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Periodic solutions to superlinear planar Hamiltonian systems

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Abstract. We prove the existence of infinitely many periodic (harmonic and subharmonic) solutions to planar Hamiltonian systems satisfying a suitable superlinearity condition at infinity. The proof relies on the Poincaré–Birkhoff fixed point theorem.

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1. Introduction

In this note, we consider the problem of the existence of periodic (harmonic and subharmonic) solutions to the planar Hamiltonian system

$$Jz' = \nabla_z H(t, z), \qquad z = (x, y) \in \mathbb{R}^2, \tag{1.1}$$

being $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ the standard symplectic matrix and $H : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ a (regular enough) function which is T-periodic in the first variable. We focus on the case in which the vector field $\nabla_z H(t, z)$ satisfies a superlinearity condition at infinity.

Superlinear planar problems have been widely investigated dealing with the scalar undamped second order equation

$$u'' + g(t, u) = 0, \qquad u \in \mathbb{R}, \tag{1.2}$$

with $g(t, x)$ satisfying the classical superlinearity condition at infinity, namely

$$\lim_{|x| \to +\infty} \frac{g(t, x)}{x} = +\infty, \qquad \text{uniformly in } t \in [0, T].$$

Classical results in this setting—[14], [15] for the "unforced" case $g(t, 0) \equiv 0$ (see Section 4) and [7], [8], [12], [19], [20] for the general one—guarantee the existence of infinitely many periodic, harmonic and subharmonic, solutions. The proofs are performed via some refined versions of the Poincaré–Birkhoff fixed point theorem and, as a consequence, the solutions found are accompanied by a sharp nodal characterization. We emphasize the fact that such a technique of proof, based on the Poincaré–Birkhoff theorem, provides a unifying setting to deal at the same time with harmonic and subharmonic solutions.

In the case of a general (superlinear) Hamiltonian system (1.1), far fewer results are available in literature. Due to the variational structure of the equation, critical point theory can be successfully applied (even in dimension greater than two) leading to the existence of infinitely many T-periodic solutions (we cite [1], [17] for the first works in this direction). Despite the great range of applicability from the point of view of the space dimension, however, results obtained via critical point techniques, when applied to planar problems, do not provide information about the nodal properties of the solutions and, as a major drawback, they usually do not consider (at least in a direct way) the problem of subharmonic solutions.

We finally remark that, when dealing with first order differential systems (and, of course, with an Hamiltonian system like (1.1)), there is not a standard definition of *superlinearity* (incidentally, notice also that when Hamiltonian systems are considered the term "superquadraticity" is sometimes preferred, referring to the assumption on $H(t, z)$). In particular, it is worth noticing that most of the superlinearity conditions considered in literature (like the Ambrosetti-Rabinowitz one (2.8), see Remark 2.4) require both the components of the vector field $J\nabla_z H(t, z)$ to be superlinear. As a consequence, results concerning superlinear Hamiltonian systems often do not apply to superlinear second order equations (1.2), which are written in Hamiltonian form as

$$x' = y, \qquad y' = -g(t, x).$$

Here, as our starting point, we take the paper [5], where, for a (possibly) non-Hamiltonian planar system, a suitable superlinearity condition is introduced and used in order to prove the existence of at least one T-periodic solution ([5], Theorem 4). Such a superlinearity condition, which is suggested by the use of some systems of modified polar coordinates in the plane, is fulfilled in the case of superlinear second order equations (see Remark 2.4). The proof of ([5], Theorem 4) exploits topological degree arguments and, as remarked in ([5], p. 389), the result is optimal from the point of view of the multiplicity, since in the general (non-Hamiltonian) case no more than one T-periodic solution can be expected.

The aim of this brief note is to show that, *in the Hamiltonian case* (1.1), the (essentially) same assumptions of [5] imply the existence of infinitely many kT-periodic solutions, for every integer $k \geq 1$. Information about the nodal properties of the solutions is provided, too. We refer to Section 2 for the precise statement of the main result, Theorem 2.3.

The proof (see Section 3) is performed via the Poincaré–Birkhoff fixed point theorem (in the version by W. Y. Ding [10]), similarly as in [8], [12], dealing with the second order case. In view of the previous discussion, hence, our Theorem 2.3 includes some classical results concerning superlinear scalar equations and extends them to a general planar Hamiltonian system (1.1).

In Section 4, we finally give some further remarks about the unforced case, namely $\nabla_z H(t,0) \equiv 0$.

Notation. If $z_1, z_2 \in \mathbb{R}^2$, we will write $\langle z_1 \,|\, z_2 \rangle$ to denote the Euclidean scalar product of z_1, z_2, and $|z_1|$ to denote the Euclidean norm in $\mathbb{R}^2$.

2. Preliminaries and statement of the main result

Throughout the paper, we will denote by $\mathscr{P}$ the class made up by the C^1-functions $V : \mathbb{R}^2 \to \mathbb{R}$ which are positive and positively homogeneous of degree 2, i.e. for every $\lambda > 0$ and $z \neq 0$,

$$0 < V(\lambda z) = \lambda^2 V(z).$$

For $V \in \mathscr{P}$, we set

$$A_V := \int_{\{V(x,y) \le 1\}} dx\, dy.$$

Such a class of functions appears in the formulation of the superlinearity condition, for $\nabla_z H(t,z)$, at infinity (see assumption (H_2) of Theorem 2.3 below).

Moreover, it allows the definition of suitable systems of deformed polar coordinates and of a modified rotation number, which plays an essential role in our proof. We recall here the precise definition, as given in ([23], p. 17).

Definition 2.1. Let $V \in \mathscr{P}$ and let $z : [t_1, t_2] \to \mathbb{R}^2$ be an absolutely continuous path, with $z(t) \neq 0$ for every $t \in [t_1, t_2]$. The V-modified rotation number of $z(t)$ is defined as

$$\mathrm{Rot}_V\big(z(t); [t_1, t_2]\big) := \frac{1}{2A_V} \int_{t_1}^{t_2} \frac{\langle Jz'(t) \,|\, z(t) \rangle}{V\big(z(t)\big)}\, dt.$$

It is clear that the standard (clockwise) rotation number (that is, the normalized clockwise angular displacement of the curve $z(t)$ around the origin, in the time interval $[t_1, t_2]$) corresponds to the choice $V(z) = |z|^2$, and it will be simply denoted by Rot.

We remark that such a modified rotation number is implicitly used already in [5], to count the number of revolutions of some closed paths. In this paper, we will estimate modified rotation numbers of non-closed paths also, so that we need the following crucial extra property, which is proved in ([2], Proposition 2.1) as a consequence of some ideas developed in [23].

Proposition 2.2. *Let $V \in \mathscr{P}$ and let $z : [t_1, t_2] \to \mathbb{R}^2$ be an absolutely continuous path, with $z(t) \neq 0$ for every $t \in [t_1, t_2]$. Then, for every $j \in \mathbb{Z}$,*

$$\mathrm{Rot}_V\big(z(t); [t_1, t_2]\big) > j \iff \mathrm{Rot}\big(z(t); [t_1, t_2]\big) > j$$
$$\mathrm{Rot}_V\big(z(t); [t_1, t_2]\big) < j \iff \mathrm{Rot}\big(z(t); [t_1, t_2]\big) < j.$$

Roughly speaking, despite the fact that, in general, the values (on the same path) of Rot_V and Rot are different one from the other, Rot_V counts the same number of *complete* clockwise turns around the origin as Rot.

We are now ready to state our main result, concerning the existence of harmonic and subharmonic solutions to the planar Hamiltonian system

$$Jz' = \nabla_z H(t, z) \qquad z = (x, y) \in \mathbb{R}^2. \tag{2.1}$$

We will always assume that $H : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is T-periodic in the first variable (with $T > 0$ fixed) and differentiable in the second one, with $\nabla_z H(t, z)$ an L^1-Carathéodory function, that is to say, $\nabla_z H(t, \cdot)$ is continuous for a.e. $t \in [0, T]$, $\nabla_z H(\cdot, z)$ is measurable for every $z \in \mathbb{R}^2$ and for every $r > 0$ there exists $\zeta_r \in L^1(0, T)$ such that $|\nabla_z H(t, z)| \le \zeta_r(t)$ for a.e. $t \in [0, T]$ and for every $z \in \mathbb{R}^2$ with $|z| \le r$.

Theorem 2.3. *Assume that the uniqueness for the solutions to the Cauchy problems associated with* (2.1) *is guaranteed. Moreover, suppose that:*

(H_1) *there exists a C^1-function $K : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$, T-periodic in the first variable, with $K(t, z) > 0$ for every $t \in \mathbb{R}$ and $z \in \mathbb{R}^2$ and such that*

$$\lim_{|z| \to +\infty} K(t, z) = +\infty, \qquad \text{uniformly in } t \in [0, T],$$

satisfying the following condition: for every $t_1, t_2 \in \mathbb{R}$ and for every $z :]t_1, t_2[\to \mathbb{R}^2$ solving (2.1)*, there exists $c \in L^1(t_1, t_2)$ such that, for a.e. $t \in]t_1, t_2[$,*

$$\left| \frac{d}{dt} K\big(t, z(t)\big) \right| \le c(t) K\big(t, z(t)\big); \tag{2.2}$$

(H_2) *there exist sequences* $(V_n)_n \subset \mathscr{P}$, $(a_n)_n \subset L^1(0,T)$ *such that, for every* $n \in \mathbb{N}_0$,

$$\liminf_{|z|\to+\infty} \frac{\langle \nabla_z H(t,z) \,|\, z\rangle}{V_n(z)} \geq a_n(t), \qquad \textit{uniformly for a.e. } t \in [0,T], \tag{2.3}$$

and

$$\lim_{n\to+\infty} \frac{\int_0^T a_n(t)\,dt}{A_{V_n}} = +\infty. \tag{2.4}$$

Then, for every integer $k \geq 1$, *equation* (2.1) *has infinitely many* kT*-periodic solutions. More precisely, for every integer* $k \geq 1$, *there exists an integer* j_k^* *such that, for every integer* $j \geq j_k^*$, *equation* (2.1) *has two* kT*-periodic solutions* $z_{k,j}^{(1)}(t)$, $z_{k,j}^{(2)}(t)$ *such that, for* $i = 1, 2$,

$$\operatorname{Rot}\big(z_{k,j}^{(i)}(t); [0,kT]\big) = j. \tag{2.5}$$

Moreover, for every $k \geq 1$ *and* $i = 1, 2$,

$$\lim_{j\to+\infty} |z_{k,j}^{(i)}(t)| = +\infty, \qquad \textit{uniformly in } t \in [0,T]. \tag{2.6}$$

Initial values, at time $t = 0$, of such kT-periodic solutions will be provided as fixed points of the k-th iterate of the Poincaré map Ψ associated with (2.1) (see Section 3 for the details); formula (2.5), of course, has to be meant as an information about the nodal properties of the periodic solutions found.

For $k = 1$, Theorem 2.3 in particular gives the existence of infinitely many T-periodic solutions (i.e., *harmonic solutions*) to the planar Hamiltonian system (2.1). On the other hand, when $k > 1$, it is easy to see that (see, for instance, [9], pp. 523–524), whenever k, j are relatively prime integers (namely, their greatest common divisor is 1), then the kT-periodic solutions $z_{k,j}^{(1)}(t)$, $z_{k,j}^{(2)}(t)$ are not lT-periodic for any integer $l = 1, \dots, k-1$. In this case, $z_{k,j}^{(1)}(t)$, $z_{k,j}^{(2)}(t)$ are said to be *subharmonic solutions of order* k to (2.1) and they correspond to fixed points of Ψ^k which are not fixed points of Ψ^l for any $l = 1, \dots, k-1$. Notice that, if we restrict ourselves to the pairs (k, j) with k, j relatively prime, the periodic solutions provided by Theorem 2.3 are pairwise distinct.

We finally remark that, as pointed out in the proof of ([22], Theorem 5), it is possible to show that the subharmonic solutions $z_{k,j}^{(1)}(t)$, $z_{k,j}^{(2)}(t)$ do not belong to the same periodicity class, i.e. $z_{k,j}^{(1)}(\cdot) \not\equiv z_{k,j}^{(2)}(\cdot + lT)$ for every integer $l = 1, \dots, k-1$. This corresponds to the fact that the orbits of $z_1 := z_{k,j}^{(1)}(0)$ and $z_2 := z_{k,j}^{(2)}(0)$, namely $\mathcal{O}_1 := \{z_1, \Psi(z_1), \dots, \Psi^{k-1}(z_2)\}$ and $\mathcal{O}_2 := \{z_2, \Psi(z_2), \dots, \Psi^{k-1}(z_2)\}$, are disjoint.

We now make some comments about the assumptions and the range of applicability of Theorem 2.3. When we refer to the scalar undamped second order equation

$$u'' + g(t, u) = 0, \qquad u \in \mathbb{R}, \tag{2.7}$$

with $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a function which is T-periodic in the first variable, we will always tacitly mean that it is written as the equivalent planar Hamiltonian system (2.1), with the position $H(t, x, y) = \frac{1}{2}y^2 + \int_0^x g(t, \xi)\, d\xi$.

Remark 2.4. Assumption (H_2) is a superlinearity condition for $\nabla_z H(t, z)$ at infinity and, as anticipated in the Introduction, it is (essentially) the same condition of ([5], Theorem 4). We also notice that the dual version of it (i.e., a sublinearity condition) has been recently used, together with the Poincaré–Birkhoff fixed point theorem, in ([2], Theorem 3.1).

For what concerns the range of applicability of (H_2), here we just limit ourselves to emphasize that it is more general than the Ambrosetti-Rabinowitz condition ($0 < k < 1/2$, $R \geq 0$)

$$0 < H(t, z) \leq k\langle \nabla_z H(t, z) \,|\, z \rangle, \qquad \text{for every } t \in [0, T],\ |z| \geq R, \tag{2.8}$$

which is often employed when system (2.1) is treated with variational techniques (see for instance [1], [17]). Indeed, from (2.8) we get that $H(t, z) \geq a|z|^{1/k}$ for every $t \in [0, T]$, $|z| \geq R$ (with $a > 0$), so that (H_2) is satisfied with $V_n(z) = |z|^2$ and $a_n(t) \equiv n$. It is also worth mentioning that from the proof of Theorem 2.3 it will be clear that condition (2.3) of (H_2) can be weakened to hold in an L^1 sense. Precisely, we can require the following:

for every $\varepsilon > 0$, there exist $R_\varepsilon > 0$ and $\eta_\varepsilon \in L^1(0, T)$, with $\int_0^T |\eta_\varepsilon(t)|\, dt \leq \varepsilon$, such that, for almost every $t \in [0, T]$, and every $|z| \geq R_\varepsilon$,

$$\frac{\langle \nabla_z H(t, z) \,|\, z \rangle}{V_n(z)} \geq a_n(t) - \eta_\varepsilon(t).$$

In particular, if (2.1) comes from the second order equation (2.7), then (this weaker version of) (H_2) is satisfied whenever $g(t, x)$ is L^1-Carathéodory and fulfills the standard superlinearity condition, namely

$$\lim_{|x| \to +\infty} \frac{g(t, x)}{x} = +\infty, \qquad \text{uniformly for a.e. } t \in [0, T]. \tag{2.9}$$

Indeed, in this case it is easy to see that, for every integer n, there exists $r_n \in L^1(0, T)$ such that, for every $z = (x, y) \in \mathbb{R}^2$

$$\langle \nabla_z H(t, z) \,|\, z \rangle \geq nx^2 + y^2 - r_n(t),$$

so that (H_2) holds true with the choices $V_n(x, y) = nx^2 + y^2$ and $a_n(t) \equiv 1$. We refer to ([5], Example 4) for a similar computation. Notice that, when $g(t, x)$ is continuous, then $r_n(t) \equiv r_n$ so that (H_2) suffices.

Remark 2.5. We now turn our attention to assumption (H_1). First of all, observe that, since $K(t, z)$ is of class C^1 and $z :]t_1, t_2[\to \mathbb{R}^2$ is (locally) absolutely continuous, then $K(t, z(t))$ is (locally) absolutely continuous too, with, for a.e. $t \in]t_1, t_2[$,

$$\frac{d}{dt}K\big(t, z(t)\big) = \frac{\partial}{\partial t}K\big(t, z(t)\big) + \big\langle \nabla_z K\big(t, z(t)\big) \,|\, z'(t)\big\rangle$$

Condition (H_1) is used here to get the global continuability for the solutions to (2.1). Indeed, it is well known that, when $\nabla_z H(t, z)$ satisfies the superlinearity condition (H_2), the global continuability can fail, even when (2.1) comes from the second order equation (2.7) (see [6]). On the other hand, it is worth noticing that when, roughly speaking, $H(t, z) \to +\infty$ for $|z| \to +\infty$, then (H_1) is often satisfied. In particular, we emphasize the following special cases.

1) If $H(t, z)$ is regular and, uniformly in $t \in [0, T]$,

$$\lim_{|z| \to +\infty} H(t, z) = +\infty, \tag{2.10}$$

then assumption (H_1) is satisfied, provided that there exists $c, M > 0$ such that, for every $t \in \mathbb{R}$ and $|z| \geq M$,

$$\left|\frac{\partial}{\partial t}H(t, z)\right| \leq cH(t, z). \tag{2.11}$$

In this case, the natural choice is $K(t, z) = H(t, z) + d$, for some positive constant d large enough. Indeed, by (2.10) we have that $H(t, z)$ is bounded from below, so that $K(t, z) > 0$ if d is suitably chosen. As a consequence, by possibly enlarging the constant c appearing in (2.11), we have that for every $t \in \mathbb{R}$ and $z \in \mathbb{R}^2$,

$$\left|\frac{\partial}{\partial t}K(t, z)\right| \leq cK(t, z).$$

Hence, for every $z :]t_1, t_2[\to \mathbb{R}^2$ solving (2.1), we have that

$$\begin{aligned}\left|\frac{d}{dt}K\big(t, z(t)\big)\right| &= \left|\frac{\partial}{\partial t}H\big(t, z(t)\big) + \big\langle \nabla_z H\big(t, z(t)\big) \,|\, z'(t)\big\rangle\right| \\ &= \left|\frac{\partial}{\partial t}K\big(t, z(t)\big) + \langle Jz'(t) \,|\, z'(t)\rangle\right| \\ &= \left|\frac{\partial}{\partial t}K\big(t, z(t)\big)\right|,\end{aligned}$$

so that (H_1) is satisfied.

For the second order equation (2.7), with $\int_0^x g(t,\xi)\,d\xi \to +\infty$ for $|x| \to +\infty$, such a choice leads (modulo regularity assumptions) to

$$\int_0^x \frac{\partial}{\partial t} g(t,\xi)\,d\xi \le c \int_0^x g(t,\xi)\,d\xi, \qquad \text{for every } |x| \ge M. \tag{2.12}$$

Hence, Theorem 2.3 applies to the superlinear second order equation (2.7) (i.e., with $g(t,x)$ satisfying (2.9)), provided that (2.12) is fulfilled, including ([15], Theorem $1'$). We remark that is seems to be an open problem to prove the same result for (2.7) with $g(t,x)$ satisfying only the superlinearity assumption (2.9).

2) If $\nabla_z H(t,z) = \nabla S(z) + f(t)$ for a C^1-function $S : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\lim_{|z| \to +\infty} S(z) = +\infty,$$

then (H_1) is satisfied provided that there exists $c \in L^1(0,T)$ and $M > 0$ such that, for every $t \in \mathbb{R}$ and $|z| \ge M$

$$|\langle f(t) \,|\, J\nabla S(z) \rangle| \le c(t) S(z). \tag{2.13}$$

In such a case the choice is given by $K(t,z) = S(z) + d$, for $d \ge 0$ large enough and the proof of this fact goes similarly as before. Indeed, this is precisely the assumption required in ([5], Theorem 4) in order to ensure the validity of the elastic property (see condition (E) in the proof of Theorem 2.3).

We finally observe that the situation considered here includes as a special case the forced Duffing equation

$$u'' + h(u) = p(t), \tag{2.14}$$

with $\int_0^x h(\xi)\,d\xi \to +\infty$ for $|x| \to +\infty$. Indeed, with the choice $S(z) = \frac{1}{2}y^2 + \int_0^x h(\xi)\,d\xi$ and $f(t) = \big(p(t), 0\big)$, relation (2.13) is easily seen to be satisfied (we refer to [8], p. 334 for a similar computation). Hence, Theorem 2.3 applies to the forced superlinear Duffing equation (2.14) (i.e., with $h(x)/x \to +\infty$ for $|x| \to +\infty$), including [7]. It has to be noticed, however, that in such a case a more general result, replacing the superlinearity condition on $h(x)$ with a weaker assumption on the time map of the autonomous equation $u'' + h(u) = 0$, has been proved in [8].

3. Proof of the main result

As a first step, we show that assumption (H_1) implies the global continuability for the solutions to the differential system (2.1). To this aim, let $z(t)$ be a solution of

(2.1) and let $J \subset \mathbb{R}$ be its maximal interval of definition. We are going to show that $\sup J = +\infty$; for the backward continuability, the argument is the same. Suppose by contradiction that $\sup J < +\infty$ and choose $t^* \in J$; in view of (2.2), Gronwall's lemma implies that, for every $t \in J \cap]t^*, +\infty[$,

$$K(t, z(t)) \le K(t^*, z(t^*)) e^{\int_{t^*}^{t} c(s)\, ds}.$$

We deduce that $K(t, z(t))$ is bounded in any (left) neighborhood of $\sup J$; since $K(t, z) \to +\infty$ for $|z| \to +\infty$, uniformly in $t \in [0, T]$, $|z(t)|$ turns out to be bounded too, a contradiction.

As a consequence, we can define the Poincaré map Ψ associated with the differential equation (2.1), that is,

$$\Psi : \mathbb{R}^2 \ni \bar{z} \mapsto z(T; \bar{z}),$$

where $z(\cdot; \bar{z})$ means the solution to (2.1) such that $z(0; \bar{z}) = \bar{z}$. It is well known that, in view of the T-periodicity of $\nabla_z H(\cdot, z)$, (initial values, at time $t = 0$, of) kT-periodic solutions to (2.1) (for every integer $k \ge 1$) correspond to fixed points of Ψ^k, the k-th iterate of Ψ. To find fixed points of Ψ^k, we will use a suitable version of the Poincaré–Birkhoff fixed point theorem, following closely the argument in [12]. Indeed, we have that Ψ^k is a global homeomorphism of the plane onto itself; moreover, in view of the Hamiltonian structure of (2.1), Liouville's theorem implies that Ψ^k is area preserving. We thus have to show that a "twist condition" is satisfied for a suitable annulus.

From now on, $k \ge 1$ is a fixed integer. As a preliminary step for our argument, we recall that the global continuability for the solutions to (2.1) implies the following "elastic property" (see for instance [12], Proposition 3.2):

(E) *for every* $t_1 \ge 0$ *and for every* $M_1 > 0$, *there exists* $M_2 > M_1$ *such that* $|z(t; \bar{z})| \ge M_1$ *for every* $t \in [0, t_1]$ *and for every* $|\bar{z}| \ge M_2$.

According to (E), fix a constant $r > 0$ such that, for $|\bar{z}| \ge r$, then $z(t; \bar{z}) \ne 0$ for every $t \in [0, kT]$ and define j_k^* as the smallest integer such that

$$\operatorname{Rot}(z(t; \bar{z}); [0, kT]) < j_k^*, \qquad \text{for every } |\bar{z}| = r \tag{3.1}$$

(in view of the continuous dependence of the solutions to (2.1) from the initial conditions, $\operatorname{Rot}(z(t; \cdot); [0, kT])$ is a continuous function).

Fix now an integer $j \ge j_k^*$: we claim that there exists $R > r$ such that

$$\operatorname{Rot}(z(t; \bar{z}); [0, kT]) > j, \qquad \text{for every } |\bar{z}| = R. \tag{3.2}$$

Indeed, in view of (2.4), let $n \in \mathbb{N}_0$ and $\eta_n > 0$ be such that

$$\frac{k}{2A_{V_n}}\Big(\int_0^T a_n(t)\,dt - \eta_n T\Big) > j.$$

By condition (2.3), there exists $\tilde{R} \geq r$ such that, for a.e. $t \in [0, T]$ and $|z| \geq \tilde{R}$,

$$\frac{\langle \nabla_z H(t,z)\,|\,z\rangle}{V_n(z)} \geq a_n(t) - \eta_n.$$

Let us choose (using (E) again) $R > \tilde{R}$ such that, for $|\bar{z}| = R$, then $|z(t;\bar{z})| \geq \tilde{R}$ for every $t \in [0, kT]$; then

$$\begin{aligned}\mathrm{Rot}_{V_n}\big(z(t;\bar{z});[0,kT]\big) &= \frac{1}{2A_{V_n}}\int_0^{kT} \frac{\big\langle \nabla_z H\big(t, z(t;\bar{z})\big)\,|\,z(t;\bar{z})\big\rangle}{V_n\big(z(t;\bar{z})\big)}\,dt \\ &\geq \frac{k}{2A_{V_n}}\int_0^T a_n(t)\,dt - \frac{\eta_n kT}{2A_{V_n}} > j.\end{aligned}$$

In view of Proposition 2.2, we have that (3.2) holds true, as claimed.

Taking into account (3.1) and (3.2), the existence of two kT-periodic solutions $z^{(1)}_{k,j}(t)$, $z^{(2)}_{k,j}(t)$ to (2.1) satisfying (2.5) follows from the Poincaré–Birkhoff fixed point theorem, in the version given in [10] (see Remark 3.1). Finally, (2.5) and the continuity of the rotation number imply that $\lim_{j\to+\infty}|z^{(i)}_{k,j}(0)| = +\infty$ (for $i = 1, 2$) so that, using again (E), relation (2.6) holds true.

Remark 3.1. It is worth noticing that we are referring, here, to the paper [10], which establishes a version of the Poincaré–Birkhoff theorem for a standard annulus, and not to the most known [11], dealing with some topological annuli also. Indeed, such a more general version of the Poincaré–Birkhoff theorem hides, probably, some difficulties, in view of the recent counterexamples given in [16] (see [13] for more information about this point). The result for a standard annulus, on the other hand, seems to be well established and we refer also to the paper [21], where an independent proof is given (see in particular [21], Corollaries 2 and 3 for a detailed description of the application of the Poincaré–Birkhoff theorem to planar Hamiltonian systems).

4. Further remarks for the unforced case

We conclude the paper with some remarks dealing with the "unforced case", namely when

$$\nabla_z H(t,0) \equiv 0. \tag{4.1}$$

For the second order equation (2.7), such a situation has been considered first by Hartman [14] and Jacobowitz [15], by the use of suitable versions of the Poincaré–Birkhoff theorem again.

As it is clear, condition (4.1) is not in contradiction with (H_1) and (H_2), so that Theorem 2.3 can still be applied, giving the existence of infinitely many kT-periodic solutions for every $k \geq 1$. On the other hand, in the unforced case, typical results are obtained by exploiting some kind of gap between the behavior of the nonlinearity at zero and at infinity. In particular, when the Poincaré–Birkhoff fixed point theorem is employed, one can choose the inner boundary of the annulus (that is, $r > 0$ in formula (3.1)) in order to reflect the behavior of small-norm solutions, leading to a more careful estimate of j_k^*. For instance, we can state the following:

assume that there exists $b \in L^1(0, T)$ *such that*

$$\limsup_{|z| \to 0} \frac{\langle \nabla_z H(t,z) \,|\, z \rangle}{|z|^2} \leq b(t), \qquad \textit{uniformly for a.e. } t \in [0, T]; \tag{4.2}$$

then $j_k^* \leq \mathscr{E}^+\big(k/(2\pi) \int_0^T b(t)\, dt\big)$, *where, for* $a \in \mathbb{R}$, *by* $\mathscr{E}^+(a)$ *we mean the least integer strictly greater then* a.

Notice that (4.2) is satisfied if $H(t,z)$ is twice differentiable in the second variable, with $D_z^2 H(t,z)$ an L^1-Caratheodory function. Other estimates for j_k^* could be given in terms of the Maslov index of the linear system $Jz' = D_z^2 H(t,0) z$, on the lines of [18].

Remark 4.1. It is worth noticing that, in order to satisfy (H_1), the second possibility described in Remark 2.5 is now no longer possible (unless $f(t) \equiv 0$, i.e. the problem is autonomous). Hence, the more natural condition of global continuability comes here from (2.10) and (2.11). Observe in particular that, in the special case of a scalar second order equation with weight like

$$u'' + q(t) f(u) = 0, \tag{4.3}$$

being $f(x)x > 0$ for every $x \neq 0$, assumption (H_1) is always satisfied whenever $q(t)$ is of class C^1 and $q(t) > 0$ for every $t \in \mathbb{R}$. Indeed, with the usual position $H(t,x,y) = \frac{1}{2} y^2 + q(t) F(x)$, being $F(x) = \int_0^x f(\xi)\, d\xi$, we have, for a suitable constant $c > 0$,

$$\left| \frac{\partial}{\partial t} H(t,z) \right| = |q'(t) F(x)| \leq c q(t) F(x) \leq c H(t,z).$$

(As shown in [6], $q(t)$ locally of bounded variation turns out to be sufficient for the global continuability of the solutions to (4.3).)

On the other hand, it has to be remarked that the hypothesis of global continuability is avoided in [14] using a priori-bounds techniques and a truncation argument, which are possible in view of (4.1). We do not know if this is the case for a general planar Hamiltonian system, without further conditions.

Remark 4.2. For a planar Hamiltonian system satisfying (4.1) the existence of infinitely many Dirichlet solutions (i.e., $x(0) = x(T) = 0$), with the same nodal characterization of Theorem 2.3, has been recently proved in [4], assuming (H_1) of Theorem 2.3 (precisely, (2.10) and (2.11) in Remark 2.5) and $D_z^2 H(t,z)$ positive definite, with

$$\lim_{|z|\to+\infty} \mu_{\min}\big(D_z^2 H(t,z)\big) = +\infty, \qquad \text{uniformly in } t \in [0,T].$$

Here by $\mu_{\min}(A)$ we mean the least (real) eigenvalue of a (real) symmetric matrix A. It is easily seen that such a condition implies (H_2) of Theorem 2.3; indeed,

$$\begin{aligned}\langle \nabla_z H(t,z) \,|\, z\rangle &= \Big\langle \Big(\int_0^1 D_z^2 H(t,sz)\, ds \Big) z \,\Big|\, z \Big\rangle \\ &\geq \mu_{\min}\Big(\int_0^1 D_z^2 H(t,sz)\, ds \Big) |z|^2,\end{aligned}$$

so that (H_2) holds true with $V_n(z) = |z|^2$ and $a_n(t) \equiv n$.

Actually, it is possible to see that ([4], Theorem 2.1) holds true also assuming the weaker condition (H_2). On the other hand, the proof in [4], which is based on global bifurcation techniques, works also for some classes of non-Hamiltonian planar systems (see [4], Remark 2.4 as well as the generalization to planar Dirac-type system given in [3]).

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