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Rate of decay to 0 of the solutions to a nonlinear parabolic equation

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Abstract. We study the decay rate to 0 as $t \to +\infty$ of the solution of equation $\psi_t - \Delta\psi + |\psi|^{p-1}\psi = 0$ with Neumann boundary conditions in a bounded smooth open connected domain of $\mathbb{R}^n$ where $p > 1$. We show that either $\psi(t, \cdot)$ converges to 0 exponentially fast or $\psi(t, \cdot)$ decreases like $t^{-1/(p-1)}$.

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1. Introduction and main results

In this paper we consider the nonlinear parabolic equation

$$\begin{cases} \psi_t - \Delta\psi + g(\psi) = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ \dfrac{\partial\psi}{\partial n} = 0 & \text{on } \mathbb{R}^+ \times \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded smooth open connected domain of $\mathbb{R}^n$ and $g \in C^1(\mathbb{R})$ satisfies

$$g(0) = 0 \tag{2}$$

and for some $p > 1$

$$\exists c > 0,\ \forall s \in \mathbb{R}, \qquad 0 \le g'(s) \le c|s|^{p-1}. \tag{3}$$

From (2)–(3) we deduce that $g(s)$ has the sign of s and

$$\forall s \in \mathbb{R}, \qquad |g(s)| \le \frac{c}{p}|s|^p. \tag{4}$$

We define the operator A by

$$D(A) = \left\{ \psi \in H^2(\Omega), \frac{\partial \psi}{\partial n} = 0 \text{ on } \partial\Omega \right\}$$

and

$$\forall \psi \in D(A), \qquad A\psi = -\Delta\psi.$$

It is well known that A is maximal monotone with compact resolvant on $L^2(\Omega)$. The first eigenvalue of A is 0 with eigenspace reduced to constants. The second eigenvalue is $\lambda_2 > 0$ and will be denoted by λ_2 through the text. Moreover, the operator B defined by

$$D(B) = \left\{ \psi \in L^2(\Omega) \,|\, -\Delta\psi + g(\psi) \in L^2(\Omega) \text{ and } \frac{\partial \psi}{\partial n} = 0 \text{ on } \partial\Omega \right\}$$

and

$$\forall \psi \in D(B), \qquad B\psi = -\Delta\psi + g(\psi)$$

is maximal monotone in $L^2(\Omega)$. As a consequence of [1], [2] for any $\psi_0 \in L^2(\Omega)$ there exists a unique weak solution of the equation

$$\psi' + B\psi = 0 \text{ on } \mathbb{R}^+; \qquad \psi(0,x) = \psi_0.$$

In addition, it is well known that if $\psi_0 \in L^\infty(\Omega)$, $\psi(t,\cdot)$ remains in $L^\infty(\Omega)$ for all $t > 0$. Finally [8] contains an estimate of the solution in $C(\overline{\Omega})$ and $C^1(\overline{\Omega})$ for $t > 0$, which is valid for any sufficiently regular domain.

Concerning the behaviour for t large, in [6], A. Haraux established in the case of a pure power nonlinearity the exponential convergence to 0 of the projection on the range of A of the solution of equation (1). Moreover in [5], the study of the equation $u'' + u' - \Delta u + g(u) = 0$ with Neumann boundary conditions and where g satisfies

$$\exists C, c > 0, \, \forall s \in \mathbb{R}, \qquad c|s|^{p-1} \leq g'(s) \leq C|s|^{p-1}$$

for some $p > 1$, showed that either $u(t)$ converges to 0 exponentially fast, or $\|u(t)\|_{H_0^1(\Omega)} \geq \gamma t^{-1/(p-1)}$ with $\gamma > 1$ for $t \geq 1$.

Several authors have treated some variants of equation (1). For example in [7] equation (1) is considered with

$$g(u) = c|u|^{p-1}u - \lambda_1 u$$

and with Dirichlet boundary conditions, and the authors studied the decay rate at the infinity of solutions to (1), where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. The result obtained there is optimal for positive solutions.

According to La Salle's invariance principle, cf. [3], [4], any solution ψ of (1), having a precompact range on $\mathbb{R}^+$ with values in $L^\infty(\Omega)$, converges to a continuum of stationary solutions of equation (1), which reduces here to the constants of some sub-interval of $g^{-1}(0)$. Since the L^2 distance of two solutions of (1) is nondecreasing, actually $\psi(t,\cdot)$ converges to some constant $a \in g^{-1}(0)$ as well; cf. [2], Théorème 3.11, for a more general result.

Our first result is valid without any additional hypothesis on g.

Theorem 1.1. *Let g satisfy* (2) *and* (3). *Then any solution* $\psi \in C\big((0,+\infty), L^\infty\big)$ *of* (1) *satisfies the following alternative as* $t \to \infty$*: either*

$$\|\psi(t,\cdot)\|_\infty \leq Ce^{-\lambda_2 t}, \tag{5}$$

or

$$\exists c' > 0,\ \forall t \geq 1, \qquad \left|\int_\Omega \psi(t,x)\,dx\right| \geq c' t^{-1/(p-1)}. \tag{6}$$

Remark. In Theorem 1.1, if the limit a of $\psi(t,\cdot)$ is not 0, (6) is automatically satisfied since $\int_\Omega \psi(t,x)\,dx$ tends to the positive limit $|\Omega|\,|a|$. One might wonder whether in this case (5)–(6) become true with ψ replaced by $\psi - a$. It is unfortunately the case only if $a \in \mathrm{Int}\big(g^{-1}(0)\big)$. The special case $g(s) = \big((s-1)^+\big)^{1+\varepsilon}$ shows that we cannot hope (6) to be true with ψ replaced by $\psi - a$ in case $a = 1$. Indeed, g satisfies (3) for any $p \geq \varepsilon + 1$ but of course (6) will not satisfied for p arbitrarily large when $\psi(0,\cdot) \geq c > 1$. Because (3) is not translation invariant, the special solution 0 plays a privileged role here. On the other hand, when $g^{-1}(0)$ is an interval $J = [\alpha, \beta]$, where $-\infty \leq \alpha < \beta \leq +\infty$, replacing (3) by

$$0 \leq g'(s) \leq c\big(\rho(s)\big)^{p-1} \tag{7}$$

where $\rho(s) = \mathrm{dist}\big(s, g^{-1}(0)\big)$, we obtain

Proposition 1.2. *Let g satisfy* (2) *and* (7) *and let a be the limit of a solution* $\psi \in C\big((0,+\infty), L^\infty\big)$. *Then*

$$\|\psi(t,\cdot) - a\|_\infty \leq Ce^{-\lambda_2 t}, \tag{8}$$

or

$$\exists c' > 0,\ \forall t \geq 1, \qquad \left|\int_\Omega \big(\psi(t,x) - a\big)\,dx\right| \geq c' t^{-1/(p-1)}, \tag{9}$$

Our second result provides a more accurate estimate when $g(\psi) = |\psi|^{p-1}\psi$.

Theorem 1.3. *Let us consider the nonlinear parabolic problem* (1) *with* $g(\psi) = |\psi|^{p-1}\psi$. *Then any solution* $\psi \in C\big((0,+\infty), L^\infty\big)$ *of* (1) *satisfies the following alternative as* $t \to \infty$*: either*

$$\|\psi(t,\cdot)\|_\infty \leq Ce^{-\lambda_2 t}, \tag{10}$$

or

$$\forall t \geq 1, \qquad \big\| |\psi(t,\cdot)| - \big((p-1)t\big)^{-1/(p-1)} \big\|_\infty \leq Kt^{-(1/(p-1))-1}, \tag{11}$$

where $K, C > 0$, $p > 1$.

In the following proposition we consider two special cases showing that both possibilities in the second result in the Theorem 1.1 can actually happen.

Proposition 1.4. *Let* g *satisfy* (2) *and* (3). *Then we have:*

(i) *If* Ω *is symmetric around* 0, g *is odd and* $\psi(0,\cdot)$ *is an odd function in* Ω, *then any solution* $\psi \in C\big((0,+\infty), L^\infty\big)$ *of* (1) *satisfies* (5).

(ii) *Any solution* $\psi \in C\big((0,+\infty), L^\infty(\Omega)\big)$ *of* (1) *such that* $\psi(t,\cdot) > 0$ *a.e. in* Ω *satisfies* (6). *In particular this is the case for the solution* $\psi_0 \in L^2(\Omega)$ *if* $\psi_0 \geq 0$ *and* ψ_0 *does not vanish a.e. in* Ω.

Finally, our last result shows that the second possibility is sharp for a class of functions g more general than the pure power.

Proposition 1.5. *Under the additional hypothesis*

$$\exists k_1 > 0, \forall s \in \mathbb{R}, \qquad |g(s)| \geq k_1|s|^p \tag{12}$$

for any solution $\psi \in C\big((0,+\infty), L^\infty\big)$ *of* (1), *we have*

$$\forall t \geq 1, \qquad \|\psi(t,\cdot)\|_\infty \leq \left\{\frac{1}{k_1(p-1)}\right\}^{1/(p-1)} t^{-1/(p-1)}.$$

2. Proof of Proposition 1.5

Proof. Up to a time translation of ε, we may assume $\psi \in C(\mathbb{R}^+, L^\infty)$, hence $\psi(0,\cdot) \in L^\infty$. If $\psi(0,\cdot) = 0$, we have $\psi(t,\cdot) \equiv 0$ and the result is obvious. Otherwise let z be defined by

$$z(t) = \left\{\frac{1}{\|\psi(0,\cdot)\|_\infty^{1-p} + k_1(p-1)t}\right\}^{1/(p-1)}.$$

Then z is a solution of the nonlinear ODE problem

$$\begin{cases} z' + k_1 z^p = 0, \\ z(0) = \|\psi(0,\cdot)\|_\infty. \end{cases}$$

Under the additional condition (12), we will show that z is a super-solution of (1). Indeed, we have

$$z_t - \Delta z + g(z) = -k_1 z^p + g(z) \geq 0.$$

Since $\psi(0,\cdot) \leq z(0)$ we deduce, by the standard comparison principle, that $\psi(t,\cdot) \leq z(t)$ for all $t \geq 1$.

A similar calculation shows that $\psi(t,\cdot) \geq -z(t)$ for all $t \geq 1$, which concludes the proof. □

3. A general result on the range component

Defining the orthogonal projection $P : H \to N$, where

$$H = L^2(\Omega), \qquad N = \ker(A) \qquad \text{and} \qquad P\psi(t,\cdot) = \frac{1}{|\Omega|}\int_\Omega \psi(t,x)\,dx,$$

as already mentioned in the introduction, it was shown in [6] that for $g(\psi) = |\psi|^{p-1}\psi$ the estimate

$$\|\psi(t) - P\psi(t)\|_{L^2(\Omega)} \leq Ke^{-\lambda_2 t},$$

holds for some constant $K > 0$. In this section, we will show that we have the same result for any function g satisfying (3). More generally we have

Proposition 3.1. *Let $\psi \in C(\mathbb{R}^+, L^\infty)$ be any solution of* (1). *Assume that g is a locally Lipschitz non-decreasing function. Then we have*

$$\|\psi(t) - P\psi(t)\|_2 \leq \|\psi(0) - P\psi(0)\|_2 e^{-\lambda_2 t}, \tag{13}$$

where $\|\cdot\|_2$ denotes the norm in $L^2(\Omega)$.

Proof. We denote by (u, v) the inner product of two functions u, v of $L^2(\Omega)$. Since g is a nondecreasing function for all $\psi \in L^\infty(\Omega)$, we have a.e. in $x \in \Omega$

$$\big(g(\psi) - g(P\psi)\big)(\psi - P\psi) \geq 0$$

and then by integrating over Ω

$$\big(g(\psi), \psi - P\psi\big) - \big(g(P\psi), \psi - P\psi\big) \geq 0. \tag{14}$$

Since $g(P\psi)$ is a constant and $(\psi - P\psi) \in N^{\perp}$, we deduce that $\big(g(P\psi), \psi - P\psi\big) = 0$. Hence from (14),

$$\big(g(\psi), \psi - P\psi\big) \geq 0.$$

Setting

$$w = \psi - P\psi,$$

we have

$$w' - \Delta w = -(I - P)g(\psi)$$

since $\Delta P\psi = P\Delta\psi = 0$. Since

$$\big(w, (I - P)g(\psi)\big) = \big((I - P)w, g(\psi)\big) = \big(g(\psi), \psi - P\psi\big)$$

we find

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2 = (w, \Delta w) - \big(g(\psi), \psi - P\psi\big) \leq -\lambda_2\|w\|_2^2.$$

By integrating we obtain (13). □

4. Proof of Theorem 1.1

We set $\psi = u + w$, where $u = P\psi$ and $w = (I - P)\psi$. By projecting (1) on N we obtain

$$u_t + P\big(g(\psi)\big) = 0, \tag{15}$$

where we have used that $P(A\psi) = 0$ since $R(A) \subset N^{\perp}$. Noticing that

$$u_t + P\big(g(u)\big) + P\big(g(\psi) - g(u)\big) = u_t + g(u) + P\big(g(\psi) - g(u)\big),$$

we can rewrite the equation (15) as

$$u_t + g(u) = -P\big(g(\psi) - g(u)\big).$$

By assumption (3), we deduce that

$$\big|P\big(g(\psi) - g(u)\big)\big| \leq \frac{1}{|\Omega|}\|g(\psi) - g(u)\|_1 \leq \frac{c}{|\Omega|}(\|\psi\|_{2p-2}^{p-1} + \|u\|_{2p-2}^{p-1})\|w\|_2.$$

But ψ and u are uniformly bounded and from Proposition 3.1 we have the estimate $\|w(t)\|_2 \leq Ke^{-\lambda_2 t}$. Therefore

$$\big|P\big(g(\psi) - g(u)\big)\big| \leq K'e^{-\lambda_2 t},$$

with $K' > 0$. This leads us to study the ODE

$$u' + g(u) = f(t) \qquad \text{in } \mathbb{R}^+, \tag{16}$$

where

$$f(t) = P\big(g(\psi) - g(u)\big) \qquad \text{and} \qquad |f(t)| \leq K'e^{-\lambda_2 t}.$$

Using the same method as in [5], we show the following result:

Lemma 4.1. *Let $c > 0$, $\gamma > 0$, $p > 1$ and g satisfying* (2) *and* (3). *Let $M > 0$ such that*

$$M \leq \left(\frac{\gamma}{2c}\right)^{1/(p-1)}, \tag{17}$$

$c_1 > 0$ with

$$c_1 \leq \frac{\gamma}{2} M.$$

Then, for every continuous function f in $(0, +\infty)$ satisfying

$$|f(t)| \leq c_1 e^{-\gamma t},$$

there exists a unique function $v \in C^1(\mathbb{R}^+)$ with

$$\forall t \geq 0, \qquad v' + g(v) = f(t) \tag{18}$$

and

$$\sup_{t \in (0,+\infty)} \{e^{\gamma t}|v(t)|\} \leq M. \tag{19}$$

Proof. Since any solution of (18), (19) satisfies the integral equation

$$v(t) = -\int_t^{+\infty} \big(f(s) - g\big(v(s)\big)\big)\, ds, \tag{20}$$

we look for a solution of (20). It is then natural to introduce the function space

$$X = \left\{ v \in C(0,+\infty) \,\middle|\, \sup_{t\in(0,+\infty)} e^{\gamma t}|v(t)| \le M \right\},$$

equipped with the distance associated to the norm

$$\|v\|_\gamma = \sup_{t\in(0,+\infty)} e^{\gamma t}|v(t)|.$$

We consider the operator $\mathscr{T} : X \to C(0,+\infty)$ defined by

$$\mathscr{T} v(t) = -\int_t^{+\infty} \big(f(s) - g\big(v(s)\big)\, ds.$$

From (4), we have the estimate

$$\forall s \in \mathbb{R}^+, \qquad \big|g\big(v(s)\big)\big| \le \frac{c}{p}|v(s)|^p.$$

First we will show that $\mathscr{T}(X) \subset X$. Let $v \in X$; then for all $t \ge 0$,

$$\begin{aligned} |\mathscr{T} v(t)| &\le \int_t^{+\infty} |f(s)|\, ds + \frac{c}{p}\int_t^{+\infty} |v(s)|^p\, ds \\ &\le \frac{c_1}{\gamma}e^{-\gamma t} + \frac{c}{p}M^p \int_t^{+\infty} e^{-p\gamma s}\, ds \\ &\le \left(\frac{c_1}{\gamma} + \frac{cM^p}{p^2\gamma}\right)e^{-\gamma t} \\ &\le \left(\frac{M}{2} + \frac{M}{2p^2}\right)e^{-\gamma t}. \end{aligned}$$

Since $p > 1$, it follows that

$$|\mathscr{T} v(t)| \le Me^{-\gamma t}.$$

Hence by (19), we obtain that $\mathscr{T} v \in X$, with

$$\|\mathscr{T} v(t)\|_\gamma \le M.$$

Secondly, we will prove that $\mathscr{T}$ is a contraction on X. In fact, for $x, \bar{x} \in X$ and for all $t \ge 0$,

$$\begin{aligned} |\mathscr{T} x(t) - \mathscr{T} \bar{x}(t)| &\le cM^{p-1}\int_t^{+\infty} e^{-p\gamma s}e^{\gamma s}|x(s) - \bar{x}(s)|\, ds \\ &\le \frac{cM^{p-1}}{p\gamma}\|x - \bar{x}\|_\gamma e^{-\gamma t}. \end{aligned}$$

Then we have

$$|\mathscr{T}x(t) - \mathscr{T}\bar{x}(t)|e^{\gamma t} \leq \frac{cM^{p-1}}{p\gamma}\|x - \bar{x}\|_\gamma.$$

Therefore, since M^{p-1} satisfies (17), we conclude that $\forall x, \bar{x} \in X$,

$$\|\mathscr{T}x - \mathscr{T}\bar{x}\|_\gamma \leq \frac{1}{2}\|x - \bar{x}\|_\gamma.$$

Thus $\mathscr{T}$ is a $\frac{1}{2}$-Lipschitz functional on the complete metric space X and the result follows from the Banach fixed point theorem. From (20) it follows easily that v satisfies (18). Then the uniqueness of v follows from the uniqueness of the solution of (20) ($\mathscr{T}$ is a contraction) and the fact that any solution of (18) satisfies (20). The existence comes from the fact that conversely any solution of (20) satisfies (18). □

Proof of Theorem 1.1 (*continued*). First we notice that if $|f(t)| \leq Ke^{\gamma t}$, we have $|f(t+T)| \leq Ke^{\gamma T}e^{\gamma t}$, and then Lemma 4.1 provides the existence of an exponentially decaying solution defined on $[T, +\infty)$ assuming T large enough. Consequently, we have a solution v that satisfies equation (18) for all $t \geq T_0$, where T_0 is a positive constant large enough, with

$$|v(t)| \leq Me^{-\lambda_2 t}, \tag{21}$$

where $M = M'e^{\gamma T_0}$ and $M' > 0$. If we subtract (18) from (16) we obtain

$$(u - v)' + g(u) - g(v) = 0.$$

Setting $z = u - v$, we complete the proof analyzing two cases.

Case 1: If $z(T_0) = 0$, then for all $t \geq T_0$, $z(t) = 0$. Hence $u \equiv v$ and from (21) it follows that

$$|u(t)| \leq Me^{-\lambda_2 t}.$$

Then, using (13), we obtain

$$\|\psi(t)\|_2 \leq M'e^{-\lambda_2 t}.$$

Finally by reasoning as in [6], [7] we obtain (5).

Case 2: If $z(T_0) \neq 0$ then $\forall t \geq T_0$, $z(t) \neq 0$ and we have

$$z'(t) + \frac{g\big(u(t)\big) - g\big(v(t)\big)}{u(t) - v(t)} z(t) = 0.$$

Since g is a monotonic function,

$$\alpha(t) := \frac{g\big(u(t)\big) - g\big(v(t)\big)}{u(t) - v(t)}$$

is a non-negative function. Moreover, there exists $\theta \in]0,1[$, such that

$$\alpha(t) = g'\big(\theta u(t) + (1-\theta)v(t)\big) \leq c|\theta u(t) + (1-\theta)v(t)|^{p-1}.$$

We distinguish two cases:

- If $p > 2$, then by convexity of the $(p-1)$-th power we have

$$|\theta u(t) + (1-\theta)v(t)|^{p-1} \leq \theta|u|^{p-1} + (1-\theta)|v|^{p-1} \leq |u|^{p-1} + |v|^{p-1}.$$

- If $1 < p < 2$, we study the function $(x+y)^a - x^a$ for $0 < a < 1$ and $x, y > 0$. We prove that $X \to (1+X)^a - X^a$ is a decreasing function on $(0, +\infty)$ and deduce that $0 < (1+X)^a - X^a < 1$. It follows by homogeneity that $(x+y)^a < x^a + y^a$ by letting $X = \frac{x}{y}$. Then we conclude that

$$|\theta u(t) + (1-\theta)v(t)|^{p-1} \leq |u|^{p-1} + |v|^{p-1}.$$

Consequently we obtain

$$\forall p > 1, \qquad \alpha(t) \leq c(|u|^{p-1} + |v|^{p-1}). \tag{22}$$

Setting $y = |z|$, we have

$$y' + \alpha(t)y \geq 0.$$

Then the estimate (22) implies that

$$y' \geq -c(|u|^{p-1} + |v|^{p-1})y = -c(|z+v|^{p-1} + |v|^{p-1})y.$$

Hence there exists some constants $c_2, c_3 > 0$ such that

$$y' \geq -c_2(|z|^{p-1})y - c_3|v|^{p-1}y.$$

Since $y = |z|$, this gives

$$y' \geq -c_2 y^p - c_3|v|^{p-1}y.$$

Putting $a(t) = c_3|v|^{p-1}$, we deduce that

$$y' + a(t)y \geq -c_2 y^p. \tag{23}$$

Let us now set

$$A(t) = -c_3 \int_t^{+\infty} |v|^{p-1}\, ds, \qquad \omega(t) = e^{A(t)} y.$$

Replacing ω in (23), we obtain

$$\omega(t) \geq \left\{ \frac{1}{\omega(0)^{1-p} + (p-1)c_4 t} \right\}^{1/(p-1)}$$

with $c_4 = \left(\int_0^{+\infty} a(s)\, ds\right)^{(p-1)}$. Then for t large enough we see that

$$\omega(t) \geq K t^{-1/(p-1)}.$$

Since $e^{A(t)}$ is a bounded function of t, we conclude that (6) holds by observing that $u = z + v$ and v tends to 0 exponentially at infinity. □

5. Proof of Theorem 1.3

Considering $g(\psi) = |\psi|^{p-1}\psi$, g satisfies (2) and (3) with $c = p$. Hence Lemma 4.1 is applicable with $c = p$. Therefore we assume that

$$M \leq \left(\frac{\lambda_2}{2p}\right)^{1/(p-1)}.$$

In (16), we replace $g(\psi) = |\psi|^{p-1}\psi$, we can subtract (18) from (16), we deduce

$$(u - v)' + |u|^{p-1}u - |v|^{p-1}v = 0. \tag{24}$$

We will study two cases.

Case 1: If $z(T_0) = 0$ then $z(t) = 0$ for all $t \geq T_0$. Hence $u \equiv v$ and from (21) it follows that

$$|u(t)| \leq M e^{-\lambda_2 t}.$$

Moreover, using (13), we obtain (10).

Case 2: If $z(T_0) \neq 0$ then $z(t) \neq 0$ for all $t \geq T_0$. We have

$$z'(t) + |u(t)|^{p-1}u(t) - |v(t)|^{p-1}v(t) = z'(t) + \alpha(t)z(t) = 0.$$

with

$$\begin{aligned}\alpha(t) &= \frac{|u(t)|^{p-1}u(t) - |v(t)|^{p-1}v(t)}{u(t)-v(t)}\\ &= \frac{|z(t)+v(t)|^{p-1}\big(z(t)+v(t)\big) - |v(t)|^{p-1}v(t)}{z(t)}\\ &= \frac{|z(t)|^{p-1}\left|1+\frac{v(t)}{z(t)}\right|^{p-1}\big(z(t)+v(t)\big) - |v(t)|^{p-1}v(t)}{z(t)}.\end{aligned}$$

In that case $\alpha(t) > 0$, indeed $t \mapsto |u(t)|^{p-1}u(t)$ is non decreasing function.

Since v satisfies (19), we have $\left|\frac{v(t)}{z(t)}\right| \leq 1$ for t large enough.

Therefore

$$\left|\frac{|v|^{p-1}v}{z}\right| \leq ce^{-p\lambda_2 t}.$$

For the other term, we apply Taylor's formula to $\left(1+\frac{v(t)}{z(t)}\right)^p$ and we use $|\frac{v}{z}| \leq Ke^{-\lambda t}$ for any $\lambda < \lambda_2$ and $t \geq T(\lambda)$. Indeed,

$$|z(t)|^{p-1}\left|1+\frac{v(t)}{z(t)}\right|^{p-1}\left(1+\frac{v(t)}{z(t)}\right) = |z(t)|^{p-1}\left(1+\left(1+\frac{v(t)}{z(t)}\right)^p - 1\right),$$

but we have

$$\left|\left(1+\frac{v(t)}{z(t)}\right)^p - 1\right| \leq (p+\varepsilon)\left|\frac{v(t)}{z(t)}\right| \leq (p+1)\left|\frac{v(t)}{z(t)}\right| \leq (p+1)Ke^{-\lambda t},$$

where $\varepsilon > 0$ and $\eta < \lambda_2$. We then obtain

$$\alpha(t) = |z(t)|^{p-1} + \beta(t), \tag{25}$$

with $|\beta(t)| \leq Be^{-\eta t}$. Replacing α by its expression in (25), equation (24) becomes

$$z' + |z|^{p-1}z + \beta(t)z = 0.$$

For $y = |z|$, we obtain

$$y' + y^p + \beta(t)y = 0. \tag{26}$$

Setting $\xi(t) = e^{A(t)}y(t)$, with $A(t) = -\int_t^{+\infty}\beta(s)\,ds$ we find

$$|A(t)| \leq \int_t^{+\infty}|\beta(s)|\,ds \leq \frac{B}{\eta}e^{-\eta t}. \tag{27}$$

For $t \geq \frac{1}{\eta} \ln \frac{B}{\eta}$ and by Taylor's formula we have

$$\forall h \in [-1, 1], \qquad |e^h - 1| \leq 2|h|. \tag{28}$$

We obtain the estimate

$$|\xi(t) - y(t)| \leq y(t)|e^{A(t)} - 1| \leq 2|A(t)|y(t) \leq \frac{2B}{\eta}\|y\|_\infty e^{-\eta t}$$

and conclude that

$$|\xi(t) - y(t)| \leq ke^{-\delta t}, \tag{29}$$

where $\delta = \eta$ and $k = \frac{2B}{\eta}\|y\|_\infty$. Replacing $\xi(t)$ in (26), we have

$$-e^{-A(t)}\xi'(t) = e^{-pA(t)}\xi^p(t).$$

The map $t \mapsto e^{-(p-1)A(t)}$ is bounded and tends to 1 at infinity by integrating over $[t_0, t]$, where $0 < t_0 < t$, ξ is given by

$$\xi(t)^{p-1} = \frac{1}{\xi(t_0)^{1-p} + (p-1)\int_{t_0}^t e^{-(p-1)A(s)}\, ds}.$$

We set

$$D(t) = \xi(t_0)^{1-p} + (p-1)\int_{t_0}^t e^{-(p-1)A(s)}\, ds \qquad \text{and} \qquad h(t) = -(p-1)A(t).$$

Then we show that $D(t) - (p-1)t$ is bounded.

From (27), we know that $t \mapsto h(t)$ is an integrable function, so $(e^{-(p-1)A(t)} - 1)$ is also integrable by (28). In order to show (11), we proceed as follows:

$$\begin{aligned}|D(t) - (p-1)t| &= \left|\xi(t_0)^{1-p} + (p-1)\int_{t_0}^t e^{-(p-1)A(s)}\, ds - (p-1)t\right| \\ &= \left|\xi(t_0)^{1-p} + (p-1)\int_{t_0}^t (e^{-(p-1)A(s)} - 1)\, ds - (p-1)t_0\right|.\end{aligned}$$

Using (28) we obtain

$$|D(t) - (p-1)t| \leq |K| + 2(p-1)^2\int_{t_0}^t |A(s)|\, ds \leq M_1 = |K| + 2(p-1)^2\frac{B}{\eta^2}$$

with $K = \xi(t_0)^{1-p} - (p-1)t_0$. Setting $d(t) = D(t) - (p-1)t$, we obtain

$$\begin{aligned}
&\left|\xi(t) - \left(\frac{1}{(p-1)t}\right)^{1/(p-1)}\right| \\
&\quad = \left|\left(\frac{1}{d(t)+(p-1)t}\right)^{1/(p-1)} - \left(\frac{1}{(p-1)t}\right)^{1/(p-1)}\right| \\
&\quad = \left|\left(\frac{1}{(p-1)t}\right)^{1/(p-1)}\left(\frac{1}{\frac{d(t)}{(p-1)t}+1}\right)^{1/(p-1)} - \left(\frac{1}{(p-1)t}\right)^{1/(p-1)}\right| \\
&\quad = \left(\frac{1}{(p-1)t}\right)^{1/(p-1)}\left|\left(\frac{d(t)}{(p-1)t}+1\right)^{-1/(p-1)} - 1\right| \\
&\quad = \left(\frac{1}{(p-1)t}\right)^{1/(p-1)}\left(1 - \left(\frac{d(t)}{(p-1)t}+1\right)^{-1/(p-1)}\right),
\end{aligned}$$

since $\left|\frac{d(t)}{(p-1)t}\right| < 1$ for t large enough. Now let $\eta(t) = \left(1 + \frac{d(t)}{(p-1)t}\right)^{-1/(p-1)}$ and suppose that $\left|\frac{d(t)}{(p-1)t}\right| \leq \frac{1}{2}$. By the mean value theorem we obtain

$$|\eta'(t)| \leq \frac{1}{p-1} \times 2^{1+(1/(p-1))}.$$

Therefore

$$\left|1 - \left(1 + \frac{d(t)}{(p-1)t}\right)^{-1/(p-1)}\right| \leq \frac{1}{p-1} \cdot 2^{1+(1/(p-1))}\left|\frac{d(t)}{(p-1)t}\right|.$$

As we have seen above, $d(t)$ is bounded by M_1, so we conclude that

$$\left|\xi(t) - \left(\frac{1}{(p-1)t}\right)^{1/(p-1)}\right| \leq Ct^{-1-(1/(p-1))},$$

with $C = \left(\frac{1}{p-1}\right)^{p/(p-1)} \times 2^{1+(1/(p-1))} M_1$.

We recall that z has a constant sign on $[T_0, +\infty[$, and z and u have the same sign. As in Section 4, we set $u = v + z$ and $\psi = u + w$, and we distinguish the cases $z > 0$ and $z < 0$.

- If $z > 0$, then $u > 0$ and $|\psi| = \psi$ for t large enough. Then

$$\left||\psi| - \left(\frac{1}{(p-1)t}\right)^{1/(p-1)}\right| \leq \left|u - \left(\frac{1}{(p-1)t}\right)^{1/(p-1)}\right| + |w| \leq Kt^{-1-(1/(p-1))}.$$

Indeed,

$$\left|u - \left(\frac{1}{(p-1)t}\right)^{1/(p-1)}\right| \le |u - z| + |z - \xi| + \left|\xi - \left(\frac{1}{(p-1)t}\right)^{1/(p-1)}\right|$$
$$\le Me^{-\lambda_2 t} + ke^{-\delta t} + Ct^{-1-(1/(p-1))}.$$

Since we have (29), we obtain (11).

- If we suppose that $z < 0$, then $u < 0$. By similar calculations we obtain the same result. Indeed, $|\psi| = -\psi$, and we have

$$\left||\psi| - \left(\frac{1}{(p-1)t}\right)^{1/(p-1)}\right| \le \left|-u - \left(\frac{1}{(p-1)t}\right)^{1/(p-1)}\right| + |w|$$
$$\le Kt^{-1-(1/(p-1))}$$

since

$$\left|-u - \left(\frac{1}{(p-1)t}\right)^{1/(p-1)}\right| \le |u - z| + |-z - \xi| + \left|\xi - \left(\frac{1}{(p-1)t}\right)^{1/(p-1)}\right|$$
$$\le Me^{-\lambda_2 t} + ke^{-\delta t} + Ct^{-1-(1/(p-1))}.$$

Also by (29), we finally obtain (11).

6. Proof of Proposition 1.4

Proof of (i). If ψ is an odd function, then $\psi(0,-x) = -\psi(0,x)$. This implies $\psi(t,-x) = -\psi(t,x)$ for all $t > 0$. In that case

$$u(t) = P\psi(t,x) = \frac{1}{|\Omega|}\int_\Omega \psi(t,x)\,dx = 0.$$

Moreover, we know that $\psi = u + w$, where $w(t) \in N^\perp$ for all $t \ge 0$. We have (cf. Proposition 3.1)

$$\|w(t)\|_{L^\infty(\Omega)} \le Ke^{-\lambda_2 t},$$

hence the solution ψ satisfies (5). □

Proof of (ii). If $\psi(0,x) \ge 0$ and ψ does not vanish a.e. in Ω, then $\psi(t,x) > 0$ for all $t \ge 0$, which implies that

$$\int_\Omega \psi(t,x)\,dx > 0. \tag{30}$$

We suppose that we have $\|\psi(t,\cdot)\|_\infty \le Ce^{-\lambda_2 t}$ and consider the problem (1). We integrate on Ω and obtain

$$\int_\Omega \psi_t(t,x)\,dx = -\int_\Omega g\big(\psi(t,x)\big)\,dx. \tag{31}$$

An elementary calculation shows that we have

$$\begin{aligned}\int_\Omega g\big(\psi(t,x)\big)\,dx &\le \frac{c}{p}\int_\Omega |\psi(t,x)|^p\,dx\\ &\le \frac{c}{p}\int_\Omega \|\psi(t,x)\|_\infty^{p-1}\psi(t,x)\,dx\\ &\le \frac{c}{p}C^{p-1}e^{-(p-1)\lambda_2 t}\int_\Omega \psi(t,x)\,dx.\end{aligned}$$

Now we set $y(t) = \int_\Omega \psi(t,x)\,dx$. From (31) we deduce that

$$y'(t) \ge -Me^{-\delta t}y(t),$$

with $M = \frac{c}{p}C^{p-1}$ and $\delta = (p-1)\lambda_2$.

Since $y(t) > 0$ by (30), we can integrate in the interval $[0,t]$ and obtain

$$y(t) \ge y(0)\exp\Big\{-M\int_0^t e^{-\delta s}\,ds\Big\} \ge y(0)\exp\Big\{-\frac{M}{\delta}\Big\} > 0. \tag{32}$$

Hence y does not tend to 0 for t large, which contradicts our hypothesis, and we conclude that y satisfies (6). □

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