\boldsymbol{u}_2 = (1,-1,-1,1)$, $\boldsymbol{u}_3 = (0,-1,2,-1)$, we see that the facets of Δ_M are $\{\boldsymbol{u}_1^+, \boldsymbol{u}_2^+\}$ and $\{\boldsymbol{u}_1^+, \boldsymbol{u}_3^+\}$. By Theorem 3.1, the facets of Δ_L^0 are $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ and $\{\boldsymbol{u}_1, \boldsymbol{u}_3\}$ and M and I_L are resolved minimally by Δ_M and Δ_L^0, respectively.

4. Some algebraic properties

By Serre's criterion, a Noetherian ring is normal if and only if it satisfies Serre's conditions S_2 and R_1. On the other hand, Hochster's theorem states that every normal toric ring is Cohen–Macaulay. The S_2 condition alone is not sufficient for Cohen–Macaulayness as an example due to Hochster (cf. [4], Exercise 6.2.7) shows. However, Goto, Watanabe and Suzuki [4], Exercise 6.2.8 (c), proved that for simplicial toric ring being Cohen–Macaulay is equivalent to satisfying S_2 condition. In this section, we will present a homological proof for a result due to Yanagawa which states that for Scarf toric rings, being Cohen–Macaulay is equivalent to satisfying S_2 condition. The combinatorially inclined reader may refer to [8], Proposition 2.6.1, for a very nice combinatorial proof of this result.

Motivated by Hochster's theorem, Sturmfels asked and conjectured the finer question that if a toric ideal is Cohen–Macaulay, does it have a Cohen–Macaulay initial ideal? In [8], Matusevich showed that for a Cohen–Macaulay generic toric ideal I_A, the initial ideals $\mathrm{in}_{-e_i}(I_A)$ $(i = 1, \ldots, n)$, are Cohen–Macaulay. In this section, we will prove a similar result for Cohen–Macaulay codimension 2 toric ideals.

Lemma 4.1. *Let I_L be a Scarf lattice ideal and $p = \text{proj-dim}_S(S/I_L) = \dim \Delta_L^0 + 1$. Then the Krull dimension of $\mathrm{Ext}_S^p(S/I_L, S)$ is equal to $n-p$ or $n-p-1$.*

Proof. By [4], Corollary 3.5.11, we have $\dim \mathrm{Ext}_S^p(S/I_L, S) \le n-p$. Let $e \in (\boldsymbol{F}_L)_p$ be a generator of $\boldsymbol{F}_L$ in homological degree p corresponding to a highest minimal syzygy of S/I_L over S, and let $e^* \in \boldsymbol{F}_L^*$ be its dual. Since e^* is a cocycle of $\boldsymbol{F}_L^*$, we have the corresponding element $\overline{e^*} \in \mathrm{Ext}_S^p(S/I_L, S)$. Setting $J = \mathrm{ann}(\overline{e^*})$, we see that $S/J \simeq S \cdot \overline{e^*} \subset \mathrm{Ext}_S^p(S/I_L, S)$. By the construction of $\boldsymbol{F}_L$ we have $\partial(e) = \sum_{i=1}^{p+1} m_i \cdot e_i$, where each m_i is a nonconstant monomial and each e_i is a generator of $\boldsymbol{F}_L$ in homological degree $p-1$. It is easy to show that $J' := \langle m_1, \ldots, m_{p+1}\rangle \supset J$. Since by Krull's theorem we have $\dim S/J' \ge n-p-1$, we conclude that $\dim \mathrm{Ext}_S^p(S/I_L, S) \ge n-p-1$. □

Theorem 4.2 (Yanagawa). *Let I_L be a Scarf lattice ideal. Then S/I_L satisfies Serre's condition S_2 if and only if it is Cohen–Macaulay.*

Proof. The "if" part is obvious. To prove the "only if" part, suppose, on the contrary, that S/I_L is not Cohen–Macaulay. Then $p := \text{proj-dim}_S(S/I_L) > \text{codim}(I_L)$. Since S/I_L satisfies Serre's condition S_2, we have $\dim \text{Ext}^j_S(S/I_L, S) \leq n - j - 2$ for all $j > \text{codim}(I_L)$ by [15], Lemma 2.9 (3). In particular, we have $\dim \text{Ext}^p_S(S/I_L, S) \leq n - p - 2$, which contradicts Lemma 4.1. □

Theorem 4.3. *Let I_L be a Scarf lattice ideal. Then S/I_L is Gorenstein if and only if I_L is a principal ideal.*

Proof. If I_L is principal, the result is obvious. Conversely, let S/I_L be Gorenstein and $p = \text{proj-dim}_S(S/I_L)$. We have $\text{Ext}^p_S(S/I_L, S) \simeq S^{\beta_p}/\text{im}\,\partial_p^T$ where $\partial_p : (\boldsymbol{F}_L)_p = S^{\beta_p} \to (\boldsymbol{F}_L)_{p-1} = S^{\beta_{p-1}}$ is the last differential in the minimal free resolution $\boldsymbol{F}_L$. Since S/I_L is Gorenstein, we have $\text{Ext}^p_S(S/I_L, S) \simeq S/I_L$ and $\beta_p = 1$. If $p \geq 2$, then the structure of differential of $\boldsymbol{F}_L$ implies that $\text{im}\,\partial_p^T$ is a monomial ideal, which is a contradiction. □

We say that S/I_L is k-Buchsbaum ($k \geq 0$ is an integer) if $\mathfrak{m}^k H^i_{\mathfrak{m}}(S/I_L) = 0$ for $i \neq \dim S/I_L$. Notice that 0-Buchsbaum is Cohen–Macaulay.

Theorem 4.4. *Let I_L be a Scarf lattice ideal and $p = \text{proj-dim}_S(S/I_L)$. If S/I_L is k-Buchsbaum (for some $k > 0$), then $p = n - 1$.*

Proof. We assume that $\mathfrak{m}^k H^{n-p}_{\mathfrak{m}}(S/I_L) = 0$ for some integer $k > 0$. By the local duality theorem, we have $\text{Ext}^p_S(S/I_L, S) \simeq \text{Hom}_S(H^{n-p}_{\mathfrak{m}}(S/I_L), E)$, where E is the injective hull of the residue field $S/\mathfrak{m}$. Thus, $\mathfrak{m}^k H^{n-p}_{\mathfrak{m}}(S/I_L) = 0$ if and only if $\mathfrak{m}^k \text{Ext}^p_S(S/I_L, S) = 0$. Hence, $\text{Ext}^p_S(S/I_L, S)$ is of finite length. Let e, J and J' be as in the proof of Lemma 4.1. We see that S/J and S/J' are of finite length. Thus, $\dim S/J' = 0$, which implies that $\dim S = p + 1$, i.e., $p = n - 1$. □

Remark 4.5. Consider a codimension 2 lattice ideal $I_L \subset \Bbbk[x_1, \ldots, x_n]$. If I_L is k-Buchsbaum ($k > 0$), then by Theorem 4.4 and [10], Theorem 2.3, we have $p = n - 1 \leq 3$. Since in a polynomial ring whose number of variables ≤ 3, I_L is Cohen–Macaulay, we conclude that k-Buchsbaumness ($k > 0$) for I_L implies that $p = n - 1 = 3$.

In the remainder of this section we will assume that the ideal I_L is a toric ideal of an integer $d \times n$-matrix A. Each column $\boldsymbol{a}_i$ of the matrix A is identified with a monomial $\boldsymbol{t}^{\boldsymbol{a}_i}$ in the polynomial ring $\Bbbk[\boldsymbol{t}] := \Bbbk[t_1, \ldots, t_d]$. Notice that $S/I_L = \Bbbk[\mathbb{N}A] = \Bbbk[\boldsymbol{t}^{\boldsymbol{a}_1}, \ldots, \boldsymbol{t}^{\boldsymbol{a}_n}] \subset \Bbbk[\boldsymbol{t}]$.

Theorem 4.6. *If I_L is a Scarf toric ideal, then the Scarf toric ring S/I_L is a Golod ring.*

Proof. This was proved for generic toric ideals in [6]. Exactly the same proof remains valid for Scarf lattice ideals. The key ingredient is that the Koszul homology $\mathrm{Tor}_*^S(S/I_L, \Bbbk)$ can be computed by the minimal free resolution of S/I_L which is the algebraic Scarf complex, and this property of generic toric ideals holds for Scarf toric ideals by their definition as well. □

Remark 4.7. For a Scarf toric ring S/I_L, Golodness also implies being Gorenstein is equivalent to being hypersurface, i.e., I_L is a principal ideal (cf. [1], Section 5.2).

Theorem 4.8. *The toric ring $R = S/I_L$ satisfies Serre's condition S_2 if and only if the ideal $I_L + \langle x_i \rangle$ is free of embedded primes for $i = 1, \dots, n$.*

Proof. Cf. [8], Proposition 2.5.2. □

Theorem 4.9. *Let I_L be an x_i-full Scarf toric ideal. Then I_L is Cohen–Macaulay if and only if* $\mathrm{in}_{-e_i}(I_L)$ *is Cohen–Macaulay.*

Proof. First, we assume that I_L is Cohen–Macaulay. Using the equality $I_L + \langle x_i \rangle = \mathrm{in}_{-e_i}(I_L) + \langle x_i \rangle$, we can see that the ideal $\mathrm{in}_{-e_i}(I_L)$ is generated by the set $(I_L + \langle x_i \rangle) \cap \Bbbk[x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n]$ in S. Therefore, the ideal $\mathrm{in}_{-e_i}(I_L)$ is free of embedded primes. Now the result follows from [14], Proposition 2.9, and the well-known fact (cf. [7]) that each initial ideal of a toric ideal is equidimensional, i.e., all of its minimal primes have the same height. The "only if" part follows from the inequalities $\mathrm{codim}(I_L) \le \text{proj-dim}_S(S/I_L) \le \text{proj-dim}_S\big(S/\mathrm{in}_{-e_i}(I_L)\big)$. □

Corollary 4.10. *Let I_L be either a codimension 2 or a generic toric ideal. If I_L is Cohen–Macaulay, then it admits a Cohen–Macaulay initial ideal.*

Proof. If I_L is generic, then Theorem 4.9 implies that $\mathrm{in}_{-e_i}(I_L)$ is Cohen–Macaulay for $i = 1, \dots, n$. For a codimension 2 lattice ideal I_L, we consider the two following cases:

Case 1: I_L is not complete intersection. In this case the ideal I_L is Scarf. If the ideal I_L is x_i-full, then by Theorem 4.9, we get the result. Otherwise, by [10], Proposition 8.3, there exists a reverse lexicographic term order $\prec$ with x_i smallest such that the reduced Gröbner basis of I_L with respect to $\prec$ is a minimal generating set. We assume that $\omega \in \mathbb{N}^n$ represents the term order $\prec$. Using [10], Algorithm 8.2, we construct a lattice ideal $I_{\tilde{L}}$ in $S[t] = \Bbbk[x_1, \dots, x_n, t]$ which is a flat deformation of I_L with respect to ω. According to the proof of [10], Proposition 8.3, the ideal $I_{\tilde{L}}$ has the same number of minimal generators as I_L. Therefore by [10], Proposition 4.1, the ideal $I_{\tilde{L}}$ is Cohen–Macaulay. Let $\prec'$ be a reverse lexicographic term order on monomials in $S[t] = \Bbbk[x_1, \ldots, x_n, t]$ with t smallest. Then by Theorem 4.9, $\mathrm{in}_{\prec'}(I_{\tilde{L}})$ is Cohen–Macaulay in $S[t]$ and so

$$S/\mathrm{in}_{\prec}(I_L) \simeq S[t]/\big(\mathrm{in}_{\prec'}(I_{\tilde{L}}), t\big)$$

is Cohen–Macaulay.

Case 2: I_L is complete intersection. Let $\prec$ be as in the previous case. Then $M = \mathrm{in}_{\prec}(I_L)$ is complete intersection and so Cohen–Macaulay. □

5. Examples

In this section we will present several examples of Scarf lattice ideals which were obtained by exhaustive and heuristic search using CoCoA [5]. In particular, we will give an example of Scarf lattice ideals which is neither codimension 2 nor generic.

Example 5.1 (Scarf monomial curves in $\mathbb{A}^4$, $\mathbb{A}^5$). If we assume that $\mathscr{M}$ is the set of all monomial curves $C_{a,b,c,d} : t \mapsto (t^a, t^b, t^c, t^d)$ in $\mathbb{A}^4$ with $1 \leq a < b < c < d \leq 100$, then exhaustive search by CoCoA shows that we have 5500 Scarf monomial curves of seven types (in terms of f-vector of Δ_L^0) as in Table 1. Furthermore, by heuristic search using CoCoA we found that the monomial curve $t \mapsto (t^{205}, t^{210}, t^{240}, t^{246}, t^{329})$ in $\mathbb{A}^5$ is Scarf.

Example 5.2 (A Scarf lattice ideal that is neither generic nor of codimension 2). Using CoCoA and by exhaustive search, we find that the matrix

$$B^T = \begin{bmatrix} 1 & -1 & -2 & -1 & 3 \\ 0 & 1 & 1 & -3 & 1 \\ 0 & -2 & 4 & -1 & -1 \end{bmatrix}$$

Table 1. Scarf monomial curves in $\mathscr{M}$.

f-vector of Δ_L^0	A typical example	Numbers in $\mathscr{M}$
$(7, 12, 6)$	$C_{20,24,25,31}$	4701
$(8, 14, 7)$	$C_{36,42,47,49}$	386
$(9, 16, 8)$	$C_{35,45,48,56}$	289
$(10, 18, 9)$	$C_{39,50,51,58}$	77
$(11, 20, 10)$	$C_{51,59,72,74}$	21
$(12, 22, 11)$	$C_{56,77,79,88}$	25
$(14, 26, 13)$	$C_{79,82,89,95}$	1

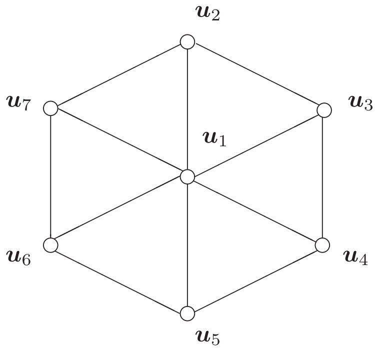

Figure 1. The linked Scarf complex Δ_L^0.

defines the lattice ideal

$$\begin{aligned} I_L = \langle & x_2x_3^2x_4 - x_1x_5^3, x_4^3 - x_2x_3x_5, x_2^3x_4^2 - x_1x_3^2x_5^2, x_2^4 - x_1x_3x_4x_5, \\ & x_2^2x_3^3 - x_1x_4^2x_5^2, x_3^4 - x_2^2x_4x_5, x_3^3x_4^2 - x_2^3x_5^2\rangle \subset S = \Bbbk[x_1, \dots, x_5], \end{aligned}$$

which has codimension 3 and is not generic. We will show that the ideal I_L is Scarf. The linked Scarf complex Δ_L^0 can be depicted as in Figure 1.

Here each vector $\boldsymbol{u}_i$ corresponds to the minimal generator $\boldsymbol{x}^{\boldsymbol{u}_i^+} - \boldsymbol{x}^{\boldsymbol{u}_i^-}$ in the ordering appeared in the above list of minimal generators, for instance $\boldsymbol{u}_1 = (-1, 1, 2, 1, -3)$. The one to one correspondence $J \mapsto C_J := \{\boldsymbol{x}^{\max(\bar{J})-\boldsymbol{u}} \mid \boldsymbol{u} \in \bar{J}\}$ between the facets of Δ_L^0 and the set of all highest basic fibers are listed as follows:

$$\begin{aligned} \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} &\mapsto \{x_2^3x_3^2x_4^3, x_1x_2^2x_4^2x_5^3, x_2^4x_3^3x_5, x_1x_3^4x_4x_5^2\}, \\ \{\boldsymbol{u}_1, \boldsymbol{u}_3, \boldsymbol{u}_4\} &\mapsto \{x_2^4x_3^2x_4^2, x_1x_2^3x_4x_5^3, x_1x_2x_3^4x_5^2, x_1x_3^3x_4^3x_5\}, \\ \{\boldsymbol{u}_1, \boldsymbol{u}_4, \boldsymbol{u}_5\} &\mapsto \{x_2^4x_3^3x_4, x_1x_2^3x_3x_5^3, x_1x_3^4x_4^2x_5, x_1x_2^2x_4^3x_5^2\}, \\ \{\boldsymbol{u}_1, \boldsymbol{u}_5, \boldsymbol{u}_6\} &\mapsto \{x_2^2x_3^4x_4, x_1x_2x_3^2x_5^3, x_1x_3x_4^3x_5^2, x_2^4x_4^2x_5\}, \\ \{\boldsymbol{u}_1, \boldsymbol{u}_6, \boldsymbol{u}_7\} &\mapsto \{x_2x_3^4x_4^2, x_1x_3^2x_4x_5^3, x_2^3x_4^3x_5, x_2^4x_3x_5^2\}, \\ \{\boldsymbol{u}_1, \boldsymbol{u}_7, \boldsymbol{u}_2\} &\mapsto \{x_2x_3^3x_4^3, x_1x_3x_4^2x_5^3, x_2^2x_3^4x_5, x_2^4x_4x_5^2\}. \end{aligned}$$

Using this correspondences, one can completely write down the Scarf chain complex $\boldsymbol{F}_L$ associated to Δ_L^0. It is of the form

$$0 \to S^6 \to S^{12} \to S^7 \to S \to 0.$$

Comparing this complex with minimal free resolution of S/I_L over S, we see that the ideal I_L is Scarf.

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