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Cross varieties of aperiodic monoids with central idempotents

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Abstract. Let $\boldsymbol{A}$ denote the class of all aperiodic monoids with central idempotents. A description of all Cross subvarieties of $\boldsymbol{A}$, based on identities that they satisfy and monoids that they cannot contain, is given. The two limit subvarieties of $\boldsymbol{A}$, published by Marcel Jackson in 2005, turn out to be the only finitely generated, almost Cross subvarieties of $\boldsymbol{A}$. It follows that it is decidable in quartic time if a finite monoid in $\boldsymbol{A}$ generates a Cross variety.

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1. Introduction

A finitely based, finitely generated variety of algebras that contains finitely many subvarieties is called a *Cross variety*. It is long established that finite algebras from the following classes generate Cross varieties: groups [17], associative rings [11], [14], Lie algebras [1], lattices [10], [15], idempotent semigroups [2], [4], [5], and commutative monoids [6]. However, this result does not hold in general. For instance, Murskiĭ's algebra, a groupoid with only three elements, generates a non-finitely based variety [16] that contains continuum many subvarieties [18].

Recall that a semigroup is *aperiodic* if all its subgroups are trivial. The present article is concerned with the class $\boldsymbol{A}$ of all aperiodic monoids with central idempotents. This class constitutes a significant source of examples in the study of the finite basis problem for semigroups and monoids. In 1969, Perkins [19] published the first two examples of non-finitely based, finite semigroups: the well-known Brandt monoid B_2^1 of order six and a certain monoid P_{25} in $\boldsymbol{A}$ of order 25. More recent work of Jackson [7], Sapir [22], and their collaboration [9] shed more light on the finite basis problem for monoids in $\boldsymbol{A}$ and demonstrated how non-finitely based monoids in $\boldsymbol{A}$ can be located. In 2005, Jackson proved that the class

$\boldsymbol{A}$ contains two finitely generated limit subvarieties $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ ([8], Proposition 5.1); recall that a *limit variety* is a variety that is minimal with respect to being non-finitely based. Not only are $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ the first published examples of limit varieties of monoids, they remain the only known explicit examples up to the present. Jackson asked if the class $\boldsymbol{A}$ contains any other finitely generated limit subvarieties ([8], Question 1), and this question was recently answered by Lee [12].

Theorem 1. *The varieties $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ are the only finitely generated limit subvarieties of $\boldsymbol{A}$.*

Consequently, any finitely generated, non-finitely based subvariety of $\boldsymbol{A}$ must contain either $\boldsymbol{J}_1$ or $\boldsymbol{J}_2$.

Now when Jackson proved that the varieties $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ are limit varieties, he also showed that they contain finitely many subvarieties that are all finitely generated ([8], Section 5). It follows that the varieties $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ are minimal with respect to being non-Cross, or *almost Cross*. In the presence of Theorem 1, it is natural to ask if $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ are the only finitely generated, almost Cross subvarieties of $\boldsymbol{A}$. It turns out that the results of Jackson [8] and Lee [12] that established Theorem 1 can easily be extended to not only answer this question affirmatively, but also provide a description of all Cross subvarieties of $\boldsymbol{A}$. This description is detailed enough to enable one to decide in quartic time when a finite monoid from $\boldsymbol{A}$ generates a Cross variety. These results are somewhat unexpected since descriptions of Cross varieties in other large classes of algebras are not very common. Apart from the classes mentioned at the beginning of this section, there are apparently only two other known large classes with well-described Cross varieties: varieties of certain commutative lattice ordered semigroups [20] and subvarieties of the variety generated by aperiodic 0-simple semigroups [13].

2. Preliminaries

For the remainder of this article, all varieties are varieties of monoids. Refer to the monograph of Burris and Sankappanavar [3] for more information on varieties and universal algebra.

Let $\mathscr{X}^*$ denote the free monoid over a countably infinite alphabet $\mathscr{X}$. Elements of $\mathscr{X}^*$ are called *words*. For any set $\mathscr{W}$ of words, let $\mathrm{S}(\mathscr{W})$ denote the Rees quotient monoid of $\mathscr{X}^*$ over the ideal of all words that are not factors of any word in $\mathscr{W}$. Equivalently, $\mathrm{S}(\mathscr{W})$ can be treated as the monoid that consists of every factor of every word in $\mathscr{W}$, together with a zero element 0, with binary operation $\cdot$ given by

$$\boldsymbol{w} \cdot \boldsymbol{w}' = \begin{cases} \boldsymbol{w}\boldsymbol{w}' & \text{if } \boldsymbol{w}\boldsymbol{w}' \text{ is a factor of some word in } \mathscr{W}, \\ 0 & \text{otherwise.} \end{cases}$$

Example 2 (Perkins [19], Section 3). The non-finitely based monoid P_{25} introduced in Section 1 is $\mathrm{S}(\{xyzyx, xzyxy, xyxy, x^2z\})$.

Example 3 (Jackson [8], Proposition 5.1). The varieties $\boldsymbol{J_1}$ and $\boldsymbol{J_2}$ are generated by the monoids $\mathrm{S}(\{xhxyty\})$ and $\mathrm{S}(\{xhytxy, xyhxty\})$, respectively.

A nonempty word $\boldsymbol{w}$ is an *isoterm* for a variety $\boldsymbol{V}$ if $\boldsymbol{V}$ does not satisfy any nontrivial identity of the form $\boldsymbol{w} \approx \boldsymbol{w}'$. For any variety $\boldsymbol{V}$, let $\mathsf{iso}(\boldsymbol{V})$ denote the set of all isoterms for $\boldsymbol{V}$.

Lemma 4 (Jackson [8], Lemma 3.3). *Suppose that $\mathscr{W}$ is any set of words and that $\boldsymbol{V}$ is any variety. Then $\mathrm{S}(\mathscr{W}) \in \boldsymbol{V}$ if and only if $\mathscr{W} \subseteq \mathsf{iso}(\boldsymbol{V})$.*

Lemma 5. *Suppose that $\boldsymbol{V}$ is any subvariety of $\boldsymbol{A}$ such that $\boldsymbol{J_1}, \boldsymbol{J_2} \notin \boldsymbol{V}$. Then $\boldsymbol{V}$ satisfies one of the following identity systems:*

$$xhxyty \approx xhyxty, \qquad xhytxy \approx xhytyx; \tag{1a}$$

$$xhxyty \approx xhyxty, \qquad xyhxty \approx yxhxty. \tag{1b}$$

Proof. If $xyx \notin \mathsf{iso}(\boldsymbol{V})$, then the variety $\boldsymbol{V}$ satisfies either $xyx \approx x^2y$ or $xyx \approx yx^2$ ([8], Lemma 4.1), whence $\boldsymbol{V}$ satisfies either (1a) or (1b). Therefore it remains to consider the case when $xyx \in \mathsf{iso}(\boldsymbol{V})$. By Example 3, Lemma 4, and the assumption that $\boldsymbol{J_1}, \boldsymbol{J_2} \notin \boldsymbol{V}$, either $xhxyty, xhytxy \notin \mathsf{iso}(\boldsymbol{V})$ or $xhxyty, xyhxty \notin \mathsf{iso}(\boldsymbol{V})$. It is then routinely shown that the variety $\boldsymbol{V}$ satisfies either (1a) or (1b). □

Lemma 6 (Straubing [23]). *Suppose that $\boldsymbol{V}$ is any finitely generated subvariety of $\boldsymbol{A}$. Then there exists some $n \geq 1$ such that $\boldsymbol{V}$ satisfies the identity system*

$$x^{n+1} \approx x^n, \qquad x^n y \approx yx^n, \qquad x^n y_1 y_2 \dots y_n \approx y_1 x y_2 x \dots y_n x. \tag{2_n}$$

3. Main results

For each $n \geq 1$, let $\boldsymbol{Z}_n$ denote the variety defined by (1a) and (2_n). Let $\boldsymbol{Z}_n^\delta$ denote the variety that is dual to $\boldsymbol{Z}_n$. It is easily seen that $\boldsymbol{Z}_n^\delta$ is defined by (1b) and (2_n).

Proposition 7. *Every variety $\boldsymbol{Z}_n$ is Cross.*

Proof. The finitely based variety $\boldsymbol{Z}_n$ is locally finite ([21], Proposition 3.1). Since $\boldsymbol{Z}_n$ contains finitely many subvarieties ([12], Corollary 4.13), it is finitely generated and hence also Cross. □

Theorem 8. *The following statements on any subvariety $\boldsymbol{V}$ of $\boldsymbol{A}$ are equivalent:*

(a) *V is Cross;*

(b) *V is finitely generated and $\boldsymbol{J_1}, \boldsymbol{J_2} \notin V$;*

(c) *V satisfies either $\{(1a), (2_n)\}$ or $\{(1b), (2_n)\}$ for some $n \geq 1$.*

Proof. The implication (a) ⇒ (b) holds since the varietal property of being Cross is inherited by subvarieties. The implication (b) ⇒ (c) follows from Lemmas 5 and 6, while the implication (c) ⇒ (a) follows from Proposition 7. □

Corollary 9. *The varieties $\boldsymbol{J_1}$ and $\boldsymbol{J_2}$ are the only finitely generated, almost Cross subvarieties of $\boldsymbol{A}$.*

Corollary 10. *The subvarieties of every $\boldsymbol{Z}_n$ and $\boldsymbol{Z}_n^{\delta}$ are precisely all Cross subvarieties of $\boldsymbol{A}$.*

Theorem 11. *It is decidable in quartic time if a finite monoid in $\boldsymbol{A}$ generates a Cross variety.*

Proof. Let M be any finite monoid in $\boldsymbol{A}$. By Lemma 6, the variety $\boldsymbol{V}$ generated by M satisfies (2_n) for some $n \geq 1$. By Theorem 8, the variety $\boldsymbol{V}$ is Cross if and only if it satisfies either (1a) or (1b). The result then follows since the identities in (1a) and (1b) involve four distinct letters. □

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