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Strong projective limit of Banach Lie algebroids

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Abstract. We define the notion of strong projective limit of Banach Lie algebroids. We study the associated structures of Fréchet bundles and the compatibility with the different morphisms. This kind of structures seems to be a convenient framework for various situations.

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1. Introduction

The notion of Lie algebroid (*algébroïde de Lie* in the French terminology) was first introduced by J. Pradines in [36] in relation with Lie groupoids. Lie algebroids are generalizations of both Lie algebras and tangent vector bundles. This notion is an adapted framework for different problems one can meet

- in geometric mechanics where a theory of Lagrangian and Hamiltonian systems can be developed on such structures (cf. [43], [9], [8]);
- in symplectic geometry in view of the symplectization of Poisson manifolds and applications to quantization ([18], [42]);
- in geometry where classifying Lie algebroids ([11]) are associated to finite type G-structures, this notion of G-structure includes most of the classical geometric structures ([28]);
- in optimal control theory where one can write a version of the Pontryagin Maximum Principle (cf. [27]).

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In finite dimensions, there exists a bijection

– between Lie algebroid structures on an anchored bundle and Poisson structures on its dual,

– between Lie algebroid structures and Lie differentials (cf. [26], [9]).

This situation is studied in [7] for almost Lie algebroids under appropriated conditions.

In this paper, we consider Lie algebroids modeled on Fréchet manifolds. Several difficulties arise when one considers manifolds modeled on Fréchet spaces $\mathbb{F}$: the lack of a general solvability theory of differential equations (cf. [16]) and the pathological structure of $\mathrm{Gl}(\mathbb{F})$ (which does not admit a reasonable Lie group structure). These problems have a solution on certain projective limits of spaces: on one hand, existence of integral curves of vector fields, autoparallel curves with respect to linear connections (cf. [4]), horizontal global section for connection on particular spaces (cf. [3]); on the other hand, existence of a generalized Lie group $H_0(\mathbb{F})$ as structural group for the tangent bundle (cf. [14]).

The study of projective (or inverse) limits of different types of spaces (manifolds, bundles, ...) was the subject of investigations by many authors:

– projective limits of tangent bundles of a finite dimensional manifold (cf. [15]) and more generally projective limits of fiber bundles (cf. [5]), a classical example being the geometry of infinite jets bundle as developed for example in [38];

– projective limits of Banach Lie groups studied in [13] linked with the ILB-groups ([31], [39]);

– universal laminated surfaces studied by Nag and Sullivan (cf. [29]) used in mathematical physics.

In this paper, we are interested in the notion of projective limits of Lie algebroids which can be endowed with Fréchet structures. One can find in [19] the notion of variational Lie algebroid, used in PDE, where the vector fields are replaced by sections of a bundle over a projective limite of finite jets.

The main result of this paper (Theorem 5.1) asserts that the strong projective limit $(\varprojlim E_i, \varprojlim \pi_i, \varprojlim M_i, \varprojlim \rho_i)$ of Banach Lie algebroids (E_i is a vector bundle over the base M_i and $\rho_i : E_i \to TM_i$ is the anchor) is a Fréchet Lie algebroid.

This paper is organized as follows. In part 2 we recall the notions of manifolds and fiber bundles modeled on convenient vector spaces as defined by Kriegel and Michor in ([20]) and different objects of such spaces. The strong projective limit of Banach fiber bundles is developed in [5] and one gets a generalization of the results obtained on the tangent bundle by Galanis in [15]; this result is recalled in part 3. The notion of Banach Lie algebroid is presented in part 4 where one can find the notions of Lie and differential derivatives and morphisms (cf. [2]). In part 5 the projective limit of this kind of algebroids is endowed with a structure of Fréchet space. In part 6 we give examples of such objects, where $E_i = TM_i$ and the anchor is a Nijenhuis tensor (framework adapted to the infinite-dimensional harmonic oscillator) and E_i is a particular sub-bundle of TM_i:

- for finite dimensional ranks, one can have the notion of diffiety,
- the inverse limit Banach (or Hilbert) setting corresponds to infinite-dimensional ranks and is an interesting framework for diverse problems in quantum field theory.

In the last part we study the projective limits of semisprays and admissible curves.

2. Infinite dimensional manifolds modeled on convenient vector spaces

Classical differential calculus is perfectly adapted to finite dimensional or even Banach manifolds (cf. [22]).

On the other hand, convenient analysis, developed in [20], provides a satisfactory solution of the question how to do analysis on a large class of locally convex spaces and in particular on projective limits of Banach manifolds or fiber bundles.

We recall the main results given in the book [20] or in the paper [21], §2.

2.1. Smooth mappings on convenient vector spaces. In order to endow some locally convex vector spaces (l.c.v.s.) E, which will be assumed Hausdorff, with a differentiable structure we first use the notion of smooth curves $c : \mathbb{R} \to E$, which poses no problems.

We denote the space $C^\infty(\mathbb{R}, E)$ by $\mathscr{C}$; the set of bounded (resp. continuous) linear functionals is denoted by E' (resp. E^*).

We then have the following characterization: a subset B of E is bounded iff $l(B)$ is bounded for any $l \in E^*$.

Definition 2.1. A locally convex vector space is said to be *convenient* if the following condition is satisfied:

if $c : \mathbb{R} \to E$ is a curve such that $l \circ c : \mathbb{R} \to \mathbb{R}$ is smooth for all $l \in E^*$, then c is smooth.

The c^∞-topology on a l.c.v.s. is the final topology with respect to all smooth curves $\mathbb{R} \to E$. Its open sets will be called c^∞-open.

For Fréchet spaces, this topology coincides with the given locally convex topology.

Let E and F be two convenient spaces and let $U \subset E$ be a c^∞-open. A map $f : E \supset U \to F$ is said to be smooth if $f \circ c \in C^\infty(\mathbb{R}, F)$ for any $c \in C^\infty(\mathbb{R}, U)$.

Moreover, cf. [21], 2.3 (5), the space $C^\infty(U, F)$ may be endowed with a structure of convenient vector space.

2.2. Differentiable manifolds

2.2.1. Structure of differentiable manifold. A chart (U, φ) on a set M is a bijection $\varphi : U \to \varphi(U) \subset E$ from a subset U of M on a c^∞-open subset of a convenient vector space E.

A family $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ of charts is called a C^∞-*atlas* if all chart changings $\varphi_{\alpha\beta} = \varphi_\alpha \circ (\varphi_\beta)^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ are smooth.

Two C^∞-atlases are called *equivalent* if their union is again a C^∞-atlas.

The set M equipped with an equivalence class of C^∞-atlases is called C^∞-*manifold*.

A subset W of the manifold M is open iff for all $\alpha \in A$ the subset $\varphi_\alpha(U_\alpha \cap W)$ of E is c^∞-open.

The so defined topology is both the final topology with respect to all inverses of chart mapping in one atlas and the final one with respect to all smooth curves.

From now on we assume that manifolds are smoothly Hausdorff, i.e., the smooth functions in $C^\infty(M, \mathbb{R})$ separate points in M.

2.2.2. Smooth mappings. A mapping $f : M \to N$ between two C^∞-manifolds is called smooth if for all $x \in M$ and for all chart (V, ψ) on N such that $f(x) \in V$ there exists a chart (U, φ) on M such that $x \in U$, $f(U) \subset V$ and such that $\psi \circ f \circ \varphi^{-1}$ is smooth.

This is the case iff $f \circ c$ is smooth for each smooth curve $c : \mathbb{R} \to M$.

We will denote by $\mathscr{F}$ the ring of smooth functions from M to $\mathbb{R}$.

2.2.3. Vector bundles. Let $p : F \to M$ be a smooth mapping between differentiable manifolds F and M.

A *vector bundle chart* on (F, p, M) is a pair (U, Φ) where U is an open subset in M and where Φ is a fiber respecting diffeomorphism as in the diagram

$$\begin{array}{ccc} F_{|U} = p^{-1}(U) & \xrightarrow{\Phi} & U \times V \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & U & \end{array}$$

where V is a fixed real convenient vector space, called the *standard fiber*.

Two charts (U_1, Φ_1) et (U_2, Φ_2) are called *compatible* if $\Phi_1 \circ (\Phi_2)^{-1}(x, v)$ may be written as $\big(x, \Phi_{1,2}(x)(v)\big)$ where $\Phi_{1,2} : U_1 \cap U_2 \to \mathrm{GL}(V)$. The mapping $\Phi_{1,2}$ is smooth in $L(V)$ where $L(V)$ is the space of bounded linear mapping (and then smooth).

A vector bundle atlas $(U_\alpha, \Phi_\alpha)_{\alpha \in A}$ for $p : F \to M$ is a set of pairwise compatible charts (U_α, Φ_α) where $(U_\alpha)_{\alpha \in A}$ is an open cover of the manifold M. The notion of *equivalent atlases* is obvious.

A *smooth vector bundle* $p : F \to M$ corresponds to manifolds F (total space), M (base) and a smooth mapping $p : F \to M$ (projection) equipped with an equivalence class of vector bundle atlases.

A *section* s of $p : F \to M$ is a smooth mapping $s : M \to F$ such that $p \circ s = \mathrm{Id}_M$.

The space $\underline{F}$ of all sections of F can be endowed with a structure of convenient vector space.

2.2.4. Vector fields. A (*kinematic*) *tangent vector* at $x \in M$ is an equivalence class for the following equivalence relation

$$c_1 \sim c_2 \text{ if and only if } \begin{cases} c_1(0) = c_2(0) = x \in U, \\ (\varphi \circ c_1)'(0) = (\varphi \circ c_2)'(0), \end{cases}$$

where (U, φ) is a chart on M.

The set of all tangent vectors at the different points of the manifold, endowed with a structure of fiber bundle, is called the (*kinematic*) *tangent bundle* and denoted by TM.

A (*kinematic*) *vector field* is a smooth section of TM. We denote the space of (kinematic) vector fields by $\mathfrak{X}(M)$. It can be equipped with a structure of convenient vector space.

For smooth regular manifolds ([20], 14), the bracket of two vector fields X and Y can be defined if M is assumed to be a c^∞-open set of a convenient vector space E by

$$[X, Y] = dY(X) - dX(Y),$$

where X and Y are seen as smooth mappings from M to E.

2.2.5. Tangent mapping. Let M and N be two differentiable manifolds and let $f : M \to N$ be a smooth mapping. f induces a linear mapping $T_x f : T_x M \to T_{f(x)} M$ which maps a tangent vector to a curve c where $c(0) = x$ to the tangent vector to the curve $f \circ c$ at $f(x)$.

The mapping $Tf : TM \to TN$ is then smooth and called *tangent mapping* of f.

2.2.6. Cotangent bundle. A (*kinematic*) 1-*form* at $x \in M$ is a bounded linear functional on the convenient vector space $T_x M$ (so it belongs to $T_x M'$). The set of all these 1-forms at the different points of M can be endowed with a structure of vector bundle called (kinematic) cotangent bundle and denoted by $T'M$.

A smooth atlas $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ of M gives rise to transition functions $x \mapsto d\left(\varphi_\beta \circ (\varphi_\alpha)^{-1}\right)_{\varphi_\alpha(x)}$.

2.2.7. Differential forms. On a manifold M a (kinematic) 1-*form* is nothing but a smooth section of $T'M$.

The set of these 1-forms can be equipped with a structure of convenient vector space.

On a smooth regular manifold, the class of differential forms ([20], 33.22) stable under Lie derivation L_X, exterior derivative d, interior product i_X and pullback f^* is the graded algebra

$$\Omega(M) = \bigoplus_{k=0}^{+\infty} \Omega^k(M)$$

where

$$\Omega^k(M) = \underline{L^k_{\text{alt}}(TM, \mathbb{R})}$$

has a structure of convenient vector space. $\Omega^0(M)$ corresponds to $\mathscr{F}$ and $\Omega^1(M) = \underline{T'M}$.

We denote by $\Omega^k(M, E) = \underline{L^k_{\text{alt}}(TM, E)}$ the space of k-forms with values in the vector bundle $p : E \to M$.

The Lie derivative $L : \mathfrak{X}(M) \times \Omega^k(M) \to \Omega^k(M)$ is a smooth mapping defined by

$$(L_X\omega)(X_1, \ldots, X_k) = X\left(\omega(X_1, \ldots, X_k)\right) - \sum_{i=1}^{k} \omega(X_1, \ldots, [X, X_i], \ldots, X_k)$$

The exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is smooth and defined by

$$\begin{aligned}(d\omega)(x)(X_0, \ldots, X_k) = {} & \sum_{i=0}^{k} (-1)^i X_i\left(\omega(X_0, \ldots, \widehat{X}_i, \ldots, X_k)\right) \\ & + \sum_{0 \le i < j \le k} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_k).\end{aligned}$$

3. Strong projective limit of Banach vector bundles

3.1. Projective limits of topological spaces. A *projective system of topological spaces* is a sequence $\left((X_i, \delta_i^j)_{j \ge i}\right)_{i \in \mathbb{N}}$ where

- for all $i \in \mathbb{N}$, X_i is a topological space,
- for all $i, j \in \mathbb{N}$ such that $j \ge i$, $\delta_i^j : X_j \to X_i$ is a continuous mapping,

- for all $i \in \mathbb{N}$, $\delta_i^i = Id_{X_i}$,
- for all integers $i \leq j \leq k$, $\delta_i^j \circ \delta_j^k = \delta_i^k$.

An element $(x_i)_{i \in \mathbb{N}}$ of the product $\prod_{i \in \mathbb{N}} X_i$ is called a *thread* if for all $j \geq i$, $\delta_i^j(x_j) = x_i$.

The set $X = \varprojlim X_i$ of such elements, endowed with the finest topology for which all the projections $\delta_i : X \to X_i$ are continuous, is called *projective limit of the sequence* $\left((X_i, \delta_i^j)_{j \geq i}\right)_{i \in \mathbb{N}}$.

A basis of the topology of X is constituted by the subsets $(\delta_i)^{-1}(U_i)$ where U_i is an open subset of X_i (and so δ_i^j is open).

Let $\left((X_i, \delta_i^j)_{j \geq i}\right)_{i \in \mathbb{N}}$ and $\left((Y_i, \gamma_i^j)_{j \geq i}\right)_{i \in \mathbb{N}}$ be two projective systems whose respective projective limits are X and Y.

A sequence $(f_i)_{i \in \mathbb{N}}$ of continuous mappings $f_i : X_i \to Y_i$, satisfying for all $i, j \in \mathbb{N}$, $j \geq i$, the condition

$$\gamma_i^j \circ f_j = f_i \circ \delta_i^j$$

is called *projective system of mappings*.

The projective limit of this sequence is the mapping

$$f : X \to Y, \qquad (x_i)_{i \in \mathbb{N}} \mapsto \left(f_i(x_i)\right)_{i \in \mathbb{N}}.$$

The mapping f is continuous and is a homeomorphism if all the f_i are homeomorphisms (cf. [1]).

3.2. Strong projective limit of Banach manifolds. The system $\left((M_i, \delta_i^j)_{j \geq i}\right)_{i \in \mathbb{N}}$ is called *strong projective system of Banach manifolds* if

- M_i is a manifold modeled on the Banach space $\mathbb{M}_i$,
- $\left((\mathbb{M}_i, \overline{\delta_i^j})_{j \geq i}\right)_{i \in \mathbb{N}}$ is a projective sequence,
- for all $x = (x_i) \in M = \varprojlim M_i$, there exists a projective system of local charts $(U_i, \varphi_i)_{i \in \mathbb{N}}$ such that $x_i \in U_i$ where one has the relation $\varphi_i \circ \delta_i^j = \overline{\delta_i^j} \circ \varphi_j$,
- $U = \varprojlim U_i$ is open in M.

The projective limit $M = \varprojlim M_i$ then has a structure of Fréchet manifold modeled on the Fréchet space $\mathbb{M} = \varprojlim \mathbb{M}_i$ where the differentiable structure is defined via the charts (U, φ) where $\varphi = \varprojlim \varphi_i : U \to \left(\varphi_i(U_i)\right)$.

φ is a homeomorphism (projective limit of homeomorphisms) and the charts changings $(\psi \circ \varphi^{-1})_{|\varphi(U)} = \varprojlim\left(\left(\psi_i \circ (\varphi_i)^{-1}\right)_{|\varphi_i(U_i)}\right)$ between open sets of Fréchet spaces are C^∞ in the sense of convenient spaces.

Example 3.1. Let $p : E \to M$ a vector bundle of finite rank over the finite dimensional manifold M. The space of infinite jets of sections of E is a *strong projective system of Banach manifolds* (cf. [38], [1]).

Example 3.2. Projective limit of Banach–Lie groups (cf. [13], [31], [1]).

A group G is called projective limit of Banach–Lie group modeled on the projective limit $\mathbb{G} = \varprojlim \mathbb{G}_i$ if

(1) $G = \varprojlim G_i$ where (G_i, δ_i^j) is a projective system of Banach–Lie groups where G_i is modeled on $\mathbb{G}_i$,

(2) for all $i \in \mathbb{N}$ there exists a chart (U_i, φ_i) centered at the unity $e_i \in G_i$ such that
   (a) $\delta_i^j(U_j) \subset U_i$ for $j \geq i$,
   (b) $\overline{\delta_i^j} \circ \varphi_j = \varphi_j \circ \delta_i^j$,
   (c) $\varprojlim \varphi_i(U_i)$ is an open set of $\mathbb{G}$ and $\varprojlim U_i$ is open in G according to the projective limit topology.

As a simple example, one can consider the space of real sequences $\mathbb{R}^{\mathbb{N}}$, equipped with the product topology; it is an abelian Lie group, projective limit of the abelian Lie groups $\mathbb{R}^j$, $j \in \mathbb{N}$.

More interesting examples correspond to compact groups because any compact group is the projective limit of a family of compact Lie groups (cf. [41]).

It is possible to define on Fréchet Lie groups G which are projective limits of sequences of Banach Lie groups the exponential $\exp_G$ as projective limit of the sequence $\exp_{G_i}$. This mapping is then continuous.

3.3. Strong projective limit of vector bundles. Let $\left((M_i, \delta_i^j)_{j \geq i}\right)_{i \in \mathbb{N}}$ be a strong projective system of Banach manifolds where each manifold M_i is modeled on the Banach space $\mathbb{M}_i$.

For any integer i let (E_i, π_i, M_i) be the Banach vector bundle whose type fiber is the Banach vector space $\mathbb{E}_i$ where $(\mathbb{E}_i, \lambda_i^j)_{j \geq i i \in \mathbb{N}}$ is a projective system of Banach spaces.

The sequence $\left((E_i, f_i^j)_{j \geq i}\right)_{i \in \mathbb{N}}$ where f_i^j is a morphism of vector bundles is called *strong projective system of Banach vector bundles* on $\left((M_i, \delta_i^j)_{j \geq i}\right)$ if for all (x_i) there exists a projective system of trivializations (U_i, τ_i) of (E_i, π_i, M_i), where $\tau_i : (\pi_i)^{-1}(U_i) \to U_i \times \mathbb{E}_i$ are local diffeomorphisms such that $x_i \in U_i$ (open in M_i), and where $U = \varprojlim U_i$ is open in M and where, for all $i, j \in \mathbb{N}$ such that $j \geq i$, we have the compatibility condition

$$(\delta_i^j \times \lambda_i^j) \circ \tau_j = \tau_i \circ f_i^j.$$

We then have the following proposition which generalizes the result of [15] about the projective limit of tangent bundles to Banach manifolds whose proof can be found in [5].

Proposition 3.3. *Let $(E_i, \pi_i, M_i)_{i \in \mathbb{N}}$ be a strong projective system of Banach vector bundles.*

Then $(\varprojlim E_i, \varprojlim \pi_i, \varprojlim M_i)$ is a Fréchet vector bundle.

Observe that $\mathrm{Gl}(\mathbb{E})$ cannot be endowed with a structure of Lie group. So it cannot play the role of structural group. We then consider, as in [14], the generalized Lie group $H^0(\mathbb{E}) = \varprojlim H_i^0(\mathbb{E})$, the projective limit of the Banach Lie groups

$$H_i^0(\mathbb{E}) = \left\{(h_1, \ldots, h_i) \in \prod_{j=1}^{i} \mathrm{Gl}(\mathbb{E}_j) : \lambda_k^j \circ h_j = h_k \circ \lambda_k^j \text{ for } k \leq j \leq i\right\}.$$

We then obtain the differentiability of the transition functions $\mathtt{T}$.

4. Banach Lie algebroids

4.1. Definition. Examples. Let $\pi : E \to M$ be a Banach vector bundle whose fiber is a Banach space $\mathbb{E}$.

A morphism of vector bundles $\rho : E \to TM$ is called *anchor*. This morphism gives rise to $\underline{\rho} : \underline{E} \to \underline{TM} = \mathfrak{X}(M)$ defined for every $x \in M$ and every section s of E by: $\big(\underline{\rho}(s)\big)(x) = \rho\big(s(x)\big)$ and still denoted by ρ.

Assume there exists a bracket $[.,.]_E$ on the space $\underline{E}$ which provides a structure of real Lie algebra on $\underline{E}$.

Definition 4.1. (E, π, M, ρ) is called a *Banach Lie algebroid* if

(1) $\rho : (\underline{E}, [.,.]_E) \to \big(\mathfrak{X}(M), [.,.]\big)$ is a Lie algebra homomorphism,

(2) $[s_1, fs_2]_E = f[s_1, s_2]_E + \big(\rho(s_1)\big)(f)s_2$ for every $f \in \mathscr{F}$ and $\mathrm{s}_1, s_2 \in \underline{E}$.

Example 4.2. $E = TM$ and $\rho = N$ is a Nijenhuis tensor, i.e., satisfying the condition

$$[NX, NY] = N\big([NX, Y] + [X, NY] - N([N, Y])\big)$$

(TM, π, M, N) is a Lie algebroid for the bracket $[.,.]_N$ defined by

$$[X, Y]_N = [NX, Y] + [X, NY] - N([X, Y]).$$

The trivial case corresponds to $N = \mathrm{Id}_{TM}$.

Example 4.3. E is an involutive distribution over a Banach manifold M. The anchor is then the canonical injection $\rho : E \to TM$.

Example 4.4. E is the cotangent bundle of a Banach manifold and $\rho = P$ is a Poisson tensor. The bracket on the sections of T^*M (cf. [25]) is defined by

$$\{\alpha, \beta\}_P = L_{P\beta}(\alpha) - L_{P\alpha}(\beta) + d\langle\beta, P\alpha\rangle$$

$(T^*M, P, M, \{.,.\}_P)$ is a Lie algebroid because, in particular, we have

$$\{\alpha, f.\beta\}_P = f.\{\alpha, \beta\}_P + L_{P\alpha}(f).\beta.$$

One can find in [34] a generalization to the Jacobi structures (which were introduced by Lichnerowicz in [23]).

Example 4.5. Let $\psi : M \times G \to M$ be a right action of a Lie group G (with Lie algebra $\mathscr{G}$) on a Banach manifold M. Then there exists a natural morphism of the trivial Banach bundle $M \times \mathscr{G}$ in TM defined by

$$\Psi(x, X) = T_{(x,e)}\psi(0, X).$$

For all X and Y in $\mathscr{G}$, we have

$$\Psi(\{X, Y\}) = [\Psi(X), \Psi(Y)],$$

where $\{\,,\}$ is the Lie bracket on $\mathscr{G}$ (cf. [20], 36.12).

$(M \times \mathscr{G}, \Psi, M, \{\,,\})$ is then a Lie algebroid.

4.2. Derivatives. On a Banach Lie algebroid the base of which is smooth regular one can define the notions of Lie derivative L_s^ρ with respect to a section s of E (this notion generalizes the Lie derivative L_X with respect to a vector field, section of the tangent bundle) and exterior derivative d_ρ (cf. [2], [7]). For the case of finite dimensional algebroid, see [26].

For every section s of the vector bundle E, there exists a unique graded endomorphism of degree 0 of the graded algebra $\Omega(M, E)$, called the *Lie derivative* with respect to s and denoted by L_s^ρ which satisfies the following properties:

(1) for a smooth function $f \in \Omega^0(M, E) = \mathscr{F}$

$$L_s^\rho(f) = L_{\rho\circ s}(f) = i_{\rho\circ s}(df)$$

where L_X denotes the usual Lie derivative with respect to the vector field X,

(2) for a q-form $\omega \in \Omega^q(M, E)$ (where $q > 0$)

$$(L_s^\rho\omega)(s_1, \dots, s_q) = L_s^\sigma\big(\omega(s_1, \dots, s_q)\big) - \sum_{i=1}^{q} \omega(s_1, \dots, s_{i-1}, [s, s_i]_E, s_{i+1}, \dots, s_q).$$

On the other hand, we can also define for any function $f \in \Omega^0(M,E) = \mathscr{F}$ the element of $\Omega^1(M,E)$, denoted $d_\rho f$, by

$$d_\rho f = t_\rho \circ df \tag{1}$$

where $t_\rho : T^*M \to E^*$ is the transpose of the anchor.

There exists a unique graded endomorphism of degree 1 of the graded algebra $\Omega(M,E)$, called $\Omega(M,E)$-*value derivative*, denoted d_ρ, which satisfies the following properties:

(1) For any function $f \in \Omega^0(M,E) = \mathscr{F}$, $d_\rho f$ is the element of $\Omega^1(M,E)$ defined by the relation (1).

(2) For any element ω of $\Omega^q(M,E)$ $(q > 0)$, $d_\rho\omega$ is the unique element of $\Omega^{q+1}(M,E)$ such that for all $s_0,\dots,s_q \in \underline{E}$,

$$\begin{aligned}(d_\rho\omega)(s_0,\dots,s_q) &= \sum_{i=0}^{q}(-1)^i L^\rho_{s_i}\big(\omega(s_0,\dots,\widehat{s_i},\dots,s_q)\big)\\ &\quad + \sum_{0\le i<j\le q}^{q}(-1)^{i+j}\big(\omega([s_i,s_j]_E,s_0,\dots,\widehat{s_i},\dots,\widehat{s_j},\dots,s_q)\big).\end{aligned}$$

We then have

$$d_\rho \circ d_\rho = 0.$$

4.3. Algebroids morphisms

Definition 4.6. A linear bundle morphism $\psi : E \to E'$ over $f : M \to M'$ is a morphism of the Lie algebroids (E,π,M,ρ) and (E',π',M',ρ') if the mapping $\psi^* : \Omega^q(M,E') \to \Omega^q(M,E)$ defined by

$$(\psi^*\alpha')_x(s_1,\dots,s_q) = \alpha'_{f(x)}(\psi\circ s_1,\dots,\psi\circ s_q)$$

commutes with the differentials

$$d_\rho\circ\psi^* = \psi^*\circ d_{\rho'}.$$

We then get the *category of Banach Lie algebroids*.

4.4. Admissible curves. In what concerns mechanics, an element a of E can be regarded as a generalized velocity and the actual velocity v is obtained when applying the anchor to a, i.e., $v = \rho(a)$.

A curve $\gamma : [0,1] \to E$ is said to be *admissible* (cf. [9]) if $m'(t) = \rho\big(\gamma(t)\big)$ where $t \mapsto m(t) = \pi\big(\gamma(t)\big)$ is the base curve.

A Lie algebroid morphism maps admissible curves to admissible curves.

4.5. Semisprays. Let (E, π, M, ρ) be a Banach Lie algebroid and let $T\pi : TE \to TM$ the tangent map of π. We denote by $\tau_E : TE \to E$ the tangent bundle of E.

The notion of semispray we give is a direct generalization of the one used when $E = TM$.

Definition 4.7. A section $S : E \to TE$ is called a semispray if

(1) $\tau_E \circ S = \mathrm{Id}_E$,

(2) $T\pi \circ S = \rho$.

We then have the following link between admissible curves and semisprays (cf. [2])

Proposition 4.8. *A vector field on E is a semispray if and only if all its integral curves are admissible curves.*

We now introduce a particular class of semisprays. For $\lambda > 0$, we denote by $h_\lambda : E \to E$ the homothety of factor λ defined by $h_\lambda(u_x) = \lambda u_x$ for any $u \in E_x$ and any $x \in M$. A semispray S is a *spray* if we have

$$S \circ h_\lambda = \lambda T h_\lambda \circ S.$$

5. Strong projective limits of Banach Lie algebroids

$(E_i, \pi_i, M_i, \rho_i)_{i \in \mathbb{N}}$ is called *strong projective system of Lie algebroids* if

- $\left((E_i, f_i^j)_{j \geq i}\right)_{i \in \mathbb{N}}$ is a strong projective system of Banach vector bundles $(\pi_i : E_i \to M_i)_{i \in \mathbb{N}}$ over the strong projective system of manifolds $\left((M_i, \delta_i^j)_{j \geq i}\right)_{i \in \mathbb{N}}$,
- for all $i, j \in \mathbb{N}$ such that $j \geq i$, one has

$$\rho_i \circ f_i^j = T\delta_i^j \circ \rho_j,$$

- $f_i^j : E_j \to E_i$ is a morphism of the Lie algebroids $(E_j, \pi_j, M_j, \rho_j)$ and $(E_i, \pi_i, M_i, \rho_i)$.

We then have the main result of this paper.

Theorem 5.1. *Let $(E_i, \pi_i, M_i, \rho_i)_{i \in \mathbb{N}}$ be a strong projective systems of Banach Lie algebroids.*

Then $(\varprojlim E_i, \varprojlim \pi_i, \varprojlim M_i, \varprojlim \rho_i)$ is a Fréchet Lie algebroid.

Proof. First observe that the projective limit $\varprojlim M_i$ is endowed with a differential manifold structure as defined in 2.1.1. Then $(\lim E_i, \varprojlim \pi_i, \varprojlim M)_i$ is a Fréchet vector bundle whose structural group is $H^0(\mathbb{E})$ (cf. Proposition 3.3). The projective limit of the (vector) tangent bundles $(\varprojlim TM_i, \varprojlim p_i, \varprojlim M_i)$ is equipped with a Fréchet vector bundle structure; we then get the result of [15], Theorem 2.1.

Let us study the properties of the sections of the vector bundles $\varprojlim TM_i$, $\varprojlim E_i$ and the projective limit of the anchors ρ_i.

For $(g_i)_{i \in \mathbb{N}}$ such that $g_j = g_i \circ \delta_i^j = (\delta_i^j)^*(g_i)$ we can define the projective limit $g = \varprojlim g_i$ which is still smooth.

First remark that if $X_i = T\delta_i^j(X_j)$, we have $X_i(g_i) = \left(T\delta_i^j(X_j)\right)(g_i) = X_j(g_i \circ \delta_i^j) = X_j(g_j)$. We can define $X = \varprojlim X_i \in \varprojlim \mathfrak{X}(M_i)$ and we get $Xg = \varprojlim X_i g_i$ where $X_i g_i \in \mathscr{F}_i$. If the sequences $(X_i^1)_{i \in \mathbb{N}}$ and $(X_i^2)_{i \in \mathbb{N}}$ where $X_i^1, X_i^2 \in \mathfrak{X}(M_i)$ are such that $X_i^1 = T\delta_i^j(X_j^1)$ (resp. $X_i^2 = T\delta_i^j(X_j^2)$), they give rise to elements $X^1, X^2 \in \varprojlim \mathfrak{X}(M_i)$. Because X_i^1 and X_j^1 are δ_i^j-related (so are X_i^2 and X_j^2), their brackets are δ_i^j-related too, i.e., $[X_i^1, X_i^2]_i = T\delta_i^j([X_j^1, X_j^2]_j)$ and we get the bracket of X^1 and X^2 as projective limit of these brackets.

Let $s = \varprojlim s_i$ be where $s_i \in \underline{E_i}$. Because the spaces $\varprojlim M_i$ and $\varprojlim E_i$ are differentiable manifolds, the section $s : (x_0, x_1, \ldots) \mapsto \left(s_0(x_0), s_1(x_1), \ldots\right)$ is smooth (cf. Definition 2.2.2).

Let us prove that we can deduce the compatibility condition

$$f_i^j \circ [s_j^1, s_j^2]_{E_j} = [s_i^1, s_i^2]_{E_i} \circ \delta_i^j$$

from the structure of morphism of f_i^j (commutativity with the differentials applied to 1-forms).

We have $\left((f_i^j)^*(d_{E_i}\alpha_i)\right)(s_j^1, s_j^2) = (d_{E_i}\alpha_i)(f_i^j \circ s_j^1, f_i^j \circ s_j^2)$ where

$$\begin{aligned}
&(d_{E_i}\alpha_i)(f_i^j \circ s_j^1, f_i^j \circ s_j^2) \\
&\quad = L_{\rho_i \circ (f_i^j \circ s_j^1)}\left(\alpha_i(f_i^j \circ s_j^2)\right) - L_{\rho_i \circ (f_i^j \circ s_j^2)}\left(\alpha_i(f_i^j \circ s_j^1)\right) - \alpha_i[f_i^j \circ s_j^1, f_i^j \circ s_j^2]_{E_i} \\
&\quad = L_{\rho_i \circ s_i^1}\left(\alpha_i(s_i^2)\right) - L_{\rho_i \circ s_i^2}\left(\alpha_i(s_i^1)\right) - \alpha_i[s_i^1, s_i^2]_{E_i} \\
&\quad = X_i^1\left(\alpha_i(s_i^2)\right) - X_i^2\left(\alpha_i(s_i^1)\right) - \alpha_i[s_i^1, s_i^2]_{E_i}.
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
&\left(d_{E_j}\left((f_i^j)^*\alpha_i\right)\right)(s_j^1, s_j^2) \\
&\quad = L_{\rho_j \circ s_j^1}\left(\left((f_i^j)^*\alpha_i\right)(s_j^2)\right) - L_{\rho_j \circ s_j^2}\left(\left((f_i^j)^*\alpha_i\right)(s_j^1)\right) - \left((f_i^j)^*\alpha_i\right)[s_j^1, s_j^2]_{E_j} \\
&\quad = X_j^1\left(\alpha_i(f_i^j \circ s_j^2)\right) - X_j^2\left(\alpha_i(f_i^j \circ s_i^1)\right) - \alpha_i[f_i^j \circ s_j^1, f_i^j \circ s_i^2]_{E_i} \\
&\quad = X_i^1\left(\alpha_i(s_i^2)\right) - X_i^2\left(\alpha_i(s_i^1)\right) - \alpha_i[f_i^j \circ s_j^1, f_i^j \circ s_i^2]_{E_i}.
\end{aligned}$$

Because f_i^j is a morphism, we get $\alpha_i[s_i^1, s_i^2]_{E_i} = \alpha_i[f_i^j \circ s_j^1, f_i^j \circ s_i^2]_{E_i}$ and then the compatibility condition.

So the bracket $[s^1, s^2]_{\varprojlim E_i}$ of projective limits of sections $s^1 = \varprojlim s_i^1$ and $s^2 = \varprojlim s_i^2$ can be defined as the projective limit of the sections $[s_i^1, s_i^2]_{E_i}$ of E_i.

The set $\varprojlim E_i$ equipped with this bracket is then a Lie algebra.

According to the condition $\rho_i \circ f_i^j = T\delta_i^j \circ \rho_j$ the projective limit $\rho = \varprojlim \rho_i$ is a linear bundle morphism.

So the mapping $\rho = \varprojlim \rho_i$ is a Lie algebra homomorphism between $(\varprojlim \underline{E_i}, [.,.]_{\varprojlim E_i})$ and $(\varprojlim \underline{TM_i}, [.,.]_i)$.

For all $i \in \mathbb{N}$, every section s_i^1 and s_i^2 of E_i and every smooth $g_i : M_i \to \mathbb{R}$, we have

$$[s_i^1, g_i s_i^2]_{E_i} = g_i[s_i^1, s_i^2]_{E_i} + \big(\rho_i(s_i^1)\big)(g_i)s_i^2.$$

In order to get the relation

$$[s_1, gs_2]_E = g[s_1, s_2]_E + \big(\rho(s_1)\big)(g)s_2$$

we have to prove that

1) $f_i^j \circ (g_j[s_j^1, s_j^2]) = g_i[s_i^1, s_i^2] \circ \delta_i^j$,
2) $f_i^j \circ \big[\big(\rho_j(s_j^1)\big)(g_j)s_j^2\big] = \big[\big(\rho_i(s_i^1)\big)(g_i)s_i^2\big] \circ \delta_i^j$.

For the first item, for any thread $(x_i)_{i \in N}$, i.e., $x_j = \delta_i^j(x_j)$, we have

$$f_i^j \circ (g_j[s_j^1, s_j^2]_{E_j})(x_j) = f_i^j\big((g_i \circ \delta_i^j \times [s_j^1, s_j^2]_{E_j})(x_j)\big).$$

Because f_i^j is a linear mapping from $\pi_j^{-1}(x_j)$ to $\pi_i^{-1}(x_i)$, this expression equals $g_i(x_i) \times f_i^j\big([s_j^1, s_j^2]_{E_j}(x_j)\big)$. Thanks to the compatibility condition $f_i^j \circ [s_j^1, s_j^2]_{E_j} = [s_i^1, s_i^2]_{E_i} \circ \delta_i^j$ we have proved the first point.

For the second item, we first use the commutativity with the differentials d_{E_i} and d_{E_j}.

$$[(f_i^j)^*(d_{E_i} g_i)](s_j)(x_j) = \big[d_{E_j}\big((\delta_i^j)^*(g_i)\big)\big](s_j)(x_j)$$

and so

$$(d_{E_i} g_i)(f_i^j \circ s_j)(x_j) = [d_{E_j}(g_j)](s_j)(x_j).$$

Using the definition of d_{E_i}, i.e., $d_{E_i} g_i = t_{\rho_i} \circ dg_i$ we have

$$dg_i\big[\rho_i\big(f_i^j\big(s_j(x_j)\big)\big)\big] = dg_j\big(\rho_j\big(s_j(x_j)\big)\big)$$

and so

$$[\rho_i(f_i^j \circ s_j)](g_i)(x_i) = [\rho_j \circ s_j](g_j)(x_j).$$

Due to the compatibility condition, we get

$$[f_i^j \circ (\rho_j \circ s_j)](g_i) = [\rho_j(s_j)]\big((\delta_i^j)^* \circ g_i\big).$$

It is then easy to obtain the second point. □

6. Examples

6.1. Nijenhuis Lie algebroid. Let $\big((M_i, \delta_i^j)_{j\geq i}\big)_{i\in\mathbb{N}}$ be a strong projective system of Banach manifolds.

For any $i \in \mathbb{N}$, consider a Nijenhuis tensor $N_i : TM_i \to TM_i$ (cf. Example 4.2). In this case, we consider $f_i^j = T\delta_i^j$ morphism from TM_j to TM_i. If we have the compatibility condition

$$N_i \circ T\delta_i^j = T\delta_i^j \circ N_j,$$

then $(\varprojlim TM_i, \varprojlim \pi_i, \varprojlim M_i, \varprojlim N_i)$ is a Fréchet Lie algebroid because we get in particular

$$(\delta_i^j)^* \circ d_{TM_i} = d_{TM_j} \circ (\delta_i^j)^*.$$

As an example we can consider the case of an infinite-dimensional harmonic oscillator which is a L-integrable Hamiltonian system (cf. [24]). We consider the projective limit $\big((\mathbb{R}^{2i}, \delta_i^j)_{j\geq i}\big)_{i\in\mathbb{N}}$ where δ_i^j is the canonical projection from $\mathbb{R}^{2j}$ onto $\mathbb{R}^{2i}$. The Nijenhuis tensor N_i which corresponds to the recursion operator can be written as

$$N_i = \sum_{k=1}^{i} (x_k^2 + y_k^2)\left(dx_k \otimes \frac{\partial}{\partial x_k} + dy_k \otimes \frac{\partial}{\partial y_k}\right),$$

where $\big((x_1, y_1), \ldots, (x_i, y_i)\big)$ are the coordinates on $\mathbb{R}^{2i}$. It is then easy to establish the compatibility condition.

6.2. Distributions. A distribution on a Banach manifold B is a smooth map $D : B \to TB$ such that for every $x \in B$, D_x is a linear subspace of T_xB. This distribution is involutive if for any vector fields X and Y tangent to D, the bracket $[X, Y]$ is still tangent to D.

Notice that the range of a Lie algebroid anchor is an involutive weak distribution (cf. [33]) if the base is smooth regular (cf. [7]).

Let $\big((M_i, \delta_i^j)_{j\geq i}\big)_{i\in\mathbb{N}}$ be a strong projective system of Banach manifolds. Consider for any $i \in \mathbb{N}$ a smooth involutive distribution E_i over the manifold M_i.

Then $(E_i, \pi_i, M_i, J_i)_{i\in\mathbb{N}}$, where $J_i : E_i \to TM_i$ is the natural injection and f_i^j is the restriction of $T\delta_i^j$ to E_i, is a strong projective system of Lie algebroids.

The projective limit $\varprojlim E_i$ can be seen as an involutive distribution of the Fréchet bundle $\varprojlim TM_i$.

6.2.1. Projective limit of finite rank distributions. Consider the case of a 1-dimensional distribution on the infinite jets of sections of a linear bundle $p : F \to N$. Let X be a vector field on F projectable on N with projection $\hat{X}$; the flow φ_t^X of X covers the flow $\varphi_t^{\hat{X}}$ of $\hat{X}$ and by prolongation to $J^\infty(F)$ we obtain a one-parameter local group $\phi_t = pr^\infty(\varphi_t^X)$ of transformations on $J^\infty(F)$ (cf. [6], [30]). The prolongation $pr^\infty(X)$ of the vector field X is the vector field on $J^\infty(p)$ associated to this flow. Moreover this flow preserves the Cartan distribution (contact ideal) $\mathscr{C}$.

One can remark that $\mathscr{C}$ is an involutive distribution on the projective limit $\varprojlim TJ^i(p)$ which appears as limit of non involutive distributions on $J^i(p)$ (cf. [38]).

If one considers a system of PDEs $\mathscr{E}$, i.e., a subvariety of the bundle $J^k(\pi)$, by infinite prolongation, we get a submanifold $i : \mathscr{E} \to J^\infty(\pi)$ of $J^\infty(\pi)$. We then have an involutive distribution on $\mathscr{E}$ by restriction of the Cartan distribution to $\mathscr{E}$ by the pull-back i (cf. [19], [10]).

Recall that an infinite-dimensional smooth Fréchet differentiable manifold equipped with a finite dimensional involutive distribution corresponds to the notion of *diffiety* (differential variety) as introduced by Vinogradov ([40]). One can find applications of such a framework in non holonomic mechanics and non linear control systems (see for instance [12]).

6.2.2. Inverse limit of Banach distributions. One considers here the case where the maps $\delta_i^{i+1} : M_{i+1} \to M_i$ are canonical injections between Banach manifolds, the distributions E_i are of corank 1 defined as $\ker \alpha_i$ where α_i is a 1-form fulfilling the different compatibility conditions.

One can meet this kind of situation for $M_i = C^i(\mathbb{S}^1)$ where $\alpha_i(u_i) = \int_{\mathbb{S}^1} u_i(x)\,dx$. The associated distribution is affine ([17]) and is linked with the first Poisson tensor of the KdV equation.

7. Strong projective limit of semisprays

Let $(E_i, \pi_i, M_i, \rho_i)_{i\in\mathbb{N}}$ be a strong projective system of Lie algebroids.

Consider a sequence $(\gamma_i)_{i\in\mathbb{N}}$ where $\gamma_i : [0,1] \to E_i$ is an admissible curve such that for all $i, j \in \mathbb{N}$ such that $j \geq i$

$$f_i^j \circ \gamma_j = \gamma_i.$$

Hence $\gamma = \varprojlim \gamma_i$ exists.

For all $i, j \in \mathbb{N}$ such that $j \geq i$ and for all $t \in [0, 1]$, using the equalities

$$(\pi_i \circ \gamma_i)'(t) = \big(\delta_i^j \circ (\pi_j \circ \gamma_j)\big)'(t) = T\delta_i^j\big((\pi_j \circ \gamma_j)'(t)\big)$$

we obtain

$$(\pi_i \circ \gamma_i)'(t) - \rho_i\big(\gamma_i(t)\big) = T\delta_i^j\big((\pi_j \circ \gamma_j)'(t) - \rho_j\big(\gamma_j(t)\big)\big).$$

So, for all $t \in [0, 1]$, we have

$$(\pi \circ \gamma)'(t) = \rho\big(\gamma(t)\big).$$

Such a curve will be called *admissible curve* in $E = \varprojlim E_i$.

Now consider a sequence $(S_i)_{i \in \mathbb{N}}$ where $S_i : E_i \to TE_i$ is a semispray such that

$$Tf_i^j \circ S_j = S_i \circ f_i^j.$$

We then can define $S : (u_0, u_1, \ldots) \mapsto \big(S_0(u_0), S_1(u_1), \ldots\big)$ which is a smooth section of $\varprojlim TE_i$. It is easy to see that we have $\tau_E \circ S = \mathrm{Id}_E$.

For all $i, j \in \mathbb{N}$ such that $j \geq i$ and for all $u_i = f_i^j(u_j)$ we have

$$\begin{aligned}(T\pi_i \circ s_i - \rho_i)(u_i) &= (T\pi_i \circ s_i \circ f_i^j - \rho_i \circ f_i^j)(u_j) \\ &= (T\pi_i \circ Tf_i^j \circ s_j - \rho_i \circ f_i^j)(u_j) \\ &= \big(T(\pi_i \circ f_i^j) \circ s_j - \rho_i \circ f_i^j\big)(u_j) \\ &= \big(T(\delta_i^j \circ \pi_j) \circ s_j - \rho_i \circ f_i^j\big)(u_j).\end{aligned}$$

Finally, we can write

$$(T\pi_i \circ s_i - \rho_i)(u_i) = T\delta_i^j(T\pi_j \circ s_j - \rho_j)(u_j).$$

So we have

$$T\pi \circ s = \rho.$$

S will be called *semispray*.

One can obviously define the notion of spray on the projective limit $\varprojlim E_i$ as projective limit of sprays.

We end this paper with a proposition which establishes the link between semisprays and admissible curves. It generalizes the result of [2] (for the case of sprays in the particular case $E = TM$ see [37]).

Proposition 7.1. *A vector field $S = \varprojlim S_i$ on $E = \varprojlim E_i$ is a semispray if and only if all its integral curves are admissible curves.*

Proof. The proof is nothing but an adaptation of the proof of Theorem 2.3 one can find in the paper [2]. Consider a semispray $S = \varprojlim S_i$ and assume that $c : [0,1] \to E$ is an integral curve of S. Then for all $i \in \mathbb{N}$, $c_i : [0,1] \to E_i$ is an integral curve of S_i (i.e., for all $t \in [0,1]$, $c_i'(t) = S_i\big(c_i(t)\big)$), where $f_i^j \circ c_j = c_i$. It follows that for all $i \in \mathbb{N}$ and for all $t \in [0,1]$ we have

$$T\pi_i \circ c_i'(t) = (T\pi_i \circ S_i)\big(c_i(t)\big).$$

Because $\pi_i \circ c'(t) = \rho_i\big(c(t)\big)$, c_i is an admissible curve and so is $c = \varprojlim c_i$.

The converse is left to the reader. □

For a projective limit of sprays it is easy to prove that for all $i, j \in \mathbb{N}$ such that $j \geq i$ and for all $u_i = f_i^j(u_j)$ using the relation $h_i^\lambda \circ f_i^j = f_i^j \circ h_j^\lambda$ and properties of tangent mappings we have

$$Tf_i^j(s_j \circ h_j^\lambda - \lambda Th_j^\lambda \circ s_j)(u_j) = (s_i \circ h_i^\lambda - \lambda Th_i^\lambda \circ s_i)(u_i).$$

So we can write $S \circ h_\lambda = \lambda Th_\lambda \circ S$ and S is a spray on E.

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