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Cutting corners in Michell trusses

Elizabeth Figueroa, Adam Hill, Denise Iusco and Rolf Ryham*

Abstract. A corner is defined to be the vertex of a strict cone locally containing the support of a truss and having locally empty intersection with the support of the applied force. A topological perturbation called corner cutting is defined in two dimensions and the process is shown to have a negative effect on the mass of planar trusses, independent of the angle of the corner. Minimal, finite, planar trusses are therefore free of corners. Applied point forces are shown to be balanced by at least one truss, thereby showing that the admissible class is nonempty. Explicit mass and geometric bounds are presented as part of this construction.

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1. Introduction

Michell trusses deal with a variational problem posed by the mechanical engineer Anthony George Maldon Michell in the early part of the twentieth century [9]; that is, what configuration of bars and cables needed to balance a system of applied forces is most economical? Force is transmitted by an axial stress; a symmetric, rank one matrix-valued measure supported on the bars and cables. The linear density of individual bars or cables is proportional to their strength. In the Michell truss problem, one minimizes the total mass subject to balancing an applied load. The stress is a solution to the force balance equations stating that the material body will remain at rest under the applied force. In the study of Michell trusses, one is interested in the construction and existence of minimal structures. This is a difficult mathematical problem with subtle existence and regularity issues where the number of bars and cables needed to withstand an applied point force may diverge to infinity [4].

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In this paper, a sufficient condition is given for determining when a truss is not minimal. Minimality can be tested by making local perturbations to the structure, but these perturbations are generally difficult to construct since satisfying the force balance equation requires a change in topology. However, a natural place to look for improvement is corners. A topological perturbation called corner cutting is introduced and it is shown to have a negative effect on the mass of finite, planar structures. Repeatedly applying the corner cutting construction yields a sequence of trusses with decreasing mass and increasing number of members.

Returning to the global question, in the direct method, it is necessary to demonstrate the existence of at least one load balancing structure. To this end, an explicit construction of a balancing truss is given which yields geometric and mass estimates in terms of the data—the applied force.

The definitions of a truss and the corner cutting construction appearing in this paper were inspired by [4]. There, a strategy to address the formation of diffuse structures was proposed by defining a class of stresses which accounts for the limits of finite trusses. A duality principle for the stress and strain tensors was formulated and the infimum over this class was shown to be the same as in Michell's problem. Solving the dual formulation is often more straightforward and leads to constructive, optimization algorithms [3], [5], [6], [10], [11].

In three dimensions, the effective stress tensor can be more complicated than presented above. From the point of view of optimal design, one must consider the need for a material with two dimensional microstructures, in addition to bars and cables. This is discussed in more detail in [1]. Among potential materials that may be used to fulfill these criteria are metallic foams and single-scale laminates [2]. However, this is beyond the scope of our discussion.

In Section 2, trusses and applied point forces are defined. The main results are stated in Section 3. The corner cutting perturbation is defined in Section 4 and the construction of a truss with geometric and mass bounds is given in Section 5.

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2. Notation

Throughout the paper, the following notation will be used.

$$\mathbb{R}^{n\times n},\ \text{linear transformations of } \mathbb{R}^n,$$

$$v \cdot w = \sum_{i=1}^{n} v_i w_i \quad \text{for } v, w \in \mathbb{R}^n, \qquad \sigma : \tau = \sum_{i,j=1}^{n} \sigma_{ij}\tau_{ij} \quad \text{for } \sigma, \tau \in \mathbb{R}^{n\times n},$$

$$v \otimes w \in \mathbb{R}^{n\times n} \qquad \text{where } (v \otimes w)_{ij} = v_i w_j \text{ for } i, j = 1, \dots, n, \ v, w \in \mathbb{R}^n.$$

For x and y distinct points in $\mathbb{R}^n$, $[xy]$ is the closed line segment and (xy) is the open line segment.

2.1. Trusses. To motivate the definition of a truss, consider an idealized one dimensional bar under compression. Let u be a unit vector parallel to the bar. Following Cauchy, slice the bar into two pieces with a plane perpendicular to the bar. Let n be the outward pointing normal vector of the first piece. By Galilean invariance, c.f. [8], the stress at the cut is a symmetric matrix. Since the force points in the direction of the outward normal, n is an eigen-vector of the stress with positive eigen-value. Finally, the plane perpendicular to the bar is the null space of the stress, since the bar withstands only compression. From the spectral theorem, it follows that the stress is of the form $\omega u \otimes u$ for some positive constant ω. The number ω is called the strength, or weight, of the bar.

Motivated by this discussion, the stress carried by a bar spanning distinct points x and y in $\mathbb{R}^n$ is $\omega\sigma_{xy}$, where ω is a positive constant,

$$d\sigma_{xy} = \frac{x-y}{|x-y|} \otimes \frac{x-y}{|x-y|} d\mu, \tag{1}$$

and μ is arc length measure with respect to (xy). Note that the Leibniz notation $d\sigma_{xy}$ is used to define σ_{xy}. Similarly, the stress carried by an idealized cable spanning x and y is $\nu\sigma_{xy}$ where ν is a negative constant.

More generally, let $\boldsymbol{T}$ be the collection of finite, $\mathrm{Ver}(n)$-valued Radon measures. An element of $\boldsymbol{T}$ is called a truss. Here, $\mathrm{Ver}(n)$ is the Veronese cone of rank one symmetric linear transformations, i.e. matrices of the form $\lambda u \otimes u$ for $\lambda \in \mathbb{R}$ and $u \in \mathbb{R}^n$. Recall that a vector or matrix valued measure is Radon if it is Borel regular and its variation measure is finite on compact sets [7]. Let $\boldsymbol{T}_0 \subset \boldsymbol{T}$ be the collection of $\mathrm{Ver}(n)$-valued Radon measures σ of the form $\sigma = \sum_{i\in I} \lambda_i \sigma_{x_i y_i}$, where I is finite and $x_i, y_i \in \mathbb{R}^n$ are distinct and λ_i is a nonzero real number for all $i \in I$. An element of $\boldsymbol{T}_0$ is called a finite truss.

If $\sigma \in \boldsymbol{T}$, then the variation measure $|\sigma|$ and mass $\|\sigma\|$ are defined by

$$|\sigma|(U) = \sup_{\substack{\phi \in C_0(U\mathbb{R}^{n\times n}),\\ \|\phi\|_* \le 1}} \int_{\mathbb{R}^n} \phi \cdot d\sigma, \qquad \|\sigma\| = |\sigma|(\mathbb{R}^n),$$

for all open subsets U of $\mathbb{R}^n$. Here, $\|\phi\|_* = \max_{x\in\mathbb{S}^{n-1}} |\phi : x \otimes x|$. In particular, if $\sigma = \sum_{i\in I} \lambda_i \sigma_{x_i y_i} \in \boldsymbol{T}_0$, and B is a Borel set, then

$$|\sigma|(B) = \sum_{i \in I} |\lambda_i| \mu([x_i y_i] \cap B), \qquad \|\sigma\| = \sum_{i \in I} |\lambda_i| \, |x_i - y_i|. \tag{2}$$

Finally, a point p is said to lie in the support of σ if $r > 0$ implies $|\sigma|\big(\boldsymbol{B}_r(p)\big) > 0$.

Equilibrated applied forces. Applied forces are vector-valued measures supported in $\mathbb{R}^n$. If a static solution to the force balance equation exists, then the linear and angular moments of the applied forces vanish. Thus, the space of equilibrated forces $\boldsymbol{F}$ is the collection of $\mathbb{R}^n$-valued Radon measures f with finite first moment measure $|x| d|f|$, and

$$\int_{\mathbb{R}^n} \Omega(x - z) \cdot df(x) = 0 \qquad \text{for all } \Omega \in \mathrm{Skw}(n) \text{ and } z \in \mathbb{R}^n. \tag{3}$$

Here, $\mathrm{Skw}(n)$ is the collection of skew symmetric linear transformations.

The space of applied point forces $\boldsymbol{F}_0 \subset \boldsymbol{F}$ is the collection of equilibrated forces with finite support. If f is an element of $\boldsymbol{F}_0$, then there is a finite set $\boldsymbol{A}$ and vectors $\{f_a\}_{a \in \boldsymbol{A}}$ in $\mathbb{R}^n$ so that

$$f = \sum_{a \in \boldsymbol{A}} f_a \delta_a, \sum_{a \in \boldsymbol{A}} \Omega(a - z) \cdot f_a = 0 \qquad \text{for all } \Omega \in \mathrm{Skw}(n), \ z \in \mathbb{R}^n, \tag{4}$$

where δ_a is the Dirac measure supported at a. If $f \in \boldsymbol{F}$, then its mass is defined by $\|f\| = |f|(\mathbb{R}^n)$, the variation measure of the whole space. In particular, if $f = \sum_{a \in \boldsymbol{A}} f_a \delta_a \in \boldsymbol{F}_0$, then,

$$\|f\| = \sum_{a \in \boldsymbol{A}} |f_a|.$$

A truss $\sigma \in \boldsymbol{T}$ balances $f \in \boldsymbol{F}$ if $\operatorname{div} \sigma + f = 0$ in the sense of distributions. This means that for all smooth, compactly supported vector fields ϕ,

$$\int_{\mathbb{R}^n} \nabla \phi : d\sigma = \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} \frac{\partial \phi_i}{\partial x_j} d\sigma_{ij} = \sum_{i=1}^{n} \int_{\mathbb{R}^n} \phi_i \, df_i = \int_{\mathbb{R}^n} \phi \cdot df. \tag{5}$$

If $\sigma = \sum_{i \in I} \lambda_i \sigma_{x_i y_i}$ is a finite truss and $f = \sum_{j \in J} f_j \delta_{x_j}$ is an equilibrated applied point force, then (5) is equivalent to the statement

$$\sum_{i \in I} \lambda_i \frac{x_i - y_i}{|x_i - y_i|} \cdot [\phi(x_i) - \phi(y_i)] = \sum_{i \in J} f_i \cdot \phi(x_i), \tag{6}$$

for all $\phi \in C^0(\mathbb{R}^n; \mathbb{R}^n)$. A truss σ is minimal if $\|\sigma\| \le \|\tau\|$ for all $\tau \in \boldsymbol{T}$ such that $\operatorname{div} \tau = \operatorname{div} \sigma$.

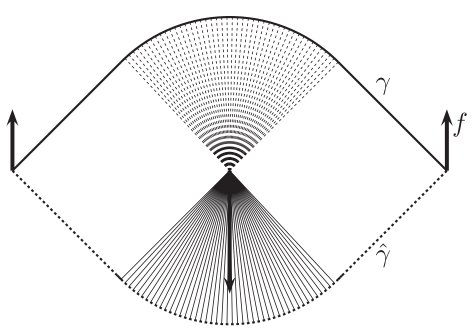

Figure 1. The Michell Bridge Truss.

As an illustration, consider the Michell bridge truss. The bridge truss, introduced in [9], is an example of a minimal truss which is infinite, curved, and has members under compression and members under tension. To describe the truss, let γ be the arc-length parametrized curve in $\mathbb{R}^2 = \mathbb{C}$ consisting of the line segment $[\sqrt{2}, e^{i\pi/4}]$, the circular arc $e^{i\theta}$ for $\pi/4 < \theta < 3\pi/4$ and the line segment $[e^{i3\pi/4}, -\sqrt{2}]$. Let $\hat{\gamma}$ be the reflection of γ across the real axis. Define $\sigma_M \in \boldsymbol{T}$ by

$$d\sigma_M = \gamma' \otimes \gamma' \, d\gamma - e^{i\alpha} \otimes e^{i\alpha} \, dr \, d\theta - \hat{\gamma}' \otimes \hat{\gamma}' \, d\hat{\gamma} + e^{i\beta} \otimes e^{i\beta} \, dr \, d\theta,$$

where $\pi/4 < \alpha$, $\beta - \pi < 3\pi/4$ and $0 < r < 1$. One verifies by direct calculation that σ_M satisfies the weak formulation (5) for the applied point force

$$f = i\sqrt{2}(\delta_{\sqrt{2}} + \delta_{-\sqrt{2}} - 2\delta_0).$$

Both f and σ are depicted in Figure 1. Solid lines represent bars while dashed lines represent cables. The force vectors are scaled for clarity. The mass of the bridge truss is $4 + 2\pi$. Using a duality argument [4], σ_M was shown to be minimal.

Definition 2.1. Let $\sigma \in \boldsymbol{T}$ with $\operatorname{div} \sigma \in \boldsymbol{F}$. Then a point p is a corner if p lies in the support of σ and there are $r > 0$, $\theta > 0$, and $v \in \mathbb{S}^{n-1}$ such that $|\operatorname{div} \sigma|\big(\boldsymbol{B}_r(p)\big) = 0$ and $|\sigma|(\{x : (x - p) \cdot v < \theta|x - p|, |x - p| < r\}) = 0$.

3. Main results

The following result gives a necessary condition for the minimality of finite planar trusses.

Theorem 3.1. *Let $\sigma \in \boldsymbol{T}_0$ on $\mathbb{R}^2$. If σ possesses a corner, then σ is not minimal.*

Remark 3.2. The conclusion of Theorem 3.1 is independent of the angle of the corner.

The next result establishes the existence of a minimizing sequence. It states that an applied point force is equilibrated if and only if it is balanced by a truss.

Theorem 3.3. *Let* $f \in \boldsymbol{F}_0$ *be an applied point force on* $\mathbb{R}^n$. *There exists a truss* $\sigma \in \boldsymbol{T}_0$ *with* $\operatorname{div}\sigma + f = 0$ *and*

$$\operatorname{Supp}(\sigma) \subset \bigcap_{q \in \mathbf{A}} \boldsymbol{B}_{\operatorname{Diam}(\mathbf{A})}(q),$$

$$\|\sigma\| \le K_n\big(\operatorname{Diam}(\mathbf{A}) + 1\big)^2 \|(|x| + 1)f\|,$$

where $\mathbf{A} = \operatorname{Supp}(f)$ *and where* $K_n = 8^n\sqrt{n!}$.

4. Cutting corners

In the remainder of the section, p is a corner of a finite truss $\sigma \in \boldsymbol{T}_0$. Lemma 4.1 states that the members of the truss meeting the corner lie inside a strict cone and that the net force on the corner is zero. Proposition 4.2 determines the behavior of weights when the members meeting the corner are duplicated and shifted in independent directions.

Lemma 4.1. *There exists a finite set* $\boldsymbol{Y} \subset \mathbb{R}^n$, $\theta > 0$, *and* $v \in \mathbb{S}^{n-1}$ *with*

$$\sigma = \tau + \hat{\tau}, \qquad \tau = \sum_{y \in \boldsymbol{Y}} \kappa_{0y}\sigma_{yp}, \qquad \hat{\tau} = \sum_{j \in J} \kappa_j \sigma_{x_j y_j}, \qquad p \notin \{x_j, y_j\}_{j \in J} \tag{7}$$

where

$$\sum_{y \in \boldsymbol{Y}} \kappa_{0y}\frac{y - p}{|y - p|} = 0, \qquad (y - p)\cdot v \ge \theta|y - p| \quad \text{for all } y \in \boldsymbol{Y}. \tag{8}$$

Proof. Let $\sigma = \sum_{i \in I}\kappa_i\sigma_{x_i y_i}$. Let K be the collection of k in I for which $p \in [x_k y_k]$. Since p lies in the support of σ, K is nonempty. If $k \in K$, then $p = \eta x_k + (1 - \eta) y_k$ for some $\eta \in [0, 1]$. By definition, there are $\theta > 0$, $r > 0$, and $v \in \mathbb{S}^{n-1}$ for which $|\sigma|\big(B_{r,\theta,v}(p)\big) = 0$ where $B_{r,\theta,v}(p) = \{x : (x - p)\cdot v < \theta|x - p|, |x - p| < r\}$. Thus, by (2),

$$|\kappa_k|\mu\big([x_k y_k] \cap B_{r,\theta,v}(p)\big) \le \sum_{i \in I}|\kappa_i|\mu\big([x_i y_i] \cap B_{r,\theta,v}(p)\big) = |\sigma|\big(B_{r,\theta,v}(p)\big) = 0.$$

Since κ_k is nonzero, it follows that $(q_h - p)\cdot v \ge \theta|q_h - p|$ when $q_h = hx_k + (1 - h)y_k$, $h \in [0, 1]$, and $|q_h - p| < r$. Expanding the previous inequality in terms of η and h,

$$(h - \eta)(x_k - y_k)\cdot v = (q_h - p)\cdot v \ge \theta|q_h - p| = \theta|h - \eta|\,|x_k - y_k|.$$

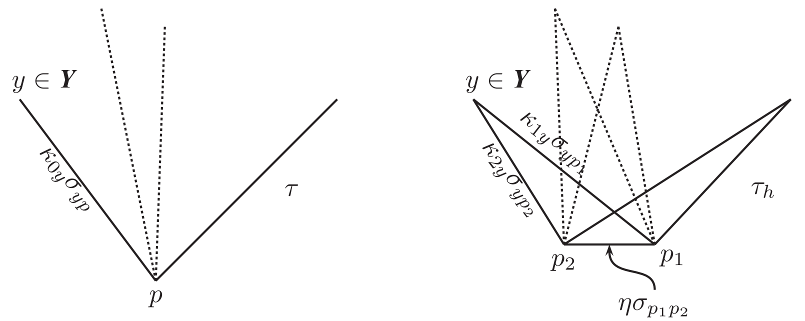

Figure 2. Corner cutting. In two dimensions, the vertex p of the cone is shifted to two points p_1 and p_2. The residual force at y is then carried by two members. The residual force at p_1 and p_2 vanishes as a consequence of the balance of forces at p.

If $\eta \in (0, 1)$, then the left-hand side changes sign for h lying in a sufficiently small neighborhood of η while the sign of the right-hand side does not change. This is a contradiction. It follows that $\eta = 0$ or $\eta = 1$. Suppose that $\eta = 1$ so that $x_k = p$. Choosing h sufficiently close to but less than 1, whereby $(h - \eta)/|h - \eta| = -1$, yields

$$(y_k - p) \cdot v \geq \theta |y_k - p| \qquad \text{for all } k \in K.$$

In the alternative case $\eta = 0$, switch the labels x_k and y_k. Let $\boldsymbol{Y} = \{y_k\}_{k \in K}$, $J = I \backslash K$, and $\kappa_{0y} = \kappa_k$ whenever $y = y_k$, $k \in K$. By construction, $p \notin \bigcup_{j \in J} [x_j, y_j]$. Then, by the definition of corner and (6), $\left|\sum_{y \in \boldsymbol{Y}} \kappa_{0y} \frac{y-p}{|y-p|}\right| \leq |\mathrm{div}\, \sigma|\big(\boldsymbol{B}_r(p)\big) = 0$. □

For simplicity, assume $p = 0$. In general this may be achieved by shifting the truss so that the corner lies at the origin. To describe the corner cutting construction as applied to planar trusses, define $v_1 = e_1 + \theta e_2$ and $v_2 = -e_1 + \theta e_2$. Here, e_α are orthonormal vectors in $\mathbb{R}^2$ and $v = e_2$ where v is the vector in Lemma 4.1. Consider the system of equations

$$p_\alpha = h v_\alpha, \qquad \alpha = 1, 2, \tag{9}$$

$$\kappa_{0y} \frac{y}{|y|} = \sum_{\alpha=1}^{2} \kappa_{\alpha y} \frac{y - p_\alpha}{|y - p_\alpha|} \qquad \text{for all } y \in \boldsymbol{Y}, \tag{10}$$

$$\eta \frac{p_1 - p_2}{|p_1 - p_2|} = \sum_{y \in \boldsymbol{Y}} \kappa_{1y} \frac{y - p_1}{|y - p_1|} = -\sum_{y \in \boldsymbol{Y}} \kappa_{2y} \frac{y - p_2}{|y - p_2|}. \tag{11}$$

Proposition 4.2. *There exist functions* $\{\kappa_{\alpha y}\}_{\alpha=1,2,y\in \boldsymbol{Y}}$, $\eta \in C^1([0,h_0))$ *satisfying* (9)–(11) *for all* $h \in (0,h_0)$ *where* $h_0 = \min_{y\in \boldsymbol{Y}}|y|$. *In particular, the sums appearing in* (11) *are equal and parallel to* $p_1 - p_2$ *for all* $h \in (0,h_0)$. *Moreover,*

$$\sum_{\alpha=1}^{2}\kappa_{\alpha y}(0) = \kappa_{0y}, \qquad \sum_{\alpha=1}^{2}|\kappa_{\alpha y}(0)| = |\kappa_{0y}|, \tag{12}$$

$$\sum_{\alpha=1}^{2}\frac{\kappa_{\alpha y}(0)}{|\kappa_{\alpha y}(0)|}\kappa'_{\alpha y}(0) = 0, \tag{13}$$

$$\sum_{\alpha=1}^{2}|\kappa_{\alpha y}(0)|v_\alpha = \frac{\theta|\kappa_{0y}|y}{y\cdot e_2} \qquad \textit{for all } y \in \boldsymbol{Y}, \tag{14}$$

$$\eta(0) = \theta\sum_{y\in Y}\frac{\kappa_{0y}(y\cdot e_1)^2}{2|y|y\cdot e_2}. \tag{15}$$

Proof. For $h \in (0,h_0)$ and $y \in \boldsymbol{Y}$, define

$$\kappa_{1y} = \kappa_{0y}\frac{(y\cdot e_2 + \theta y\cdot e_1)|y - p_1|}{2|y|(y\cdot e_2 - \theta h)}, \qquad \kappa_{2y} = \kappa_{0y}\frac{(y\cdot e_2 - \theta y\cdot e_1)|y - p_2|}{2|y|(y\cdot e_2 - \theta h)}. \tag{16}$$

The numerator and denominator in the above two expressions is positive. This is a consequence of the inequality in (8), where $p = 0$, $v = e_2$, and $0 \le h < \min_{y\in \boldsymbol{Y}}|y|$. It follows that $\{\kappa_{\alpha y}\}_{\alpha=1,2,y\in \boldsymbol{Y}} \in C^1([0,h_0))$. Moreover, the sign of $\kappa_{\alpha y}$ is independent of h and α and is the same as the sign of κ_{0y} (by definition, κ_{0y} is nonzero). To see that $\kappa_{\alpha y}$ satisfies (10), let $y \in \boldsymbol{Y}$ and consider

$$\begin{aligned}
&\left(\kappa_{0y}\frac{y}{|y|} - \sum_{\alpha=1}^{2}\kappa_{\alpha y}\frac{y - p_\alpha}{|y - p_\alpha|}\right)\frac{2|y|(y\cdot e_2 - \theta h)}{\kappa_{0y}}\\
&\quad = 2(y\cdot e_2 - \theta h)y - (y\cdot e_2 + \theta y\cdot e_1)(y - p_1) - (y\cdot e_2 - \theta y\cdot e_1)(y - p_2)\\
&\quad = 2(y\cdot e_2 - \theta h)y - 2y\cdot e_2(y - h\theta e_2) + 2h\theta y\cdot e_1 e_1 = 0.
\end{aligned}$$

To derive (12), set $h = 0$ in (16) and add the two expressions and their absolute value respectively. To derive (13), differentiate (10) with respect to h and evaluate at $h = 0$; noting that κ_{0y} and y are independent of h,

$$\sum_{\alpha=1}^{2}\kappa'_{\alpha y}(0)\frac{y}{|y|} - \kappa_{\alpha y}(0)\left(\frac{v_\alpha}{|y|} - \frac{y\cdot v_\alpha y}{|y|^3}\right) = 0.$$

The right summands are orthogonal projections onto $y^\perp$. Forming the inner product of this expression with y then gives $\sum_{\alpha=1}^{2}\kappa'_{\alpha y}(0) = 0$, implying (13) since the sign of $\kappa_{\alpha y}$ is independent of α for $h \in [0,h_0)$.

The derivation of (14) follows by evaluation. Using (16), where all terms other than κ_{0y} are positive,

$$\sum_{\alpha=1}^{2} |\kappa_{\alpha y}(0)| v_\alpha = \frac{|\kappa_{0y}|}{2y \cdot e_2}[(y \cdot e_2 + \theta y \cdot e_1)(e_1 + \theta e_2) + (y \cdot e_2 - \theta y \cdot e_1)(-e_1 + \theta e_2)]$$

$$= \frac{|\kappa_{0y}|}{y \cdot e_2}[\theta y \cdot e_1 e_1 + \theta y \cdot e_2 e_2] = \frac{\theta |\kappa_{0y}| y}{y \cdot e_2}.$$

To define $\eta(h)$, note that $\sum_{y \in \boldsymbol{Y}} \kappa_{1y} \frac{y - p_1}{|y - p_1|}$ is parallel to $p_1 - p_2$ for all $h \in (0, h_0)$. To see this, let $\Omega = e_2 \otimes e_1 - e_1 \otimes e_2$. By (8) (with $p = 0$) and the skew symmetric property $\Omega y \cdot y = 0$,

$$0 = \Omega p_2 \cdot \sum_{y \in \boldsymbol{Y}} \kappa_{0y} \frac{y}{|y|} = \sum_{y \in \boldsymbol{Y}} \Omega(p_2 - y) \cdot \kappa_{0y} \frac{y}{|y|}.$$

Then, expanding $\kappa_{0y} \frac{y}{|y|}$ using (10),

$$0 = \sum_{\alpha=1}^{2} \sum_{y \in \boldsymbol{Y}} \Omega(y - p_2) \cdot \kappa_{\alpha y} \frac{y - p_\alpha}{|y - p_\alpha|} = \sum_{\alpha=1}^{2} \sum_{y \in \boldsymbol{Y}} \Omega(y - p_\alpha + p_\alpha - p_2) \cdot \kappa_{\alpha y} \frac{y - p_\alpha}{|y - p_\alpha|}$$

$$= \sum_{\alpha=1}^{2} \sum_{y \in \boldsymbol{Y}} \Omega(p_\alpha - p_2) \cdot \kappa_{\alpha y} \frac{y - p_\alpha}{|y - p_\alpha|} = \sum_{y \in \boldsymbol{Y}} \Omega(p_1 - p_2) \cdot \kappa_{1y} \frac{y - p_1}{|y - p_1|}$$

$$= 2he_2 \cdot \sum_{y \in \boldsymbol{Y}} \kappa_{1y} \frac{y - p_1}{|y - p_1|}.$$

The skew symmetric property $\Omega(y - p_\alpha) \cdot (y - p_\alpha) = 0$ was used in the third equation. In the fourth equation, the left-hand summand vanishes when $\alpha = 2$. The fifth equation follows from the identities $\Omega e_1 = e_2$ and $p_1 - p_2 = 2he_1$. The fact that $\sum_{y \in \boldsymbol{Y}} \kappa_{1y} \frac{y - p_1}{|y - p_1|}$ is perpendicular to e_2 shows that it is also parallel to $p_1 - p_2$. The analogous argument implies that $\sum_{y \in \boldsymbol{Y}} \kappa_{2y} \frac{y - p_2}{|y - p_2|}$ is parallel to $p_2 - p_1$. Finally, combining (8) and (10),

$$\sum_{y \in \boldsymbol{Y}} \kappa_{1y} \frac{y - p_1}{|y - p_1|} + \sum_{y \in \boldsymbol{Y}} \kappa_{2y} \frac{y - p_2}{|y - p_2|} = \sum_{y \in \boldsymbol{Y}} \kappa_{0y} \frac{y}{|y|} = 0.$$

The existence of the function $\eta(h)$ for $h \in (0, h_0)$ is now established since all the terms appearing in (11) are parallel and two rightmost expressions are equal. In particular, forming the inner product of (11) with e_1 gives

$$-\eta(h) + \sum_{y \in \boldsymbol{Y}} \kappa_{1y} \frac{y \cdot e_1 + h}{|y - p_1|} = 0.$$

This formula shows that $\eta(h)$ extends to a continuously differentiable function on $h \in [0, h_0)$. Setting $h = 0$ and using (16),

$$\eta(0) = \sum_{y \in Y} \frac{\kappa_{0y}(y \cdot e_2 + \theta y \cdot e_1)}{2y \cdot e_2} \frac{y \cdot e_1}{|y|} = \theta \sum_{y \in Y} \frac{\kappa_{0y}(y \cdot e_1)^2}{2|y| y \cdot e_2}.$$

Note that the first summands in the middle term vanish due to (8). This last equality implies (15), thus completing the proof. □

Theorem 3.1 is now a corollary of Proposition 4.2.

Proof of Theorem 3.1. Applying the above notation, let $\sigma = \tau + \hat{\tau}$ be the decomposition provided by (7). To show that σ is not minimal, there is a one parameter family of trusses σ^h for $h \in (0, h_0)$ and a continuously differentiable function $m(h)$ for $h \in [0, h_0)$ so that

$$\operatorname{div} \sigma^h = \operatorname{div} \sigma, \qquad m(h) = \|\sigma^h\|, \qquad \forall h \in (0, h_0),$$
$$m(0) = \|\sigma\|, \qquad m'(0) < 0.$$

Pending the existence of σ^h and $m(h)$, the proof of the theorem follows since $\|\sigma^h\|(h) = m(h) < m(0) = \|\sigma\|$ for h sufficiently small. Thus, σ is not minimal.

To define σ^h, let $\sigma^h = \tau^h + \hat{\tau}$ where

$$\tau_h = \eta \sigma_{p_1 p_2} + \sum_{\alpha=1}^{2} \sum_{y \in Y} \kappa_{\alpha y} \sigma_{y p_\alpha}, \qquad h \in (0, h_0)$$

and where $\{\kappa_{\alpha y}\}_{\alpha=1,2, y \in Y}$ and η are found in Proposition 4.2. To define $m(h)$, let

$$m(h) = 2|\eta(h)| h + \sum_{\alpha=1}^{2} \sum_{y \in Y} |\kappa_{\alpha y}(h)| \, |y - p_\alpha(h)| + \|\hat{\tau}\|, \qquad h \in [0, h_0).$$

Proposition 4.2 guarantees that $m(h) \in C^1([0, h_0))$ while (2) shows that $m(h) = \|\sigma^h\|$ for $h \in (0, h_0)$. Evaluating the function $m(h)$ at $h = 0$,

$$\begin{aligned} m(0) &= 2|\eta(0)| 0 + \sum_{\alpha=1}^{2} \sum_{y \in Y} |\kappa_{\alpha y}(0)| \, |y - p_\alpha(0)| + \|\hat{\tau}\| \\ &= \sum_{y \in Y} \sum_{\alpha=1}^{2} |\kappa_{\alpha y}(0)| \, |y - p| + \|\tau\| = \sum_{y \in Y} |\kappa_{0y}(0)| \, |y - p| + \|\hat{\tau}\| = \|\sigma\|. \end{aligned}$$

Here (15) implies that $\eta(0)$ is finite, the identity $p_\alpha(0) = p$ was used in the second equation, and (12) was used in the third equation.

To show that $\operatorname{div}\sigma^h = \operatorname{div}\sigma$, consider the distributional definition of the divergence (6) of τ^h : for $\phi \in C^0(\mathbb{R}^2;\mathbb{R}^2)$,

$$\begin{aligned}
&\eta\frac{p_1-p_2}{|p_1-p_2|}\cdot\big(\phi(p_1)-\phi(p_2)\big)+\sum_{\alpha=1}^{2}\sum_{y\in\boldsymbol{Y}}\kappa_{\alpha y}\frac{y-p_\alpha}{|y-p_\alpha|}\cdot\big(\phi(y)-\phi(p_\alpha)\big)\\
&\quad=\sum_{y\in\boldsymbol{Y}}\kappa_{1y}\frac{y-p_1}{|y-p_1|}\cdot\phi(p_1)+\sum_{y\in\boldsymbol{Y}}\kappa_{2y}\frac{y-p_2}{|y-p_2|}\cdot\phi(p_2)\\
&\qquad+\sum_{\alpha=1}^{2}\sum_{y\in\boldsymbol{Y}}\kappa_{\alpha y}\frac{y-p_\alpha}{|y-p_\alpha|}\cdot\big(\phi(y)-\phi(p_\alpha)\big)\\
&\quad=\sum_{\alpha=1}^{2}\sum_{y\in\boldsymbol{Y}}\kappa_{\alpha y}\frac{y-p_\alpha}{|y-p_\alpha|}\cdot\phi(y)=\sum_{y\in\boldsymbol{Y}}\kappa_{0y}\frac{y-p}{|y-p|}\cdot\phi(y).
\end{aligned}$$

Here, (11) was used to substitute the first term on the left with the two sums on the right of the first equation and then (10) was used in the third equation. Since ϕ was arbitrary, this shows that $\operatorname{div}\tau^h = \operatorname{div}\sigma - \operatorname{div}\hat{\tau}$, proving the claim $\operatorname{div}\sigma^h = \operatorname{div}\sigma$.

Finally, to show that the mass of σ^h increases to the mass of σ as h tends to zero, note that $m(h)$ is a differentiable. In particular,

$$m'(h)=2h\frac{\eta'}{|\eta|}+2|\eta|+\sum_{\alpha=1}^{2}\sum_{y\in\boldsymbol{Y}}\left(\frac{\kappa'_{\alpha y}}{|\kappa_{\alpha y}|}|y-p_\alpha|-|\kappa_{\alpha y}|\frac{y-p_\alpha}{|y-p_\alpha|}\cdot v_\alpha\right).$$

Since $\eta'(0)$ is finite, $\eta(0)$ is nonzero and $p_\alpha(0)=p(=0)$,

$$m'(0)=2|\eta(0)|+\sum_{y\in\boldsymbol{Y}}\Big(\sum_{\alpha=1}^{2}\frac{\kappa'_{\alpha y}(0)}{|\kappa_{\alpha y}(0)|}|y|-\sum_{\alpha=1}^{2}|\kappa_{\alpha y}(0)|v_\alpha\cdot\frac{y}{|y|}\Big).$$

Since $|y|$ is now independent of α, substituting these three expressions using the identities found in (13), (14), and (15) yields

$$\begin{aligned}
m'(0)&=2\theta\Big|\sum_{y\in\boldsymbol{Y}}\frac{\kappa_{0y}(y\cdot e_1)^2}{2|y|y\cdot e_2}\Big|+\sum_{y\in\boldsymbol{Y}}\left(0|y|-\frac{\theta|\kappa_{0y}|y}{y\cdot e_2}\cdot\frac{y}{|y|}\right)\\
&\le\theta\sum_{y\in\boldsymbol{Y}}\frac{|\kappa_{0y}|(y\cdot e_1)^2}{|y|y\cdot e_2}-\frac{|\kappa_{0y}|\,|y|}{y\cdot e_2}=-\theta\sum_{y\in\boldsymbol{Y}}\frac{|\kappa_{0y}|}{|y|}y\cdot e_2<0.
\end{aligned}$$

The first inequality follows from the triangular inequality while the second equality uses the identity $(y\cdot e_1)^2-|y|^2=-(y\cdot e_2)^2$. The final inequality follows from the positivity of θ and the fact that the members of the truss meeting p lie in a strict cone as defined by (8). □

5. Existence and mass estimates

In the remainder of the section, $f = \sum_{i=1}^{k} f_i \delta_{a_i} \in \boldsymbol{F}_0$ for $f_i, a_i \in \mathbf{R}^n$. By rotation and translation, we assume without loss of generality that $a_k = 0$ and f_k is parallel to e_n. Here, $e_1, e_2, \ldots, e_n$ are the orthonormal basis vectors of $\mathbb{R}^n$. We will write $\boldsymbol{A} = \{a_1, a_2, \ldots, a_k\}$ to denote the points of application of the applied force f. We will use the natural identification of the hyperplanes of $\mathbb{R}^n$,

$$\{0\} \subset \mathbb{R} \subset \mathbb{R}^2 \subset \cdots \subset \mathbb{R}^n.$$

For $v \in \mathbb{R}^n$, we write $v = v^H + v^\perp$ where $v^H \in \mathbb{R}^{n-1}$ and $v^\perp$ is parallel to e_n. The following constructions were suggested by Wilfrid Gangbo (personal communication).

Lemma 5.1. *Assume either*

(i) $a_i, f_i \subset \mathbb{R}$ *for* $i = 1, \ldots, k$ *or*

(ii) $a_i \in \mathbb{R}^{n-1}$ *with* f_i *parallel to* e_n *for* $i = 1, \ldots, k$.

Then there exists a truss σ *with* $\operatorname{div} \sigma = f$ *and*

$$\operatorname{Supp}(\sigma) \subset \boldsymbol{B}_{\operatorname{Diam}(\boldsymbol{A})}(0), \tag{17}$$

$$\|\sigma\| \leq 2 \operatorname{Diam}(\boldsymbol{A}) \|(|x| + 1) f\|. \tag{18}$$

Proof. Assume (i). Then there are real numbers λ_i so that $f_i = -\lambda_i e_1$ and $a_i - 0$ is parallel to e_1 for $i = 1, \ldots, k$. Using (4), note that

$$\begin{aligned} f &= \sum_{i=1}^{k} f_i \delta_{a_i} - 0\delta_0 = \sum_{i=1}^{k} f_i (\delta_{a_i} - \delta_0) \\ &= -\sum_{i=1}^{k} \lambda_i e_1 (\delta_{a_i} - \delta_0) = -\sum_{i=1}^{k} \lambda_i \frac{a_i - 0}{|a_i - 0|} (\delta_{a_i} - \delta_0). \end{aligned}$$

Letting $x_i = a_i$ and $y_i = 0$, the above equation along with (6) shows $\sigma := \sum_{i=1}^{k} \omega_i \sigma_{x_i y_i}$ satisfies $\operatorname{div} \sigma = f$. Moreover,

$$\|\sigma\| = \sum_{i=1}^{k} |\lambda_i| \, |x_i - y_i| = \sum_{i=1}^{k} |f_i| \, |a_i| = \|(|x| f)\|$$

as claimed.

Now assume (ii). Let $x = (0, \ldots, 0, a)$ be a point on the line perpendicular to $\mathbb{R}^{n-1}$. Let $f_i = \phi_i e_n$. We decompose the force vectors as follows,

$$f_i = f_i^o + f_i^p := \left(f_i - \frac{\phi_i}{a} a_i \right) + \frac{\phi_i}{a} a_i, \qquad i = 1, \ldots, k.$$

Note that in this way, f_i^p is parallel with $a_i - a_k$ (we're assuming $a_k = 0$) and $f_i^o = \frac{\phi_i}{a}(x - a_i)$. We claim that the linear moment of f_i^o as well as the linear moment of f_i^p vanish. For the moment, assume this to be true. Then

$$\begin{aligned} f &= \sum_{i=1}^k (f_i^o + f_i^p)\delta_{a_i} = \Big(\sum_{i=1}^k f_i^o (\delta_{a_i} - \delta_x) \Big) + \Big(\sum_{i=1}^k f_i^p (\delta_{a_i} - \delta_0) \Big) \\ &= -\Big(\sum_{i=1}^k \frac{|x - a_i|\phi_i}{a} \frac{a_i - x}{|a_i - x|} (\delta_{a_i} - \delta_x) \Big) + \Big(\sum_{i=1}^k \frac{|a_i|\phi_i}{a} \frac{a_i - 0}{|a_i - 0|} (\delta_{a_i} - \delta_0) \Big). \end{aligned}$$

For $i = 1, \ldots, k$, letting

$$\lambda_i = \frac{|x - a_i|\phi_i}{a}, \quad x_i = a_i,\ y_i = x, \quad \tilde{\lambda}_i = -\frac{|a_i|\phi_i}{a}, \quad \tilde{x}_i = a_i,\ \tilde{y}_i = 0,$$

we see that $\sigma = \sum_{i=1}^k \lambda_i \sigma_{x_i y_i} + \sum_{i=1}^k \tilde{\lambda}_i \sigma_{\tilde{x}_i \tilde{y}_i}$ satisfies $\operatorname{div} \sigma = f$. Moreover,

$$\|\sigma\| = \sum_{i=1}^k |\lambda_i|\, |x_i - y_i| + |\tilde{\lambda}_i|\, |\tilde{x}_i - \tilde{y}_i| = \frac{1}{a} \sum_{i=1}^k |f_i|(|x - a_i|^2 + |a_i|^2).$$

Letting $a = \operatorname{Diam}(\boldsymbol{A})$, the triangular inequality gives (17) and (18).

It remains to be shown that the linear moment of f_i^o as well as the linear moment of f_i^p vanish. This is proved using the vanishing angular and linear moment of f. Let $v \in \mathbb{R}^{n-1}$ and define the skew symmetric matrix $\Omega = v \otimes e_n - e_n \otimes v \in \operatorname{Skw}(n)$. By (4),

$$0 = \sum_{i=1}^k \Omega a_i \cdot f_i = v \cdot \sum_{i=1}^k \phi_i a_i = av \cdot \sum_{i=1}^k f_i^p.$$

Furthermore, $e_n \cdot \sum_{i=1}^k f_i^p = 0$. Since v was arbitrary, this shows that $\sum_{i=1}^k f_i^p = 0$. Note that one consequence of (4) is the identity $\sum_{i=1}^k f_i = 0$. The claim now follows since two out three of the linear moments vanish. □

Lemma 5.2. *There exist equilibrated point forces* $h = \sum_{i=1}^{(n-1)k} h_i \delta_{y_i}$ *and* $g = \sum_{i=1}^{nk} g_i \delta_{z_i}$ *and a truss* τ *with*

(i) $h_i, y_i \in \mathbb{R}^{n-1}$,

(ii) $z_i \in \mathbb{R}^{n-1}$ *and* g_i *parallel to* e_n,

$f = \operatorname{div} \tau - h - g$ *and*

$$\operatorname{Diam}(\boldsymbol{Y}) \le \operatorname{Diam}(\boldsymbol{A}), \qquad \operatorname{Diam}(\mathbf{Z}) \le \operatorname{Diam}(\boldsymbol{A}), \tag{19}$$

$$\|(|x|+1)h\| \le 2\sqrt{n}\|(|x|+1)f\|, \tag{20}$$

$$\|(|x|+1)g\| \le 3\sqrt{n}\big(\operatorname{Diam}(\boldsymbol{A})+1\big)\|(|x|+1)f\|, \tag{21}$$

$$\operatorname{Supp}(\tau) \subset \boldsymbol{B}_{\operatorname{Diam}(\boldsymbol{A})}(a_k), \tag{22}$$

$$\|\tau\| \le 2\sqrt{n}\big(\operatorname{Diam}(\boldsymbol{A})+1\big)\|(|x|+1)f\|, \tag{23}$$

where $\boldsymbol{Y} = \{y_1, \ldots, y_{(n-1)k}\}$ *and* $\mathbf{Z} = \{z_1, \ldots, z_{nk}\}$.

Proof. For $i = 1, \ldots, k-1$, define

$$\begin{aligned} a_i &= a_i^\perp + a_i^H, \\ x_{i\alpha} &= a_i^H + e_\alpha, \qquad \alpha = 1, \ldots, n-1, \\ \tau_i &= \sum_{\alpha=1}^{n-1} f_i \cdot e_\alpha |x_{i\alpha} - a_i| \sigma_{x_{i\alpha} a_i} - \Big(\sum_{\alpha=1}^{n-1} f_i \cdot e_\alpha |a_i^\perp| + f_i \cdot e_n \Big) \sigma_{a_i a_i^H}. \end{aligned}$$

We may, without loss generality, assume $a_i^\perp \cdot e_n \ge 0$ and $a_i^H \cdot e_\alpha \le 0$ so that $|a_i^\perp| e_n = a_i^\perp$ and $|a_k - x_{i\alpha}| \le |a_i|$. Then, using (6),

$$\begin{aligned} \operatorname{div} \tau_i = (f_i \delta_{a_i}) &+ \Big(-\sum_{\alpha=1}^{n-1} f_i \cdot e_\alpha e_\alpha \delta_{x_{i\alpha}} \Big) \\ &+ \Big(\sum_{\alpha=1}^{n-1} f_i \cdot e_\alpha |a_i^\perp| e_n \delta_{x_{i\alpha}} - \Big[\sum_{\alpha=1}^{n-1} f_i \cdot e_\alpha |a_i^\perp| + f_i \cdot e_n \Big] e_n \delta_{a_i^H} \Big). \end{aligned}$$

Note that of the three parenthetical expressions in the last expression, the first is simply the force f applied to a_i, the second are horizontal forces with points of application in $\mathbb{R}^{n-1}$ while the third are perpendicular forces with points of application in $\mathbb{R}^{n-1}$. Accordingly we define

$$\tau = \sum_{i=1}^{k} \tau_i, \qquad (\tau_k = 0),$$

$$h = \sum_{i=1}^{k} \sum_{\alpha=1}^{n-1} f_i \cdot e_\alpha e_\alpha \delta_{x_{i\alpha}},$$

$$g = \sum_{i=1}^{k} \sum_{\alpha=1}^{n-1} f_i \cdot e_\alpha |a_i^\perp| e_n \delta_{x_{i\alpha}} - \Big[\sum_{\alpha=1}^{n-1} f_i \cdot e_\alpha |a_i^\perp| + f_i \cdot e_n \Big] e_n \delta_{a_i^H}.$$

The above shows that

$$f = \operatorname{div} \tau - h - g.$$

We claim g and h are equilibrated. Note that $\operatorname{div} \tau$ is equilibrated. By assumption, f is equilibrated. Therefore it suffices to only prove that h is equilibrated. Clearly, if $v \in \mathbb{R}^{n-1}$,

$$v \cdot \int_{\mathbb{R}^n} dh = \sum_{i=1}^{k} \sum_{\alpha=1}^{n-1} f_i \cdot e_\alpha v \cdot e_\alpha = \sum_{i=1}^{k} f_i \cdot v = 0.$$

Since h is a horizontal force, this shows that the linear moment of h vanishes. To prove the angular moment also vanishes, let $\Omega \in \operatorname{Skw}(n)$. Since h is horizontal, we may restrict ourselves to the cases when $\Omega e_n = 0$. Consider

$$\begin{aligned}
\int_{\mathbb{R}^n} \Omega x \cdot dh(x) &= \sum_{i=1}^{k} \sum_{\alpha=1}^{n-1} f_i \cdot e_\alpha \Omega x_{i\alpha} \cdot e_\alpha \\
&= \sum_{i=1}^{k} \sum_{\alpha=1}^{n-1} f_i \cdot e_\alpha \Omega e_\alpha \cdot e_\alpha + \sum_{i=1}^{k} \sum_{\alpha=1}^{n-1} f_i \cdot e_\alpha \Omega a_i^H \cdot e_\alpha \\
&= 0 + \sum_{i=1}^{k} \Omega a_i^h \cdot f_i^H = \sum_{i=1}^{k} \Omega a_i \cdot f_i = 0.
\end{aligned}$$

Note that the first sum vanished because Ω is skew symmetric while in the second sum adding the perpendicular component of f_i and a_i leaves the summands unchanged. This proves the claim.

Using the estimates $\sum_{i=1}^{n} |x \cdot e_i| \le \sqrt{n}|x|$ and $|x_{i\alpha} - a_i|^2 = 1 + |a_i^\perp|^2$, it is straightforward that

$$\|\tau\| \le \sum_{i=1}^{k} \sqrt{n}(2|a_i|^2 + |a_i| + 1)|f_i| \le 2\sqrt{n}\big(\operatorname{Diam}(A) + 1\big)\|(|x| + 1)f\|.$$

Writing $h = \sum_{i=1}^{(n-1)k} h_i \delta_{y_i}$ and $g = \sum_{i=1}^{nk} g_i \delta_{z_i}$,

$$\begin{aligned}\|(|x|+1)h\| &= \sum_{i=1}^{k}\sum_{\alpha=1}^{n-1}(|x_{i\alpha}|+1)|f_i \cdot e_\alpha| \le \sqrt{n}\sum_{i=1}^{k}(|a_i|+2)|f_i| \\ &\le 2\sqrt{n}\|(|x|+1)f\|.\end{aligned}$$

and

$$\begin{aligned}\|(|x|+1)g\| &= \sum_{i=1}^{k}\Big(\sum_{\alpha=1}^{n-1}|a_i^{\perp}|(|x_{i\alpha}|+|a_i^h|+2)|f_i\cdot e_\alpha|\Big)+(|a_i^h|+1)|f_i\cdot e_n| \\ &\le \sqrt{n}\sum_{i=1}^{k}(|a_i|+1)(2|a_i|+3)|f_i| \\ &\le 3\sqrt{n}\big(\mathrm{Diam}(\boldsymbol{A})+1\big)\|(|x|+1)f\|. \qquad \square\end{aligned}$$

With the help of Lemmas 5.1 and 5.2, we have the following result.

Proof of Theorem 3.3. The proof proceeds by induction. The inductive hypothesis reads as follows:

$$\text{If } h = \sum_{i=1}^{l} h_i\delta_{y_i} \in \boldsymbol{F}_0 \text{ with } h_i, y_i \in \mathbb{R}^m, \text{ then there exists } \sigma^H \in \boldsymbol{T}_0$$
$$\text{with } \operatorname{div}\sigma^H = h, \qquad \mathrm{Supp}(\sigma^H) \subset \bigcap_{q\in\boldsymbol{Y}} \boldsymbol{B}_{\mathrm{Diam}(\boldsymbol{Y})}(q),$$
$$\|\sigma^H\| \le K_m\big(\mathrm{Diam}(\boldsymbol{Y})+1\big)^2\|(|x|+1)h\|.$$

Here $\boldsymbol{Y} = \{y_1, \dots, y_l\}$. We conclude the case $m = 1$ from part 1 of Lemma 5.1. We assume the inductive hypothesis to hold true for $m = n-1$. Let $f = \sum_{i=1}^{k} f_i\delta_{a_i}$ be a equilibrated applied force with forces and points of application in $\mathbb{R}^n$. Choose $p \in \boldsymbol{A} = \{a_1, \dots, a_k\}$. Reordering if neccesary, we assume without loss of generality that $p = a_k$. Let

$$f = \operatorname{div}\tau - \tilde{h} - g \tag{24}$$

be the decomposition given by Lemma 5.2. The given g has points of application $\mathbf{Z}$ in $\mathbb{R}^{n-1}$ with forces parallel to e_n. Applying Lemma 5.1 to g, there is a truss $\sigma^{\perp}$ satisfying

$$\begin{aligned}&\operatorname{div}\sigma^{\perp} = g, \qquad \mathrm{Supp}(\sigma^{\perp}) \subset \boldsymbol{B}_{\mathrm{Diam}(\mathbf{Z})}(p), \\ &\|\sigma^{\perp}\| \le 2\,\mathrm{Diam}(\mathbf{Z})\|(|x|+1)g\|.\end{aligned} \tag{25}$$

The given $\tilde{h}$ has points of application $\tilde{\boldsymbol{Y}}$ in $\mathbb{R}^{n-1}$ with forces also in $\mathbb{R}^{n-1}$. Let $h := \tilde{h} + 0\delta_p$ and $\boldsymbol{Y} := \tilde{\boldsymbol{Y}} \cup \{p\}$. Note that from the construction in Lemma 5.1, $\mathrm{Diam}(\boldsymbol{Y}) = \mathrm{Diam}(\tilde{\boldsymbol{Y}})$. Applying the inductive hypothesis to h, there is a truss σ^H with

$$\begin{aligned} &\operatorname{div} \sigma^H = h, \qquad \mathrm{Supp}(\sigma^\perp) \subset \boldsymbol{B}_{\mathrm{Diam}(\boldsymbol{Y})}(p), \\ &\|\sigma^H\| \le K_{n-1}\big(\mathrm{Diam}(\boldsymbol{Y}) + 1\big)^2 \|(|x| + 1)h\|. \end{aligned} \tag{26}$$

Setting $\sigma = \sigma^\perp + \sigma^H - \tau$ and applying (24), we have shown that there exists a truss σ equilibrating f, namely $\operatorname{div} \sigma + f = 0$.

Using estimates (19–21) and (25),

$$\begin{aligned} \|\sigma^\perp\| &\le 6\sqrt{n}\big(\mathrm{Diam}(\boldsymbol{A}) + 1\big)^2 \|(|x| + 1)f\|, \\ \|\sigma^H\| &\le 2\sqrt{n}K_{n-1}\big(\mathrm{Diam}(\boldsymbol{A}) + 1\big)^2 \|(|x| + 1)f\|. \end{aligned}$$

In total, along with the estimate (23) for $\|\tau\|$,

$$\|\sigma\| \le (8\sqrt{n} + 2\sqrt{n}K_{n-1})\big(\mathrm{Diam}(\boldsymbol{A}) + 1\big)^2 \|(|x| + 1)f\|.$$

Furthermore,

$$\mathrm{Supp}(\sigma) \subset \boldsymbol{B}_{\mathrm{Diam}(\mathbf{Z})}(p) \cup \boldsymbol{B}_{\mathrm{Diam}(\boldsymbol{Y})}(p) \cup \boldsymbol{B}_{\mathrm{Diam}(\boldsymbol{A})}(p) \subset \boldsymbol{B}_{\mathrm{Diam}(\boldsymbol{A})}(p).$$

Since $p \in \boldsymbol{A}$ was arbitrary, we conclude the inductive hypothesis for the case $m = n$ as well. □

6. Conclusion

It has been shown that any finite, planar truss containing corners cannot be the cost-minimizing solution to the Michell Truss Problem, irrespective of the angle. A process is presented which always has a negative effect on the cost function of the parent truss. The method can potentially be applied to more complicated cost functions.

The corner cutting construction explicitly defines a topological perturbation of a truss and therefore does not rely on duality arguments in the optimization procedure. A generalization to dimensions three and higher seems plausible.

In Problem 5.2 of [4], it is conjectured that a minimizing truss is supported in a bounded region. Theorem 3.3 of the present work merely guarantees the existence of a minimizing sequence. As described in the introduction, the Michell bridge truss is not included in the convex hull of the applied force. In future work, the role of corner cutting in forming barriers as to define a priori bounds on the support of a minimizing sequence will be investigated.

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E. Figueroa, Department of Bioengineering, Rice University, Houston, TX 77030, USA
E-mail: lizfig@rice.edu

A. Hill, 4214 Southwestern Street, Houston, TX 77005, USA
E-mail: adam.r.hill@gmail.com

D. Iusco, Rice University, 6360 Main Street, Houston, TX 77005, USA
E-mail: denichii@gmail.com

R. Ryham, Department of Mathematics, Fordham University, Bronx, NY 10458, USA
E-mail: rryham@fordham.edu