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Two results on the rank partition of a matroid

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Abstract. The rank partition of a matroid M is the maximum dominance ordered partition ρ such that the ground set of M can be partitioned into independent sets of sizes $\rho_1, \rho_2, \dots$. We prove two structural results on this partition, both motivated by representation theory of the general linear group. The first result characterizes the rank partition in terms of standard Young tableaux with a certain matroidal property. The second result says that the rank partition interacts nicely with certain polytopal decompositions of the matroid polytope of M. We also describe the representation theoretical motivation of these results.

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1. Introduction and statement of results

The rank partition of a matroid M is an isomorphism invariant that describes when a matroid can be partitioned into independent sets of various sizes. It was defined by J. A. Dias da Silva in 1990 [5], and is related to matroid partitioning theorems of Edmonds [6], and Edmonds–Fulkerson [7]. The goal of this paper is to prove two new results on the rank partition of a general matroid.

Given a matroid M with ground set $[n]$, its *rank partition* is the sequence $\rho(M) = (\rho_1, \rho_2, \dots, \rho_\ell)$ defined by the condition that for every positive integer $k \leq \ell$, the partial sum $\sum_{i=1}^{k} \rho_i$ is the size of the largest union of k independent sets from M.

Our main results on the rank partition are motivated by algebraic reformulations that have easy proofs for matroids realizable over a field $\mathbf{k}$ of characteristic zero. We indicate here how these results arise, giving the details in Section 5.

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Let $v = (v_1, \ldots, v_n)$ be a configuration of vectors realizing M in a vector space V over $\mathbf{k}$, a field of characteristic zero. Define,

$$G(v) := \text{span}_{\mathbf{k}}\{gv_1 \otimes gv_2 \otimes \cdots \otimes gv_n : g \in \text{GL}(V)\}.$$

It is shown in [3] how this module is related to the rank partition of M: $G(v)$ contains an irreducible representation of highest weight λ^t if and only if $\lambda \leq \rho(M)$ (dominance order). This is essentially equivalent to results of Dias da Silva and Gamas on vanishing of symmetrized tensors [5].

By considering the projections of $G(v)$ to a fixed irreducible representation of $\text{GL}(V)$, we were able to formulate and prove the following combinatorial result.

Theorem 1. *Let M be a loopless matroid with ground set $[n]$. There is a standard tableau of shape λ whose rows index independent sets of M if and only if $\lambda \leq \rho(M)$ in dominance order.*

The matroid basis polytope of M is the convex hull $P(M)$ in $\mathbb{R}^n$ of the characteristic vectors of the bases of M. Polyhedral subdivisions of $P(M)$ sometimes arise by taking one parameter family of projective configurations $v(t)$ and taking the limit as $t \to 0$, as first observed by Kapranov [9]. Studying how $G(v(t))$ behaves in the limit $t \to 0$, we were able to formulate and prove the following combinatorial result.

Theorem 2. *Suppose that M is a loopless matroid and*

$$P(M) = \bigcup_i P(M_i)$$

is a polyhedral subdivision. Then there is an index j such that $\rho(M) = \rho(M_j)$. Further, for any tableau whose rows index independent sets of M, there is an index j such that the rows of this tableau index independent sets of M_j.

After giving preliminary results on matroids, rank partitions and matroid polytopes, we give combinatorial proofs of these results. We conclude by showing how these results are obtained from $G(v)$ using the algebraic methods suggested above.

2. Preliminaries

Here we collect several definitions and basic results about matroids, rank partitions and matroid polytopes.

2.1. Basic matroid definitions. We always take M to be a matroid with ground set $[n] := \{1, 2, \dots, n\}$. That is, M is a simplicial complex on $[n]$ whose collection of faces $\mathscr{I}(M)$, called independent sets, satisfy the exchange axiom: For any pair of independent sets $I, I' \in \mathscr{I}(M)$ with $|I| < |I'|$, there is some $e \in I' - I$ such that $I \cup \{e\} \in \mathscr{I}(M)$.

The bases of M are its maximal independent sets, all of which have the same cardinality $r(M)$. We denote the set of bases of M by $\mathscr{B}(M)$. The rank of an arbitrary subset $A \subset [n]$ is the size of a maximal independent set contained in A, and this is denoted $r_M(A)$.

Examples of matroids abound, the most prominent example coming from linear algebra: If $v_1, v_2, \dots, v_n$ are vectors in a vector space V, then the independent sets of the associated matroid are those $I \subset [n]$ such that $\{v_i : i \in I\}$ is a linearly independent list of vectors. Such matroids are said to be realizable over the field of V.

2.2. The rank partition. For each matroid M we define a sequence of numbers $\rho(M) = (\rho_1, \rho_2, \dots, \rho_k, \dots)$ by the condition that, for each $k \geq 1$,

$$\rho_1 + \cdots + \rho_k = \max\Big\{|J| : J = \bigcup_{j=1}^{k} I_j, I_j \in \mathscr{I}(M)\Big\}.$$

This sequence is called the *rank partition* of M. The choice of terminology is justified in the following result of Dias da Silva.

Proposition 3 (Dias da Silva [5]). *For every matroid M, $\rho(M)$ is a partition, i.e., $\rho_1 \geq \rho_2 \geq \cdots$.*

If every singleton is independent in M, then we say that M is *loopless*. A moment's thought reveals that when M is loopless, $\rho(M)$ is a partition of n, i.e., $\rho_1 + \rho_2 + \cdots = n$.

Example 4. The definition of the rank partition is valid for any simplicial complex Δ. The following example (due originally to Seth Sullivant) shows that, in general, the rank partition of Δ may fail to be a partition. Let Δ be the simplicial complex on [6] with facets

$$\{1, 2, 3\}, \qquad \{1, 4\}, \qquad \{2, 5\}, \qquad \{3, 6\}.$$

Then $\rho(\Delta) = (3, 1, 2)$, which is not a partition.

The *union* of two matroids M and N, both with ground set $[n]$, is defined to be the matroid $M \cup N$ whose independent sets are unions $I \cup J$ of independent sets

$I \in \mathscr{I}(M)$ and $J \in \mathscr{I}(N)$. We denote the k-fold union of M with itself by $M^{(k)}$. The rank partition of M is at once seen to be capture the ranks of the unions $M, M^{(2)}, M^{(3)}, \ldots$.

Let M and N be matroids with common ground set $[n]$. A *weak map* of matroids is a bijection $w : [n] \to [n]$ such that $w\mathscr{I}(N) \subset I(M)$. We denote the existence of a weak map by writing $M \to N$ and say that N is a weak image of M. Clearly, whenever $M \to N$ the rank of N is at most the rank of M. There is a more general notion of weak map that allows some elements of M to "be mapped to zero", however we will not use this notion here.

Proposition 5. *If M and N are loopless matroids on $[n]$ and $M \to N$ is a weak map, then $\rho(N) \leq \rho(M)$ in dominance order.*

Recall that the *dominance order* on partition is defined so that $\lambda \leq \mu$ if and only if $|\lambda| = |\mu|$ and for every $k \geq 1$, $\sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i$.

Proof. If $M \to N$ is a weak map, then for every $k \geq 1$ there are weak maps

$$M^{(k)} \to N^{(k)}.$$

The resulting inequality of the ranks of these matroids is exactly the statement that $\rho(N) \leq \rho(M)$. □

Example 6. Consider the matroid M obtained from $U_{2,3}$ by doing a parallel extension once to each element.

This matroid has rank partition $(2, 2, 2)$. Consider the matroid N obtained from $U_{2,2}$ by doing a parallel extension twice.

Then $\rho(N) = (2, 2, 2)$. Both M and N are weak order minimal with rank partition $(2, 2, 2)$. The fibers of the poset map

$$\text{(Matroids, weak order)} \to \text{(Partitions, dominance order)},$$

do not seem to be easy to describe.

A *tableau* is a filling of the numbers $1, \ldots, n$ into the Young diagram of a partition $\lambda \vdash n$ (drawn in English notation), with each number used exactly once. A tableau whose rows index independent set of M is said to be *M-independent*. A tableau is said to be *standard* if the numbers in each row and column form an increasing sequence. The following result connects M-independent tableaux with the rank partition.

Theorem 7 (Dias da Silva). *There is an M-independent tableau of shape λ if and only if $\lambda \le \rho(M)$ in dominance order.*

The content of Theorem 1 is to strengthen the above result by adding a standardness constraint to the tableaux.

2.3. Matroid polytopes. If M is a matroid on $[n]$ with bases $\mathscr{B}(M)$, then the *matroid basis polytope* of M is defined to be the convex hull of the points $\sum_{i \in B} e_i \in \mathbb{R}^n$, where B ranges over all elements of $\mathscr{B}(M)$. Denote the matroid base polytope of M by $P(M)$. This polytope lives in the hyperplane of $\mathbb{R}^n$ where the sum of the coordinates is $r(M)$. Each base $B \in \mathscr{B}(M)$ gives a vertex $\sum_{i \in B} e_i$ of $P(M)$.

There is the related notion of the *independence polytope* of M, denoted $Q(M)$, which is the convex hull of the incidence vectors of independent sets of M. The matroid base polytope is the face of $Q(M)$ determined by maximizing the sum of the coordinates. For all $I \in \mathscr{I}(M)$, $\sum_{i \in I} e_i$ is a vertex of $Q(M)$.

Here we give a simple interpretation of the independence polytope of the k-fold union of M with itself, that does not appear to be recorded in the literature.

Proposition 8. *For any matroid M,*

$$Q(M^{(k)}) = kQ(M) \cap [0,1]^n.$$

Proof. We see that $kQ(M)$ is the Minkowski sum of $Q(M)$ with itself k times. The containment "$\subset$" follows since every vertex of $Q(M^{(k)})$ is in the intersection. For the reverse containment, we work with an inequality description of the independence polytope. The rank function of the union is known to be

$$r_{M^{(k)}}(A) = \min_{B \subset A} \big(|A - B| + k \cdot r_M(B) \big).$$

An inequality description of $P(M^{(k)})$ is thus given by $x_i \ge 0$ for all i and for every flag of subsets $B \subset A \subset [n]$,

$$\sum_{i \in A} x_i \le |A - B| + k \cdot r_M(B).$$

We need to check whether these inequalities are valid on $kQ(M) \cap [0,1]^n$. Certainly each of the inequalities $x_i \ge 0$ is valid. Pick $x \in kQ(M) \cap [0,1]^n$ and subsets

$B \subset A \subset [n]$. Then

$$\begin{aligned}\sum_{i\in A} x_i/k &= \sum_{i\in A-B} x_i/k + \sum_{j\in B} x_j/k \\ &\le \sum_{i\in A-B} 1/k + \sum_{j\in B} x_j/k = |A-B|/k + \sum_{j\in B} x_j/k \\ &\le |A-B|/k + r_M(B).\end{aligned}$$

The first inequality follows since $x \in [0,1]^n$ and the second inequality follows since $x/k \in Q(M)$. Multiplying both sides by k proves that $kQ(M) \cap [0,1]^n \subset Q(M^{(k)})$. □

As noted by a referee, the analogous statement holds for an arbitrary union of matroids:

$$Q(M_1 \cup \cdots \cup M_\ell) = \big(Q(M_1) + \cdots + Q(M_\ell)\big) \cap [0,1]^n.$$

3. Proof of Theorem 1

The idea of the proof of Theorem 1 is as follows: Take an M-independent tableau of shape λ, and if it is not standard apply some local moves to its entries that preserve the M-independence and bring the tableau closer to being standard. When M is a uniform matroid, this can be done with the usual straightening algorithm of the representation theory of $\mathfrak{S}_n$ and $\mathrm{GL}_r(\mathbb{C})$. The goal of this section is to show, when interpreted properly, the straightening algorithm works for all matroids.

We will need the *alternating basis exchange* property of matroids, proved by Kung.

Lemma 9 (Kung [10]). *Let $A = A_1 \sqcup A_2$ and $B = B_1 \sqcup B_2$ be bases of a matroid M such that $A_1 \cap B_2 = \emptyset$ and $A_1 \cup B_2$ is dependent. Then, there are non-empty subsets $C \subset A_1$ and $D \subset B_2$ such that $A - C \cup D$ and $B - D \cup C$ are both bases of M.*

To remember the slightly odd numbering of the sets being exchanged here, we draw the mnemonic

$$\begin{matrix} B_1 & B_2 \\ A_1 & A_2 \end{matrix}$$

and remember that the straightening algorithm for tableaux exchanges elements of A_1 with elements of B_2 and does nothing to B_1 and A_2.

Corollary 10. *Let $A = A_1 \sqcup A_2$ and $B = B_1 \sqcup B_2$ be independent sets of a matroid M such that $A_1 \cap B_2 = \emptyset$, $|B| \geq |A|$ and $|A_1 \cup B_2| = |B| + 1$.*

Then there are non-empty subsets $C \subset A_1$ and $D \subset B_2$ of the same size such that $A - C \cup D$ and $B - D \cup C$ are both independent in M.

In the proof we use the *truncation* of M to a given rank $s \leq r(M)$. This is the matroid whose bases are those independent sets of M of size s. We will also "add generic elements" to M without increasing its rank. That is, we will add new elements to M that are not contained in a circuit of size less than $r(M) + 1$.

Proof. We may assume that B is base of M. If not, we can truncate M to the rank of B, so that B is a base. Since the truncation of M is a weak image of M, the result will follow by proving the truncated version.

Assume that B is a base of M. By adding generic elements with large labels to A_2 (without increasing the rank of M), we obtain a set $A_2' \supset A_2$ such that $A_1 \cup A_2'$ is a base of M too. Apply the alternating basis exchange lemma to $A_1 \sqcup A_2'$ and $B_1 \sqcup B_2$, noting that $A_1 \cup B_2$ is dependent as it has cardinality larger than the rank of M. □

Proof of Theorem 1. We use the straightening algorithm that arises in the representation theory of the symmetric group, see [11], Chapter 2.

Consider the violation to T being a standard tableau that occurs in the south-most row and east-most column of this row, say cell (j, k) of T. That is, rows $j - 1$ and j of T look as follows

$$x_1 < x_2 < \cdots < x_k < x_{k+1} < \cdots\cdots\cdots < x_r,$$
$$y_1 < y_2 < \cdots < y_k < y_{k+1} < \cdots < y_s,$$

where $y_k < x_k$ and $x_j < y_j$ for $k < j \leq s$. Let B denote the elements in row $j - 1$ of T and A denote the elements in row j of T. We partition B and A as

$$B = B_1 \cup B_2, \qquad B_1 := \{x_1, x_2, \ldots, x_{k-1}\}, \qquad B_2 := \{x_k, \ldots, x_r\},$$
$$A = A_1 \cup A_2, \qquad A_1 := \{y_1, y_2, \ldots, y_k\}, \qquad A_2 := \{y_{k+1}, \ldots, y_s\}.$$

Since $|A_1 \cup B_2| = |B| + 1$, we apply the corollary to the alternating basis exchange lemma to conclude that there are non-trivial subsets $C \subset A_1$ and $D \subset B_2$ such that $A - C \cup D$ and $B - D \cup C$ are independent sets in M. Let S be the tableau obtained by exchanging the numbers in C with D, and then sorting the rows of the result. It follows from the dominance lemma for row tabloids [11], Chapter 2, that the south-east most violation in S of being standard is north-west of (j,k). We conclude that S is closer to being a standard tableau than T, hence the theorem is proved by induction. □

Example 11. The number of M-independent standard tableaux of shape λ depends on the ordering of the ground set of M. This can be seen in rank two already by taking M to be $U_{2,2}$ and adding a parallel element. Suppose that the elements of $U_{2,2}$ are labeled 1 and 2, and 3 is parallel to 1. The only M-independent standard filling is

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\,.$$

If we relabel so that 2 and 3 are taken to be parallel in M then both of

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\,, \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$

are M-independent.

At least for *hook shapes*[1] there is an explanation for the disparity seen here: The number of standard tableaux of a fixed hook shape whose rows are non-broken circuit sets of M does not depend on an order of the ground set. This is proved in [2]. In the second case above, the first row of $\tiny\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$ is a broken circuit.

4. A valuation and the proof of Theorem 2

Recall that a function f from the set of labeled matroids (say with labels $[n]$) to an abelian group is said to be valuative if, for any matroid basis polytope decomposition $P(M) = \bigcup_i P(M_i)$,

$$f(M) = \sum_i f(M_i) - \sum_{i<j} f(M_{i,j}) + \sum_{i<j<k} f(M_{i,j,k}) - \cdots,$$

where $M_{i_1,\ldots,i_j}$ is the matroid whose bases are those bases common to $M_{i_1},\ldots,M_{i_j}$. Derksen and Fink [8] have given a complete description of the module of matroid invariants that behave valuatively.

We will write $A_\bullet$ for a set partition $\{A_1,\ldots,A_\ell\}$ of $[n]$, i.e., $[n] = A_1 \sqcup \cdots \sqcup A_\ell$ and $A_i \neq \emptyset$.

[1]A hook is a partition with at most one part not equal to 1.

Lemma 12. *Consider a matroid M on $[n]$ with rank function r_M. Define the formal sum*

$$H(M) = \sum_{A_\bullet} A_\bullet,$$

where the sum ranges over set partitions $A_\bullet$ of $[n]$ and A_i is independent for all i. Then H is a valuative invariant of matroids.

Proof. We follow the idea of Ardila, Fink and Rincon [1]. It is sufficient to prove that the function $f_{A_\bullet}(M)$, that is 1 if every A_i is independent in M and 0 otherwise, is a valuative invariant. Indeed,

$$H(M) = \sum_{A_\bullet} f_{A_\bullet}(M)(A_1, \dots, A_\ell),$$

the sum over all set partitions of $[n]$.

Define $P_{S,s} = \{x \in [0,1]^n : \sum_{j \in S} x_j \geq s\}$ along with its characteristic function $\chi_{S,s}$, which is 1 on $P_{S,s}$ and 0 otherwise. The statement $P(M) \cap \bigcap_i P_{A_i, |A_i|} \neq \emptyset$ is equivalent to $f_{A_\bullet}(M) = 1$. It follows that

$$f_{A_\bullet} = \prod_{j=1}^{\ell} \chi_{A_j, |A_j|}$$

and this is a valuation by a result of Ardila, Fink and Rincon [1]. □

We can now prove the second of the main theorems.

Proof of Theorem 2. Suppose that $P(M) = \bigcup_{j=1}^{\ell} P(M_j)$ is a matroid base polytope decomposition. Then there is a set partition $A_\bullet$ of $[n]$ into independent sets of sizes $\rho_1, \rho_2, \dots,$ where $\rho(M) = (\rho_1, \rho_2, \dots)$.

Thus, there is some j such that $H(M_j)$ contains the term $A_\bullet$, which proves the second part of the theorem. It follows that $\rho(M_j) \geq \rho(M)$. However, $\rho(M_j) \leq \rho(M)$, since M_j is a weak image of M. We conclude that $\rho(M) = \rho(M_j)$. □

Corollary 13. *Let M be a loopless matroid on $[n]$. Suppose that for every weak map $M \to N$ we have $\rho(M) \neq \rho(N)$. There are at most finitely many realizations of M over a given field, up to projective equivalence.*

Proof. This follows from a result of Lafforgue, who proves that matroids whose polytopes have no matroid basis polytope decompositions have finitely many projectively distinct realizations. □

Example 14. Let $M(K_4)$ be the matroid of the complete graph on four vertices, K_4. This matroid is realizable over any field. Since K_4 can be covered by two disjoint spanning trees, $\rho\big(M(K_4)\big) = (3,3)$. Any rank 3 matroid that is a weak image of $M(K_4)$ is a direct sum of a rank two matroid and a rank one matroid, hence $M(K_4)$ has at most finitely projective realizations over *any field*. In fact, $M(K_4)$ is projectively unique.

Example 15. Alfonsin and Chatelain [4] introduced the notion of a matroid subdivision. This was a decomposition of the bases of matroid M as a union $\mathscr{B}(M) = \bigcup_i \mathscr{B}(M^i)$ where $M^i \neq M$ for all i, and for all i, j, the intersections $\mathscr{B}(M^i) \cap \mathscr{B}(M^j)$, were the bases of a matroid.

Let M have bases $\mathscr{B}(M) = \{13, 14, 23, 24, 34\}$. Define M_1 to have bases $\mathscr{B}(M_1) = \{13, 34, 14\}$ and M_2 to have bases $\mathscr{B}(M_2) = \{23, 34, 24\}$. The intersection is $\{34\}$, which is the collection of bases of a matroid. However, the rank partitions of these matroids are

$$\rho(M) = (2,2), \qquad \rho(M_1) = \rho(M_2) = (2,1,1).$$

Hence the convexity implicit in the statement of Theorem 2 is essential in making the statement true.

5. Representation theory motivation

The results above were inspired by the representation theory and geometry that surrounds certain general linear group orbits in $(\mathbf{k}^r)^{\otimes n}$ and torus orbit closures in Grassmannians.

Let $v = (v_1, v_2, \ldots, v_n)$ be a realization of M by vectors $v_i \in V$, where $V = \mathbf{k}^r$ is the r-dimensional vector space over a field $\mathbf{k}$ of characteristic zero. The object that motivated Theorems 1 and 2 is the $\mathbf{k}$-linear span of the set of tensors

$$\{gv_1 \otimes gv_2 \otimes \cdots \otimes gv_n : g \in \mathrm{GL}(V)\} \subset V^{\otimes n}.$$

We denote this span by $G(v)$; this is a module for the diagonal action of the general linear group $\mathrm{GL}(V)$ on the tensor product. The combinatorics of M is subtly tied to the irreducible decomposition of the $\mathrm{GL}(V)$-module $G(v)$.

Recall that the irreducible representations of $\mathrm{GL}(V)$ that occur in $V^{\otimes n}$ are indexed by partitions λ that fit in a $\dim(V)$-by-n box. Thus, to describe the isomorphism type of $G(v)$ it is sufficient give a list of partitions along with their multiplicities in $G(v)$.

A *Young symmetrizer* is a particular element of the symmetric group algebra $\mathbf{k}\mathfrak{S}_n$ (whose precise definition we do not need here) that acts (up to a scalar) as a

$\mathrm{GL}(V)$-module projection of $V^{\otimes n}$ onto an irreducible submodule. Young symmetrizers are indexed by tableaux, and if c_T is the symmetrizer associated to the tableau T, then the image of c_T on $V^{\otimes n}$ is either zero or the irreducible $\mathrm{GL}(V)$-module associated to the shape of T. We write the action of the symmetric group $\mathfrak{S}_n$ on $V^{\otimes n}$, by permuting tensor factors, on the right. In this way, $V^{\otimes n}$ is a left $\mathrm{GL}(V)$-module and a right $\mathbf{k}\mathfrak{S}_n$-module.

See [3], Theorem 2, for the following result.

Theorem 16. *The image of a Young symmetrizer c_T on $G(v)$ is not zero if and only if the columns of T index independent sets of the matroid M.*

It is a consequence of Schur–Weyl duality that the Young symmetrizers of standard tableaux of shape λ span the space of $\mathrm{GL}(V)$-module homomorphisms $V^{\otimes n} \to$ (the irreducible $\mathrm{GL}(V)$-module of shape λ). Thus, we obtain the following result.

Proposition 17. *The multiplicity of λ in $G(v)$ is positive if and only if there is a standard M-independent tableaux of shape λ.*

The irreducible decomposition of $G(v)$ is far from having a complete description in terms of the combinatorics of M. However, we offer a result on how $G(v)$ changes as v varies along a one-parameter curve of configurations.

Here it is appropriate to work in the field of Laurent series $K = \mathbf{k}((t))$, which is a field with a valuation $v : K \to \mathbf{Z}$, sending $at^n +$ (higher degree terms) $\mapsto n$, for $a \neq 0$. Let $R = \mathbf{k}[[t]]$ be the ring of elements with non-negative valuation. The field k is an R-module, where t acts by zero on k, and the functor $k \otimes_R -$ can be thought of as "evaluation at zero." Let $v(t) = \big(v_1(t), \dots, v_n(t)\big)$ be a collection of elements $v_i \in R^r$.

Then $G\big(v(t)\big)$ is a representation of $\mathrm{GL}(K \otimes_k V)$.

Lemma 18. *The modules $G\big(v(t)\big)$ and*

$$\mathbf{k} \otimes_R \big(G\big(v(t)\big) \cap R^{\otimes n}\big)$$

determine each other in the sense that the multiplicity of a partition λ in one is equal to the multiplicity of λ in the other.

Proof. Take a complete set of highest weight vectors of weight λ for $G\big(v(t)\big)$. The intersection of the K-vector space these generate with $(R^r)^{\otimes n}$ is a free R-module, since $(R^r)^{\otimes n}$ is free and R is a torsion free principal ideal domain. Extend a basis of this submodule to a basis of $(R^r)^{\otimes n}$ and let A be the change of basis matrix between this basis and the standard basis of $(R^r)^{\otimes n}$. The determinant of A is invertible over R which means it is invertible at $t = 0$. Hence the basis for the submodule remains independent at $t = 0$. This proves that the multiplicity of λ in $\mathbf{k} \otimes_R \big(G(v(t)) \cap R^{\otimes n}\big)$ is at least the multiplicity of λ in $G(v(t))$, and since the multiplicity of λ could only go down upon setting $t = 0$, we are done. □

We now sketch how Theorem 2 arose. To do this, we project the full rank configurations of the $\mathrm{GL}_r(K) \times (K^\times)^n$-orbit closure (Zariski closure) of $\big(v_1(t), \dots, v_n(t)\big)$ into the Grassmannian of r planes in n-space. Viewing this configuration as an r-by-n matrix, the projection map is taking the row span. The image is a torus orbit closure that is known to break into finitely many torus orbit closures at $t = 0$, as described by Kapranov [9] or Speyer [12]. Further, if $x^1, \dots, x^\ell$ are points of the Grassmannian belonging to the interiors of these torus orbit closures, then we can find configurations of n-vectors, $v^1, \dots, v^\ell$ such that v^i projects to x^i and

$$\mathbf{k} \otimes_R \big(G(v(t)) \cap R^{\otimes n}\big) = \sum_i G(v^i) \subset V^{\otimes n}.$$

By [12], Proposition 12.2, of Speyer,

$$P\big(M(v(t))\big) = \bigcup_i^{\ell} P\big(M(v^i)\big)$$

is a matroid polytope subdivision. Since the rank partition ρ of $M\big(v(t)\big)$ appears with positive multiplicity on the left in the equality,

$$\mathbf{k} \otimes_R \big(G(v(t)) \cap R^{\otimes n}\big) = \sum_i G(v^i)$$

there is some i such that the ρ appears with positive multiplicity in $G(v^i)$. This means that $\rho\big(M(v^i)\big) = \rho$.

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